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Gevrey Asymptotic Implicit Function Theorem

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Abstract

We prove an Asymptotic Implicit Function Theorem in the setting of Gevrey asymptotics with respect to a parameter. The unique implicitly defined solution admits a Gevrey asymptotic expansion and furthermore it is the Borel resummation of the corresponding implicitly defined formal power series solution. The main theorem can therefore be rephrased as an Implicit Function Theorem for Borel summable power series. As an application, we give a diagonal or Jordan decomposition for holomorphic matrices in Gevrey asymptotic families.

Contents

§ 1. Introduction

Suppose $F(x, \hbar, z)$ is a holomorphic, perhaps vector-valued, function of several complex variables x and z and a small complex perturbation parameter $\hbar$ constrained to some sector at the origin in the $\hbar$-plane where F admits an asymptotic expansion $\widehat{F}(x, \hbar, z)$ as $\hbar \to 0$. This paper arose from the following question: what is the meaning of a formal $\hbar$-power series solution $z = \widehat{f}(x, \hbar)$ of the formal equation $\widehat{F}(x, \hbar, z) = 0$? The answer we find is that, provided sufficient control on the asymptotics of F, the formal solution $\widehat{f}$ is the asymptotic expansion of an actual solution $z = f(x, \hbar)$ of the analytic equation $F(x, \hbar, z) = 0$, and furthermore f is the Borel resummation of $\widehat{f}$. Thus, the purpose of this article is to prove the following version of the Implicit Function Theorem in the setting of Gevrey asymptotics.

1.1. Theorem (Gevrey Asymptotic Implicit Function Theorem).
Fix a point $(x_0, z_0) \in \mathbb{C}_x^d \times \mathbb{C}_z^N$ with $d \geqslant 0$ and $N \geqslant 1$. Let $\mathsf{X} \subset \mathbb{C}_x^d$ be a domain containing x_0 and $\mathsf{S} \subset \mathbb{C}_\hbar$ a sectorial domain with vertex at the origin and opening arc Θ with opening angle $|\Theta| = \pi$. Suppose F is a holomorphic map $\mathsf{X} \times \mathsf{S} \times \mathbb{C}_z^N \to \mathbb{C}^N$ which admits a Gevrey asymptotic expansion

$$F(x, \hbar, z) \simeq \widehat{F}(x, \hbar, z) = \sum_{k=0}^{\infty} F_k(x, z) \hbar^k \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta} \quad , \tag{1}$$

uniformly for all $x \in \mathsf{X}$ and locally uniformly for all $z \in \mathbb{C}_z^N$. Suppose its leading-order part in $\hbar$ satisfies $F_0(x_0, z_0) = 0$ and the Jacobian $\partial F_0 / \partial z$ is invertible at (x_0, z_0).

Then there is a subdomain $\mathsf{X}_0 \subset \mathsf{X}$ containing x_0 and a sectorial subdomain $\mathsf{S}_0 \subset \mathsf{S}$ with the same opening Θ such that there is a unique holomorphic map $f : \mathsf{X}_0 \times \mathsf{S}_0 \to \mathbb{C}^N$ which admits a Gevrey asymptotic expansion

$$f(x, \hbar) \simeq \widehat{f}(x, \hbar) = \sum_{n=0}^{\infty} f_n(x) \hbar^n \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta} \quad , \tag{2}$$

uniformly for all $x \in \mathsf{X}_0$, and such that

$$f_0(x_0) = z_0 \qquad \text{and} \qquad F\big(x, \hbar, f(x, \hbar)\big) = 0 \qquad \forall (x, \hbar) \in \mathsf{X}_0 \times \mathsf{S}_0 \ . \tag{3}$$

Furthermore, f is the uniform Borel resummation of $\widehat{f}$ in the direction θ that bisects the arc Θ: for all $(x, \hbar) \in \mathsf{X}_0 \times \mathsf{S}_0$,

$$f(x, \hbar) = \mathcal{S}_\theta \big[\, \widehat{f}\, \big](x, \hbar) \ . \tag{4}$$

This theorem provides a general answer in a large class of problems to the question of developing a theory of asymptotic implicit function theorems. Such a question in a specialised setting was posed by Gérard and Jurkat in [GJ92, p.45], but to the best of our knowledge has not been addressed[1]. In addition, our techniques give a much more refined information about the implicit function f, chiefly its uniform Borel summability properties.

1.2. Application: Linear Algebra in Gevrey Asymptotic Families. As an application, which serves as the main source of motivation for us, Theorem 1.1 can be used to diagonalise holomorphic matrices $A(x, \hbar)$ in uniform Gevrey asymptotic families; i.e., via transformations with the same regularity as A. This means that the eigenvalues and the eigenspaces of A are guaranteed to have the same asymptotic behaviour as $\hbar \to 0$ as the matrix A itself. More precisely, we prove the following diagonalisation theorem when the leading-order eigenvalues of A are all distinct, which follows from the more general Jordan block decomposition Theorem 3.1.

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[1]In particular, the promised second part of their 1992 paper [GJ92] has not appeared.

1.3. Theorem (Diagonalisation in Gevrey Asymptotic Families).

Fix a domain $\mathsf{X} \subset \mathbb{C}^d_x$ and a point $x_0 \in \mathsf{X}$. Let $\mathsf{S} \subset \mathbb{C}_\hbar$ be a sectorial domain at the origin and opening arc Θ with opening angle $|\Theta| = \pi$. Let $A = A(x, \hbar)$ be a holomorphic $n{\times}n$-matrix on $\mathsf{X} \times \mathsf{S}$ which admits a uniform Gevrey asymptotic expansion

$$A(x, \hbar) \simeq \widehat{A}(x, \hbar) = \sum_{k=0}^\infty A_k(x)\hbar^k \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta}, \text{ unif. } \forall x \in \mathsf{X} \ . \tag{5}$$

Suppose that the $\hbar$-leading-order part $A_{00} := A_0(x_0)$ at the point x_0 has distinct eigenvalues $a_1, \dots, a_n \in \mathbb{C}$. Let P_{00} be an invertible $n{\times}n$-matrix that diagonalises A_{00}:

$$P_{00}A_{00}P_{00}^{-1} = \mathrm{diag}\,(a_1, \dots, a_n) \ . \tag{6}$$

Then there is a subdomain $\mathsf{X}_0 \subset \mathsf{X}$ containing x_0 and a sectorial subdomain $\mathsf{S}_0 \subset \mathsf{S}$ with the same opening Θ such that there is a unique holomorphic invertible $n{\times}n$-matrix $P = P(x, \hbar)$ on $\mathsf{X}_0 \times \mathsf{S}_0$ that admits a uniform Gevrey asymptotic expansion

$$P(x, \hbar) \simeq \widehat{P}(x, \hbar) = \sum_{k=0}^\infty P_k(x)\hbar^k \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta}, \text{ unif. } \forall x \in \mathsf{X}_0 \ , \tag{7}$$

such that $P_0(x_0) = P_{00}$ and which diagonalises A; i.e.,

$$PAP^{-1} = \mathrm{diag}\,(\lambda_1, \dots, \lambda_n) \ . \tag{8}$$

Furthermore, the transformation P is the uniform Borel resummation of its asymptotic power series $\widehat{P}$ in the direction θ that bisects the arc Θ: for all $(x, \hbar) \in \mathsf{X}_0 \times \mathsf{S}_0$,

$$P = \mathcal{S}_\theta\big[\widehat{P}\big] \ . \tag{9}$$

In addition:

(1) *The eigenvalues $\lambda_i = \lambda_i(x, \hbar)$ of A are holomorphic functions on $\mathsf{X}_0 \times \mathsf{S}_0$ that admit uniform Gevrey asymptotic expansions*

$$\lambda_i(x, \hbar) \simeq \widehat{\lambda}_i(x, \hbar) = \sum_{k=0}^\infty \lambda_{i,k}(x)\hbar^k \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta}, \text{ unif. } \forall x \in \mathsf{X}_0 \ , \tag{10}$$

with $\lambda_{i,0}(x_0) = a_i$. Moreover, each eigenvalue λ_i is the uniform Borel resummation of its asymptotic series $\widehat{\lambda}_i$ in the direction θ: for all $(x, \hbar) \in \mathsf{X}_0 \times \mathsf{S}_0$,

$$\lambda_i = \mathcal{S}_\theta\big[\widehat{\lambda}_i\big] \ . \tag{11}$$

(2) *Given an eigenbasis $v_1, \dots, v_n \in \mathbb{C}^n$ for A_{00}, there is a unique eigenbasis $e_1, \dots, e_n$ for A consisting of holomorphic vectors $e_i = e_i(x, \hbar)$ on $\mathsf{X}_0 \times \mathsf{S}_0$ that admit uniform Gevrey asymptotic expansions*

$$e_i(x, \hbar) \simeq \widehat{e}_i(x, \hbar) = \sum_{k=0}^\infty e_{i,k}(x)\hbar^k \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta}, \text{ unif. } \forall x \in \mathsf{X}_0 \ , \tag{12}$$

with $e_{i,0}(x_0) = v_i$. Moreover, each eigenvector e_i is the uniform Borel resummation of its asymptotic series $\widehat{e}_i$ in the direction θ: for all $(x, \hbar) \in \mathsf{X}_0 \times \mathsf{S}_0$,

$$e_i = \mathcal{S}_\theta\big[\widehat{e}_i\big] \ . \tag{13}$$

Such results are useful in the exact WKB analysis of singularly perturbed meromorphic differential systems and more generally singularly perturbed meromorphic connections on holomorphic vector bundles over Riemann surfaces. There, the role

of $A(x, \hbar)$ is played by the principal part of the differential system at a pole. For example, see [Nik19] for the analysis of rank-two systems near a logarithmic pole.

1.4. Scalar polynomial case. Particularly notable for its simplicity and utility is the special case of Theorem 1.1 where $N = 1$ and F is a polynomial in the single variable z. We restate it under these assumptions for ease of reference.

1.5. Corollary. *Fix a domain* $\mathsf{X} \subset \mathbb{C}_x^d$. *Let* $\mathsf{S} \subset \mathbb{C}_\hbar$ *be a sectorial domain at the origin and opening arc* Θ *with opening angle* $|\Theta| = \pi$. *Consider a polynomial*

$$F = a_0 + a_1 z + \ldots + a_n z^n \tag{14}$$

whose coefficients $a_0, \ldots, a_m$ *are holomorphic functions of* $(x, \hbar) \in \mathsf{X} \times \mathsf{S}$ *which admit uniform Gevrey asymptotic expansions*

$$a_i(x, \hbar) \simeq \widehat{a}_i(x, \hbar) = \sum_{k=0}^{\infty} a_{i,k}(x) \hbar^k \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta}, \text{ unif.} \forall x \in \mathsf{X} . \tag{15}$$

Suppose that the leading-order discriminant

$$D_0 = D_0(x) := \mathrm{Disc}_x\left(F_0\right) = \mathrm{Disc}_x\left(a_{0,0} + a_{1,0} z + \ldots + a_{n,0} z^n\right)$$

is nonvanishing on X. *Let* $z = f_0$ *be a leading-order solution on* X; *i.e., a holomorphic function* $f_0(x)$ *on* X *such that* $F_0\left(x, f_0(x)\right)$ *for all* $x \in \mathsf{X}'$. *Then for any compactly contained subdomain* $\mathsf{X}_0 \subset \mathsf{X}$, *there is a sectorial subdomain* $\mathsf{S}_0 \subset \mathsf{S}$ *with the same opening* Θ *such that the polynomial* F *has a unique root* $z = f(x, \hbar)$ *which is a holomorphic function on* $\mathsf{X}_0 \times \mathsf{S}_0$ *and admits a uniform Gevrey asymptotic expansion* (2) *with leading-order being the leading-order solution* f_0. *Furthermore,* f *is the uniform Borel resummation of* $\widehat{f}$ *in the direction* θ *that bisects* Θ.

1.6. The asymptotic conditions (1) and (2) mean that both formal power series $\widehat{F}$ and $\widehat{f}$ are Borel-summable series in the direction θ, so Theorem 1.1 can be rephrased as an Implicit Function Theorem in the setting of Borel-summable series.

1.7. Corollary (Implicit Function Theorem for Borel-Summable Series).
Fix a point $(x_0, z_0) \in \mathbb{C}_x^d \times \mathbb{C}_z^N$, *a domain* $\mathsf{X} \subset \mathbb{C}_x^d$ *containing* x_0, *and a direction* θ. *Let*

$$\widehat{F} = \widehat{F}(x, \hbar, z) = \sum_{k=0}^{\infty} F_k(x, z) \hbar^k \tag{16}$$

be a formal power series in $\hbar$ *whose coefficients* F_k *are holomorphic maps* $\mathsf{X} \times \mathbb{C}_z^N \to \mathbb{C}^N$ *such that* $F_0(x_0, z_0) = 0$ *and the Jacobian* $\partial F_0 / \partial z$ *is invertible at* (x_0, z_0). *Suppose* $\widehat{F}$ *is Borel-summable in the direction* θ *uniformly for all* $x \in \mathsf{X}$ *and locally uniformly for all* $z \in \mathbb{C}_z^N$. *Then there is a subdomain* $\mathsf{X}_0 \subset \mathsf{X}$ *such that the unique formal series*

$$\widehat{f} = \widehat{f}(x, \hbar) = \sum_{n=0}^{\infty} f_n(x) \hbar^n , \tag{17}$$

with holomorphic coefficients $f_n : \mathsf{X}_0 \to \mathbb{C}^N$, *which satisfies* $f_0(x_0) = z_0$ *as well as* $\widehat{F}\left(x, \hbar, \widehat{f}(x, \hbar)\right) = 0$, *is Borel-summable in the direction* θ *uniformly for all* $x \in \mathsf{X}_0$.

After the initial release of this manuscript on the arXiv, we were alerted that a special case of this corollary (with $d = 0, N = 1$) was proved earlier by Kamimoto and Koike in [KK11, Appendix A] using a slightly different strategy.

1.8. Remarks and discussion. Our construction of the implicit function f employs relatively basic and classical techniques from complex analysis which form the basis for the more modern and sophisticated theory of resurgent asymptotic analysis à la Écalle [Éca85]; see also for instance [Cos09, Sau14, LR16]. Namely, we use the *Borel-Laplace method*, also known as the theory of *Borel-Laplace summability*. We stress that the Borel-Laplace method "is nothing other than the theory of Laplace transforms, written in slightly different variables", echoing the words of Alan Sokal [Sok80]. As such, we have tried to keep our presentation very hands-on and self-contained, so the knowledge of basic complex analysis should be sufficient to follow.

We emphasise that the asymptotic condition (1) on F is required to hold over the *closed* arc $\overline{\Theta} = [\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}]$, which is stronger than ordinary Gevrey asymptotics along an open arc Θ (see §A.1 or [Nik20, Appendix A.5 and A.16] for more details). This type of condition is exactly adapted to the Borel-Laplace method, see §A.2. Similar methods are also used in the construction of exact WKB solutions for singularly perturbed ODEs such as the Schrödinger equation [Nik21].

What we call *Gevrey asymptotics* is often called 1-*Gevrey asymptotics*. It is part of an entire hierarchy of asymptotic regularity classes [Ram78, Ram80]; see also [LR16, §1.2]. However, arguments about other Gevrey classes can usually be reduced to arguments about 1-Gevrey asymptotics via a simple fractional transformation in the $\hbar$-space. Therefore, we believe it is not difficult to extend our results to all other Gevrey asymptotic classes. We leave this as a natural open problem.

1.9. Structure of the Paper. The proof of Theorem 1.1 makes up all of §2. Then in §3, we prove Theorem 1.3 as well as its generalisation to Jordan blocks. For pedagogical reasons, in Appendix B we present the entire proof in the scalar case $N = 1$. Although the contents of §2 are strictly more general than those of Appendix B, the two sections have been written in an entirely independent manner without any reference to each other. We recommend the reader to begin with Appendix B because it contains more or less all the essential ideas in the proof of the general vectorial case but without the added complication of having to keep track of many extra indices.

1.10. Notation and conventions. Our notation, conventions, and definitions from Gevrey asymptotics and Borel-Laplace theory are consistent with those given in Appendices A and B in [Nik20]. A brief summary can be found in Appendix A.

Throughout, we fix integers $d \geqslant 0$ and $N \geqslant 1$, and we write the vector components as $x = (x_1, \ldots, x_d)$, $z = (z_1, \ldots, z_N)$, $F = (F^1, \ldots, F^N)$, $f = (f^1, \ldots, f^N)$. The symbol $\mathbb{N}$ stands for nonnegative integers $0, 1, 2, \ldots$. We use boldface letters to denote index vectors; i.e., $\boldsymbol{m} := (m_1, \ldots, m_N) \in \mathbb{N}^N$, etc., and we put $|\boldsymbol{m}| := m_1 + \cdots + m_N$. Unless otherwise indicated, all sums over unbolded indices $n, m, \ldots$ are taken to run over $\mathbb{N}$, and all sums over boldface letters $\boldsymbol{n}, \boldsymbol{m}, \ldots$ are taken to run over $\mathbb{N}^N$. Throughout this paper, we often suppress the explicit dependance on x in the notation in the interest of brevity.

