

Tropical linear algebra with the Łukasiewicz T-norm

Gavalec, Martin; Němcová, Zuzana; Sergeev, Sergey

DOI:

[10.1016/j.fss.2014.11.008](https://doi.org/10.1016/j.fss.2014.11.008)

License:

Creative Commons: Attribution (CC BY)

Document Version

Publisher's PDF, also known as Version of record

Citation for published version (Harvard):

Gavalec, M, Němcová, Z & Sergeev, S 2015, 'Tropical linear algebra with the Łukasiewicz T-norm', *Fuzzy Sets and Systems*, vol. 276, pp. 131-148. <https://doi.org/10.1016/j.fss.2014.11.008>

[Link to publication on Research at Birmingham portal](#)

Publisher Rights Statement:

Eligibility for repository : checked 14/01/2015

General rights

Unless a licence is specified above, all rights (including copyright and moral rights) in this document are retained by the authors and/or the copyright holders. The express permission of the copyright holder must be obtained for any use of this material other than for purposes permitted by law.

- Users may freely distribute the URL that is used to identify this publication.
- Users may download and/or print one copy of the publication from the University of Birmingham research portal for the purpose of private study or non-commercial research.
- User may use extracts from the document in line with the concept of 'fair dealing' under the Copyright, Designs and Patents Act 1988 (?)
- Users may not further distribute the material nor use it for the purposes of commercial gain.

Where a licence is displayed above, please note the terms and conditions of the licence govern your use of this document.

When citing, please reference the published version.

Take down policy

While the University of Birmingham exercises care and attention in making items available there are rare occasions when an item has been uploaded in error or has been deemed to be commercially or otherwise sensitive.

If you believe that this is the case for this document, please contact UBIRA@lists.bham.ac.uk providing details and we will remove access to the work immediately and investigate.

Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems ●●● (●●●●) ●●●—●●●

FUZZY
sets and systemswww.elsevier.com/locate/fss

Tropical linear algebra with the Łukasiewicz T-norm

Martin Gavalec^{a,1}, Zuzana Němcová^{a,1}, Sergej Sergeev^{b,*,2}^a University of Hradec Králové, Faculty of Informatics and Management, Rokytanského 62, Hradec Králové 3, 500 03, Czech Republic^b University of Birmingham, School of Mathematics, Edgbaston B15 2TT, United Kingdom

Received 29 June 2013; received in revised form 27 August 2014; accepted 7 November 2014

Abstract

The max-Łukasiewicz semiring is defined as the unit interval $[0, 1]$ equipped with the arithmetics “ $a + b$ ” = $\max(a, b)$ and “ ab ” = $\max(0, a + b - 1)$. Linear algebra over this semiring can be developed in the usual way. We observe that any problem of the max-Łukasiewicz linear algebra can be equivalently formulated as a problem of the tropical (max-plus) linear algebra. Based on this equivalence, we develop a theory of the matrix powers and the eigenproblem over the max-Łukasiewicz semiring.

© 2014 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license

<http://creativecommons.org/licenses/by/3.0/>.

Keywords: Tropical; Max-plus; Łukasiewicz; Eigenvector; Matrix power

1. Introduction

The max-Łukasiewicz semiring is defined over the interval $[0, 1]$ equipped with the operations of addition $a \oplus b = \max(a, b)$ and multiplication $a \otimes_L b = \max(0, a + b - 1)$. These operations are extended to matrices and vectors in the usual way: $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$ and $(A \otimes_L B)_{ik} = \bigoplus_j a_{ij} \otimes_L b_{jk}$. We consider the max-Łukasiewicz powers

of matrices $A^{\otimes_L k} := \overbrace{A \otimes_L \cdots \otimes_L A}^k$, and the spectral problem over max-Łukasiewicz semiring: given $\lambda \in [0, 1]$, find $x \in [0, 1]^n$ with not all components 0 such that $A \otimes_L x = \lambda \otimes_L x$.

Our study of max-Łukasiewicz semiring is motivated by the recent success of tropical linear algebra, developed over the max-plus semiring $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ equipped with operations of “addition” $a \oplus b = \max(a, b)$ and “multiplication” $a \otimes b = a + b$. We convert the problems of max-Łukasiewicz linear algebra, i.e., the linear algebra over max-Łukasiewicz semiring, to the problems of tropical (max-plus) linear algebra. Then we can take advantage of the well-developed theory and algorithms of the latter, see [2,4,5,8,14,24,25], to mention only a few possible sources. Our main ingredients are the theory of spectral problem $A \otimes x = \lambda \otimes x$, Bellman equations or Z-equations $x = A \otimes x \oplus b$,

* Corresponding author.

E-mail addresses: martin.gavalec@uhk.cz (M. Gavalec), zuzana.nemcova@uhk.cz (Z. Němcová), sergiej@gmail.com (S. Sergeev).¹ Supported by the Czech Science Foundation grant #14-02424S and the Grant Agency of Excellence UHK FIM #2204.² Supported by EPSRC grant EP/J00829X/1, RFBR grant 12-01-00886 and RFBR–CNRS grant 11-0193106. This research was initiated when the third author was with INRIA and CMAP École Polytechnique, 91128 Palaiseau Cedex, France.<http://dx.doi.org/10.1016/j.fss.2014.11.008>

0165-0114/© 2014 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license

<http://creativecommons.org/licenses/by/3.0/>.

one-sided systems $A \otimes x = b$ and, occasionally, the tropical linear programming. Recall that the basic theory of one-sided systems $A \otimes x = b$ involves residuation theory [15] as well as set coverings [5]. Also see Rashid et al. [36] for a preliminary work on the max-Łukasiewicz linear algebra exploring details of the three-dimensional case.

A basic idea behind this paper is that each fuzzy triangular norm (t-norm) as described, for instance, in Klement, Mesiar and Pap [31], leads to an idempotent semiring, which we call a max-t semiring. There are directions of abstract fuzzy sets theory which are related to the present work. Some connections of multivalued logic and fuzzy algebra with idempotent mathematics have been developed by Di Nola et al. [18–21]. These works develop certain aspects of algebra over semirings arising from fuzzy logic (MV-algebras, Łukasiewicz transform), which currently seem most interesting and useful for the fuzzy sets theory.

However, neither general MV-algebras [19] nor even the special case of Łukasiewicz MV-algebra are considered in this paper. We are rather motivated by the basic problems of the tropical linear algebra, which we are going to consider here in the context of max-Łukasiewicz semiring. We also remark that max-Łukasiewicz semiring can be seen as a special case of incline algebras of Cao, Kim and Roush [9], see also, e.g., Han–Li [28] and Tan [40]. One of the main problems considered in that algebra is to study the periodicity of matrix powers over a larger class of semirings, using lattice theory, lattice ordered groups and residuations. Let us also recall the distributive lattices as another special case of incline algebras, although this special case does not include the max-Łukasiewicz semiring. Powers of matrices over distributive lattices are studied, e.g., by Cechlárová [11].

The approach which we develop here, does not apply to incline algebras in general (or to distributive lattices in particular), but it allows to study the linear-algebraic problems over max-Łukasiewicz algebra in much more detail.

Aggressive network. Let us also recall the basic network motivation to study tropical (max-plus) linear algebra and, more generally, linear algebra over semirings (Gondran–Minoux [26], see also Litvinov–Maslov [34]). This motivation suggests a directed graph \mathcal{D} with nodes $N = \{1, \dots, n\}$ and edges $E = \{(i, j) : i, j \in N\}$, where each edge (i, j) is weighted by a_{ij} . Sarah is an agent travelling in the network. She is given 1 unit of money before entering it at node i (say, 1 thousand of GBP), and a_{ij} expresses the amount of money left on her bank account after she moves from i to j . The quantity $c_{ij} := 1 - a_{ij}$ expresses the cost of moving from i to j . More generally, if Sarah is given x_i units of money, with $0 \leq x_i \leq 1$, then there are two cases: when $x_i - c_{ij} \geq 0$ and when $x_i - c_{ij} < 0$. In the first case, $x_i - c_{ij} = x_i + a_{ij} - 1$ will be the money left on her account after she moves from i to j . In the second case, Sarah's account will be frozen, in other words, her balance will be set to 0 forever and with no excuse. In any case, $x_i \otimes_L a_{ij}$ expresses the amount of money left on Sarah's account if she goes from i to j . More generally, if Sarah is given 1 unit of money and follows a walk $P = (i_1, \dots, i_k)$ on \mathcal{D} , then the Łukasiewicz weight of P computed as $w_L(P) = a_{i_1 i_2} \otimes_L \dots \otimes_L a_{i_{k-1} i_k}$ will show how much money will remain on her bank account. Computing matrix powers over Łukasiewicz semiring, it can be seen that the entry $(A^{\otimes_L t})_{ij}$ shows Sarah's funds at j if she chooses an optimal walk from i to j . Computing the left orbit of a vector, $(x \otimes_L A^{\otimes_L t})_i$ also shows Sarah's funds if 1) for each ℓ , x_ℓ is the amount of money given to her if she enters the network in state ℓ , 2) she chooses an optimal starting node and an optimal walk from that node to i , where the Łukasiewicz weight of a walk $P = (i_1, \dots, i_k)$ is now computed as $x_{i_1} \otimes_L a_{i_1 i_2} \otimes_L \dots \otimes_L a_{i_{k-1} i_k}$.

In the context of aggressive network, we can pose the Łukasiewicz spectral problem $x \otimes_L A = \lambda \otimes_L x$ if we want to control the dynamics of Sarah's funds, for instance, to know precisely when the game will be over. As $(\lambda \otimes_L x)_i = \max(\lambda - 1 + x_i, 0)$, it is also natural to take a partition (K, L) of $\{1, \dots, n\}$, that is, the subsets $K, L \subseteq \{1, \dots, n\}$ such that $K \cup L = \{1, \dots, n\}$ and $K \cap L = \emptyset$, and impose that $x_i \leq 1 - \lambda$ for $i \in L$ and $x_i \geq 1 - \lambda$ for $i \in K$. The Łukasiewicz eigenvectors satisfying these conditions are called (K, L) -Łukasiewicz eigenvectors. As we shall see, when K and L are proper subsets of $\{1, \dots, n\}$, the existence of (K, L) eigenvectors is equivalent to (K, L) being a “secure partition” of the network where we subtract λ from each edge, see Definition 3.2. This establishes a connection between the Łukasiewicz spectral problem and the combinatorics of weighted digraphs. The network sense of (K, L) -eigenvectors is also clear if we require the strict inequalities $x_i < 1 - \lambda$ for $i \in L$ and $x_i > 1 - \lambda$ for $i \in K$: in this case we know how much Sarah should be given in each state, in order that the game will be over after one step if Sarah starts in any node of L , and in order that she has a chance to live longer if she starts in a node of K and chooses an optimal trajectory.