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§ 2. Proof of Theorem 1.1

This section is dedicated to proving our main result, the Gevrey Asymptotic Implicit Function Theorem (Theorem 1.1). The overall strategy of the proof is as follows. First, we construct a formal solution $z = \widehat{f}$ of the equation $F(x, \hbar, z) = 0$ using the ordinary Holomorphic Implicit Function Theorem at the leading-order in $\hbar$ and then using a recursion to determine all higher-order corrections. We then want to apply the Borel resummation to $\widehat{f}$ to get f. To do so, we first make a convenient change of variables $z \mapsto w$ in order to put our equation into a certain standard form which is more amenable to the Borel transform. Applying the Borel transform, we obtain a first-order ordinary differential equation for $\sigma = \mathfrak{B}[w]$, albeit nonlinear and with convolution. Nevertheless, this ODE is easy to convert into an integral equation, which we then proceed to solve using the method of successive approximations. To show that this sequence of approximations converges to an actual solution σ, we give an estimate on the terms of this sequence by employing in an interesting way the ordinary Holomorphic Implicit Function Theorem. This estimate also allows us to conclude that the Laplace transform $g = \mathfrak{L}[\sigma]$ of the obtained solution σ exists and defines a holomorphic solution of our equation in standard form. Undoing the change of variables $z \mapsto w$ sends g to the desired solution f.

The proof is split into several intermediate lemmas. All this work is finally put together on page 21.

§ 2.1. Formal Perturbation Theory

The starting point is the following classical result whose proof is supplied below for completeness and in order to introduce some helpful notation.

2.1. Proposition (Formal Implicit Function Theorem).
Fix a domain $\mathsf{X} \subset \mathbb{C}_x^d$ and a point $(x_0, z_0) \in \mathsf{X} \times \mathbb{C}_z^N$. Let

$$\widehat{F} = \widehat{F}(x, \hbar, z) = \sum_{k=0}^{\infty} F_k(x, z) \hbar^k \tag{18}$$

be a formal power series in $\hbar$ whose coefficients F_k are holomorphic maps $\mathsf{X} \times \mathbb{C}_z^N \to \mathbb{C}^N$ such that $F_0(x_0, z_0) = 0$ and the Jacobian matrix $\partial F_0 / \partial z$ is invertible at (x_0, z_0).

Then there is subdomain $\mathsf{X}_0 \subset \mathsf{X}$ containing x_0 such that there is a unique formal power series

$$\widehat{f} = \widehat{f}(x, \hbar) = \sum_{n=0}^{\infty} f_n(x) \hbar^n \tag{19}$$

whose coefficients f_n are holomorphic maps $\mathsf{X}_0 \to \mathbb{C}^N$, satisfying

$$f_0(x_0) = z_0 \quad \text{and} \quad \widehat{F}\big(x, \hbar, \widehat{f}(x, \hbar)\big) = 0 \quad \forall x \in \mathsf{X}_0 \,. \tag{20}$$

In other words, the equation $\widehat{F}(x, \hbar, z) = 0$ has a unique solution $z = \widehat{f}$ defined near the point x_0 such that $f_0(x_0) = z_0$. In fact, all the higher-order coefficients f_k are uniquely determined by f_0.

In particular, if $\mathsf{S} \subset \mathbb{C}_\hbar$ is a sectorial domain at the origin, and F is a holomorphic map $\mathsf{X} \times \mathsf{S} \times \mathbb{C}_z^N \to \mathbb{C}^N$ which admits the power series $\widehat{F}$ as an asymptotic expansion as $\hbar \to 0$ in S, uniformly in x and locally uniformly in z, then the equation $F(x, \hbar, z) = 0$ has a unique formal power series solution $z = \widehat{f}$ near x_0 such that $f_0(x_0) = z_0$.

Proof. The proof amounts to plugging the solution ansatz (19) into the formal equation $\widehat{F}(x, \hbar, z) = 0$ and solving order-by-order in $\hbar$. First, let us note down a few formulas in order to proceed with the calculation. See also Part 1.10.

STEP 0: COLLECT SOME FORMULAS. Write the double power series expansion of each component $\widehat{F}^i$ as

$$\widehat{F}^i(x, \hbar, z) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} F_{k\boldsymbol{m}}^i(x)\hbar^k z^{\boldsymbol{m}} \ , \tag{21}$$

where $F_{k\boldsymbol{m}}^i z^{\boldsymbol{m}} := F_{km_1\cdots m_N}^i z_1^{m_1}\cdots z_N^{m_N}$. In particular, the expansion of the leading-order part F_0 is

$$F_0^i(x, z) = \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} F_{0\boldsymbol{m}}^i(x) z^{\boldsymbol{m}} \ . \tag{22}$$

For every $\boldsymbol{m} \in \mathbb{N}^N$, we have $\frac{\partial}{\partial z_j} z^{\boldsymbol{m}} = \frac{m_j}{z_j} z^{\boldsymbol{m}}$, so the (i,j)-component of the Jacobian matrix $\partial F_0/\partial z$ can be written as

$$\left[\frac{\partial F_0}{\partial z}\right]_{ij} = \frac{\partial F_0^i}{\partial z_j} = \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} F_{0\boldsymbol{m}}^i(x)\frac{\partial}{\partial z_j} z^{\boldsymbol{m}} = \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \frac{m_j}{z_j} F_{0\boldsymbol{m}}^i(x) z^{\boldsymbol{m}} \ . \tag{23}$$

Next, the $\boldsymbol{m}$-th power $\widehat{f}^{\boldsymbol{m}}$ of the power series ansatz (19) expands as follows:

$$\left(\sum_{n=0}^{\infty} f_n \hbar^n\right)^{\boldsymbol{m}} = \left(\sum_{n_1=0}^{\infty} f_{n_1}^1 \hbar^{n_1}\right)^{m_1} \cdots \left(\sum_{n_N=0}^{\infty} f_{n_N}^N \hbar^{n_N}\right)^{m_N}$$

$$= \left(\sum_{n_1=0}^{\infty} \sum_{|\boldsymbol{j}_1|=n_1}^{\boldsymbol{j}_1 \in \mathbb{N}^{m_1}} f_{j_{1,1}}^1 \cdots f_{j_{1,m_1}}^1 \hbar^{n_1}\right) \cdots \left(\sum_{n_N=0}^{\infty} \sum_{|\boldsymbol{j}_N|=n_N}^{\boldsymbol{j}_N \in \mathbb{N}^{m_N}} f_{j_{N,1}}^N \cdots f_{j_{N,m_N}}^N \hbar^{n_N}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \left(\sum_{|\boldsymbol{j}_1|=n_1}^{\boldsymbol{j}_1 \in \mathbb{N}^{m_1}} f_{j_{1,1}}^1 \cdots f_{j_{1,m_1}}^1\right) \cdots \left(\sum_{|\boldsymbol{j}_N|=n_N}^{\boldsymbol{j}_N \in \mathbb{N}^{m_N}} f_{j_{N,1}}^N \cdots f_{j_{N,m_N}}^N\right) \hbar^n$$

In these formulas, we have denoted the components of each vector $\boldsymbol{j}_i \in \mathbb{N}^{m_i}$ by $(j_{i,1}, \ldots, j_{i,m_i})$. Let us introduce the following shorthand notation:

$$\boldsymbol{f_n^m} := \left(\sum_{|\boldsymbol{j}_1|=n_1}^{\boldsymbol{j}_1 \in \mathbb{N}^{m_1}} f_{j_{1,1}}^1 \cdots f_{j_{1,m_1}}^1\right) \cdots \left(\sum_{|\boldsymbol{j}_N|=n_N}^{\boldsymbol{j}_N \in \mathbb{N}^{m_N}} f_{j_{N,1}}^N \cdots f_{j_{N,m_N}}^N\right) \ . \tag{24}$$

We note the following simple but useful identities:

$$\boldsymbol{f_0^0} = 1 \ ; \quad \boldsymbol{f_0^m} = f_0^{\boldsymbol{m}} = (f_0^1)^{m_1}\cdots(f_0^N)^{m_N} \ ; \quad \boldsymbol{f_n^m} = 0 \text{ whenever } |\boldsymbol{n}| > 0 \ . \tag{25}$$

Using this notation, the formula for $\widehat{f}^{\boldsymbol{m}}$ can be written much more compactly:

$$\widehat{f}^{\boldsymbol{m}} = \left(\sum_{n=0}^{\infty} f_n \hbar^n\right)^{\boldsymbol{m}} = \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{f_n^m} \hbar^n \ . \tag{26}$$

STEP 1: EXPAND ORDER-BY-ORDER. Now, we plug the solution ansatz (19) into the equation $\widehat{F}(x, \hbar, z) = 0$. Using (21) and (26), we find:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{|\boldsymbol{n}|=n-k} \sum_{|\boldsymbol{m}|=m} F_{k\boldsymbol{m}}^{i} \boldsymbol{f_n^m} \hbar^n = 0 \qquad (i = 1, \dots, N) . \qquad (27)$$

We solve (27) for the coefficients f_n order-by-order in $\hbar$.

STEP 2: LEADING-ORDER PART. First, at order $n = 0$, equation (27) yields:

$$\sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} F_{0\boldsymbol{m}}^{i}(x) \boldsymbol{f_0^m} = 0 \qquad (i = 1, \dots, N) . \qquad (28)$$

Comparing with (22), these equations are simply the components of the equation $F_0(x, f_0) = 0$. By the Holomorphic Implicit Function Theorem, there is a domain $\mathsf{X}_0 \subset \mathsf{X}$ containing x_0 such that there is a unique holomorphic map $f_0 : \mathsf{X}_0 \to \mathbb{C}^N$ that satisfies $F_0\big(x, f_0(x)\big) = 0$ and $f_0(x_0) = z_0$. In fact, the domain X_0 can be chosen so small that the Jacobian $\partial F_0 / \partial z$ remains invertible at the point $\big(x, f_0(x)\big)$ for all $x \in \mathsf{X}_0$. Thus, we can define a holomorphic invertible $N{\times}N$-matrix J_0 on X_0 by

$$J_0(x) := \left. \frac{\partial F_0}{\partial z} \right|_{\big(x, f_0(x)\big)} \qquad (29)$$

The (i, j)-component of J_0 is:

$$[J_0]_{ij} = \left. \frac{\partial F_0^i}{\partial z_j} \right|_{\big(x, f_0(x)\big)} = \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \frac{m_j}{f_0^j} F_{0\boldsymbol{m}}^{i} \boldsymbol{f_0^m} . \qquad (30)$$

STEP 3: NEXT-TO-LEADING-ORDER PART. For clarity, let us also examine equation (27) at order $n = 1$. First, let us note that if $|\boldsymbol{n}| = 1$, then $\boldsymbol{n} = (0, \dots, 1, \dots, 0)$ with the only 1 in some position i, in which case the notation (24) reduces to:

$$\boldsymbol{f_n^m} = (f_0^1)^{m_1} \cdots \Big(m_j f_1^j \Big)(f_0^j)^{m_j - 1} \cdots (f_0^N)^{m_N} = \frac{m_j}{f_0^j} \boldsymbol{f_0^m} f_1^j . \qquad (31)$$

Then at order $n = 1$, equation (27) comprises two main summands corresponding to $k = 0$ and $k = 1$, which simplify using identities (30) and (31):

$$\sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \sum_{|\boldsymbol{n}|=1} F_{0\boldsymbol{m}}^{i} \boldsymbol{f_n^m} + \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} F_{1\boldsymbol{m}}^{i} \boldsymbol{f_0^m} = 0 ,$$

$$\sum_{j=1}^{N} \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \frac{m_j}{f_0^j} F_{0\boldsymbol{m}}^{i} \boldsymbol{f_0^m} f_1^j + \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} F_{1\boldsymbol{m}}^{i} \boldsymbol{f_0^m} = 0 ,$$

$$\sum_{j=1}^{N} [J_0]_{ij} f_1^j + \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} F_{1\boldsymbol{m}}^{i} \boldsymbol{f_0^m} = 0 . \qquad (32)$$

Observe that the blue term is nothing but the i-th component of the vector $J_0 f_1$. Since J_0 is an invertible matrix, multiplying the system of N equations (32) on the left by J_0^{-1}, we solve uniquely for a holomorphic vector f_1 on X_0.

STEP 4: INDUCTIVE STEP. Suppose now that $n \geqslant 1$ and we have already solved equation (27) for holomorphic vectors $f_0, f_1, \ldots, f_{n-1}$ on X_0. Similar to (31), we have that if $\boldsymbol{n} = (0, \ldots, n, \ldots, 0)$ with the only nonzero entry in some position j, then

$$\boldsymbol{f_n^m} = (f_0^1)^{m_1} \cdots \left(m_j f_n^j\right)(f_0^j)^{m_j - 1} \cdots (f_0^N)^{m_N} = \frac{m_j}{f_0^j} \boldsymbol{f_0^m} f_n^j \ . \tag{33}$$

Then at order n in $\hbar$, we separate out the $k = 0$ summand and simplify using the identities (30) and (33):

$$\sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{|\boldsymbol{n}|=n-k} \sum_{|\boldsymbol{m}|=m} F_{k\boldsymbol{m}}^i \boldsymbol{f_n^m} = 0 \ ,$$

$$\sum_{m=0}^{\infty} \left(\sum_{|\boldsymbol{n}|=n} \sum_{|\boldsymbol{m}|=m} F_{0\boldsymbol{m}}^i \boldsymbol{f_n^m} + \sum_{k=1}^{n} \sum_{|\boldsymbol{n}|=n-k} \sum_{|\boldsymbol{m}|=m} F_{k\boldsymbol{m}}^i \boldsymbol{f_n^m} \right) = 0 \ ,$$

$$\color{blue}{\sum_{j=1}^{N} \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \frac{m_j}{f_0^j} F_{0\boldsymbol{m}}^i \boldsymbol{f_0^m} f_n^j}$$
$$+ \sum_{m=0}^{\infty} \left(\sum_{\substack{|\boldsymbol{n}|=n \\ n_1, \ldots, n_N \neq n}} \sum_{|\boldsymbol{m}|=m} F_{0\boldsymbol{m}}^i \boldsymbol{f_n^m} + \sum_{k=1}^{n} \sum_{|\boldsymbol{n}|=n-k} \sum_{|\boldsymbol{m}|=m} F_{k\boldsymbol{m}}^i \boldsymbol{f_n^m} \right) = 0 \ ,$$

$$\color{blue}{\sum_{j=1}^{N} [J_0]_{ij} f_n^j} + \sum_{m=0}^{\infty} \left(\sum_{\substack{|\boldsymbol{n}|=n \\ n_1, \ldots, n_N \neq n}} \sum_{|\boldsymbol{m}|=m} F_{0\boldsymbol{m}}^i \boldsymbol{f_n^m} + \sum_{k=1}^{n} \sum_{|\boldsymbol{n}|=n-k} \sum_{|\boldsymbol{m}|=m} F_{k\boldsymbol{m}}^i \boldsymbol{f_n^m} \right) = 0 \ .$$

The term in blue is nothing but the i-th component of the vector $J_0 f_n$. Observe that the remaining part of this expression involves only the already-known components of the lower-order vectors $f_0, \ldots, f_{n-1}$. Therefore, since J_0 is invertible, multiplying this system of N equations on the left by J_0^{-1}, we can solve uniquely for the holomorphic vector f_n on X_0. ■

§ 2.2. Transformation to the Standard Form

Next, we make a convenient change of variables in order to bring the given equation $F(x, \hbar, z) = 0$ to a standard form that is more easily handled using the Borel-Laplace method. This transformation and the standard form are fully determined by the leading-order solution f_0 of the equation $F_0(x, z) = 0$ and can always be achieved under our hypotheses. Namely, we have the following statement.