For convenience, in the paper we will consider **right** Łukasiewicz eigenvectors and orbits.

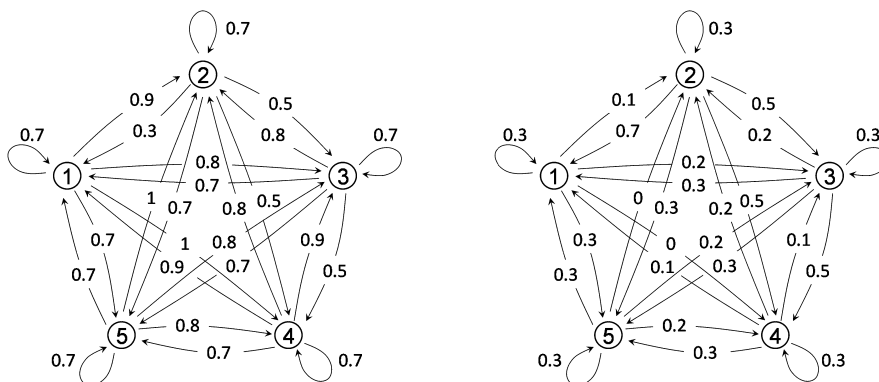


Fig. 1. An example of aggressive network: values of travel costs (left) and account balance after one step (right).

Fig. 1 represents an example of aggressive network. The weights of edges on the left denote Sarah’s debts, or payments if she is able to make them, and the weights on the right stand for her balance after moving along the corresponding edge.

The rest of the paper is organized as follows. Section 2 is occupied with necessary preliminaries on tropical convexity and tropical linear algebra. Section 3 develops the theory of (K, L) -Łukasiewicz eigenvectors and secure partitions. Here we give an explicit description of generating sets of (K, L) -Łukasiewicz eigenspaces, establish the connection with the problem of finding secure partitions of weighted digraphs, and provide an algorithm for enumeration of all possible secure partitions. In Section 4 we examine the powers and orbits of matrices in Łukasiewicz semiring, whose theory is closely related to its well-known tropical (max-plus) counterpart.

2. Basics of the tropical linear algebra

The main idea of this paper is that the max-Łukasiewicz linear algebra is closely related to the tropical (max-plus) linear algebra. A key observation relating the Łukasiewicz linear algebra with the tropical linear algebra is that, given a matrix $A \in [0, 1]^{m \times n}$ and a vector $x \in [0, 1]^n$, we have

$$A \otimes_L x = A^{(1)} \otimes x \oplus \mathbf{0}, \tag{1}$$

where $A^{(1)}$ denotes the matrix with entries $a_{ij} - 1$, and $\mathbf{0}$ is the vector with m entries all equal to 0. Note that the entries of $A^{(1)}$ belong to $[-1, 0]$, while the entries of x are required to be in $[0, 1]^n$. More generally, we denote by $A^{(\alpha)}$ the matrix with entries $a_{ij} - \alpha$.

Sometimes we will also use the min-plus version of the tropical linear algebra, in particular, the min-plus addition $a \wedge b := \min(a, b)$ and the min-plus matrix product $(A \otimes' B)_{ik} = \bigwedge_j a_{ij} + b_{jk}$. Denoting by A^\sharp the matrix with entries $(-a_{ji})$, known as *Cuninghame-Green inverse*, one obtains the following duality law:

$$A \otimes x \leq b \Leftrightarrow x \leq A^\sharp \otimes' b \tag{2}$$

In the sequel we usually omit the \otimes sign for tropical matrix multiplication, unlike the \otimes_L sign for the max-Łukasiewicz multiplication. We are only interested in matrices and vectors with real entries, that is, with no $-\infty$ entries.

We note that we only need the special case of tropical (max-plus) semiring here. See, e.g., Gondran and Minoux [26, 27] for more general idempotent algebras (dioids).

2.1. Elements of tropical convexity

A set $C \in (\mathbb{R} \cup \{-\infty\})^n$ is called tropically convex if together with any two points $x, y \in C$ it contains the whole tropical segment

$$[x, y]_\oplus := \{\lambda x \oplus \mu y : \lambda \oplus \mu = 0\}. \tag{3}$$

Note that $x \oplus y \in [x, y]_\oplus$.

The **tropical convex hull** of a set $X \subseteq (\mathbb{R} \cup \{-\infty\})^n$ is defined as

$$\text{tconv}(X) := \left\{ \bigoplus_{\mu} \lambda_{\mu} x_{\mu} : \bigoplus_{\mu} \lambda_{\mu} = 0, x_{\mu} \in X \right\}, \tag{4}$$

where only a finite number of λ_{μ} are not $-\infty$. In this case, the *tropical Carathéodory theorem* states that in (4), we can restrict without loss of generality to tropical convex combinations of no more than $n + 1$ points x_{μ} .

As in the case of the usual convexity, there exists an *internal description* of tropical convex sets in terms of extremal points and recessive rays, and *external description* as intersection of (tropical) halfspaces. See [3,13,17,24] for some of the recent references. Here we will be mostly interested in the internal description of a compact tropical convex set C based on *extreme points*: if represented as a point in a tropical segment of C , such a point should coincide with one of its ends.

It follows that the linear equations over max-Łukasiewicz semiring are affine equations over max-plus semiring, with the solutions confined in the hypercube $[0, 1]^n$. The solution sets to systems of such equations are compact and tropically convex. Moreover, they are *tropical polyhedra*, i.e. tropical convex hulls of a finite number of points.

A tropical analogue of a theorem of Minkowski was proved in full generality by Gaubert and Katz [24], see also Butkovič et al. [7] for a part of this result. Here we are interested in the particular case of compact tropically convex sets in \mathbb{R}^n .

Theorem 2.1. *Let $C \subseteq \mathbb{R}^n$ be a compact tropical convex set. Further let $u^{(\mu)}$ be a (possibly infinite) set of its extreme points. Then,*

$$C = \left\{ \bigoplus_{\mu} \lambda_{\mu} u^{(\mu)}, \lambda_{\mu} \in \mathbb{R} \cup \{-\infty\}, \bigoplus_{\mu} \lambda_{\mu} = 0 \right\}, \tag{5}$$

where in (5), only a finite number of λ_{μ} are not equal to $-\infty$. In words, any compact tropically convex subset of \mathbb{R}^n is generated by its extreme points.

2.2. Cyclicity theorem

Starting from this subsection, all matrices have only finite entries, no $-\infty$. We now consider the sequence of

max-plus matrix powers $A^k = \overbrace{A \otimes \dots \otimes A}^k$, for $A \in \mathbb{R}^{n \times n}$.

The max-algebraic *cyclicity theorem* states that if the maximum cycle mean of $A \in \mathbb{R}^{n \times n}$ (to be defined later) equals 0, then the sequence of max-plus matrix powers A^k becomes periodic after some finite transient time $T(A)$, and that the ultimate period of A^k is equal to the cyclicity of the critical graph. Cohen et al. [12] seem to be the first to discover this, see also [2,4,5,14,30]. Generalizations to reducible case, computational complexity issues and important special cases of the cyclicity theorem have been extensively studied in [5,16,22,23,35,38].

Below we need this theorem in the form of CSR-representations as in [37,39]. To formulate it precisely we need the following concepts and notation. The notions of the associated digraph, the maximum cycle mean, the critical graph and the Kleene star are of general importance in the tropical linear algebra.

For a matrix $A \in \mathbb{R}^{n \times n}$, define the associated weighted digraph $\mathcal{D}(A) = (N, E)$, where $N = \{1, \dots, n\}$, $E = N \times N$ and each edge $(i, j) \in E$ has weight a_{ij} . Conversely, each weighted digraph with real entries on n nodes corresponds to an $n \times n$ real matrix.

Let $\rho(A)$ denote the **maximum cycle mean** of A , i.e.,

$$\rho(A) = \max_{k=1}^n \max_{i_1, \dots, i_k} \frac{a_{i_1 i_2} + \dots + a_{i_k i_1}}{k}. \tag{6}$$

The cycles (i_1, \dots, i_k) where the maximum cycle mean is attained are called **critical**. Further, the **critical graph**, denoted by $\mathcal{C}(A)$, consists of all nodes and edges belonging to critical cycles. As it will be emphasized later, $\rho(A)$ also plays the role of the (unique) tropical eigenvalue of A .

The sum of formal series

$$A^* := I \oplus A \oplus A^2 \oplus \dots \tag{7}$$

is called the **Kleene star** of $A \in \mathbb{R}^{n \times n}$. Here I denotes the tropical identity matrix, i.e., the matrix with diagonal entries equal to 0 and the off-diagonal entries equal to $-\infty$. Series (7) converges if and only if $\rho(A) \leq 0$, in which case $A^* = I \oplus A \oplus \dots \oplus A^{n-1}$. The Kleene star satisfies

$$A^* = AA^* \oplus I. \quad (8)$$

Defining the **additive weight** of a walk P on $\mathcal{D}(A)$ as sum of the weights of all edges contained in the walk, observe that the following **optimal walk interpretation** of tropical matrix powers A^k and Kleene star A^* : 1) for each pair i, j , the (i, j) entry of A^k is equal to the greatest additive weight of a walk connecting i to j with length k , 2) for each pair i, j with $i \neq j$, the (i, j) entry of A^* is equal to the greatest additive weight of a walk connecting i to j (with no length restriction).³

We introduce the notation related to *CSR*-representation. Let $C \in \mathbb{R}^{n \times c}$ and $R \in \mathbb{R}^{c \times n}$ be matrices extracted from the critical columns (resp. rows) of $((A - \rho(A))^\gamma)^*$, where γ is the **cyclicity** of $\mathcal{C}(A)$. To calculate γ by definition, one needs to take the g.c.d. of the lengths of all simple cycles in each strongly connected component of $\mathcal{C}(A)$, and then to take the l.c.m. of these. Without loss of generality we are assuming that $\mathcal{C}(A)$ occupies the first c nodes of the associated graph.

Let S be defined by

$$s_{ij} = \begin{cases} a_{ij} - \rho(A), & \text{if } (i, j) \in \mathcal{C}(A), \\ -\infty, & \text{otherwise.} \end{cases} \quad (9)$$

By $A_{\cdot i}$ and A_i we denote the i th column, respectively the i th row of A .