2.2. Lemma. *Suppose F is a holomorphic map $\mathsf{X} \times \mathsf{S} \times \mathbb{C}_z^N \to \mathbb{C}^N$ satisfying the hypotheses of Proposition 2.1. Let f_0 and f_1 be the leading- and the next-to-leading-order parts of the formal solution $\widehat{f}$ defined on $\mathsf{X}_0 \subset \mathsf{X}$. Then the change of the unknown variable $z \mapsto w$ given by*

$$z = f_0 + \hbar(f_1 + w) \tag{34}$$

transforms the equation $F(x, \hbar, z) = 0$ into an equation in w of the form

$$w = \hbar G(x, \hbar, w) \ , \tag{35}$$

where G is a holomorphic map $\mathsf{X}_0 \times \mathsf{S} \times \mathbb{C}_w^N \to \mathbb{C}^N$ uniquely determined by f_0 and F. Furthermore, if F admits a Gevrey asymptotic expansion as $\hbar \to 0$ along $\overline{\Theta}$ uniformly for all $x \in \mathsf{X}$ and locally uniformly for all $z \in \mathbb{C}_z^N$ and the domain X_0 is chosen so small that all the eigenvalues of J_0 (where J_0 is the invertible holomorphic matrix on X_0 given by (29)) are bounded from below on X_0, then G also admits a Gevrey

asymptotic expansion as $\hbar \to 0$ along $\overline{\Theta}$ uniformly for all $x \in \mathsf{X}_0$ and locally uniformly for all $z \in \mathbb{C}_z^N$. Specifically, G is defined by

$$G(x, \hbar, w) := \hbar^{-1}\left(w - \hbar^{-1}J_0^{-1}(x)F\big(x, \hbar, f_0(x) + \hbar f_1(x) + \hbar w\big)\right) . \quad (36)$$

Proof. The only thing to check is that the righthand side of (36) has no negative powers in $\hbar$. In particular, since each component of F is an entire function in the variables $z_1, \ldots, z_N$, identity (36) makes it obvious that G admits a uniform Gevrey asymptotic expansion $\widehat{G}$ as $\hbar \to 0$ along $\overline{\Theta}$ whenever the eigenvalues of J_0 are bounded from below and F admits uniform Gevrey asymptotics.

Let us now verify that G has no negative powers in $\hbar$. Clearly, the leading-order part of $F\big(\hbar, f_0 + \hbar f_1 + \hbar w\big)$ is simply $F_0\big(x, f_0(x)\big)$ which is zero because f_0 is the leading-order solution. Therefore, the righthand side of (36) is at worst of order $\hbar^{-1}$. We argue that the next-to-leading-order part of $F\big(\hbar, f_0 + \hbar f_1 + \hbar w\big)$ is equal to $J_0 w$. Evidently,

$$\left[F\big(\hbar, f_0 + \hbar f_1 + \hbar w\big)\right]^{O(\hbar)} = F_1(f_0) + \left[F_0\big(f_0 + \hbar f_1 + \hbar w\big)\right]^{O(\hbar)} . \quad (37)$$

The i-th component of $F_1(f_0)$ is easy to write down:

$$F_1^i(f_0) = \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} F_{1\boldsymbol{m}}^i \boldsymbol{f_0^m} . \quad (38)$$

To expand the term $\left[F_0\big(f_0 + \hbar f_1 + \hbar w\big)\right]^{O(\hbar)}$, consider first the following calculation:

$$\begin{aligned}
&\left(f_0 + \hbar(f_1 + w)\right)^{\boldsymbol{m}} \\
&= \left(f_0^1 + \hbar(f_1^1 + w_1)\right)^{m_1} \cdots \left(f_0^N + \hbar(f_1^N + w_N)\right)^{m_N} \\
&= \left(\sum_{i_1+j_1=m_1} \binom{m_1}{i_1, j_1}(f_0^1)^{i_1}(f_1^1 + w_1)^{j_1} \hbar^{j_1}\right) \cdots \left(\sum_{i_N+j_N=m_N} \binom{m_N}{i_N, j_N}(f_0^N)^{i_N}(f_1^N + w_N)^{j_N} \hbar^{j_N}\right) \\
&= \sum_{\substack{i_1+j_1=m_1 \\ \cdots \\ i_N+j_N=m_N}}^{\boldsymbol{i}, \boldsymbol{j} \in \mathbb{N}^N} \binom{m_1}{i_1, j_1} \cdots \binom{m_N}{i_N, j_N} f_0^{\boldsymbol{i}}(f_1 + w)^{\boldsymbol{j}} \hbar^{|\boldsymbol{j}|} .
\end{aligned}$$

We are only interested in the $|\boldsymbol{j}| = 1$ part of this sum. This means $\boldsymbol{j} = (0, \ldots, 1, \ldots, 0)$; i.e., for each $k = 1, \ldots, N$, we have $j_k = 1, i_k = m_k - 1$, and $j_{k'} = 0, i_{k'} = m_1$ for all $k' \neq k$. Since $\binom{m_k}{m_k-1, 1} = m_k$ and $\binom{m_{k'}}{m_{k'}, 0} = 1$, the coefficient of $\hbar$ in the above expression simplifies as follows:

$$\sum_{k=1}^{N} \frac{m_k}{f_0^k} \boldsymbol{f_0^m}(f_1^k + w_k) .$$

Therefore, continuing (37) and using the above calculation together with (30) and (38), we find for every $i = 1, \ldots, N$:

$$\left[F^i\big(\hbar, f_0 + \hbar f_1 + \hbar w\big)\right]^{O(\hbar)} = \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} F_{1\boldsymbol{m}}^i \boldsymbol{f_0^m} + \sum_{k=1}^{N} \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} F_{0\boldsymbol{m}}^i \frac{m_k}{f_0^k} \boldsymbol{f_0^m}(f_1^k + w_k)$$

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$$= \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} F_{1\boldsymbol{m}}^i \boldsymbol{f_0^m} + \sum_{k=1}^{N} [J_0]_{ki} f_1^k + \sum_{k=1}^{N} [J_0]_{ki} w_k \ .$$

Using (32), it is now clear this this expression equals the i-th component of $J_0 w$. ∎

The analogue of the Formal Implicit Function Theorem (Proposition 2.1) for equations of the form (19) is especially easy to formulate.

2.3. Lemma. *Let*

$$\widehat{G} = \widehat{G}(x, \hbar, w) := \sum_{k=0}^{\infty} G_k(x, w) \hbar^k \tag{39}$$

be any formal power series in $\hbar$ with holomorphic coefficients $G_k : \mathsf{X}_0 \times \mathbb{C}_w^N \to \mathbb{C}^N$ for some domain $\mathsf{X}_0 \subset \mathbb{C}_x^d$. Then there is a unique formal power series

$$\widehat{g} = \widehat{g}(x, \hbar) = \sum_{n=0}^{\infty} g_n(x) \hbar^n \tag{40}$$

with holomorphic coefficients $g_k : \mathsf{X}_0 \to \mathbb{C}^N$, which satisfies $\widehat{g}(x, \hbar) = \widehat{G}\big(x, \hbar, \widehat{g}(x, \hbar)\big)$ for all $x \in \mathsf{X}_0$. In other words, the equation $w = \hbar \widehat{G}(x, \hbar, w)$ has a unique formal power series solution $w = \widehat{g}(x, \hbar)$.

In particular, if $\mathsf{S} \subset \mathbb{C}_{\hbar}$ is a sectorial domain at the origin and G is a holomorphic map $\mathsf{X}_0 \times \mathsf{S} \times \mathbb{C}_w^N \to \mathbb{C}^N$ which admits the power series $\widehat{G}$ as a locally uniform asymptotic expansion as $\hbar \to 0$ in S, then the equation $w = G(x, \hbar, w) = 0$ has a unique formal power series solution $w = \widehat{g}(x, \hbar)$ as above.

Moreover, $g_0 \equiv 0$ and all the higher-order coefficients g_n are given by the following recursive formula: for every $i = 1, \dots, N$,

$$g_{n+1}^i = \sum_{k=0}^{n} \sum_{m=0}^{n-k} \sum_{|\boldsymbol{m}|=m} \sum_{|\boldsymbol{n}|=n-k} G_{k\boldsymbol{m}}^i \boldsymbol{g_n^m} \ , \tag{41}$$

where

$$\boldsymbol{g_n^m} := \left(\sum_{|\boldsymbol{j_1}|=n_1}^{\boldsymbol{j_1} \in \mathbb{N}^{m_1}} g_{j_{1,1}}^1 \cdots g_{j_{1,m_1}}^1 \right) \cdots \left(\sum_{|\boldsymbol{j_N}|=n_N}^{\boldsymbol{j_N} \in \mathbb{N}^{m_N}} g_{j_{N,1}}^N \cdots g_{j_{N,m_N}}^N \right) , \tag{42}$$

and where $G_{k\boldsymbol{m}}^i = G_{k\boldsymbol{m}}^i(x)$ are the coefficients of the double power series expansion

$$\widehat{G}^i(x, \hbar, w) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} G_{k\boldsymbol{m}}^i(x) \hbar^k w^{\boldsymbol{m}} \ . \tag{43}$$

Proof. The proof is a computation very similar to the one in the proof of Proposition 2.1. Plugging the solution ansatz (40) into the double power series expansion (43) of $\widehat{G}^i$, the righthand side of the equation $w = \hbar \widehat{G}(x, \hbar, w)$ becomes:

$$\hbar \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} G_{k\boldsymbol{m}}^i \hbar^k \left(\sum_{n=0}^{\infty} g_n \hbar^n \right)^{\boldsymbol{m}} = \hbar \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} G_{k\boldsymbol{m}}^i \boldsymbol{g_n^m} \hbar^{k+n}$$

$$= \hbar \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \sum_{|\boldsymbol{n}|=n-k} G_{k\boldsymbol{m}}^i \boldsymbol{g_n^m} \hbar^n$$

$$= \hbar \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{n-k} \sum_{|\boldsymbol{m}|=m} \sum_{|\boldsymbol{n}|=n-k} G_{k\boldsymbol{m}}^i \boldsymbol{g_n^m} \hbar^n \ ,$$

where in the last step we noticed that all terms with $m > |\boldsymbol{n}| = n - k$ are zero because $g_0 \equiv 0$; cf. (42). ∎

§ 2.3. Gevrey Regularity of the Formal Solution

Now we show that the formal Borel transform of the formal solution $\widehat{f}$ is a convergent power series in the Borel variable ξ; that is, the coefficients f_n grow not faster than $n!$. More precisely, we prove the following proposition.

2.4. Proposition (Gevrey Formal Implicit Function Theorem).
Assume all the hypotheses of Proposition 2.1 and suppose in addition that the power series $\widehat{F}$ is locally uniformly Gevrey on $\mathsf{X} \times \mathbb{C}_z^N$. Then $\mathsf{X}_0 \subset \mathsf{X}$ can be chosen so small that the formal power series $\widehat{f}$ is uniformly Gevrey on X_0. In particular, the formal Borel transform

$$\widehat{\varphi}(x,\xi) = \widehat{\mathfrak{B}}[\widehat{f}](x,\xi) := \sum_{n=0}^{\infty} \tfrac{1}{n!} f_{n+1}(x)\xi^n \tag{44}$$

is a uniformly convergent power series in ξ. Concretely, if $\mathsf{X}_0 \subset \mathsf{X}$ is any subset where all eigenvalues of J_0 are bounded from below and such that there are $A, B > 0$ such that $|F_k(x,z)| \leqslant AB^k k!$ for all $k \geqslant 0$, uniformly for all $x \in \mathsf{X}_0$ and for all $z \in \mathbb{C}_z^N$ with $|z| < R$ for some $R > 0$, then there are constants $C, M > 0$ such that

$$\left| f_k(x) \right| \leqslant CM^k k! \qquad \forall x \in \mathsf{X}_0, \forall k \ . \tag{45}$$

Proof. Let $\mathsf{X}_0 \subset \mathsf{X}$ be such that all the eigenvalues of the invertible holomorphic matrix J_0 from (29) are bounded from below. Then, by Lemma 2.2, the proof boils down to proving the following claim.

Claim. *Assume all the hypotheses of Lemma 2.3 and suppose that the power series $\widehat{G}$ is Gevrey uniformly for all $x \in \mathsf{X}_0$ and locally uniformly for all $w \in \mathbb{C}_w^N$. Then the formal solution $\widehat{g}$ is also uniformly Gevrey on X_0.*

Let $A, B > 0$ be constants such that, for all $i = 1, \ldots, N$, all $k, m \in \mathbb{N}$, all $\boldsymbol{m} \in \mathbb{N}^N$ such that $|\boldsymbol{m}| = m$, and all $x \in \mathsf{X}_0$,

$$\left| G_{k\boldsymbol{m}}^i(x) \right| \leqslant \rho_m AB^{k+m} k! \ , \tag{46}$$

where ρ_m is a normalisation constant defined as[2]

$$\frac{1}{\rho_m} := \sum_{|\boldsymbol{m}|=m} 1 = \binom{m+N-1}{N-1} \ . \tag{47}$$

We will show that there is a constant $M > 0$ such that

$$\left| g_{n+1}^i(x) \right| \leqslant M^{n+1} n! \qquad \forall x \in \mathsf{X}_0, \ \forall n \in \mathbb{N} \ . \tag{48}$$

This bound will be demonstrated in two main steps. First, we will recursively construct a sequence $\{M_n\}_{n=0}^{\infty}$ of nonnegative real numbers such that

$$\left| g_{n+1}^i(x) \right| \leqslant M_{n+1} n! \qquad \forall x \in \mathsf{X}_0, \ \forall n \in \mathbb{N} \ . \tag{49}$$

Then we will show that there is a constant $M > 0$ such that $M_n \leqslant M^n$ for all n.

---

[2]The righthand side is the total number of weak compositions of m into N parts.

STEP 1: CONSTRUCTION OF $\{M_n\}_{n=0}^\infty$. Let $M_0 := 0$. We can take $M_1 := A$ because $g_1^i = G_{0\mathbf{0}}^i$. Now we use induction on n and formula (41), which is more convenient to rewrite as follows:

$$g_{n+1}^i = \sum_{m=0}^\infty \sum_{k=0}^n \sum_{|\boldsymbol{m}|=m} \sum_{|\boldsymbol{n}|=n-k} G_{k\boldsymbol{m}}^i \boldsymbol{g_n^m} \tag{50}$$

Notice that $\boldsymbol{g_n^m} = 0$ whenever $m = |\boldsymbol{m}| > |\boldsymbol{n}| = n - k$, so this expression really is the same as (41). Assume that we have already constructed $M_0, \ldots, M_n$ such that $\left|g_j^i\right| \leqslant M_j(j-1)!$ for all $j = 1, \ldots, n$ and all $x \in \mathsf{X}_0$.

Let us write down an estimate for $\boldsymbol{g_n^m}$ using formula (42):

$$\left|\boldsymbol{g_n^m}\right| \leqslant \left(\sum_{|\boldsymbol{j}_1|=n_1}^{\boldsymbol{j}_1 \in \mathbb{N}^{m_1}} \left|g_{j_{1,1}}^1\right| \cdots \left|g_{j_{1,m_1}}^1\right|\right) \cdots \left(\sum_{|\boldsymbol{j}_N|=n_N}^{\boldsymbol{j}_N \in \mathbb{N}^{m_N}} \left|g_{j_{N,1}}^N\right| \cdots \left|g_{j_{N,m_N}}^N\right|\right)$$

$$\leqslant \left(\sum_{|\boldsymbol{j}_1|=n_1}^{\boldsymbol{j}_1 \in \mathbb{N}^{m_1}} M_{j_{1,1}} \cdots M_{j_{1,m_1}}\right) \cdots \left(\sum_{|\boldsymbol{j}_N|=n_N}^{\boldsymbol{j}_N \in \mathbb{N}^{m_N}} M_{j_{N,1}} \cdots M_{j_{N,m_N}}\right) (|\boldsymbol{n}| - |\boldsymbol{m}|)! \, ,$$

where we repeatedly used the inequality $i!j! \leqslant (i+j)!$. Introduce the following shorthand:

$$\boldsymbol{M_n^m} := \left(\sum_{|\boldsymbol{j}_1|=n_1}^{\boldsymbol{j}_1 \in \mathbb{N}^{m_1}} M_{j_{1,1}} \cdots M_{j_{1,m_1}}\right) \cdots \left(\sum_{|\boldsymbol{j}_N|=n_N}^{\boldsymbol{j}_N \in \mathbb{N}^{m_N}} M_{j_{N,1}} \cdots M_{j_{N,m_N}}\right) \tag{51}$$

Then the estimate for $\boldsymbol{g_n^m}$ becomes simply $\left|\boldsymbol{g_n^m}\right| \leqslant \boldsymbol{M_n^m}(|\boldsymbol{n}| - |\boldsymbol{m}|)!$. Now we can estimate g_{n+1}^i using formula (50):

$$\left|g_{n+1}^i\right| \leqslant \sum_{k=0}^n \sum_{m=0}^\infty \sum_{|\boldsymbol{m}|=m} \sum_{|\boldsymbol{n}|=n-k} \rho_m A B^{k+m} k! \boldsymbol{M_n^m}(n-k-m)!$$

$$\leqslant A \sum_{k=0}^n B^k \sum_{m=0}^\infty \sum_{|\boldsymbol{m}|=m} \sum_{|\boldsymbol{n}|=n-k} \rho_m B^m \boldsymbol{M_n^m} n! \, .$$

Thus, we can define

$$M_{n+1} := A \sum_{k=0}^n B^k \sum_{m=0}^\infty \sum_{|\boldsymbol{m}|=m} \sum_{|\boldsymbol{n}|=n-k} \rho_m B^m \boldsymbol{M_n^m} \tag{52}$$

STEP 2: CONSTRUCTION OF M. To see that $M_n \leqslant M^n$ for some $M > 0$, we argue as follows. Consider the following pair of power series in an abstract variable t:

$$\widehat{p}(t) := \sum_{n=0}^\infty M_n t^n \quad \text{and} \quad Q(t) := \sum_{m=0}^\infty B^m t^m \tag{53}$$

Notice that $\widehat{p}(0) = M_0 = 0$ and that $Q(t)$ is convergent. We will show that $\widehat{p}(t)$ is also convergent. The key is the observation that they satisfy the following equation:

$$\widehat{p}(t) = AtQ(t)Q(\widehat{p}(t)) = AtQ(t) \sum_{m=0}^\infty B^m \widehat{p}(t)^m \tag{54}$$

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This equation was found by trial and error. In order to verify it, we rewrite the power series $Q(t)$ in the following strange way:

$$Q(t) = \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \rho_m B^m t^{\boldsymbol{m}} \ ,$$

where $t^{\boldsymbol{m}} := t^{m_1} \cdots t^{m_N} = t^m$. Then (54) is straightforward to verify directly by substituting the power series $\widehat{p}(t)$ and $Q(t)$ and comparing the coefficients of t^{n+1} using the defining formula (52) for M_{n+1}. Indeed, using the notation introduced in (51), we see that

$$
\begin{aligned}
\widehat{p}(t)^{\boldsymbol{m}} &= \widehat{p}(t)^{m_1} \cdots \widehat{p}(t)^{m_N} \\
&= \left(\sum_{n_1=0}^{\infty} M_{n_1} t^{n_1} \right)^{m_1} \cdots \left(\sum_{n_N=0}^{\infty} M_{n_N} t^{n_N} \right)^{m_N} \\
&= \left(\sum_{n_1=0}^{\infty} \sum_{\substack{\boldsymbol{i}_1 \in \mathbb{N}^{m_1} \\ |\boldsymbol{i}_1|=n_1}} M_{i_{1,1}} \cdots M_{i_{1,m_1}} t^{n_1} \right) \cdots \left(\sum_{n_N=0}^{\infty} \sum_{\substack{\boldsymbol{i}_N \in \mathbb{N}^{m_N} \\ |\boldsymbol{i}_N|=n_N}} M_{i_{N,1}} \cdots M_{i_{N,m_N}} t^{n_N} \right) \\
&= \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \left(\sum_{\substack{\boldsymbol{i}_1 \in \mathbb{N}^{m_1} \\ |\boldsymbol{i}_1|=n_1}} M_{i_{1,1}} \cdots M_{i_{1,m_1}} \right) \cdots \left(\sum_{\substack{\boldsymbol{i}_N \in \mathbb{N}^{m_N} \\ |\boldsymbol{i}_N|=n_N}} M_{i_{N,1}} \cdots M_{i_{N,m_N}} \right) t^n \\
&= \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{M_n^m} t^n \ .
\end{aligned}
$$

Then the righthand side of (54) expands as follows:

$$
\begin{aligned}
& A t \left(\sum_{k=0}^{\infty} B^k t^k \right) \left(\sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \rho_m B^m \big(\widehat{p}(t)\big)^{\boldsymbol{m}} \right) \\
&= A t \left(\sum_{k=0}^{\infty} B^k t^k \right) \left(\sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \rho_m B^m \left(\sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{M_n^m} t^n \right) \right) \\
&= A t \left(\sum_{k=0}^{\infty} B^k t^k \right) \left(\sum_{n=0}^{\infty} C_n t^n \right) \quad \text{where} \quad C_n := \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \sum_{|\boldsymbol{n}|=n} \rho_m B^m \boldsymbol{M_n^m} t^n \\
&= A t \sum_{n=0}^{\infty} \sum_{k=0}^{n} B^k C_{n-k} t^n = \sum_{n=0}^{\infty} \left(A \sum_{k=0}^{n} B^k \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \sum_{|\boldsymbol{n}|=n} \rho_m B^m \boldsymbol{M_n^m} \right) t^{n+1} \ ,
\end{aligned}
$$

which matches with (52). Now, consider the following holomorphic function in two variables (t, p):

$$F(t, p) := -p + A t Q(t) Q(p) \ .$$

It has the following properties:

$$F(0, 0) = 0 \qquad \text{and} \qquad \left. \frac{\partial F}{\partial p} \right|_{(t,p)=(0,0)} = -1 \neq 0 \ .$$

By the Holomorphic Implicit Function Theorem, there exists a unique holomorphic function $p(t)$ near $t = 0$ such that $p(t) = 0$ and $F\big(t, p(t)\big) = 0$. Thus, $\widehat{p}(t)$ must be the convergent Taylor series expansion at $t = 0$ for $p(t)$, so its coefficients grow at most exponentially: i.e., there is a constant $M > 0$ such that $M_n \leqslant M^n$. ∎

§ 2.4. Exact Perturbation Theory

Now we show that the convergent Borel transform $\widehat{\varphi}(x, \xi)$ of the formal solution admits an analytic continuation along a ray in the Borel ξ-plane and furthermore its Laplace transform is well-defined. First, we prove the following lemma.

2.5. Lemma. *Let* $\mathsf{X}_0 \subset \mathbb{C}_x^d$ *be a domain. Let* $\mathsf{S} := \left\{ \hbar \mid \mathrm{Re}(1/\hbar) > 1/R \right\} \subset \mathbb{C}_\hbar$ *be the Borel disc of some diameter* $R > 0$. *Recall that its opening is* $\mathsf{A}_+ := (-\pi/2, +\pi/2)$. *Let* $G : \mathsf{X}_0 \times \mathsf{S} \times \mathbb{C}_w^N \to \mathbb{C}^N$ *be a holomorphic map which admits a Gevrey asymptotic expansion*

$$G(x, \hbar, w) \simeq \widehat{G}(x, \hbar, w) \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta}_+ , \tag{55}$$

uniformly for all $x \in \mathsf{X}_0$ *and locally uniformly for all* $w \in \mathbb{C}_w^N$. *Then there is a Borel disc* $\mathsf{S}_0 := \left\{ \hbar \mid \mathrm{Re}(1/\hbar) > 1/R_0 \right\} \subset \mathsf{S}$ *of possibly smaller diameter* $R_0 \in (0, R]$ *such that there is a unique holomorphic map* $g : \mathsf{X}_0 \times \mathsf{S}_0 \to \mathbb{C}^N$ *which admits a uniform Gevrey asymptotic expansion*

$$g(x, \hbar) \simeq \widehat{g}(x, \hbar) \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta}_+, \text{ unif. } \forall x \in \mathsf{X}_0 , \tag{56}$$

and such that $g(x, \hbar) = \hbar G\big(x, \hbar, g(x, \hbar)\big) = 0$ *for all* $(x, \hbar) \in \mathsf{X}_0 \times \mathsf{S}_0$. *Furthermore,* g *is the uniform Borel resummation of* $\widehat{g}$: *for all* $(x, \hbar) \in \mathsf{X}_0 \times \mathsf{S}_0$,

$$g(x, \hbar) = \mathcal{S}\big[\, \widehat{g} \,\big](x, \hbar) . \tag{57}$$

Proof. First, uniqueness of g follows from the asymptotic property (56). Indeed, suppose g' is another such map. Then $g - g'$ is a holomorphic map $\mathsf{X}_0 \times \mathsf{S}_0 \to \mathbb{C}^N$ whose components are uniformly Gevrey asymptotic to 0 as $\hbar \to 0$ along the closed arc $\overline{\Theta}_+$ of opening angle π. By Nevanlinna's Theorem ([Nev18, pp.44-45] and [Sok80]; see also [Nik20, Theorem B.11]), there can only be one holomorphic function on S_0 (namely, the constant function 0) which is Gevrey asymptotic to 0 as $\hbar \to 0$ along $\overline{\Theta}_+$. Thus, each component of $g - g'$ must be identically zero.

To construct g, we start by expanding G as a power series in w. Each component G^i of G can be expressed as the following uniformly convergent multipower series in the components $w_1, \ldots, w_N$ of w:

$$G^i(x, \hbar, w) = \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} A_{\boldsymbol{m}}^i(x, \hbar) w^{\boldsymbol{m}} \qquad (\, i = 1, \ldots, N \,) , \tag{58}$$

where $A_{\boldsymbol{m}}^i w^{\boldsymbol{m}} := A_{m_1 \cdots m_N}^i w_1^{m_1} \cdots w_N^{m_N}$. Then the vectorial equation $w = \hbar G(x, \hbar, w)$ can be written as the following coupled system of N scalar equations:

$$w_i = \hbar \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} A_{\boldsymbol{m}}^i w^{\boldsymbol{m}} \qquad (\, i = 1, \ldots, N \,) . \tag{59}$$

It is convenient to separate the $m = 1$ term from the sum:

$$w_i = \hbar A_{\boldsymbol{0}}^i + \hbar \sum_{m=1}^{\infty} \sum_{|\boldsymbol{m}|=m} A_{\boldsymbol{m}}^i w^{\boldsymbol{m}} \qquad (\, i = 1, \ldots, N \,) . \tag{60}$$

Step 1: The Borel Transform. Let $a_{\boldsymbol{m}}^i = a_{\boldsymbol{m}}^i(x)$ be the $\hbar$-leading-order part of $A_{\boldsymbol{m}}^i$ and let $\alpha_{\boldsymbol{m}}^i(x,\xi) := \mathfrak{B}\big[A_{\boldsymbol{m}}^i\big](x,\xi)$. By the assumption (55), there is some $\varepsilon > 0$ such that each $\alpha_{\boldsymbol{m}}^i$ is a holomorphic function on $\mathsf{X}_0 \times \Xi$, where

$$\Xi := \big\{\xi \mid \operatorname{dist}(\xi, \mathbb{R}_+) < \varepsilon\big\} , \tag{61}$$

with uniformly at-most-exponential growth at infinity in ξ (cf. Part A.6), and

$$A_{\boldsymbol{m}}^i(x,\hbar) = a_{\boldsymbol{m}}^i(x) + \mathfrak{L}\big[\alpha_{\boldsymbol{m}}^i\big](x,\hbar) \tag{62}$$

for all $(x,\hbar) \in \mathsf{X}_0 \times \mathsf{S}$ provided that the diameter R is sufficiently small.

Dividing each equation (60) by $\hbar$ and applying the analytic Borel transform, we obtain the following system of N coupled nonlinear ordinary differential equations with convolution:

$$\partial_\xi \sigma^i = \alpha_{\boldsymbol{0}}^i + \sum_{m=1}^{\infty} \sum_{|\boldsymbol{m}|=m} \Big(a_{\boldsymbol{m}}^i \sigma^{*\boldsymbol{m}} + \alpha_{\boldsymbol{m}}^i * \sigma^{*\boldsymbol{m}}\Big) \qquad (i = 1, \ldots, N) , \tag{63}$$

where $\sigma^{*\boldsymbol{m}} := (\sigma^1)^{*m_1} * \cdots * (\sigma^N)^{*m_N}$ and the unknown variables w_i and σ^i are related by $\sigma^i = \mathfrak{B}[w_i]$ and $w_i = \mathfrak{L}[\sigma^i]$. A solution of the system (63) with initial condition $\sigma(x,0) = a_{\boldsymbol{0}}(x)$ is equivalently the solution of the following system of N coupled integral equations:

$$\sigma^i = a_{\boldsymbol{0}}^i + \int_0^\xi \left[\alpha_{\boldsymbol{0}}^i + \sum_{m=1}^{\infty} \sum_{|\boldsymbol{m}|=m} \Big(a_{\boldsymbol{m}}^i \sigma^{*\boldsymbol{m}} + \alpha_{\boldsymbol{m}}^i * \sigma^{*\boldsymbol{m}}\Big) \right] \mathrm{d}t \quad (i = 1, \ldots, N) , \tag{64}$$

where the integral is taken along the straight line segment from 0 to ξ.

Step 2: Method of Successive Approximations. We solve this integral equation using the method of successive approximations. To this end, define a sequence of holomorphic maps $\big\{\sigma_n = (\sigma_n^1, \ldots, \sigma_n^N) : \mathsf{X}_0 \times \Xi \to \mathbb{C}^N\big\}_{n=0}^{\infty}$, as follows: for each $i = 1, \ldots, N$, let

$$\sigma_0^i := a_{\boldsymbol{0}}^i, \qquad \sigma_1^i := \int_0^\xi \left[\alpha_{\boldsymbol{0}}^j + \sum_{|\boldsymbol{m}|=1} a_{\boldsymbol{m}}^i \sigma_0^{\boldsymbol{m}} \right] \mathrm{d}t , \tag{65}$$

and for all $n \geqslant 2$,

$$\sigma_n^i := \int_0^\xi \sum_{m=1}^{n} \sum_{|\boldsymbol{m}|=m} \left[a_{\boldsymbol{m}}^i \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \alpha_{\boldsymbol{m}}^i * \sum_{|\boldsymbol{n}|=n-m-1} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} \right] \mathrm{d}t . \tag{66}$$

Here, for any $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^N$, we have introduced the notation

$$\boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} := \left(\sum_{|\boldsymbol{j}_1|=n_1}^{\boldsymbol{j}_1 \in \mathbb{N}^{m_1}} \sigma_{j_{1,1}}^1 * \cdots * \sigma_{j_{1,m_1}}^1 \right) * \cdots * \left(\sum_{|\boldsymbol{j}_N|=n_N}^{\boldsymbol{j}_N \in \mathbb{N}^{m_N}} \sigma_{j_{N,1}}^N * \cdots * \sigma_{j_{N,m_N}}^N \right) . \tag{67}$$

Let us also note the following simple but useful identities:

$$\boldsymbol{\sigma}_{\boldsymbol{0}}^{\boldsymbol{0}} = 1 ; \quad \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{0}} = 0 \text{ whenever } |\boldsymbol{n}| > 0 ; \tag{68}$$

$$\boldsymbol{\sigma}_{\boldsymbol{0}}^{\boldsymbol{m}} = (\sigma_0^1)^{*m_1} * \cdots * (\sigma_0^N)^{*m_N} = \tfrac{1}{(m-1)!} \sigma_0^{\boldsymbol{m}} \xi^{m-1} . \tag{69}$$

Main Technical Claim. *The infinite series*

$$\sigma(x,\xi) := \sum_{n=0}^{\infty} \sigma_n(x,\xi) \tag{70}$$

converges uniformly for all $(x,\xi) \in \mathsf{X}_0 \times \Xi$ and defines a holomorphic solution of the integral equation (64) with uniformly at-most-exponential growth at infinity in ξ; that is, there are constants $D, K > 0$ such that

$$\left| \sigma(x,\xi) \right| \leqslant De^{K|\xi|} \qquad \forall (x,\xi) \in \mathsf{X}_0 \times \Xi \ . \tag{71}$$

Furthermore, the convergent formal Borel transform

$$\widehat{\sigma}(x,\xi) = \widehat{\mathfrak{B}}[\widehat{g}](x,\xi) = \sum_{n=0}^{\infty} \tfrac{1}{n!} g_{n+1}(x)\xi^n \tag{72}$$

of the unique formal solution $\widehat{g}$ is the Taylor series expansion of σ at $\xi = 0$.

The assertions of Lemma 2.5 follow from this claim by defining

$$g(x,\hbar) := \mathfrak{L}[\sigma](x,\hbar) = \int\limits_0^{+\infty} e^{-\xi/\hbar}\sigma(x,\xi) \, \mathrm{d}\xi \ . \tag{73}$$

Indeed, the exponential estimate (71) implies that the Laplace transform (73) is uniformly convergent for all $(x,\hbar) \in \mathsf{X}_0 \times \mathsf{S}_0$ where $\mathsf{S}_0 = \left\{ \hbar \mid \mathrm{Re}(1/\hbar) > 1/R_0 \right\}$ as long as $R_0 < K^{-1}$. We now turn to the proof of the Main Technical Claim.