Theorem 2.2. For any matrix $A \in \mathbb{R}^{n \times n}$ there exists a number $T(A)$ such that for all $t \geq T(A)$

$$A^t = \rho^t(A)CS^tR. \quad (10)$$

Moreover, $(A^t)_{\cdot i} = \rho^t(A)S^tR_i$ and $(A^t)_i = \rho^t(A)C_iS^t$ for all $i = 1, \dots, c$.

This *CSR* form of the Cyclicity Theorem [12] was obtained in [37], see also [39].

2.3. Eigenproblem and Bellman equation

Let $A \in \mathbb{R}^{n \times n}$. Vector $x \in \mathbb{R}^n$ is called a tropical eigenvector of A associated with λ if it satisfies $A \otimes x = \lambda \otimes x$ for some λ . See [2,4,5,14,26,30] for general references.

Theorem 2.3. Let $A \in \mathbb{R}^{n \times n}$, then the maximum cycle mean $\rho(A)$ is the unique tropical eigenvalue of A .

The set of all eigenvectors of A with eigenvalue $\rho(A)$ is denoted by $V(A, \rho)$ and called the **eigencone** of A . It can be described in terms of the critical graph, by the following procedure. The critical graph of $A^{(\rho)}$, the matrix with entries $a_{ij} - \rho(A)$, consists of several strongly connected components, isolated from each other. In each of the components, one selects an arbitrary index i and picks the column $(A^{(\rho)})_{\cdot i}^*$ of the Kleene star. It can be shown that for any other choice of an index i' in the same strongly connected component, the column will be “proportional”, i.e. $(A^{(\rho)})_{\cdot i'}^* = \alpha \otimes (A^{(\rho)})_{\cdot i}^* = \alpha + (A^{(\rho)})_{\cdot i}^*$. In what follows, by $\tilde{N}_C(A)$ we denote an index set containing exactly one index from each strongly connected component of $\mathcal{C}(A)$. The following (standard) description of $V(A, \rho)$ is standard, see Krivulin [32,33].

Theorem 2.4. Let $A \in \mathbb{R}^{n \times n}$. Then $V(A, \rho)$ consists of all vectors $(A^{(\rho)})^*z$, where z is any vector in \mathbb{R}_{\max}^n satisfying

$$i \notin \tilde{N}_C(A, \lambda) \Rightarrow z_i = -\infty \quad (11)$$

³ In what follows, by the weight of a walk we mean this additive weight, and not the Łukasiewicz weight defined in Introduction

The tropical spectral theory can be further applied to the equation

$$x = Ax \oplus b, \tag{12}$$

which has been studied, e.g., in [4,10,34]. It is called Bellman equation due to its relations with dynamic optimization on graphs and in particular, the Bellman optimality principle [34]. Its nonnegative analogue is known as Z-matrix equation, see [8].

We will make use of the following basic result, formulated only recently in [8,32,33]. The proof is given for the reader's convenience. We consider only the case when A and b have real entries, since this is the only case that we will encounter. The solution set of (12) will be denoted by $S(A, b)$.

Theorem 2.5. *Let $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. Eq. (12) has nontrivial solutions if and only if $\rho(A) \leq 0$. In this case,*

$$S(A, b) = \{A^*b \oplus v : Av = v\} \tag{13}$$

*In particular, A^*b is the only solution if $\rho(A) < 0$.*

Proof. First it can be verified that any vector like on the r.h.s. of (13) satisfies $x = Ax \oplus b$, using that $A^* = AA^* \oplus I$ (8).

Iterating the equation $x = Ax \oplus b$ we obtain

$$\begin{aligned} x = Ax \oplus b &= A(Ax \oplus b) \oplus b = \dots \\ &= A^k x \oplus (A^{k-1} \oplus \dots \oplus I)b = A^k x \oplus A^*b \end{aligned} \tag{14}$$

for all $k \geq n$.

This implies $x \geq A^*b$. In particular, a solution of (12) exists if and only if A^* converges, that is, if and only if $\rho(A) \leq 0$.

Further, x satisfies $Ax \leq x$, hence $x \geq Ax \geq \dots \geq A^k x \geq \dots$

If $\rho(A) < 0$, then by the cyclicity theorem, vectors of $\{A^k x\}_{k \geq 1}$ start to fall with the constant rate $\rho(A)$, starting from some k . This shows that for large enough k , $A^k x \leq A^*b$ and A^*b is the only solution.

If $\rho(A) = 0$, then the orbit $\{A^k x\}_{k \geq 1}$ starts to cycle from some k . But as $Ax \leq x$, we have $x \geq Ax \geq \dots \geq A^k x \geq \dots$, and it is only possible that the sequence $\{A^k x\}_{k \geq 1}$ stabilizes starting from some k . That is, starting from some k , vector $v = A^k x$ satisfies $Av = v$. The proof is complete. \square

We will rather need the following formulation of the above result, implied by Theorem 2.4.

Corollary 2.1. *Let $A \in \mathbb{R}^{n \times n}$. Vector x solves $x = Ax \oplus b$ if and only if it can be written $x = A^*(b \oplus z(x))$ where $z(x)$ is a vector such that*

$$i \notin \tilde{N}_C(A, 0) \Rightarrow z(x)_i = -\infty. \tag{15}$$

3. Max-Łukasiewicz eigenproblem

The problem is to find λ such that there exist nonzero x solving $A \otimes_L x = \lambda \otimes_L x$. Using (1) we convert this problem to the following one:

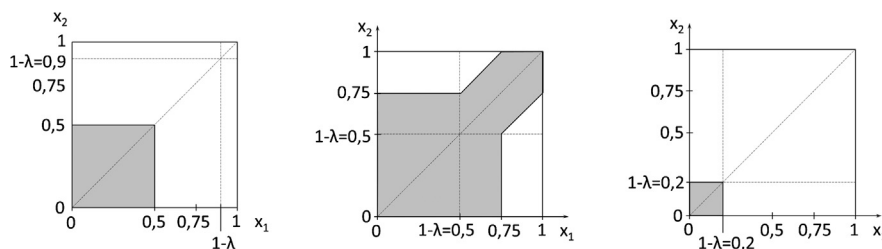
$$A^{(1)}x \oplus \mathbf{0} = (\lambda - 1)x \oplus \mathbf{0}, \quad 0 \leq x_i \leq 1. \tag{16}$$

Before developing any general theory, let us look at some two-dimensional examples. Take

$$A = \begin{pmatrix} 0.5 & 0.25 \\ 0.25 & 0.5 \end{pmatrix} \tag{17}$$

Then $A \otimes_L x = \lambda \otimes_L x$ is equivalent to

$$\begin{aligned} \max(-0.5 + x_1, -0.75 + x_2, 0) &= \max(\lambda - 1 + x_1, 0) \\ \max(-0.75 + x_1, -0.5 + x_2, 0) &= \max(\lambda - 1 + x_2, 0) \end{aligned} \tag{18}$$

Fig. 2. Cases $\lambda = 0.1$, $\lambda = 0.5$ and $\lambda = 0.8$.

Take $\lambda < 0.5$. Then the terms with λ can be cancelled, and we obtain the system of inequalities

$$\begin{aligned} \max(-0.5 + x_1, -0.75 + x_2) &\leq 0 \\ \max(-0.75 + x_1, -0.5 + x_2) &\leq 0, \end{aligned} \quad (19)$$

which has solution set

$$\{x: x_1 \leq 0.5 \wedge x_2 \leq 0.5\}. \quad (20)$$

This is the Łukasiewicz eigenspace⁴ associated with any $\lambda < 0.5$.

If $\lambda > 0.5$, then the diagonal terms on the l.h.s. of (18) can be cancelled and we obtain the system

$$\begin{aligned} \max(-0.75 + x_2, 0) &= \max(\lambda - 1 + x_1, 0) \\ \max(-0.75 + x_1, 0) &= \max(\lambda - 1 + x_2, 0) \end{aligned} \quad (21)$$

The solutions to equations of (21) can be written as

$$\{x: (x_2 \leq 0.75 \wedge x_1 \leq 1 - \lambda) \vee (x_2 \geq 0.75 \wedge x_1 \geq 1 - \lambda \wedge x_2 = \lambda - 0.25 + x_1)\} \quad (22)$$

and, respectively,

$$\{x: (x_1 \leq 0.75 \wedge x_2 \leq 1 - \lambda) \vee (x_1 \geq 0.75 \wedge x_2 \geq 1 - \lambda \wedge x_2 = 0.25 - \lambda + x_1)\} \quad (23)$$

Intersecting these sets we obtain the Łukasiewicz eigenspace

$$\{x: x_1 \leq 1 - \lambda \wedge x_2 \leq 1 - \lambda\}. \quad (24)$$

If $\lambda = 0.5$ then we solve

$$\begin{aligned} \max(-0.5 + x_1, -0.75 + x_2, 0) &= \max(-0.5 + x_1, 0) \\ \max(-0.75 + x_1, -0.5 + x_2, 0) &= \max(-0.5 + x_2, 0) \end{aligned} \quad (25)$$

which is equivalent to

$$\begin{aligned} -0.75 + x_2 &\leq \max(-0.5 + x_1, 0) \\ -0.75 + x_1 &\leq \max(-0.5 + x_2, 0). \end{aligned} \quad (26)$$

The solution set is

$$\{x: (x_2 \leq 0.25 + x_1 \vee x_2 \leq 0.75) \wedge (x_2 \geq -0.25 + x_1 \vee x_1 \leq 0.75)\}. \quad (27)$$

Łukasiewicz eigenspaces (20), (24) and (27) for $\lambda = 0.1$, $\lambda = 0.5$ and $\lambda = 0.8$ are displayed on Fig. 2.

The relatively simple structure of the eigenspace in the previous example is influenced by the symmetry of the matrix A , which reduces the number of conditions that are to be considered. Now we show another two-dimensional example, in which all entries of the matrix have different values

⁴ In what follows, we omit the prefix “max-” by the abuse of language.

$$A = \begin{pmatrix} 0.2 & 0.1 \\ 0.7 & 0.4 \end{pmatrix} \quad (28)$$

There are five possible positions of the parameter λ with respect to the diagonal entries 0.2 and 0.4, namely: $\lambda < 0.2$, $\lambda = 0.2$, $0.2 < \lambda < 0.4$, $\lambda = 0.4$ and $\lambda > 0.4$.