Step 3: Solution Check. First, assuming that the infinite series σ is uniformly convergent for all $(x,\xi) \in \mathsf{X}_0 \times \Xi$, we verify that it satisfies the integral equation (64) by direct substitution. Thus, the righthand side of (64) becomes:

$$a_{\boldsymbol{0}}^i + \int\limits_0^\xi \left[\alpha_{\boldsymbol{0}}^i + \sum_{m=1}^{\infty} \sum_{|\boldsymbol{m}|=m} a_{\boldsymbol{m}}^i \left(\sum_{n=0}^{\infty} \sigma_n \right)^{*\boldsymbol{m}} + \sum_{m=1}^{\infty} \sum_{|\boldsymbol{m}|=m} \alpha_{\boldsymbol{m}}^i * \left(\sum_{n=0}^{\infty} \sigma_n \right)^{*\boldsymbol{m}} \right] \mathrm{d}t \ . \tag{74}$$

Using the notation introduced in (67), the $\boldsymbol{m}$-fold convolution product of the infinite series σ expands as follows:

$$\left(\sum_{n=0}^{\infty} \sigma_n \right)^{*\boldsymbol{m}}$$

$$= \left(\sum_{n_1=0}^{\infty} \sigma_{n_1}^1 \right)^{*m_1} * \cdots * \left(\sum_{n_N=0}^{\infty} \sigma_{n_N}^N \right)^{*m_N}$$

$$= \left(\sum_{n_1=0}^{\infty} \sum_{|\boldsymbol{j}_1|=n_1}^{\boldsymbol{j}_1 \in \mathbb{N}^{m_1}} \sigma_{j_{1,1}}^1 * \cdots * \sigma_{j_{1,m_1}}^1 \right) * \cdots * \left(\sum_{n_N=0}^{\infty} \sum_{|\boldsymbol{j}_N|=n_N}^{\boldsymbol{j}_N \in \mathbb{N}^{m_N}} \sigma_{j_{N,1}}^N * \cdots * \sigma_{j_{N,m_N}}^N \right)$$

$$= \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \left(\sum_{|\boldsymbol{j}_1|=n_1}^{\boldsymbol{j}_1 \in \mathbb{N}^{m_1}} \sigma_{j_{1,1}}^1 * \cdots * \sigma_{j_{1,m_1}}^1 \right) * \cdots * \left(\sum_{|\boldsymbol{j}_N|=n_N}^{\boldsymbol{j}_N \in \mathbb{N}^{m_N}} \sigma_{j_{N,1}}^N * \cdots * \sigma_{j_{N,m_N}}^N \right)$$

$$= \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \sigma_{\boldsymbol{n}}^{\boldsymbol{m}} \ .$$

Use this to rewrite the blue terms in (74), separating out first the $m = 1$ part and then the $(m, n) = (1, 1)$ part using the identity (69):

$$\sum_{m=1}^{\infty} \sum_{|\boldsymbol{m}|=m} a_{\boldsymbol{m}}^i \left(\sum_{n=0}^{\infty} \sigma_n \right)^{*\boldsymbol{m}}$$

$$= \sum_{|\boldsymbol{m}|=1} a_{\boldsymbol{m}}^i \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \sum_{m=2}^{\infty} \sum_{|\boldsymbol{m}|=m} a_{\boldsymbol{m}}^i \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}}$$

$$= \sum_{|\boldsymbol{m}|=1} a_{\boldsymbol{m}}^i \sigma_0^{\boldsymbol{m}} + \sum_{|\boldsymbol{m}|=1} a_{\boldsymbol{m}}^i \sum_{n=1}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \sum_{m=2}^{\infty} \sum_{|\boldsymbol{m}|=m} a_{\boldsymbol{m}}^i \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} .$$

Substituting this back into (74) and using (65), we find:

$$\sigma_0^i + \sigma_1^i + \int_0^\xi \left[\sum_{|\boldsymbol{m}|=1} a_{\boldsymbol{m}}^i \sum_{n=1}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \sum_{m=2}^{\infty} \sum_{|\boldsymbol{m}|=m} a_{\boldsymbol{m}}^i \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} \right.$$
$$\left. + \sum_{m=1}^{\infty} \sum_{|\boldsymbol{m}|=m} \alpha_{\boldsymbol{m}}^i * \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} \right] \mathrm{d}t . \quad (75)$$

The goal is to show that the the integral in (75) is equal to $\sum_{n \geqslant 2} \sigma_n^i$. Focus on the expression inside the integral:

$$\sum_{|\boldsymbol{m}|=1} a_{\boldsymbol{m}}^i \sum_{n=1}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \sum_{m=2}^{\infty} \sum_{|\boldsymbol{m}|=m} a_{\boldsymbol{m}}^i \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \sum_{m=1}^{\infty} \sum_{|\boldsymbol{m}|=m} \alpha_{\boldsymbol{m}}^i * \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} .$$

Shift the summation index n up by 1 in the black sum, by m in the blue sum, and by $m + 1$ in the green sum:

$$\sum_{|\boldsymbol{m}|=1} a_{\boldsymbol{m}}^i \sum_{n=2}^{\infty} \sum_{|\boldsymbol{n}|=n-1} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \sum_{m=2}^{\infty} \sum_{|\boldsymbol{m}|=m} a_{\boldsymbol{m}}^i \sum_{n=m}^{\infty} \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \sum_{m=1}^{\infty} \sum_{|\boldsymbol{m}|=m} \alpha_{\boldsymbol{m}}^i * \sum_{n=m+1}^{\infty} \sum_{|\boldsymbol{n}|=n-m-1} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} .$$

Notice that all terms in the blue sum with $n < m$ are zero, so we can start the summation over n from $n = 2$ (which is the lowest possible value of m) without altering the result. Similarly, all terms in the green sum with $n < m + 1$ are zero, so we may as well start from $n = 2$. The black sum is left unaltered. Thus, we get:

$$\sum_{|\boldsymbol{m}|=1} a_{\boldsymbol{m}}^i \sum_{n=2}^{\infty} \sum_{|\boldsymbol{n}|=n-1} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \sum_{m=2}^{\infty} \sum_{|\boldsymbol{m}|=m} a_{\boldsymbol{m}}^i \sum_{n=2}^{\infty} \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \sum_{m=1}^{\infty} \sum_{|\boldsymbol{m}|=m} \alpha_{\boldsymbol{m}}^i * \sum_{n=2}^{\infty} \sum_{|\boldsymbol{n}|=n-m-1} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} .$$

The advantage of this way of expressing the sums is that we can now interchange the summations over m and n to obtain:

$$\sum_{n=2}^{\infty} \left\{ \sum_{|\boldsymbol{m}|=1} a_{\boldsymbol{m}}^i \sum_{|\boldsymbol{n}|=n-1} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \sum_{m=2}^{\infty} \sum_{|\boldsymbol{m}|=m} a_{\boldsymbol{m}}^i \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \sum_{m=1}^{\infty} \sum_{|\boldsymbol{m}|=m} \alpha_{\boldsymbol{m}}^i * \sum_{|\boldsymbol{n}|=n-m-1} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} \right\} .$$

Observe that the black sum fits well into the blue sum over m to give the $m = 1$ term. So we get:

$$\sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \sum_{|\boldsymbol{m}|=m} \left\{ a_{\boldsymbol{m}}^i \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \alpha_{\boldsymbol{m}}^i * \sum_{|\boldsymbol{n}|=n-m-1} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} \right\} .$$

Finally, notice that both sums are empty for $m > n$, so we get:

$$\sum_{n=2}^{\infty} \sum_{m=1}^{n} \sum_{|\boldsymbol{m}|=m} \left\{ a_{\boldsymbol{m}}^i \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} + \alpha_{\boldsymbol{m}}^i * \sum_{|\boldsymbol{n}|=n-m-1} \boldsymbol{\sigma}_{\boldsymbol{n}}^{\boldsymbol{m}} \right\} .$$

The sum over m is precisely the expression inside the integral in (66) defining σ_n^i. This shows that σ satisfies the integral equation (64).

Step 4: Convergence. Now we show that σ is a uniformly convergent series on $\mathsf{X}_0 \times \Xi$ and therefore defines a holomorphic map $\mathsf{X}_0 \times \Xi \to \mathbb{C}^N$. In the process, we also establish the estimate (71).

Let $B, C, L > 0$ be such that for all $(x, \xi) \in \mathsf{X}_0 \times \Xi$, all $i = 1, \ldots, N$, and all $\boldsymbol{m} \in \mathbb{N}^N$,

$$\left| a_{\boldsymbol{m}}^i(x) \right| \leqslant \rho_m C B^m \qquad \text{and} \qquad \left| \alpha_{\boldsymbol{m}}^i(x, \xi) \right| \leqslant \rho_m C B^m e^{L|\xi|} , \tag{76}$$

where $m = |\boldsymbol{m}|$ and ρ_m is the normalisation constant (47). We claim that there are constants $D, M > 0$ such that for all $(x, \xi) \in \mathsf{X}_0 \times \Xi$ and all $n \in \mathbb{N}$,

$$\left| \sigma_n^i(x, \xi) \right| \leqslant D M^n \frac{|\xi|^n}{n!} e^{L|\xi|} . \tag{77}$$

If we achieve (77), then the uniform convergence and the exponential estimate (71) both follow at once because

$$\left| \sigma^i(x, \xi) \right| \leqslant \sum_{n=0}^{\infty} \left| \sigma_n^i(x, \xi) \right| \leqslant \sum_{n=0}^{\infty} D M^n \frac{|\xi|^n}{n!} e^{L|\xi|} \leqslant D e^{(M+L)|\xi|} .$$

To demonstrate (77), we proceed in two steps. First, we construct a sequence of positive real numbers $\{M_n\}_{n=0}^{\infty}$ such that for all $n \in \mathbb{N}$ and all $(x, \xi) \in \mathsf{X}_0 \times \Xi$,

$$\left| \sigma_n^i(x, \xi) \right| \leqslant M_n \frac{|\xi|^n}{n!} e^{L|\xi|} . \tag{78}$$

We will then show that there are constants D, M such that $M_n \leqslant D M^n$ for all n.

Step 4.1: Construction of $\{M_n\}$. We can take $M_0 := C$ and $M_1 := C(1 + B M_0)$ because $\sigma_0^i = a_{\boldsymbol{0}}^i$ and

$$\left| \sigma_1^i \right| \leqslant \int_0^{\xi} \left(|\alpha_{\boldsymbol{0}}^i| + \sum_{|\boldsymbol{m}|=1} |a_{\boldsymbol{m}}^i| |\sigma_0^{\boldsymbol{m}}| \right) |\,\mathrm{d}t| \leqslant \int_0^{\xi} \left(C e^{L|t|} + C^2 B \rho_1 \sum_{|\boldsymbol{m}|=1} 1 \right) |\,\mathrm{d}t|$$

$$\leqslant C(1 + B M_0) \int_0^{|\xi|} e^{Ls} \,\mathrm{d}s \leqslant C(1 + B M_0) |\xi| e^{L|\xi|} ,$$

where in the final step we used Lemma A.10. Now, let us assume that we have already constructed the constants $M_0, \ldots, M_{n-1}$ such that $|\sigma_k^i| \leqslant M_k \frac{|\xi|^k}{k!} e^{L|\xi|}$ for all $k = 0, \ldots, n-1$ and all $i = 1, \ldots, N$. Then we use formula (66) together with Lemma A.10 and Lemma A.11 in order to derive an estimate for σ_n.

First, let us write down an estimate for $\boldsymbol{\sigma_n^m}$ using formula (67). Thanks to Lemma A.11, we have for each $i = 1, \ldots, N$ and all n_i, m_i:

$$\sum_{\substack{\boldsymbol{j_i} \in \mathbb{N}^{m_i} \\ |\boldsymbol{j_i}| = n_j}} \left| \sigma_{j_{i,1}}^i * \cdots * \sigma_{j_{i,m_i}}^i \right| \leqslant \sum_{\substack{\boldsymbol{j_i} \in \mathbb{N}^{m_i} \\ |\boldsymbol{j_i}| = n_i}} M_{j_{i,1}} \cdots M_{j_{i,m_i}} \frac{|\xi|^{n_i + m_i - 1}}{(n_i + m_i - 1)!} e^{L|\xi|} .$$

Then, for all $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^N$ with $|\boldsymbol{n}| + |\boldsymbol{m}| \geqslant 1$,

$$\left| \boldsymbol{\sigma_n^m} \right| \leqslant \boldsymbol{M_n^m} \frac{|\xi|^{|\boldsymbol{n}| + |\boldsymbol{m}| - 1}}{(|\boldsymbol{n}| + |\boldsymbol{m}| - 1)!} e^{L|\xi|} . \tag{79}$$

where $\boldsymbol{M_n^m}$ is the shorthand introduced in (51). Therefore, formula (66) gives the following estimate:

$$
\begin{aligned}
|\sigma_n^i| &\leqslant \int_0^\xi \sum_{m=1}^n \sum_{|\boldsymbol{m}|=m} \left\{ |a_{\boldsymbol{m}}^i| \sum_{|\boldsymbol{n}|=n-m} |\boldsymbol{\sigma_n^m}| + \sum_{|\boldsymbol{n}|=n-m-1} |\alpha_{\boldsymbol{m}}^i * \boldsymbol{\sigma_n^m}| \right\} |\,\mathrm{d}t\,| \\
&\leqslant \sum_{m=1}^n \sum_{|\boldsymbol{m}|=m} \left\{ \rho_m C B^m \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{M_n^m} + \rho_m C B^m \sum_{|\boldsymbol{n}|=n-m-1} \boldsymbol{M_n^m} \right\} \int_0^\xi \frac{|t|^{n-1}}{(n-1)!} e^{L|t|} |\,\mathrm{d}t\,| \\
&\leqslant \sum_{m=1}^n \rho_m C B^m \sum_{|\boldsymbol{m}|=m} \left\{ \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{M_n^m} + \sum_{|\boldsymbol{n}|=n-m-1} \boldsymbol{M_n^m} \right\} \frac{|\xi|^n}{n!} e^{L|\xi|}
\end{aligned}
$$

Thus, this expression allows us to define the constant M_n for $n \geqslant 2$. In fact, a quick glance at this formula reveals that it can be extended to $n = 0, 1$ by defining

$$M_n := \sum_{m=0}^n \rho_m C B^m \sum_{|\boldsymbol{m}|=m} \left\{ \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{M_n^m} + \sum_{|\boldsymbol{n}|=n-m-1} \boldsymbol{M_n^m} \right\} \qquad \forall n \in \mathbb{N} . \tag{80}$$

Indeed, if $m = 0$, then the two sums inside the brackets can only possibly be nonzero when $n = 0$, in which case the second sum is empty and the first sum is 1, so we recover $M_0 = C$. Likewise, if $n = 1$, then the $m = 0$ term is $0 + C$ and the $m = 1$ term is $CBM_0 + 0$, so again we recover the constant M_1 defined previously.

Step 4.2: Bounding M_n. To see that $M_n \leqslant DM^n$ for some $D, M > 0$, consider the following two power series in an abstract variable t:

$$\widehat{p}(t) := \sum_{n=0}^\infty M_n t^n \qquad \text{and} \qquad Q(t) := \sum_{m=0}^\infty C B^m t^m . \tag{81}$$

Notice that $Q(t)$ is convergent and $Q(0) = C = M_0$. We will show that $\widehat{p}(t)$ is also a convergent power series. The key observation is that $\widehat{p}$ satisfies the following functional equation:

$$\widehat{p}(t) = (1+t)Q\big(t\widehat{p}(t)\big) . \tag{82}$$

This equation was found by trial and error. In order to verify it, we rewrite the power series $Q(t)$ in the following way:

$$Q(t) = \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \rho_m CB^m t^{\boldsymbol{m}} \quad . \tag{83}$$

Then (82) is straightforward to verify by direct substitution and comparing the coefficients of t^n using the defining formula (80) for M_n. Thus, the righthand side of (82) expands as follows:

$$(1+t) \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \rho_m CB^m \left(t \sum_{n=0}^{\infty} M_n t^n \right)^{\boldsymbol{m}}$$

$$= (1+t) \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \rho_m CB^m t^m \left(\sum_{n_1=0}^{\infty} M_{n_1} t^{n_1} \right)^{m_1} \cdots \left(\sum_{n_N=0}^{\infty} M_{n_N} t^{n_N} \right)^{m_N}$$

$$= (1+t) \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \rho_m CB^m \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n} \boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{m}} t^{n+m}$$

$$= (1+t) \sum_{m=0}^{\infty} \sum_{|\boldsymbol{m}|=m} \rho_m CB^m \sum_{n=0}^{\infty} \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{m}} t^{n}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \rho_m CB^m \sum_{|\boldsymbol{m}|=m} \left\{ \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{m}} t^{n} + \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{m}} t^{n+1} \right\}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \rho_m CB^m \sum_{|\boldsymbol{m}|=m} \left\{ \sum_{|\boldsymbol{n}|=n-m} \boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{m}} + \sum_{|\boldsymbol{n}|=n-m-1} \boldsymbol{M}_{\boldsymbol{n}}^{\boldsymbol{m}} \right\} t^{n}$$

In the final equality, we once again noticed that both sums inside the curly brackets are zero whenever $m > n$.

Now, consider the following holomorphic function in two variables (t, p):

$$F(t, p) := -p + (1+t)Q(tp) \quad . \tag{84}$$

It has the following properties:

$$F(0, C) = 0 \qquad \text{and} \qquad \left. \frac{\partial P}{\partial p} \right|_{(t,p)=(0,C)} = -1 \neq 0 \quad .$$

By the Holomorphic Implicit Function Theorem, there exists a unique holomorphic function $p(t)$ near $t = 0$ such that $p(0) = C$ and $F\big(t, p(t)\big) = 0$. Therefore, $\widehat{p}(t)$ must be the convergent Taylor series expansion of $p(t)$ at $t = 0$, so its coefficients grow at most exponentially: i.e., there are constants $D, M > 0$ such that $M_n \leqslant DM^n$. This completes the proof of the Main Technical Claim and hence of Lemma 2.5. ∎

At last, we are able to collect all our work in order to finish the proof of the Gevrey Asymptotic Implicit Function Theorem (Theorem 1.1).