For $\lambda < 0.4$, i.e. in the first three cases, the solution set is

$$\{x: x_1 \leq 0.3 \wedge x_2 \leq 0.6\}. \quad (29)$$

In fact, it is the solution set for the second equation, whereas solution sets for the first equation in cases $\lambda < 0.2$, $\lambda = 0.2$ and $0.2 < \lambda < 0.4$ are: $\{x: x_1 \leq 0.8 \wedge x_2 \leq 0.9\}$, $\{x: (x_1 \leq 0.8 \wedge x_2 \leq 0.9) \vee (x_1 > 0.8 \wedge -0.1 + x_2 \leq x_1)\}$ and $\{x: (x_1 \leq 1 - \lambda \wedge x_2 \leq 0.9) \vee (x_1 > 1 - \lambda \wedge 0.1 - \lambda + x_2 = x_1)\}$, respectively. All these sets contain the solution set (29), hence the intersection of the solutions for both equations is described by (29).

For $\lambda = 0.4$ we obtain the solution sets for the first and the second equation

$$\begin{aligned} & \{x: (x_1 \leq 0.6 \wedge x_2 \leq 0.9) \vee (x_1 > 0.6 \wedge -0.3 + x_2 = x_1)\} \\ & \{x: (x_2 \leq 0.6 \wedge x_1 \leq 0.3) \vee (x_2 > 0.6 \wedge 0.3 + x_1 \leq x_2)\} \end{aligned} \quad (30)$$

and their intersection is the solution set

$$\begin{aligned} & \{x: (x_1 \leq 0.3 \wedge x_2 \leq 0.6) \vee (x_1 \leq 0.6 \wedge 0.6 < x_2 \leq 0.9 \wedge 0.3 + x_1 \leq x_2) \\ & \vee (x_1 > 0.6 \wedge x_2 > 0.6 \wedge 0.3 + x_1 = x_2)\}. \end{aligned} \quad (31)$$

For $\lambda > 0.4$ we obtain the solution sets for the equations

$$\begin{aligned} & \{x: (x_1 \leq 1 - \lambda \wedge x_2 \leq 0.9) \vee (x_1 > 1 - \lambda \wedge 0.1 - \lambda + x_2 = x_1)\} \\ & \{x: (x_2 \leq 1 - \lambda \wedge x_1 \leq 0.3) \vee (x_2 > 1 - \lambda \wedge 0.7 - \lambda + x_1 = x_2)\}. \end{aligned} \quad (32)$$

Their intersection (the solution set) for $0.4 < \lambda < 0.7$ has the form

$$\{x: (x_1 \leq 1 - \lambda \wedge 1 - \lambda < x_2 \leq 0.9 \wedge 0.7 - \lambda + x_1 = x_2) \vee (x_1 \leq 0.3 \wedge x_2 \leq 1 - \lambda)\}, \quad (33)$$

while for $\lambda \geq 0.7$ the solution set is simply

$$\{x: x_1 \leq 1 - \lambda \wedge x_2 \leq 1 - \lambda\}. \quad (34)$$

All described eigenspaces (29), (31), (33) and (34) for $\lambda = 0.35$, $\lambda = 0.4$, $\lambda = 0.55$ and $\lambda = 0.85$ are displayed on Fig. 3.

We now going to describe certain parts of the Łukasiewicz eigenspace, determined by the pattern of maxima on the r.h.s. of (16), whose generating sets (containing no more than $n + 1$ points) can be written explicitly and computed in polynomial time.

3.1. Background eigenvectors

These are the eigenvectors that satisfy $x_i \leq 1 - \lambda$ for all i . In this case (16) becomes $A^{(1)}x \leq \mathbf{0}$, and using (2) we obtain that the solutions are given by

$$x_j \leq \min\left(1 - \lambda, \min_i(1 - a_{ij})\right) \quad (35)$$

If $\lambda = 1$ then there are no nonzero background eigenvectors.

If $\lambda = 0$ then all Łukasiewicz eigenvectors are background.

If $\lambda < 1$ then nonzero background eigenvectors exist if and only if there is j such that $\max_i a_{ij} < 1$. In other words, background eigenvectors do not exist if and only if in each column of A there is an entry equal to 1. It can be seen that in this case $\rho(A) = 1$.

In particular, we obtain the following.

Proposition 3.1. *A matrix $A \in [0, 1]^{n \times n}$ with $\rho(A) < 1$ has nonzero background Łukasiewicz eigenvectors for all $\lambda < 1$.*

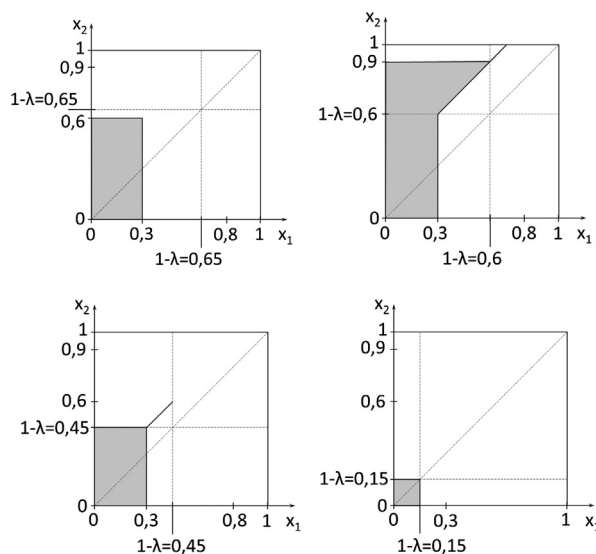


Fig. 3. Cases $\lambda = 0.35$, $\lambda = 0.4$, $\lambda = 0.55$ and $\lambda = 0.85$.

3.2. Pure eigenvectors

These are the vectors that satisfy $x_i \geq 1 - \lambda$ for all i . In this case (16) transforms to

$$x = A^{(\lambda)}x \oplus (1 - \lambda) \otimes \mathbf{0}, \quad 1 - \lambda \leq x_i \leq 1. \tag{36}$$

Evidently, the set of vectors satisfying (36) is tropically convex.

We obtain the following description of the pure eigenvectors.

Theorem 3.1. Matrix $A \in [0, 1]^{n \times n}$ has a pure eigenvector associated with $\lambda \in [0, 1]$ if and only if $\rho(A) \leq \lambda$ and $\max((A^{(\lambda)})^* \mathbf{0}) \leq \lambda$. Vector $x \in \mathbb{R}^n$ is a pure Łukasiewicz eigenvector associated with λ if and only if it is of the form

$$x = (A^{(\lambda)})^* ((1 - \lambda) \otimes \mathbf{0} \oplus z(x)) \tag{37}$$

where $z(x)$ is a vector satisfying

$$\begin{aligned} z(x) &\leq ((A^{(\lambda)})^*)^\sharp \mathbf{1}, \\ i \notin \tilde{N}_C(A, \lambda) &\Rightarrow z(x)_i = -\infty. \end{aligned} \tag{38}$$

In particular, $z(x) = -\infty$ whenever $\lambda \neq \rho(A)$.

Proof. As any pure eigenvector is a solution to (36) we can use Corollary 2.1. We obtain that $x \in \mathbb{R}^n$ is a pure eigenvector if and only if it can be represented as in (37) with $z(x)$ satisfying the second constraint in (38), and has all coordinates between $1 - \lambda$ and 1. From (37) we conclude that $x \geq (1 - \lambda) \otimes (A^{(\lambda)})^* \mathbf{0} \geq (1 - \lambda) \otimes \mathbf{0}$, so $x_i \geq 1 - \lambda$ holds for all i . Substituting (37) in $x \leq \mathbf{1}$ and using the duality law (2) we obtain the condition $\max((A^{(\lambda)})^* \mathbf{0}) \leq \lambda$ and the first constraint in (38). \square

Definition 3.1. For $\lambda > 0$, a weighted graph \mathcal{D} is called λ -secure if for each node of \mathcal{D} and each walk P issuing from that node, $-\lambda + w(P) \leq 0$.

Corollary 3.1. A matrix $A \in [0, 1]^{n \times n}$ has a pure eigenvector associated with $\lambda \in [0, 1]$ if and only if $\mathcal{D}(A^{(\lambda)})$ is λ -secure.

Proof. By Theorem 3.1, pure Łukasiewicz eigenvector associated with λ exists if and only if 1) $\rho(A^{(\lambda)}) \leq 0$ and 2) $\max((A^{(\lambda)})^*) \leq \lambda$. The first condition implies that $(A^{(\lambda)})^*$ exists, and the second condition implies that $\mathcal{D}(A^{(\lambda)})$

is λ -secure, by the walk interpretation $(A^{(\lambda)})^*$. Conversely, the λ -security is impossible if $\mathcal{D}(A^{(\lambda)})$ has a cycle with positive weight, and the condition $\max((A^{(\lambda)})^*) \leq \lambda$ follows from the optimal walk interpretation of $(A^{(\lambda)})^*$. \square

We now describe the generating set of pure Łukasiewicz eigenvectors.

Corollary 3.2. *The set of pure Łukasiewicz eigenvectors associated with $\lambda \in [0, 1]$ is the tropical convex hull of vectors u and $v^{(k)}$ for $k \in \tilde{N}_C(A, \lambda)$ given by (37) where*

1. $z(u) = -\infty$,
2. $z(v^{(k)})_k = (((A^{(\lambda)})^*)^\sharp \otimes' \mathbf{1})_k$ and $z(v^{(k)})_i = -\infty$ for $i \neq k$.

Proof. By Theorem 3.1, u and $v^{(k)}$ are pure Łukasiewicz eigenvectors associated with λ , and so is any tropically convex combination of them. Further, all $z(x)$ satisfying (38) are tropical convex combinations of $z(u)$ and $z(v^{(k)})$. Using max-linearity of (37), we express any pure Łukasiewicz eigenvector as a tropical convex combination of u and $v^{(k)}$. \square

This result also leads to a description of all Łukasiewicz eigenvectors when $\lambda = 1$.

Corollary 3.3. *When $\lambda = 1$, all nontrivial Łukasiewicz eigenvectors are pure. Further, nontrivial eigenvectors exist if and only if $\rho(A) = 1$. In this case, the Łukasiewicz eigenspace is the tropical convex hull of $\mathbf{0}$ and the columns of $\mathbf{1} \otimes (A^{(1)})^*$ with indices in $\tilde{N}_C(A, 1)$ (i.e., the fundamental eigenvectors of A shifted by one).*

Proof. When $\lambda = 1$ we have $\lambda + x_i - 1 = x_i \geq 0$, hence all nontrivial eigenvectors are pure. Next we apply Theorem 3.1 with $\lambda = 1$. Vector $(1 - \lambda) \otimes (A^{(\lambda)})^* \mathbf{0}$ becomes $(A^{(1)})^* \mathbf{0} = \mathbf{0}$, and the condition $\max(A^{(\lambda)})^* \mathbf{0} \leq \lambda$ is always true. It remains to apply Corollary 3.2 observing that $(((A^{(1)})^*)^\sharp \otimes' \mathbf{1})_i = 1$ for all i . \square

Note that in this case it can be shown that $\mathbf{0}$ and $v^{(i)}$ for $i = 1, \dots, l$, are the extreme points of the set of Łukasiewicz eigenvectors.