Proof of Theorem 1.1. By the Formal Implicit Function Theorem (Proposition 2.1), there is a subdomain $\mathsf{X}_0 \subset \mathsf{X}$ containing x_0 such that the equation $F(x, \hbar, z) = 0$ has a unique formal solution $\widehat{f}$ satisfying $f_0(x_0) = z_0$. Let f_0, f_1 be its leading-

and next-to-leading-order parts in $\hbar$. As in Lemma 2.2, we change variables as $z = f_0 + \hbar f_1 + \hbar w$ to transform the equation $F(x, \hbar, z) = 0$ into $w = \hbar G(x, \hbar, w)$. By Lemma 2.5, this equation has a unique holomorphic solution $w = g(x, \hbar)$ on $\mathsf{X}_0 \times \mathsf{S}_0$ for some sectorial subdomain $\mathsf{S}_0 \subset \mathsf{S}$ still with opening Θ and admitting a uniform Gevrey asymptotic expansion as $\hbar \to 0$ along $\overline{\Theta}$. Finally, we define $f := f_0 + \hbar f_1 + \hbar g$ which is readily seen to have all the desired properties. ∎

§ 3. Linear Algebra in Gevrey Asymptotic Families

In this section, we provide an application of our main theorem to the study of asymptotic families of holomorphic matrices over a sector. Namely, we prove the following theorem, from which Theorem 1.3 follows immediately as a special case.

3.1. Theorem (Jordan Decomposition in Gevrey Asymptotic Families).

Fix a domain $\mathsf{X} \subset \mathbb{C}_x^d$ and a point $x_0 \in \mathsf{X}$. Let $\mathsf{S} \subset \mathbb{C}_\hbar$ be a sectorial domain at the origin and opening arc Θ with opening angle $|\Theta| = \pi$. Let $A = A(x, \hbar)$ be a holomorphic $n{\times}n$-matrix on $\mathsf{X} \times \mathsf{S}$ which admits a uniform Gevrey asymptotic expansion

$$A(x, \hbar) \simeq \widehat{A}(x, \hbar) \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta}, \text{ unif. } \forall x \in \mathsf{X} \, . \tag{85}$$

Suppose that the distinct eigenvalues of its $\hbar$-leading-order part $A_{00} := A_0(x_0)$ at the point x_0 are $a_1, \ldots, a_m \in \mathbb{C}$ with respective multiplicities $n_1, \ldots, n_m$. Let P_{00} be a constant invertible $n{\times}n$-matrix that puts A_{00} into a Jordan normal form:

$$P_{00} A_{00} P_{00}^{-1} = \Lambda_{00} := \operatorname{diag}\left(a_1 I_{n_1} + N_1, \ldots, a_m I_{n_m} + N_m\right) \, ,$$

where I_{n_i} is the identity $n_i{\times}n_i$-matrix and N_i is a nilpotent $n_i{\times}n_i$-matrix containing zeros in all positions except those in the first superdiagonal, which may contain either zeros or ones.

Then there is a subdomain $\mathsf{X}_0 \subset \mathsf{X}$ containing x_0 and a sectorial subdomain $\mathsf{S}_0 \subset \mathsf{S}$ with the same opening Θ such that there is an invertible $n{\times}n$-matrix $P = P(x, \hbar)$ on $\mathsf{X}_0 \times \mathsf{S}_0$ that admits a uniform Gevrey asymptotic expansion

$$P(x, \hbar) \simeq \widehat{P}(x, \hbar) = \sum_{k=0}^{\infty} P_k(x)\hbar^k \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta}, \text{ unif. } \forall x \in \mathsf{X}_0 \, , \tag{86}$$

such that $P_0(x_0) = P_{00}$ and which block-diagonalises the matrix A:

$$PAP^{-1} = \Lambda = \operatorname{diag}\left(\Lambda_1, \ldots, \Lambda_m\right) \, , \tag{87}$$

where each $\Lambda_i = \Lambda_i(x, \hbar)$ is an $n_i{\times}n_i$-matrix which admits a uniform Gevrey asymptotic expansion

$$\Lambda_i(x, \hbar) \simeq \widehat{\Lambda}_i(x, \hbar) = \sum_{k=0}^{\infty} \Lambda_{i,k}(x)\hbar^k \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta}, \text{ unif. } \forall x \in \mathsf{X}_0 \, , \tag{88}$$

with $\Lambda_{i,0}(x_0) = a_i I_{n_1} + N_i$. Furthermore, the transformation P is the uniform Borel resummation of its asymptotic power series $\widehat{P}$ in the direction θ that bisects the arc Θ: for all $(x, \hbar) \in \mathsf{X}_0 \times \mathsf{S}_0$,

$$P = \mathcal{S}_\theta\big[\widehat{P}\big] \, . \tag{89}$$

Proof. Using standard theory (see, e.g., [Was76, §25.2]), we can find a holomorphic invertible matrix $P_0(x)$ on a domain $\mathsf{X}_0 \subset \mathsf{X}$ such that

$$P_0 A_0 P_0^{-1} = \mathrm{diag}(\Lambda_{1,0}, \ldots, \Lambda_{m,0})$$

and $P_0(x_0) = P_{00}$ where each $\Lambda_{i,0} = \Lambda_{i,0}(x)$ is a holomorphic $n_i \times n_i$-matrix on X_0 with the property that $\Lambda_{i,0}(x_0) = a_i I_{n_1} + N_i$. To simplify notation, let us assume that the leading-order matrix A_0 has already been diagonalised over X_0, so

$$P_0 = I \quad \text{and} \quad A_0 = \mathrm{diag}\left(\Lambda_{1,0}, \ldots, \Lambda_{m,0}\right) = \Lambda_0 .$$

Our goal is to find a holomorphic matrix $P = P(x, \hbar)$ whose leading order is P_0 and a holomorphic block-diagonal matrix $\Lambda = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_m)$, with blocks $\Lambda_i = \Lambda_i(x, \hbar)$ of size $n_i \times n_i$, whose leading order is Λ_0, such that

$$PA = \Lambda P . \tag{90}$$

Let us break up the eigenvalues $a_1, \ldots, a_m$ arbitrarily into two separate groups. Pick any $p \in \{1, \ldots, m\}$ and let

$$\Lambda_0' := \mathrm{diag}(\Lambda_{1,0}, \ldots, \Lambda_{p,0}) \quad \text{and} \quad \Lambda_0'' := \mathrm{diag}(\Lambda_{p+1,0}, \ldots, \Lambda_{m,0}) . \tag{91}$$

Note that Λ_0' and Λ_0'' have no eignvalues in common. Put $n' := n_1 + \ldots + n_p$ and $n'' := n_{p+1} + \ldots + n_m$. Then we block-partition the matrices Λ and A accordingly:

$$\Lambda = \left[\begin{array}{c|c} \Lambda' & 0 \\ \hline 0 & \Lambda'' \end{array}\right] \quad \text{and} \quad A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{c|c} n' \times n' & n' \times n'' \\ \hline n'' \times n' & n'' \times n'' \end{array}\right] , \tag{92}$$

where we have indicated the sizes of the blocks A_{ij}. Inspired by techniques in [Sib58, RS66, RS68], we search for P in the following block-matrix form:

$$P = \left[\begin{array}{c|c} I_{n'} & S \\ \hline T & I_{n''} \end{array}\right] = \left[\begin{array}{c|c} n' \times n' & n' \times n'' \\ \hline n'' \times n' & n'' \times n'' \end{array}\right] , \tag{93}$$

where $I_{n'}, I_{n''}$ are respectively the identity $n' \times n'$- and $n'' \times n''$-matrices. Substituting the block-partitions (92) and the ansatz (93) into equation (90) yields four conditions:

$$\begin{aligned} A_{11} + SA_{21} &= \Lambda' ; & A_{12} + SA_{22} &= \Lambda'S ; \\ A_{22} + TA_{12} &= \Lambda'' ; & A_{21} + TA_{11} &= \Lambda''T . \end{aligned} \tag{94}$$

Matrices Λ', Λ'' can be eliminated from the two equations on the right. This leads to two uncoupled matrix quadratic equations for S and T:

$$A_{12} + SA_{22} - A_{11}S - SA_{21}S = 0 \quad \text{and} \quad A_{21} + TA_{11} - A_{22}T - SA_{12}T = 0 . \tag{95}$$

Let us focus on solving the equation for S. Observe that its leading-order in $\hbar$ is simply

$$S_0 \Lambda_0'' - \Lambda_0' S_0 = 0 . \tag{96}$$

A simple but remarkable fact from linear algebra (see, e.g., [Was76, Theorem 4.1])

says that this equation possess solutions other than $S_0 = 0$ if and only if Λ_0' and Λ_0'' have at least one eigenvalue in common, which is contrary to how the matrices Λ_0' and Λ_0'' were defined. Thus, $S_0 = 0$.

Now, put $N := n'n''$, and let $w = (w_1, \ldots, w_N)$ be the N-dimensional vector whose components are the entries of S in some order. Then the quadratic equation (95) for S can be written in the form

$$w = \hbar G(x, \hbar, w) \tag{97}$$

where G is a holomorphic map $\mathsf{X}_0 \times \mathsf{S} \times \mathbb{C}_w^N \to \mathbb{C}^N$ which is quadratic in the components of w. By the Gevrey Asymptotic Implicit Function Theorem (Theorem 1.1) (or more specifically by Lemma 2.5), there is a sectorial subdomain $\mathsf{S}_0 \subset \mathsf{S}$ with the same opening Θ such that there is a unique holomorphic map $g : \mathsf{X}_0 \times \mathsf{S}_0 \to \mathbb{C}^N$ which admits a uniform Gevrey asymptotic expansion as $\hbar \to 0$ along $\overline{\Theta}$ and such that $g(x, \hbar) = \hbar G\big(x, \hbar, g(x, \hbar)\big) = 0$. This implies the existence and uniqueness of a holomorphic matrix $S = S(x, \hbar)$ on $\mathsf{X}_0 \times \mathsf{S}_0$ satisfying (95) and admitting a uniform Gevrey asymptotic expansion as $\hbar \to 0$ along $\overline{\Theta}$.

Using exactly the same argument, we can derive a unique solution $T = T(x, \hbar)$ of (95) on $\mathsf{X}_0 \times \mathsf{S}_0$ at the expense of only possibly having to shrink the radial size of S_0 (but not the opening Θ). As a result, we have found a unique transformation P on $\mathsf{X}_0 \times \mathsf{S}_0$ defined by (93) and a unique block-diagonal matrix $\Lambda = \mathrm{diag}(\Lambda', \Lambda'')$ on $\mathsf{X}_0 \times \mathsf{S}_0$ defined by the two equations on the left in (94), which satisfy (90) and have the desired asymptotic properties.

If the total number of distinct eigenvalues m is 2, then we can proceed no further: Λ, P are the desired matrices for the assertions of the theorem. Otherwise, the procedure outlined above should now be iterated to finally find the matrix P that brings A to the desired form Λ from (87). For example, in the next step after we have found the matrices Λ', Λ'' above, we may search for a transformation that further block-diagonalises Λ' (that is, if Λ_0' has at least two distinct eigenvalues; otherwise, proceed to block-diagonalise Λ''). We may break the eigenvalues $a_1, \ldots, a_p$ up further into two groups and transform Λ from (92) to a block-diagonal matrix $\tilde{\Lambda} = \mathrm{diag}(\tilde{\Lambda}', \tilde{\Lambda}'', \Lambda'')$ by searching for a transformation in the form

$$\begin{bmatrix} I & S & \\ T & I & \\ & & I \end{bmatrix} . \qquad \blacksquare$$

3.2. Remark. Under the assumptions of Theorem 1.3, we can use a slightly different argument to the one used to prove Theorem 3.1 in order to show that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are well-defined and have the asserted properties. Indeed, the characteristic polynomial of A is a holomorphic function $F = F(x, \hbar, z)$ on $\mathsf{X} \times \mathsf{S} \times \mathbb{C}_z$, which is a polynomial in z whose coefficients admit a uniform Gevrey asymptotic expansion as $\hbar \to 0$ along $\overline{\Theta}$. Its leading-order part $F_0 = F_0(x, z)$ is the characteristic polynomial of the leading-order part A_0, so $F_0(x_0, a_i) = 0$ for each $i = 1, \ldots, n$. By assumption, the eigenvalues of A_{00} are distinct, which means the discriminant of the polynomial F_0 at $x = a$ is nonzero. As a result, the derivative

$\partial F_0 / \partial z$ is nonzero at each (x_0, a_i). By the Gevrey Asymptotic Implicit Function Theorem (Theorem 1.1, or more specifically Corollary 1.5), there is a subdomain $\mathsf{X}_0 \subset \mathsf{X}$ containing x_0 and a sectorial subdomain $\mathsf{S}_0 \subset \mathsf{S}$ with the same opening Θ such that there are unique holomorphic functions $\lambda_i = \lambda_i(x, \hbar)$ on $\mathsf{X}_0 \times \mathsf{S}_0$ that admit uniform Gevrey asymptotic expansions (10) with $\hbar$-leading-orders satisfying $\lambda_{i,0}(x_0) = a_i$.

Appendix A. Background Information

§ A.1. Gevrey Asymptotics

A.1. A ***sectorial domain*** at the origin in $\mathbb{C}_\hbar$ is a simply connected domain $\mathsf{S} \subset \mathbb{C}_\hbar^* = \mathbb{C}_\hbar \setminus \{0\}$ whose closure $\overline{\mathsf{S}}$ in the real-oriented blowup $[\mathbb{C}_\hbar : 0]$ intersects the boundary circle $\mathbb{S}^1$ in a closed arc $\overline{\Theta} \subset \mathbb{S}^1$ with nonzero length. The open arc Θ is called the ***opening*** of S, and its length $|\Theta|$ is called the ***opening angle*** of S. A ***Borel disc*** of ***diameter*** $R > 0$ is the sectorial domain $\mathsf{S} = \{ \hbar \in \mathbb{C}_\hbar \mid \mathrm{Re}(1/\hbar) > 1/R \}$. Its opening is $\Theta = (-\frac{\pi}{2}, +\frac{\pi}{2})$. Likewise, a Borel disc bisected by a direction $\theta \in \mathbb{S}^1$ is the sectorial domain $\mathsf{S} = \{ \hbar \in \mathbb{C}_\hbar \mid \mathrm{Re}(e^{i\theta}/\hbar) > 1/R \}$. Its opening is $\Theta = (\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2})$.

A.2. A holomorphic function $f(\hbar)$ on a sectorial domain S is admits a power series $\widehat{f}(\hbar)$ as its ***asymptotic expansion as $\hbar \to 0$ along Θ*** (or ***as $\hbar \to 0$ in S***) if, for every $n \geqslant 0$ and every compactly contained subarc $\Theta_0 \Subset \Theta$, there is a sectorial subdomain $\mathsf{S}_0 \subset \mathsf{S}$ with opening Θ_0 and a real constant $C_{n,0} > 0$ such that

$$\left| f(\hbar) - \sum_{k=0}^{n-1} f_k \hbar^k \right| \leqslant C_{n,0} |\hbar|^n \tag{98}$$

for all $\hbar \in \mathsf{S}_0$. The constants $C_{n,0}$ may depend on n and the opening Θ_0. If this is the case, we write

$$f(\hbar) \sim \widehat{f}(\hbar) \qquad \text{as } \hbar \to 0 \text{ along } \Theta \,. \tag{99}$$

If the constants $C_{n,0}$ in (98) can be chosen uniformly for all compactly contained subarcs $\Theta_0 \Subset \Theta$ (i.e., independent of Θ_0 so that $C_{n,0} = C_n$ for all n), then we write

$$f(\hbar) \sim \widehat{f}(\hbar) \qquad \text{as } \hbar \to 0 \text{ along } \overline{\Theta} \,. \tag{100}$$

A.3. We also say that the holomorphic function f admits $\widehat{f}$ as its ***Gevrey asymptotic expansion as $\hbar \to 0$ along Θ*** if the constants $C_{n,0}$ in (98) depend on n like $C_0 M_0^n n!$. More explicitly, for every compactly contained subarc $\Theta_0 \Subset \Theta$, there is a sectorial domain $\mathsf{S}_0 \subset \mathsf{S}$ with opening $\Theta_0 \Subset \Theta$ and real constants $C_0, M_0 > 0$ which give the bounds

$$\left| f(\hbar) - \sum_{k=0}^{n-1} f_k \hbar^k \right| \leqslant C_0 M_0^n n! |\hbar|^n \tag{101}$$

for all $\hbar \in \mathsf{S}_0$ and all $n \geqslant 0$. In this case, we write

$$f(\hbar) \simeq \widehat{f}(\hbar) \qquad \text{as } \hbar \to 0 \text{ along } \Theta \,. \tag{102}$$

If in addition to (101), the constants C_0, M_0 can be chosen uniformly for all $\Theta_0 \Subset \Theta$, then we will write

$$f(\hbar) \simeq \widehat{f}(\hbar) \qquad \text{as } \hbar \to 0 \text{ along } \overline{\Theta} \ . \tag{103}$$

A.4. A formal power series $\widehat{f}(\hbar) = \sum f_n \hbar^n$ is a ***Gevrey power series*** if there are constants $C, M > 0$ such that for all $n \geqslant 0$,

$$|f_n| \leqslant C M^n n! \ . \tag{104}$$

A.5. All the above definitions translate immediately to cover vector-valued holomorphic functions on S by using, say, the Euclidean norm in all the above estimates.