3.3. (K, L) eigenvectors

Now we consider (K, L) Łukasiewicz eigenvectors, i.e., $x \in [0, 1]^n$ such that $A \otimes_L x = \lambda \otimes_L x$, $x_i \geq (1 - \lambda)$ for $i \in K$ and $x_i \leq (1 - \lambda)$ for $i \in L$, where $K, L \subseteq \{1, \dots, n\}$ are such that $K \cup L = \{1, \dots, n\}$ and $K \cap L = \emptyset$.

We obtain the following description of (K, L) -eigenvectors, which uses the concept of tropical Schur complement, introduced by Akian, Bapat and Gaubert [1, Definition 2.13]:

$$\text{Schur}(K, \lambda, A) = A_{LL} \oplus A_{LK} (A_{KK}^{(\lambda)})^* A_{KL}^{(\lambda)}. \tag{39}$$

Observe that $\text{Schur}^{(1)}(K, \lambda, A) = A_{LL}^{(1)} \oplus A_{LK}^{(1)} (A_{KK}^{(\lambda)})^* A_{KL}^{(\lambda)}$.

Theorem 3.2. *Let λ satisfy $0 < \lambda < 1$, and let (K, L) be a partition of $\{1, \dots, n\}$ where K and L are proper subsets of $\{1, \dots, n\}$. Matrix $A \in [0, 1]^{n \times n}$ has a (K, L) Łukasiewicz eigenvector associated with λ if and only if $\rho(A_{KK}) \leq \lambda$ and $A_{LK}^{(\lambda)} (A_{KK}^{(\lambda)})^* \mathbf{0}_K \leq \mathbf{0}_L$. In this case the set of (K, L) -eigenvectors is given by*

$$\mathbf{0}_L \leq x_L \leq \min((\text{Schur}^{(1)}(K, \lambda, A))^\sharp \otimes' \mathbf{0}_L, (1 - \lambda) \otimes \mathbf{0}_L), \tag{40}$$

$$x_K = (A_{KK}^{(\lambda)})^* (A_{KL}^{(\lambda)} x_L \oplus (1 - \lambda) \mathbf{0}_K \oplus z_K(x)), \tag{41}$$

where $z_K(x)$ is any vector with components in K satisfying

$$\begin{aligned} z_K(x) &\leq (A_{LK}^{(1)} (A_{KK}^{(\lambda)})^*)^\sharp \otimes' \mathbf{0}_L, \\ i \notin N_C(A_{KK}, \lambda) &\Rightarrow z_K(x)_i = -\infty, \end{aligned} \tag{42}$$

and x_K , resp. x_L are restrictions of x to the index set K , resp. to L .

Proof. By definition, (K, L) Łukasiewicz eigenvectors satisfy the following system

$$A_{KL}^{(1)}x_L \oplus A_{KK}^{(1)}x_K \oplus \mathbf{0}_K = (\lambda - 1) \otimes x_K, \tag{43}$$

$$A_{LL}^{(1)}x_L \oplus A_{LK}^{(1)}x_K \oplus \mathbf{0}_L = \mathbf{0}_L. \tag{44}$$

We start by writing out the solution of (43). Subtracting $(\lambda - 1)$ from both sides of this equation we obtain

$$A_{KL}^{(\lambda)}x_L \oplus A_{KK}^{(\lambda)}x_K \oplus (1 - \lambda) \otimes \mathbf{0}_K = x_K \tag{45}$$

Eq. (45) has solutions if and only if $\rho(A_{KK}) \leq \lambda$. Let z_K be any K -vector (that is, a vector with all components in K) such that

$$i \notin N_C(A_{KK}, \lambda) \Rightarrow (z_K)_i = -\infty. \tag{46}$$

Then vector x satisfies (45) if and only if it satisfies

$$x_K = (A_{KK}^{(\lambda)})^* (A_{KL}^{(\lambda)}x_L \oplus (1 - \lambda)\mathbf{0}_K \oplus z_K). \tag{47}$$

Observe that if $\rho(A_{KK}) < \lambda$ then λ is not the (unique) eigenvalue of A_{KK} , and $N_C(A_{KK}, \lambda)$ is empty so that $z_K = -\infty$.

Eq. (44) can be rewritten as inequality

$$A_{LL}^{(1)}x_L \oplus A_{LK}^{(1)}x_K \leq \mathbf{0}_L, \tag{48}$$

and substituting x_K from (47) we get

$$(A_{LL}^{(1)} \oplus A_{LK}^{(1)} (A_{KK}^{(\lambda)})^* A_{KL}^{(\lambda)})x_L \oplus A_{LK}^{(\lambda)} (A_{KK}^{(\lambda)})^* \mathbf{0}_K \oplus A_{LK}^{(1)} (A_{KK}^{(\lambda)})^* z_K \leq \mathbf{0}_L \tag{49}$$

Expression in the brackets equals $\text{Schur}^{(1)}(K, \lambda, A)$. The inequality (49) is equivalent to the following three inequalities:

$$\begin{aligned} x_L &\leq (\text{Schur}^{(1)}(K, \lambda, A))^{\sharp} \otimes' \mathbf{0}_L, & A_{LK}^{(\lambda)} (A_{KK}^{(\lambda)})^* \mathbf{0}_K &\leq \mathbf{0}_L, \\ z_K &\leq (A_{LK}^{(1)} (A_{KK}^{(\lambda)})^*)^{\sharp} \otimes' \mathbf{0}_L \end{aligned} \tag{50}$$

We also must require that $(1 - \lambda) \otimes \mathbf{0}_K \leq x_K \leq \mathbf{1}_K$ and $\mathbf{0}_L \leq x_L \leq (1 - \lambda) \otimes \mathbf{0}_L$.

However, from (47) we conclude that $x_K \geq (1 - \lambda)(A_{KK}^{(\lambda)})^* \mathbf{0}_K \geq (1 - \lambda) \otimes \mathbf{0}_K$, so the inequality $x_K \geq (1 - \lambda) \otimes \mathbf{0}_K$ is automatically fulfilled. In particular, this implies the existence of a nontrivial (K, L) eigenvector associated with $\lambda < 1$, provided that $\rho(A_{KK}^{(\lambda)}) \leq 0$ and $A_{LK}^{(\lambda)} (A_{KK}^{(\lambda)})^* \mathbf{0}_K \leq \mathbf{0}_L$.

If x_K and x_L satisfy (47) and (50) then in particular we have (48), and hence $x_K \leq (A^{(1)})_{KL}^{\sharp} \otimes' \mathbf{0}_L \leq \mathbf{1}_K$. Thus $(1 - \lambda) \otimes \mathbf{0}_K \leq x_K \leq \mathbf{1}_K$ is satisfied automatically, and the claim of the theorem follows. \square

We now give a graph-theoretic interpretation for the existence conditions in Theorem 3.2.

Definition 3.2. Let \mathcal{D} be a weighted digraph with nodes $\{1, \dots, n\}$. Let K and L be proper subsets of $\{1, \dots, n\}$ such that (K, L) is a partition of $\{1, \dots, n\}$. This partition is called secure, with respect to \mathcal{D} , if every walk in \mathcal{D} , that starts in a node $\ell \in L$ and has all other nodes in K , has a non-positive weight.

We say that the partition $([n], \emptyset)$ is secure if $\mathcal{D}(A^{(\lambda)})$ is λ -secure, and that the partition $(\emptyset, [n])$ is secure.

For example, consider the Łukasiewicz eigenproblem with

$$A = \begin{pmatrix} 0.3 & 0.1 & 0.2 & 0 & 0.3 \\ 0.7 & 0.3 & 0.5 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.3 & 0.5 & 0.3 \\ 0.1 & 0.2 & 0.1 & 0.3 & 0.3 \\ 0.3 & 0 & 0.2 & 0.2 & 0.3 \end{pmatrix}$$

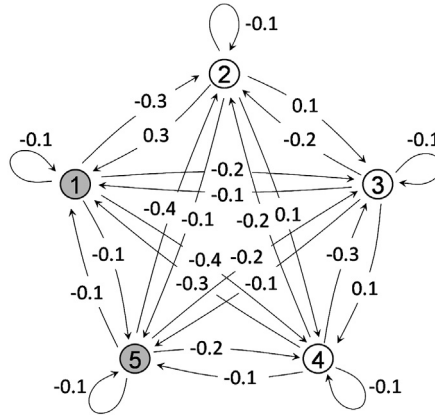


Fig. 4. Secure partition $L = \{1, 5\}$, $K = \{2, 3, 4\}$ ($\lambda = 0, 4$).

and $\lambda = 0.4$. This is the associated matrix of the graph on Fig. 1 (right). Consider secure partitions of the graph associated with $A^{(\lambda)}$. One of them has $L = \{1, 5\}$, $K = \{2, 3, 4\}$ and is shown on Fig. 4. In the graph associated with $A^{(\lambda)}$, it is secure to travel from any node that belongs to L to any node belonging to K and to continue a walk in K without having any deficit of the budget. On the other hand, the partition $L = \{5, 4\}$, $K = \{1, 2, 3\}$ is not secure, because there is a walk from index $4 \in L$ to index $1 \in K$ with positive weight (the walk $p = (4, 2, 1)$ with $w(p) = 0.1$).

Proposition 3.2. *Let λ satisfy $0 < \lambda < 1$, and let (K, L) be a partition of $\{1, \dots, n\}$ where K and L are proper subsets of $\{1, \dots, n\}$. Matrix $A \in [0, 1]^{n \times n}$ has a (K, L) Łukasiewicz eigenvector associated with λ if and only if (K, L) is a secure partition.*

Proof. We have the following necessary and sufficient conditions for existence of a nontrivial Łukasiewicz eigenvector with a given λ : 1) $\rho(A_{KK}^{(\lambda)}) \leq 0$ and 2) $A_{LK}^{(\lambda)}(A_{KK}^{(\lambda)})^* \mathbf{0}_K \leq \mathbf{0}_L$. Using the walk interpretation of $(A_{KK}^{(\lambda)})^*$ we see that 1) and 2) imply that (K, L) is secure. Conversely, (K, L) cannot be secure if $\rho(A_{KK}^{(\lambda)}) > 0$. Indeed, then there exists a cycle in K with weight exceeding 0. Any walk connecting to this cycle and following it will have a positive weight, starting from some length, thus $\rho(A_{KK}^{(\lambda)}) \leq 0$. In this case $(A_{KK}^{(\lambda)})^*$ exists and we have $A_{LK}^{(\lambda)}(A_{KK}^{(\lambda)})^* \mathbf{0}_K \leq \mathbf{0}_L$ since (K, L) is secure. \square

As a corollary of Theorem 3.2, we can describe the set of (K, L) -eigenvectors as a tropical convex hull of no more than $n + 1$ explicitly defined points. Both statement and proof are analogous to those of Corollary 3.2.

Corollary 3.4. *The set of (K, L) Łukasiewicz eigenvectors is the tropical convex hull of vectors u , $v^{(\ell)}$ for $\ell \in L$, and $w^{(k)}$ for $k \in K$ such that $k \in N_C(A_{KK}, \lambda)$, defined by (41) and the following settings:*

1. $u_L = \mathbf{0}_L$ and $z_K(u) = -\infty$,
2. $v_i^{(\ell)} = 0$ for $i \in L$ and $i \neq \ell$, $v_\ell^{(\ell)} = \min(((\text{Schur}^{(1)}(K, \lambda, A))^\# \otimes' \mathbf{0}_L)_\ell, 1 - \lambda)$ and $z_K(v^{(\ell)}) = -\infty$,
3. $w_L^{(k)} = \mathbf{0}_L$, $z_K(w^{(k)})_k = ((A_{LK}^{(1)}(A_{KK}^{(\lambda)})^*)^\# \otimes' \mathbf{0}_L)_k$ and $z_K(w^{(k)})_i = -\infty$ for $i \neq k$,

and the K -subvectors of u , $v^{(\ell)}$ and $w^{(k)}$ are given by (41).

Proof. By Theorem 3.2, u , $v^{(\ell)}$, $w^{(k)}$ are Łukasiewicz K, L -eigenvectors associated with λ , and so is any tropically convex combination of them. Further, all $z_K(x)$ satisfying (42) are tropical convex combinations of $z_K(u) = -\infty$ and $z_K(v^{(\ell)})$. Using max-linearity of (41), we express any pure Łukasiewicz eigenvector as a tropical convex combination of u , $v^{(\ell)}$ and $w^{(k)}$. \square

3.4. Finding all secure partitions

The problem of describing all Łukasiewicz eigenvectors associated with λ can be associated with the following combinatorial problem: 1) check whether graph $\mathcal{D}(A^{(\lambda)})$ is λ -secure, 2) describe all (K, L) partitions which are secure in $\mathcal{D}(A^{(\lambda)})$.

Node i of a weighted digraph is called **secure** if the weight of every walk starting in i is nonpositive.

Theorem 3.3. *Let (K_1, L_1) and (K_2, L_2) be two different secure partitions such that $L_1 \subseteq L_2$.*

1. *There is a node $k \in L_2 \setminus L_1 = K_1 \setminus K_2$, secure in $\mathcal{D}(A_{K_1 K_1}^{(\lambda)})$.*
2. *Partition $(K_1 - k, L_1 + k)$ is secure if and only if k is secure in $\mathcal{D}(A_{K_1 K_1}^{(\lambda)})$.*

Proof. 1. Consider the set $M := K_1 \setminus K_2$ and assume that no node of M is secure in K_1 . Then for each node $i \in M$ there exists a walk P in $\mathcal{D}(A_{K_1 K_1}^{(\lambda)})$ starting in i , whose weight is positive, and such that the weights of all its proper subwalks starting in i are non-positive. Let ℓ be the end node of the walk, and let k be the last node of M visited by the walk. If $k \neq \ell$ then the subwalk of P starting in k and ending in ℓ has a positive weight, which contradicts with the security of (K_2, L_2) . We conclude that for all walks in $\mathcal{D}(A_{K_1 K_1}^{(\lambda)})$ starting in M and having positive weight, the end node also belongs to M . However, using such walks and the finiteness of M we obtain that $\mathcal{D}(A_{K_1 K_1}^{(\lambda)})$ has a cycle with positive weight, contradicting the security of (K_1, L_1) . Hence M contains a node that is secure in K_1 .

2. The “if” part: Observe that all walks starting in L_1 and continuing in $K_1 - k$ have nonpositive weight since the partition (K_1, L_1) is secure. All walks starting in k and continuing in $K_1 - k$ have nonpositive weight since k is secure in $\mathcal{D}(A_{K_1 K_1}^{(\lambda)})$.

The “only if” part: If k is not secure, then there is a walk in $\mathcal{D}(A_{K_1 K_1}^{(\lambda)})$ going out of k and having a positive weight. If this walk ends in k then we obtain a cycle with positive weight, in contradiction with the security of (K_1, L_1) . Otherwise, take a subwalk which contains k only as the starting node, and then its weight must be also positive. This shows that $(K_1 - k, L_1 + k)$ is not secure. \square

Let us emphasize that the above theorem also holds in the case of $K_1 = [n]$ when we require that $\mathcal{D}(A^{(\lambda)})$ is λ -secure, since this condition (by [Theorem 3.1](#) and [Corollary 3.2](#)) implies that $\rho(A^{(\lambda)}) \leq 0$. The case when $L_2 = [n]$, in which case (K_2, L_2) is secure by definition, yields the following corollary.

Corollary 3.5. *Let (K, L) be a secure partition, then*

1. *There is a node $k \in K$, secure in $\mathcal{D}(A_{KK}^{(\lambda)})$.*
2. *Partition $(K - k, L + k)$ is secure if and only if k is secure in $\mathcal{D}(A_{KK}^{(\lambda)})$.*

The equivalence of security of partition (K, L) with existence of (K, L) -Łukasiewicz eigenvector implies that there is a unique minimal secure partition.

Proposition 3.3. *Digraph $\mathcal{D}(A^{(\lambda)})$ has the least secure partition, which corresponds to the greatest Łukasiewicz eigenvector associated with λ .*

Proof. First observe that there is the set of Łukasiewicz eigenvectors contains the greatest element: this is a tropical convex hull of a finite number of extremal points, and the greatest element is precisely the “sum” (\oplus) of all extremal points.

Let z be this greatest element, and let (K, L) be the partition such that $z_i < 1 - \lambda$ for all $i \in L$ and $z_i \geq 1 - \lambda$ for all $i \in K$. This partition is secure. For any other secure partition (K', L') there exists a (K', L') -eigenvector y (associated with λ), and we have $y \leq z$ and hence $y_i < 1 - \lambda$ for all $i \in L$ implying $L \subseteq L'$. \square

[Theorem 3.3](#), [Corollary 3.5](#) and [Proposition 3.3](#) show that all secure partitions of $\mathcal{D}(A^{(\lambda)})$ can be described by means of the following algorithm:

Algorithm 3.1 (Describing all secure partitions of $\mathcal{D}(A^{(\lambda)})$).

Part 1. For each $i \in \{1, \dots, n\}$, solve

$$\begin{aligned} \max x_i, \quad & 0 \leq x_i \leq 1 \quad \text{s.t.} \quad \forall i, \\ & A^{(1)} \otimes x \oplus \mathbf{0} = (\lambda - 1) \otimes x \oplus \mathbf{0} \end{aligned} \tag{51}$$

The vector \bar{x} whose components are solutions of (51) is the greatest Łukasiewicz eigenvector associated with λ . The partition (K, L) such that $\bar{x}_i < 1 - \lambda$ for all $i \in L$ and $\bar{x}_k \geq 1 - \lambda$ for $k \in K$ is secure partition with the least L .

Problem (51) is an instance of the tropical linear programming problems treated by Butkovic and Aminu [5,6] and Gaubert et al. [25]. It can be solved in pseudopolynomial time [5,6], by means of a pseudopolynomial number of calls to a mean-payoff game oracle [25].

Part 2. For each secure partition (K, L) , all secure partitions of the form $(K - k, L + k)$ can be found by means of shortest path algorithms with complexity at most $O(n^3)$ (for the classical Floyd–Warshall algorithm). All secure partitions can be then identified by means of a Depth First Search procedure.

Proof. Part 1. is based on Proposition 3.3.

For part 2., note that by Corollary 3.5, for each secure (K, L) with K nonempty, there exist greater secure partitions. By Theorem 3.3 part 2, for each secure partition (K, L) and for each $k \in K$, partition $(K - k, L + k)$ is secure if and only if k is secure $\mathcal{D}(A_{KK}^{(\lambda)})$. This holds if and only if the k th row of $(A_{KK}^{(\lambda)})^*$ has all components nonpositive. Matrix $(A_{KK}^{(\lambda)})^*$ can be computed in $O(n^3)$ operations. \square

Example. The example shows the work of the algorithm — describing all secure partitions of $\mathcal{D}(A^{(\lambda)})$.

Let $\lambda = 0.6$ and consider

$$A = \begin{pmatrix} 0.6 & 0.7 & 0.2 \\ 0.4 & 0.5 & 0.7 \\ 0.3 & 0.2 & 0.4 \end{pmatrix}. \tag{52}$$

Then we have

$$A^{(1)} = \begin{pmatrix} -0.4 & -0.3 & -0.8 \\ -0.6 & -0.5 & -0.3 \\ -0.7 & -0.8 & -0.6 \end{pmatrix}, \quad A^{(0.6)} = \begin{pmatrix} 0 & 0.1 & -0.4 \\ -0.2 & -0.1 & 0.1 \\ -0.3 & -0.4 & -0.2 \end{pmatrix}. \tag{53}$$

The Łukasiewicz eigenproblem with $\lambda = 0.6$ amounts to solving $A^{(1)}x \oplus \mathbf{0} = (-0.4 + x) \oplus \mathbf{0}$ subject to $0 \leq x_i \leq 1$ for all i .

According to the algorithm, we first need to find the greatest max-Łukasiewicz eigenvector. It can be verified that $(1 \ 0.8 \ 0.7)$ is a pure Łukasiewicz eigenvector (note that all components are greater than 0.4). As there is a pure eigenvector, it follows that the greatest max-Łukasiewicz eigenvector is also pure, and the minimal partition is the least partition $K = \{1, 2, 3\}$, $L = \{\emptyset\}$. In fact, $(1 \ 0.8 \ 0.7)$ is the greatest solution of the tropical Z-matrix equation $x = A^{(0.6)}x \oplus (0.4 + \mathbf{0})$ satisfying $x_i \leq 1$, hence this is exactly the greatest (pure) max-Łukasiewicz eigenvector. Moreover, it can be found that the pure eigenspace is the max-plus segment with extremal points $(1 \ 0.8 \ 0.7)$ and $(0.6 \ 0.5 \ 0.4)$.