§ A.2. Borel-Laplace Theory

A.6. Let $\Xi_\theta := \big\{ \xi \in \mathbb{C}_\xi \mid \operatorname{dist}(\xi, e^{i\theta}\mathbb{R}_+) < \varepsilon \big\}$, where $e^{i\theta}\mathbb{R}_+$ is the real ray in the direction θ. Let $\varphi = \varphi(\xi)$ be a holomorphic function on Ξ_θ. Its ***Laplace transform*** in the direction θ is defined by the formula:

$$\mathfrak{L}_\theta[\varphi](x, \hbar) := \int_{e^{i\theta}\mathbb{R}_+} \varphi(x, \xi) e^{-\xi/\hbar} \, \mathrm{d}\xi \ . \tag{105}$$

When $\theta = 0$, we write simply $\mathfrak{L}$. Clearly, φ is Laplace-transformable in the direction θ if φ has ***at-most-exponential growth*** as $|\xi| \to +\infty$ along the ray $e^{i\theta}\mathbb{R}_+$. Explicitly, this means there are constants $A, L > 0$ such that for all $\xi \in \Xi_\theta$,

$$\big|\varphi(\xi)\big| \leqslant A e^{L|\xi|} \ . \tag{106}$$

A.7. The convolution product of two holomorphic functions φ, ψ is defined by the following formula:

$$\varphi * \psi(\xi) := \int_0^\xi \varphi(\xi - y)\psi(y) \, \mathrm{d}y \ , \tag{107}$$

where the path of integration is a straight line segment from 0 to ξ.

A.8. Let f be a holomorphic function on a Borel disc $\mathsf{S} = \big\{ \hbar \in \mathbb{C}_\hbar \mid \operatorname{Re}(e^{i\theta}/\hbar) > 1/R \big\}$. The (analytic) ***Borel transform*** (a.k.a., the ***inverse Laplace transform***) of f in the direction θ is defined by the following formula:

$$\mathfrak{B}_\theta[f](x, \xi) := \frac{1}{2\pi i} \oint_\theta f(x, \hbar) e^{\xi/\hbar} \frac{\mathrm{d}\hbar}{\hbar^2} \ , \tag{108}$$

where the integral is taken along the boundary of any Borel disc

$$\mathsf{S}' = \Big\{ \hbar \in \mathbb{C}_\hbar \mid \operatorname{Re}(e^{i\theta}/\hbar) > 1/R' \Big\} \subset \mathsf{S}$$

of strictly smaller diameter $R' < R$, traversed anticlockwise (i.e., emanating from the singular point $\hbar = 0$ in the direction $\theta - \pi/2$ and reentering in the direction $\theta + \pi/2$). When $\theta = 0$, we write simply $\mathfrak{B}$.

The fundamental fact that connects Gevrey asymptotics and the Borel transform is the following (cf. [Nik20, Lemma B.5]). If $f = f(\hbar)$ is a holomorphic function

defined on a sectorial domain S with opening angle $|\Theta| = \pi$ and f admits Gevrey asymptotics as $\hbar \to 0$ along the *closed* arc $\overline{\Theta}$, then the analytic Borel transform $\varphi(\xi) = \mathfrak{B}_\theta[f](\xi)$ defines a holomorphic function on a tubular neighbourhood Ξ_θ of some thickness $\varepsilon > 0$. Moreover, its Laplace transform in the direction θ is well-defined and satisfies $\mathfrak{L}_\theta[\varphi] = f$.

A.9. Similarly, for a power series $\widehat{f}(\hbar)$, the (formal) **Borel transform** is defined by

$$\widehat{\varphi}(\xi) = \widehat{\mathfrak{B}}[\widehat{f}](\xi) := \sum_{k=0}^{\infty} \varphi_k \xi^k \quad \text{where} \quad \varphi_k := \tfrac{1}{k!} f_{k+1} \; . \tag{109}$$

The fundamental fact that connects Gevrey power series and the formal Borel transform is the following (cf. [Nik20, Lemma B.8]). If $\widehat{f}$ is a Gevrey power series, then its formal Borel transform $\widehat{\varphi}$ is a convergent power series in ξ. Furthermore, a Gevrey power series $\widehat{f}(\hbar)$ is called a **Borel summable series** in the direction θ if its convergent Borel transform $\widehat{\varphi}(\xi)$ admits an analytic continuation $\varphi(\xi) = \mathrm{AnCont}_\theta[\widehat{\varphi}](\xi)$ to a tubular neighbourhood Ξ_θ of the ray $e^{i\theta}\mathbb{R}_+$ with at-most-exponential growth in ξ at infinity in Ξ_θ. If this is the case, the Laplace transform $\mathfrak{L}_\theta[\varphi](\hbar)$ is well-defined and defines a holomorphic function $f(\hbar)$ on some Borel disc S bisected by the direction θ, and we say that $f(\hbar)$ is the **Borel resummation** in direction θ of the formal power series $\widehat{f}(\hbar)$, and we write

$$f(\hbar) = \mathcal{S}_\theta\big[\,\widehat{f}(\hbar)\,\big](\hbar) \; .$$

If $\theta = 0$, we write simply $\mathcal{S}$. Expressly, we have the following formulas:

$$\mathcal{S}_\theta\big[\,\widehat{f}(\hbar)\,\big](\hbar) = \mathfrak{L}_\theta\big[\,\varphi\,\big](\hbar) = \mathfrak{L}_\theta\big[\,\mathrm{AnCont}_\theta[\widehat{\varphi}]\,\big](\hbar) \; .$$

Thus, Borel resummation $\mathcal{S}_\theta$ can be seen as a map from the set of (germs of) holomorphic functions f on S with $|\Theta| = \pi$ satisfying (103) to the set of Borel summable power series. One of the most fundamental theorems in Gevrey asymptotics and Borel-Laplace theory is a theorem of Nevanlinna [Nev18, pp.44-45][3], which says that this map $\mathcal{S}_\theta$ is invertible and its inverse is the asymptotic expansion æ.

§ A.3. Some Useful Elementary Estimates

Here, for reference, we collect some elementary estimates used in this paper. Their proofs are straightforward (see [Nik20, Appendix C.4]).

A.10. Lemma. *For any $R \geqslant 0$, any $L \geqslant 0$, and any nonnegative integer n,*

$$\int_0^R \frac{r^n}{n!} e^{Lr} \, \mathrm{d}r \leqslant \frac{R^{n+1}}{(n+1)!} e^{LR} \; .$$

A.11. Lemma. *Let $i_1, \ldots, i_m$ be nonnegative integers and put $n := i_1 + \cdots + i_m$. Let $f_{i_1}, \ldots, f_{i_j}$ be holomorphic functions on $\Xi := \big\{\xi \mid \mathrm{dist}(\xi, \mathbb{R}_+) < \varepsilon\big\}$ for some $\varepsilon > 0$. If*

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[3]It was rediscovered and clarified decades later by Sokal [Sok80]; see also [Mal95, p.182], [LR16, Theorem 5.3.9], as well as [Nik20, §B.3].

there are constants $M_{i_1}, \ldots, M_{i_m}, L \geqslant 0$ such that

$$\left| f_{i_k}(\xi) \right| \leqslant M_{i_k} \frac{|\xi|^{i_k}}{i_k!} e^{L|\xi|} \qquad \forall \xi \in \Xi \ ,$$

then their total convolution product satisfies the following bound:

$$\left| f_{i_1} * \cdots * f_{i_m}(\xi) \right| \leqslant M_{i_1} \cdots M_{i_m} \frac{|\xi|^{i+m-1}}{(i+m-1)!} e^{L|\xi|} \qquad \forall \xi \in \Xi \ .$$

Appendix B. Proof of Theorem 1.1: Scalar Case

This section is dedicated to proving our main result, the Gevrey Asymptotic Implicit Function Theorem (Theorem 1.1), in the scalar case, $N = 1$. The overall strategy of the proof is to first construct a formal solution $z = \widehat{f}$ of the equation $F(x, \hbar, z) = 0$ using the ordinary Holomorphic Implicit Function Theorem at the leading-order in $\hbar$ and then use a recursion to determine all higher-order corrections. We then want to apply the Borel resummation to $\widehat{f}$ to get f. To do so, we first make a convenient change of variables $z \mapsto w$ in order to put our equation into a certain standard form which is more amenable to the Borel transform. Applying the Borel transform, we obtain a first-order ordinary differential equation for $\sigma = \mathfrak{B}[w]$, albeit nonlinear and with convolution. Nevertheless, this ODE is easy to convert into an integral equation, which we then proceed to solve using the method of successive approximations. To show that this sequence of approximations converges to an actual solution σ, we give an estimate on the terms of this sequence by employing in an interesting way the ordinary Holomorphic Implicit Function Theorem. This estimate also allows us to conclude that the Laplace transform $g = \mathfrak{L}[\sigma]$ of the obtained solution σ exists and defines a holomorphic solution of our equation in standard form. Undoing the change of variables $z \mapsto w$ sends g to the desired solution f.

§ B.1. Formal Perturbation Theory

The starting point is the following classical result whose proof is supplied below for completeness.

B.1. Proposition (Formal Implicit Function Theorem).
Fix a domain $\mathsf{X} \subset \mathbb{C}_x^d$ *and a point* $(x_0, z_0) \in \mathsf{X} \times \mathbb{C}_z$. *Let*

$$\widehat{F} = \widehat{F}(x, \hbar, z) = \sum_{k=0}^{\infty} F_k(x, z) \hbar^k \tag{110}$$

be a formal power series in $\hbar$ *whose coefficients* F_k *are holomorphic functions on* $\mathsf{X} \times \mathbb{C}_z$ *such that* $F_0(x_0, z_0) = 0$ *and the derivative* $\partial F_0 / \partial z$ *is nonzero at* (x_0, z_0).

Then there is subdomain $\mathsf{X}_0 \subset \mathsf{X}$ *containing* x_0 *such that there is a unique formal power series*

$$\widehat{f} = \widehat{f}(x, \hbar) = \sum_{n=0}^{\infty} f_n(x) \hbar^n \tag{111}$$

whose coefficients f_n *are holomorphic functions on* X_0, *satisfying*

$$f_0(x_0) = z_0 \qquad \text{and} \qquad \widehat{F}\big(x, \hbar, \widehat{f}(x, \hbar)\big) = 0 \qquad \forall x \in \mathsf{X}_0 \ . \tag{112}$$

In other words, the equation $\widehat{F}(x, \hbar, z) = 0$ has a unique formal power series solution $z = \widehat{f}$ defined near the point x_0 and satisfying $f_0(x_0) = z_0$. In fact, all the higher-order coefficients f_k are uniquely determined by f_0.

In particular, if $\mathsf{S} \subset \mathbb{C}_\hbar$ is a sectorial domain at the origin, and F is a holomorphic function on $\mathsf{X} \times \mathsf{S} \times \mathbb{C}_z$ which admits the power series $\widehat{F}$ as a uniform asymptotic expansion as $\hbar \to 0$ in S, then the equation $F(x, \hbar, z) = 0$ has a unique formal power series solution $z = \widehat{f}$ near x_0 such that $f_0(x_0) = z_0$.

Proof. The proof is a calculation that amounts to plugging the solution ansatz into the equation solving it order-by-order in $\hbar$. Write the double power series expansion of $\widehat{F}$ in $(\hbar, z)$ as

$$\widehat{F}(x, \hbar, z) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} F_{km}(x) \hbar^k z^m . \tag{113}$$

STEP 1: EXPAND ORDER-BY-ORDER. Plugging the solution ansatz (111) into the equation $\widehat{F}(x, \hbar, z) = 0$ yields:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{i_1+\ldots+i_m=n-k} F_{km} f_{i_1} \cdots f_{i_m} \hbar^n = 0 . \tag{114}$$

Now we solve it order-by-order in $\hbar$.

STEP 2: LEADING-ORDER PART. At order $n = 0$, we get:

$$F_0(x, f_0) = \sum_{m=0}^{\infty} F_{0m}(x) f_0^m = 0 . \tag{115}$$

By the Holomorphic Implicit Function Theorem, there is a domain $\mathsf{X}_0 \subset \mathsf{X}$ containing x_0 such that there is a unique holomorphic function $f_0 : \mathsf{X}_0 \to \mathbb{C}$ that satisfies

$$f_0(x_0) = z_0 \quad \text{and} \quad F_0\big(x, f_0(x)\big) = 0 \qquad \forall x \in \mathsf{X}_0 . \tag{116}$$

In fact, the domain X_0 can be chosen so small that

$$J_0 = J_0(x) := \left. \frac{\partial F_0}{\partial z} \right|_{\big(x, f_0(x)\big)} = \sum_{m=0}^{\infty} m F_{0m} f_0^{m-1} \neq 0 \qquad \forall x \in \mathsf{X}_0 . \tag{117}$$

STEP 3: SUBLEADING-ORDER PART. For clarity, let us examine equation (114) at low orders in $\hbar$. At order $n = 1$, it yields:

$$0 = \sum_{m=0}^{\infty} \left(\sum_{i_1+\ldots+i_m=1} F_{0m} f_{i_1} \cdots f_{i_m} + F_{1m} f_0^m \right) = \sum_{m=0}^{\infty} \left(m F_{0m} f_0^{m-1} f_1 + F_{1m} f_0^m \right)$$

$$= \left(\sum_{m=0}^{\infty} m F_{0m} f_0^{m-1} \right) f_1 + \sum_{m=0}^{\infty} F_{1m} f_0^m$$

$$= J_0 f_1 + \sum_{m=0}^{\infty} F_{1m} f_0^m .$$

Therefore, we are able to solve for f_1 uniquely:

$$f_1 := -J_0^{-1} \sum_{m=0}^{\infty} F_{1m} f_0^m . \tag{118}$$

Similarly, at order $n = 2$ in $\hbar$, we find:

$$0 = \sum_{m=0}^{\infty} \left(\sum_{i_1+\ldots+i_m=2} F_{0m} f_{i_1} \cdots f_{i_m} + \sum_{i_1+\ldots+i_m=1} F_{1m} f_{i_1} \cdots f_{i_m} + F_{m2} f_0^m \right)$$

$$= \sum_{m=0}^{\infty} \left(m F_{0m} f_0^{m-1} f_2 + \sum_{i_1+\ldots+i_m=2} F_{0m} f_{i_1} \cdots f_{i_m} + m F_{1m} f_0^{m-1} f_1 + F_{m2} f_0^m \right) .$$

Thus, once again, we are able to solve this equation for f_2 uniquely:

$$f_2 := -J_0^{-1} \sum_{m=0}^{\infty} \left(\sum_{i_1+\ldots+i_m=2} F_{0m} f_{i_1} \cdots f_{i_m} + m F_{1m} f_0^{m-1} f_1 + F_{m2} f_0^m \right) . \tag{119}$$

STEP 4: INDUCTIVE STEP. More generally, at order $n \geqslant 1$ in $\hbar$, when we have already solved uniquely for functions $f_0, \ldots, f_{n-1}$, equation (114) yields:

$$0 = \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{i_1+\ldots+i_m=n-k} F_{km} f_{i_1} \cdots f_{i_m}$$

$$= \left(\sum_{m=0}^{\infty} m F_{0m} f_0^{m-1} \right) f_n + \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{i_1+\ldots+i_m=n-k} F_{km} f_{i_1} \cdots f_{i_m} ,$$

which has a unique solution f_n given by formula

$$f_n := -J_0^{-1} \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{\substack{0 \leqslant i_1,\ldots,i_m \leqslant n-1 \\ i_1+\ldots+i_m=n-k}} F_{km} f_{i_1} \cdots f_{i_m} . \tag{120} \blacksquare$$

§ B.2. Transformation to the Standard Form

Next, we make a convenient change of variables in order to bring the given equation $F(x, \hbar, z) = 0$ to a standard form that is more easily handled using the Borel-Laplace method. This transformation and the standard form are fully determined by the leading-order solution f_0 of the equation $F_0(x, z) = 0$ and can always be achieved under our hypotheses. Namely, we have the following statement.