The second part of the algorithm describes how to find all secure partitions. The algorithm starts with the secure partition with minimal L , according to (3.5), and tries to increase L by adding indices from K . In the beginning we have $A_{KK}^{(\lambda)} = A^{(0.6)}$ written above.

To add an index $i \in K$ to L , we have to check, whether i is secure in $\mathcal{D}(A_{KK}^{(\lambda)})$, i.e., whether the weight of every walk starting in i is nonpositive in $\mathcal{D}(A_{KK}^{(\lambda)})$. Index 1 is not secure, because the walk $p = (1, 2)$ has positive weight $w(p) = 0.1$, that is, 1 cannot be added to L . Similarly, 2 is not secure and cannot be added to L . Index 3 is secure, because any walk starting in 3 has non-positive weight. Hence the next secure partition is $K = \{1, 2\}$ and $L = \{3\}$. We find that

$$A_{KK}^{(0.6)} = \begin{pmatrix} 0 & 0.1 \\ -0.2 & -0.1 \end{pmatrix}, \quad (A_{KK}^{(0.6)})^* = \begin{pmatrix} 0 & 0.1 \\ -0.2 & 0 \end{pmatrix}. \tag{54}$$

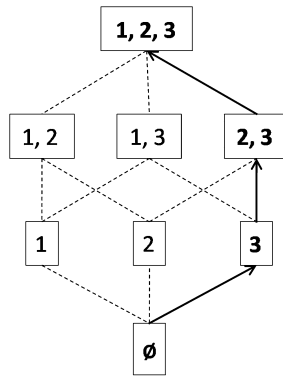


Fig. 5. Secure partition of $\mathcal{D}(A^{(\lambda)})$ (only sets L are shown).

Using Theorem 3.2 and Corollary 3.4 we find that the (K, L) eigenspace is the tropical convex hull of $u = (0.5 \ 0.4 \ 0)$, $v^{(3)} = (0.6 \ 0.5 \ 0.4)$ and $w^{(1)} = (0.7 \ 0.5 \ 0)$. (Note that $v^{(3)}$ is also a pure eigenvector. In general, some vectors can be considered as (K, L) eigenvectors for several choices of (K, L) .)

Similarly as above, index 1 is not secure in $\mathcal{D}(A_{KK}^{(0.6)})$, because of the walk $p = (1, 2)$ with $w(p) = 0.1$. On the other hand, 2 is secure in $\mathcal{D}(A_{KK}^{(0.6)})$. Thus index 2 can be added and we get further secure partition (K, L) with $K = \{1\}$ and $L = \{2, 3\}$. Then $A_{KK}^{(0.6)} = (A_{KK}^{(0.6)})^*$ consists just of one entry 0. Using Theorem 3.2 and Corollary 3.4 we find that the (K, L) eigenspace is the tropical convex hull of $u = (0.4 \ 0 \ 0)$, $v^{(2)} = (0.5 \ 0.4 \ 0)$, $v^{(3)} = (0.4 \ 0 \ 0.3)$ and $w^{(1)} = (0.6 \ 0 \ 0)$.

Index $1 \in K$ is secure in $\mathcal{D}(A_{KK}^{(0.6)})$ and we get the last secure partition (K, L) with $L = \{1, 2, 3\}$, giving the background eigenvectors (see Fig. 5). The greatest background eigenvector is $(0.4 \ 0.3 \ 0.3)$.

4. Powers and orbits

As we are going to show, any vector orbit in max-Łukasiewicz algebra is ultimately periodic. This ultimate periodicity can be described in terms of CSR representation of periodic powers $(A^{(1)})^t$. We assume the critical graph of A occupies the first c nodes. Note that the CSR terms computed for A^t and $(A^{(1)})^t$ are the same.

Observe that if $\rho(A) = 1$ then the critical graph $\mathcal{C}(A^{(1)})$ consists of all cycles, such that all entries of these cycles have zero weight. Using Theorem 2.5 we obtain the following.

Theorem 4.1. For any $A \in [0, 1]^{n \times n}$ and $x \in [0, 1]^n$ there exists a number $T(A)$ such that for all $t \geq T(A)$,

1. $A^{\otimes Lt} \otimes_L x = \mathbf{0}$ if $\rho(A) < 1$,
2. $A^{\otimes Lt} \otimes_L x = CS^t Rx \oplus \mathbf{0}$ if $\rho(A) = 1$. In this case $(A^{\otimes Lt} \otimes_L x)_i = S^t R(x \oplus \mathbf{0})_i$ for all $i = 1, \dots, c$.

Proof. We start by showing the following identity:

$$A^{\otimes Lt} \otimes_L x = (A^{(1)})^t x \oplus \mathbf{0}. \tag{55}$$

Indeed, iterating (1) we obtain

$$\begin{aligned} A^{\otimes Lt} \otimes_L x &= (A^{(1)}) (A^{\otimes Lt-1} \otimes_L x) \oplus \mathbf{0} \\ &= \dots \\ &= (A^{(1)})^t x \oplus ((A^{(1)})^{(t-1)} \mathbf{0} \oplus \dots \oplus (A^{(1)}) \mathbf{0} \oplus \mathbf{0}) \\ &= (A^{(1)})^t x \oplus \mathbf{0}. \end{aligned}$$

For the final reduction we used that all entries of $A^{(1)}$ are nonpositive, hence $(A^{(1)})^l \otimes \mathbf{0} \leq \mathbf{0}$ for all $l \geq 0$. Now we substitute the result of Theorem 2.2 observing that $\rho^t(A)CS^t Rx \leq \mathbf{0}$ for all large enough t if $\rho(A) < 1$.

If $\rho(A) = 1$ then for each t and $i = 1, \dots, c$, the i th row of $(A^{(1)})^t$ has a zero entry, from which it follows that $((A^{(1)})^t \otimes \mathbf{0})_i = 0$ for all large enough t and $i = 1, \dots, c$ implying that $((A^{(1)})^t \otimes x \oplus \mathbf{0})_i$ equals $(A^{(1)})^t(x \oplus \mathbf{0})_i$ and hence $S^t R(x \oplus \mathbf{0})_i$ for all big enough t . \square

It also follows that

$$A^{\otimes Lt} = (A^{(1)})^{t-1} A \oplus \mathbf{0}, \tag{56}$$

where $\mathbf{0}$ is the $n \times n$ matrix consisting of all zeros. Consequently, we obtain that if $\rho(A) < 1$ then $A^{\otimes Lt} = \mathbf{0}$ for sufficiently large t , and if not then

$$A^{\otimes Lt} = CS^{t-1}RA \oplus \mathbf{0} \tag{57}$$

For any $n \times n$ matrix A in max-Łukasiewicz algebra, we can define the “matrix of ones” $A^{[1]}$ by

$$a_{ij}^{[1]} = \begin{cases} 1, & \text{if } a_{ij} = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{58}$$

Theorem 4.2. *Orbits of vectors and matrix powers of a matrix A in max-Łukasiewicz algebra are ultimately periodic. They are ultimately zero if $\rho(A) < 1$. Otherwise if $\rho(A) = 1$,*

1. *The ultimate period of $\{A^{\otimes Lt} \otimes_L x\}$ divides the cyclicity γ of the critical graph $C(A)$.*
2. *The ultimate period of $\{A^{\otimes Lt}\}$ is equal to γ .*

Proof. 1.: Follows from Theorem 4.1. 2.: Observe that $A \geq A^{[1]}$, and that $(A^{\otimes Lt})^{[1]} = (A^{[1]})^{\otimes Lt}$. Also, the Łukasiewicz product of two $\{0, 1\}$ matrices coincides with their product in the Boolean algebra. This implies that if $\rho(A) = 1$ (i.e., the “graph of ones” has nontrivial strongly connected components), then $(A^{\otimes Lt})^{[1]}$ is nonzero for any t , and so is $A^{\otimes Lt}$. As the period of $(A^{\otimes Lt})^{[1]} = (A^{[1]})^{\otimes Lt}$ is γ , the period of $A^{\otimes Lt}$ cannot be less. Eq. (57), expressing the ultimate powers of A in the CSR form, assures that it cannot be more than γ . \square

Proposition 4.1. *The exact ultimate period of $\{A^{\otimes Lt} \otimes_L x\}$ can be computed in $O(n^3 \log n)$ time.*

Proof. First we find $\rho(A)$, in no more than $O(n^3)$ time. If $\rho(A) < 1$ then all orbits converge to $\mathbf{0}$. Otherwise, by Theorem 4.1, the subvector of $A^{\otimes Lt} \otimes_L x$ extracted from the first c indices equals $S^t R(x \oplus \mathbf{0})$. More generally,

$$\begin{aligned} A^{\otimes Lt} \otimes_L x &= CS^t Rx \oplus \mathbf{0} \\ &= C(S^t R(x \oplus \mathbf{0})) \oplus \mathbf{0}, \end{aligned} \tag{59}$$

since $CS^t R\mathbf{0} \leq \mathbf{0}$ (observe that none of the entries of $A^{(1)}$ and hence any of its powers exceed zero). This shows that the last $(n - c)$ components of $A^{\otimes Lt} \otimes_L x$ are tropical affine combinations of the first (critical) coordinates, and hence we only need to determine the period of $S^t R(x \oplus \mathbf{0})$, which is the ultimate period of the first c coordinates of $(A^{(1)})^t(x \oplus \mathbf{0})$. The latter period can be computed in $O(n^3 \log n)$ time by means of an algorithm described in [37], see also [5,29]. \square

Butkovič [5] and Sergeev [38] studied the so-called attraction cones in tropical algebra, that is, sets of vectors x such that the orbit $A^t x$ hits an eigenvector of A at some t .

In the case of max-Łukasiewicz algebra, **attraction sets** can be defined similarly. Then either $\rho(A) < 1$ and then all orbits converge to 0, or $\rho(A) = 1$ and then there may be a non-trivial periodic regime, in which the non-critical components of $A^{\otimes Lt} \otimes_L x$ are tropical affine combinations of the critical ones. It follows that (like in max algebra) the convergence of $A^{\otimes Lt} \otimes_L x$ to an eigenvector with eigenvalue 1 is determined by critical components only. Since these are given by $S^t R(x \oplus \mathbf{0})$, the attraction sets in max-Łukasiewicz algebra are solution sets to

$$Rx \oplus \mathbf{0} = SRx \oplus \mathbf{0}, \quad 0 \leq x_i \leq 1 \forall i. \tag{60}$$

This system of equations can be analysed as in [38], a task that we postpone to a future work.

Example. Let us consider the following matrix

$$A = \begin{pmatrix} 0.62 & 1.00 & 0.57 & 0.14 \\ 1.00 & 0.18 & 0.17 & 0.18 \\ 0.38 & 0.59 & 0.65 & 0.43 \\ 0.10 & 0.18 & 0.25 & 0.33 \end{pmatrix} \quad (61)$$

Its max-Łukasiewicz powers proceed as follows:

$$A^{\otimes_L 2} = \begin{pmatrix} 1 & 0.62 & 0.22 & 0.18 \\ 0.62 & 1 & 0.57 & 0.14 \\ 0.59 & 0.38 & 0.30 & 0.08 \\ 0.18 & 0.10 & 0 & 0 \end{pmatrix}, \quad A^{\otimes_L 3} = \begin{pmatrix} 0.62 & 1 & 0.57 & 0.14 \\ 1 & 0.62 & 0.22 & 0.18 \\ 0.38 & 0.59 & 0.16 & 0 \\ 0.10 & 0.18 & 0 & 0 \end{pmatrix},$$

$$A^{\otimes_L 4} = \begin{pmatrix} 1 & 0.62 & 0.22 & 0.18 \\ 0.62 & 1 & 0.57 & 0.14 \\ 0.59 & 0.38 & 0 & 0 \\ 0.18 & 0.10 & 0 & 0 \end{pmatrix}, \quad A^{\otimes_L 5} = \begin{pmatrix} 0.62 & 1 & 0.57 & 0.14 \\ 1 & 0.62 & 0.22 & 0.18 \\ 0.38 & 0.59 & 0.16 & 0 \\ 0.10 & 0.18 & 0 & 0 \end{pmatrix}$$

We see that the sequence $\{A^{\otimes_L t}\}_{t \geq 1}$ is ultimately periodic with period 2, and the periodicity starts at $t = 3$. The term $(A^{(1)})^{t-1}A = CS^{t-1}RA$ of (56) dominates in all the columns and rows of $A^{\otimes_L t}$ with critical indices $(1, 2)$. In the periodic regime ($t \geq 3$), the zero term dominates in almost all entries of the non-critical $A_{MM}^{\otimes_L t}$ where $M = \{3, 4\}$.

Acknowledgements

We would like to thank Dr. Imran Rashid for useful discussions that have led to the idea of using the max-plus linear algebra and tropical convexity in this context. We also thank the anonymous referee for a number of useful comments.

References

- [1] M. Akian, R. Bapat, S. Gaubert, Min-plus methods in eigenvalue perturbation theory and generalized Lidskii–Višik–Ljusternik theorem, 2004–2006, E-print: [arXiv:math/0402090v3](http://arxiv.org/abs/math/0402090v3).
- [2] M. Akian, R. Bapat, S. Gaubert, Max-plus algebras, in: L. Hogben (Ed.), *Handbook of Linear Algebra*, in: *Discrete Mathematics and Its Applications*, vol. 39, Chapman and Hall/CRC, 2006, chapter 25.
- [3] X. Allamigeon, Static analysis of memory manipulations by abstract interpretation — algorithmics of tropical polyhedra, and application to abstract interpretation, PhD thesis École Polytechnique, Palaiseau, France, November 2009, <http://www.lix.polytechnique.fr/Labo/Xavier.Allamigeon/papers/thesis.pdf>.
- [4] F.L. Baccelli, G. Cohen, G.J. Olsder, J.P. Quadrat, *Synchronization and Linearity: An Algebra for Discrete Event Systems*, Wiley, 1992.
- [5] P. Butkovič, *Max-Linear Systems: Theory and Algorithms*, Springer, 2010.
- [6] P. Butkovič, A. Aminu, Introduction to max-linear programming, *IMA J. Manag. Math.* (2008).
- [7] P. Butkovič, H. Schneider, S. Sergeev, Generators, extremals and bases of max cones, *Linear Algebra Appl.* 421 (2007) 394–406.
- [8] P. Butkovič, H. Schneider, S. Sergeev, Z-matrix equations in max-algebra, nonnegative linear algebra and other semirings, *Linear Multilinear Algebra* 60 (10) (2012) 1191–1210, [arXiv:1110.4564](http://arxiv.org/abs/1110.4564).
- [9] Z.-Q. Cao, K.H. Kim, F.W. Roush, *Incline Algebra and Its Applications*, Chichester, 1984.
- [10] B.A. Carré, An algebra for network routing problems, *J. Inst. Math. Appl.* 7 (1971).
- [11] K. Ceclárová, Powers of matrices over distributive lattices — a review, *Fuzzy Sets Syst.* 138 (3) (2003) 627–641.
- [12] G. Cohen, D. Dubois, J.P. Quadrat, M. Viot, A linear-system theoretic view of discrete-event processes and its use for performance evaluation in manufacturing, *IEEE Trans. Autom. Control* 30 (3) (1985) 210–220.
- [13] G. Cohen, S. Gaubert, J.P. Quadrat, I. Singer, Max-plus convex sets and functions, in: G. Litvinov, V. Maslov (Eds.), *Idempotent Mathematics and Mathematical Physics*, in: *Contemporary Mathematics*, vol. 377, AMS, Providence, 2005, pp. 105–129, E-print: [arXiv:math.FA/0308166](http://arxiv.org/abs/math.FA/0308166).
- [14] R.A. Cuninghame-Green, *Minimax Algebra*, *Lecture Notes in Economics and Mathematical Systems*, vol. 166, Springer, Berlin, 1979.
- [15] R.A. Cuninghame-Green, K. Ceclárová, Residuation in fuzzy algebra and some applications, *Fuzzy Sets Syst.* 71 (2) (1995) 227–239.
- [16] B. De Schutter, On the ultimate behavior of the sequence of consecutive powers of a matrix in the max-plus algebra, *Linear Algebra Appl.* 307 (2000) 103–117.
- [17] M. Develin, B. Sturmfels, Tropical convexity, *Doc. Math.* 9 (2004) 1–27, E-print: [arXiv:math.MG/0308254](http://arxiv.org/abs/math.MG/0308254).

- [18] A. Di Nola, B. Gerla, Algebras of Łukasiewicz's logic and their semiring reducts, in: G.L. Litvinov, V.P. Maslov (Eds.), *Idempotent Mathematics and Mathematical Physics*, 2005, pp. 131–144.
- [19] A. Di Nola, A. Lettieri, I. Perfilieva, V. Novák, Algebraic analysis of fuzzy systems, *Fuzzy Sets Syst.* 158 (1) (2007) 1–22.
- [20] A. Di Nola, C. Russo, Łukasiewicz transform and its application to compression and reconstruction of digital images, *Inf. Sci.* 177 (2007) 1481–1498.
- [21] A. Di Nola, C. Russo, Semiring and semimodule issues in MV-algebras, *Commun. Algebra* 41 (2013) 1017–1048.
- [22] M. Gavalec, Linear matrix period in max-plus algebra, *Linear Algebra Appl.* 307 (2000) 167–182.
- [23] M. Gavalec, Periodicity in Extremal Algebras, *Gaudeamus, Hradec Králové*, 2004.
- [24] S. Gaubert, R.D. Katz, The Minkowski theorem for max-plus convex sets, *Linear Algebra Appl.* 421 (2–3) (2007) 356–369, E-print: arXiv:math.GM/0605078.
- [25] S. Gaubert, R.D. Katz, S. Sergeev, Tropical linear-fractional programming and parametric mean-payoff games, *J. Symb. Comput.* 47 (12) (2012) 1447–1478, E-print: arXiv:1101.3431.
- [26] M. Gondran, M. Minoux, *Graphs, Dioids and Semirings: New Applications and Algorithms*, Springer, 2008.
- [27] M. Gondran, M. Minoux, Dioids and semirings: links to fuzzy sets and other applications, *Fuzzy Sets Syst.* 158 (2008) 1273–1294.
- [28] S.-C. Han, H.-X. Li, Invertible incline matrices and Cramer's rule over inclines, *Linear Algebra Appl.* 389 (2004) 121–138.
- [29] M. Hartmann, C. Arguelles, Transience bounds for long walks, *Math. Oper. Res.* 24 (2) (1999) 414–439.
- [30] B. Heidergott, G.-J. Olsder, J. van der Woude, *Max-Plus at Work*, Princeton Univ. Press, 2005.
- [31] E.P. Klement, R. Mesiar, E. Pap, *Triangular Norms*, Kluwer Academic Publ., Dordrecht, 2000.
- [32] N.K. Krivulin, On solution of generalized linear vector equations in idempotent algebra, *Vestn. St. Petersburg. Univ., Math.* 39 (2006) 16–26.
- [33] N.K. Krivulin, Methods OPF Idempotent Algebra in the Problems of Modeling and Analysis of Complex Systems, *St.-Petersburg Univ. Publ.*, 2009 (in Russian).
- [34] G.L. Litvinov, V.P. Maslov, The correspondence principle for idempotent calculus and some computer applications, in: J. Gunawardena (Ed.), *Idempotency*, Cambridge Univ. Press, 1998, pp. 420–443, E-print: arXiv:math/0101021.
- [35] M. Molnárová, Generalized matrix period in max-plus algebra, *Linear Algebra Appl.* 404 (2005) 345–366.
- [36] I. Rashid, M. Gavalec, S. Sergeev, Eigenspace of a three-dimensional max-Łukasiewicz fuzzy matrix, *Kybernetika* 48 (2) (2012) 309–328.
- [37] S. Sergeev, Max algebraic powers of irreducible matrices in the periodic regime: an application of cyclic classes, *Linear Algebra Appl.* 431 (6) (2009) 1325–1339.
- [38] S. Sergeev, Max-algebraic attraction cones of nonnegative irreducible matrices, *Linear Algebra Appl.* 435 (7) (2011) 1736–1757.
- [39] S. Sergeev, H. Schneider, CSR expansions of matrix powers in max algebra, *Trans. Am. Math. Soc.* 364 (2012) 5969–5994, E-print: arXiv:0912.2534.
- [40] Y.-J. Tan, On generalized fuzzy matrices with periods, *Fuzzy Sets Syst.* 172 (1) (2011) 87–103.