B.2. Lemma. *Suppose F is a holomorphic function on $\mathsf{X}_0 \times \mathsf{S} \times \mathbb{C}_z$ satisfying the hypotheses of Proposition B.1. Let f_0 and f_1 be the leading- and the next-to-leading-order parts of the formal solution $\widehat{f}$ defined on $\mathsf{X}_0 \subset \mathsf{X}$. Then the change of the unknown variable $z \mapsto w$ given by*

$$z = f_0 + \hbar(f_1 + w) \tag{121}$$

transforms the equation $F(x, \hbar, z) = 0$ into an equation in w of the form

$$w = \hbar G(x, \hbar, w) , \tag{122}$$

where G is a holomorphic function uniquely determined by f_0 and F. Furthermore, if F admits a Gevrey asymptotic expansion as $\hbar \to 0$ along $\overline{\Theta}$ uniformly for all $x \in \mathsf{X}$ and locally uniformly for all $z \in \mathbb{C}_z$ and the domain X_0 is chosen so small that J_0 (where J_0 is the invertible holomorphic function on X_0 given by (117)) is bounded from below

on X_0, then G also admits a Gevrey asymptotic expansion as $\hbar \to 0$ along $\overline{\Theta}$ uniformly for all $x \in \mathsf{X}_0$ and locally uniformly for all $z \in \mathbb{C}_z$. Specifically, G is defined by

$$G(x, \hbar, w) := \hbar^{-1}\left(w - \hbar^{-1}J_0^{-1}(x)F\big(x, \hbar, f_0(x) + \hbar f_1(x) + \hbar w\big)\right). \tag{123}$$

Proof. The only thing to check is that the righthand side of (123) has no negative powers in $\hbar$. In particular, since F is an entire function in the variable z, identity (123) makes it obvious that G admits a uniform Gevrey asymptotic expansion $\widehat{G}$ as $\hbar \to 0$ along $\overline{\Theta}$ whenever J_0^{-1} is bounded and F admits uniform Gevrey asymptotics. A more explicit formula for the function G is derived in Part B.4.

Let us now verify that G has no negative powers in $\hbar$. Clearly, the leading-order part of $F\big(\hbar, f_0 + \hbar f_1 + \hbar w\big)$ is simply $F_0\big(x, f_0(x)\big)$ which is zero because f_0 is the leading-order solution. Therefore, the righthand side of (123) is at worst of order $\hbar^{-1}$. We argue that the next-to-leading-order part of $F\big(\hbar, f_0 + \hbar f_1 + \hbar w\big)$ is equal to $J_0 w$. We expand it as follows:

$$
\begin{aligned}
\left[F\big(\hbar, f_0 + \hbar f_1 + \hbar w\big)\right]^{O(\hbar)} &= F_1(f_0) + \left[F_0\big(f_0 + \hbar f_1 + \hbar w\big)\right]^{O(\hbar)} \\
&= \sum_{m=0}^{\infty} F_{1m}f_0^m + \left[\sum_{m=0}^{\infty}F_{0m}\big(f_0 + \hbar(f_1 + w)\big)^m\right]^{O(\hbar)} \\
&= \sum_{m=0}^{\infty} F_{1m}f_0^m + \left[\sum_{m=0}^{\infty}\sum_{i+j=m}\binom{m}{i,j}F_{0m}\big(f_0\big)^i\big(f_1 + w\big)^j\hbar^j\right]^{O(\hbar)} \\
&= \sum_{m=0}^{\infty} F_{1m}f_0^m + \sum_{m=0}^{\infty}\sum_{i+j=m}mF_{0m}\big(f_0\big)^{m-1}\big(f_1 + w\big) \\
&= \sum_{m=0}^{\infty} F_{1m}f_0^m + J_0\big(f_1 + w\big).
\end{aligned}
$$

Identity (118) shows that this expression equals $J_0 w$, as desired. ∎

The analogue of the Formal Implicit Function Theorem (Proposition B.1) for equations of the form (122) is especially easy to formulate.

B.3. Lemma. *Let*

$$\widehat{G} = \widehat{G}(x, \hbar, w) := \sum_{k=0}^{\infty}G_k(x, w)\hbar^k \tag{124}$$

be any formal power series in $\hbar$ with holomorphic function coefficients on $\mathsf{X}_0 \times \mathbb{C}_w$ for some domain $\mathsf{X}_0 \subset \mathbb{C}_x^d$. Then there is a unique formal power series

$$\widehat{g} = \widehat{g}(x, \hbar) = \sum_{n=0}^{\infty}g_n(x)\hbar^n \tag{125}$$

with holomorphic coefficients $g_k : \mathsf{X}_0 \to \mathbb{C}$, which satisfies $\widehat{g}(x, \hbar) = \widehat{G}\big(x, \hbar, \widehat{g}(x, \hbar)\big)$ for all $x \in \mathsf{X}_0$. In other words, the equation $w = \hbar\widehat{G}(x, \hbar, w)$ has a unique formal power series solution $w = \widehat{g}$.

In particular, if $\mathsf{S} \subset \mathbb{C}_\hbar$ is a sectorial domain at the origin and G is a holomorphic function $\mathsf{X}_0 \times \mathsf{S} \times \mathbb{C}_w \to \mathbb{C}$ which admits the power series $\widehat{G}$ as a locally uniform asymptotic expansion as $\hbar \to 0$ in S, then the equation $w = G(x, \hbar, w) = 0$ has a unique formal power series solution $w = \widehat{g}(x, \hbar)$ as above.

Moreover, $g_0 \equiv 0$ and all the higher-order coefficients g_n are given by the following recursive formula:

$$g_{n+1} = \sum_{k=0}^{n} \sum_{m=0}^{n-k} \sum_{i_1 + \cdots + i_m = n-k} G_{km} g_{i_1} \cdots g_{i_m} , \qquad (126)$$

where $G_{km} = G_{km}(x)$ are the coefficients of the double power series expansion

$$\widehat{G}(x, \hbar, w) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G_{km}(x) \hbar^k w^m . \qquad (127)$$

Proof. The proof is a computation very similar to the one in the proof of Proposition B.1. Plugging the solution ansatz (125) into the double power series expansion (127) of $\widehat{G}$, the righthand side of the equation $w = \hbar \widehat{G}(x, \hbar, w)$ becomes:

$$\hbar \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} G_{km} \hbar^k \left(\sum_{n=0}^{\infty} g_n \hbar^n \right)^m = \hbar \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{i_1 + \cdots + i_m = n} G_{km} g_{i_1} \cdots g_{i_m} \hbar^{k+n}$$

$$= \hbar \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{n-k} \sum_{i_1 + \cdots + i_m = n-k} G_{km} g_{i_1} \cdots g_{i_m} \hbar^n . \quad \blacksquare$$

B.4. Explicit formula for the transformation. We can derive a more explicit formula for the holomorphic function G from Lemma B.2. First, let us write F as a uniformly convergent power series in z:

$$F(x, \hbar, z) = \sum_{m=0}^{\infty} A_m(x, \hbar) z^m . \qquad (128)$$

For every $m \in \mathbb{N}$, let $B_m = B_m(x, \hbar)$ be a holomorphic function on $\mathsf{X} \times \mathsf{S}$ defined by the identity

$$A_m = F_{0m} + \hbar F_{1m} + \hbar^2 B_m . \qquad (129)$$

Then we substitute $z = f_0 + \hbar(f_1 + w)$ into $F(x, \hbar, z) = 0$ and expand using (128) and (129):

$$0 = F(\hbar, f_0 + \hbar f_1 + \hbar w)$$

$$= \sum_{m=0}^{\infty} \sum_{i+j+k=m} \binom{m}{i,j,k} F_{0m} f_0^i f_1^j w^k \hbar^{m-i} + \hbar \sum_{m=0}^{\infty} \sum_{i+j+k=m} \binom{m}{i,j,k} F_{1m} f_0^i f_1^j w^k \hbar^{m-i}$$

$$+ \hbar^2 \sum_{m=0}^{\infty} \sum_{i+j+k=m} \binom{m}{i,j,k} B_m f_0^i f_1^j w^k \hbar^{m-i} .$$

Let us split the green and blue sums according to $[(i,j,k) = (m,0,0)] + [(i,j,k) = (m-1,1,0)] + [(i,j,k) = (m-1,0,1)] + [\text{the rest}]$ and $[(i,j,k) = (m,0,0)] + [\text{the rest}]$, respectively. We leave the black sum alone. Thus, noting that

$$\binom{m}{m,0,0} = 1 \quad \text{and} \quad \binom{m}{m-1,1,0} = \binom{m}{m-1,0,1} = m ,$$

we get:

$$0 = \sum_{m=0}^{\infty} F_{0m} f_0^m + \left(\sum_{m=0}^{\infty} m F_{0m} f_0^{m-1} \right) f_1 \hbar + \left(\sum_{m=0}^{\infty} m F_{0m} f_0^{m-1} \right) w \hbar$$

$$+ \sum_{\substack{m=0 \\ }}^{\infty} \sum_{\substack{i+j+k=m \\ 0 \leqslant i \leqslant m-2}} \tbinom{m}{i,j,k} F_{0m} f_0^i f_1^j w^k \hbar^{m-i} + \hbar \sum_{m=0}^{\infty} F_{1m} f_0^m$$

$$+ \hbar \sum_{\substack{m=0 \\ }}^{\infty} \sum_{\substack{i+j+k=m \\ 0 \leqslant i \leqslant m-1}} \tbinom{m}{i,j,k} F_{1m} f_0^i f_1^j w^k \hbar^{m-i} + \hbar^2 \sum_{\substack{m=0 \\ }}^{\infty} \sum_{i+j+k=m} \tbinom{m}{i,j,k} B_m f_0^i f_1^j w^k \hbar^{m-i} .$$

Now, the first green sum vanishes because f_0 is a leading-order solution, see (115). Likewise, the second green sum and the first blue sum add up to zero because f_1 is the next-to-leading-order solution, see (118). In the third green sum, the factor in front of $w\hbar$ is $-J_0^{-1}$. Let us also factorise $\hbar^2$ and another $\hbar$ out of the fourth green and second blue sums. Altogether, we obtain:

$$0 = -J_0^{-1} w \hbar + \hbar^2 \sum_{\substack{m=0 \\ }}^{\infty} \sum_{\substack{i+j+k=m \\ 0 \leqslant i \leqslant m-2}} \tbinom{m}{i,j,k} F_{0m} f_0^i f_1^j w^k \hbar^{m-2-i}$$

$$+ \hbar^2 \sum_{\substack{m=0 \\ }}^{\infty} \sum_{\substack{i+j+k=m \\ 0 \leqslant i \leqslant m-1}} \tbinom{m}{i,j,k} F_{1m} f_0^i f_1^j w^k \hbar^{m-1-i} + \hbar^2 \sum_{\substack{m=0 \\ }}^{\infty} \sum_{i+j+k=m} \tbinom{m}{i,j,k} B_m f_0^i f_1^j w^k \hbar^{m-i} .$$

Next, it is convenient to rearrange the summations as follows:

$$\sum_{m=0}^{\infty} \sum_{i+j+k=m} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \sum_{i+j=m-k} = \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \sum_{i+j=m-k} .$$

Thus, we get:

$$0 = -J_0^{-1} w \hbar + \hbar^2 \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \left(\sum_{\substack{i+j=m-k \\ 0 \leqslant i \leqslant m-2}} \tbinom{m}{i,j,k} F_{0m} f_0^i f_1^j \hbar^{m-2-i} + \sum_{\substack{i+j=m-k \\ 0 \leqslant i \leqslant m-1}} \tbinom{m}{i,j,k} F_{1m} f_0^i f_1^j \hbar^{m-1-i} \right.$$

$$\left. + \sum_{i+j=m-k} \tbinom{m}{i,j,k} B_m f_0^i f_1^j \hbar^{m-i} \right) w^k .$$

Multiplying through by $J_0 \hbar^{-1}$ and taking w over to the lefthand side, we obtain the equation $w = \hbar G(\hbar, w)$ where G is defined as the power series

$$G(x, \hbar, w) := \sum_{k=0}^{\infty} C_k(x, \hbar) w^k , \tag{130}$$

with coefficients C_k given by the following formula:

$$C_k := J_0 \sum_{m=k}^{\infty} \left(\sum_{\substack{i+j=m-k \\ 0 \leqslant i \leqslant m-2}} \tbinom{m}{i,j,k} F_{0m} f_0^i f_1^j \hbar^{m-2-i} + \sum_{\substack{i+j=m-k \\ 0 \leqslant i \leqslant m-1}} \tbinom{m}{i,j,k} F_{1m} f_0^i f_1^j \hbar^{m-1-i} \right.$$

$$\left. + \sum_{i+j=m-k} \tbinom{m}{i,j,k} B_m f_0^i f_1^j \hbar^{m-i} \right) . \tag{131}$$

§ B.3. Gevrey Regularity of the Formal Solution

Now we show that the formal Borel transform of the formal solution $\widehat{f}$ is a convergent power series in the Borel variable ξ; that is, the coefficients f_n grow not faster than $n!$. More precisely, we prove the following proposition.

B.5. Proposition (Gevrey Formal Implicit Function Theorem).
Assume all the hypotheses of Proposition B.1 and suppose in addition that the power series $\widehat{F}$ is locally uniformly Gevrey on $\mathsf{X} \times \mathbb{C}_z$. Then $\mathsf{X}_0 \subset \mathsf{X}$ can be chosen so small that the formal power series $\widehat{f}$ is uniformly Gevrey on X_0. In particular, the formal Borel transform

$$\widehat{\varphi}(x,\xi) = \widehat{\mathfrak{B}}[\widehat{f}](x,\xi) := \sum_{n=0}^{\infty} \tfrac{1}{n!} f_{n+1}(x)\xi^n \tag{132}$$

is a uniformly convergent power series in ξ. Concretely, if $\mathsf{X}_0 \subset \mathsf{X}$ is any subset where the function J_0 is bounded from below and such that there are $A, B > 0$ such that $|F_k(x,z)| \leqslant AB^k k!$ for all $k \geqslant 0$, uniformly for all $x \in \mathsf{X}_0$ and for all $z \in \mathbb{C}_z$ with $|z| < R$ for some $R > 0$, then there are constants $C, M > 0$ such that

$$\left| f_k(x) \right| \leqslant CM^k k! \qquad \forall x \in \mathsf{X}_0, \forall k \ . \tag{133}$$

Proof. Let $\mathsf{X}_0 \subset \mathsf{X}$ be such that the function J_0 is bounded from below on X_0. Then, by Lemma B.2, the proof boils down to proving the following claim.

Claim. *Assume all the hypotheses of Lemma B.3 and suppose that the power series $\widehat{G}$ is Gevrey uniformly for all $x \in \mathsf{X}_0$ and locally uniformly for all $w \in \mathbb{C}_w$. Then the formal solution $\widehat{g}$ is also uniformly Gevrey on X_0.*

Let $A, B > 0$ be constants such that

$$\left| G_{km}(x) \right| \leqslant AB^{k+m} k! \qquad \forall x \in \mathsf{X}_0, \ \forall k, m \in \mathbb{N} \ . \tag{134}$$

We will show that there is a constant $M > 0$ such that

$$\left| g_{n+1}(x) \right| \leqslant M^{n+1} n! \qquad \forall x \in \mathsf{X}_0, \ \forall n \in \mathbb{N} \ . \tag{135}$$

This bound will be demonstrated in two main steps. First, we will recursively construct a sequence $\{M_n\}_{n=0}^{\infty}$ of nonnegative real numbers such that

$$\left| g_{n+1}(x) \right| \leqslant M_{n+1} n! \qquad \forall x \in \mathsf{X}_0, \ \forall n \in \mathbb{N} \ . \tag{136}$$

Then we will show that there is a constant $M > 0$ such that $M_n \leqslant M^n$ for all n.

STEP 1: CONSTRUCTION OF $\{M_n\}_{n=0}^{\infty}$. Let $M_0 := 0$. We can take $M_1 := A$ because $g_1 = G_{00}$. Now we use induction on n and formula (126), which is more convenient to rewrite as follows:

$$g_{n+1} = \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{i_1+\cdots+i_m=n-k} G_{km} g_{i_1} \cdots g_{i_m} \ . \tag{137}$$

Notice that, since $g_0 \equiv 0$, the sum over $i_1, \ldots, i_m$ is empty whenever $m > n - k$, so this expression really is the same as (126). Assume that we have already constructed

$M_0, \ldots, M_n$ such that $\left| g_i \right| \leqslant M_i (i-1)!$ for all $i = 0, \ldots, n$ and all $x \in \mathsf{X}_0$. Then we estimate g_{n+1} using (137):

$$
\begin{aligned}
\left| g_{n+1} \right| &\leqslant \sum_{m=0}^{\infty} \sum_{k=0}^{n} \sum_{i_1 + \cdots + i_m = n-k} A B^{k+m} k! M_{i_1} \cdots M_{i_m} (i_1 - 1)! \cdots (i_m - 1)! \\
&\leqslant \sum_{m=0}^{\infty} \sum_{k=0}^{n} A B^k \sum_{i_1 + \cdots + i_m = n-k} B^m M_{i_1} \cdots M_{i_m} n!
\end{aligned}
$$

Here, we used the fact that $i_1 + \cdots + i_m = n - k$ and the inequality $i! j! \leqslant (i+j)!$. Thus, we can define

$$
M_{n+1} := \sum_{m=0}^{\infty} \sum_{k=0}^{n} A B^k \sum_{i_1 + \cdots + i_m = n-k} B^m M_{i_1} \cdots M_{i_m} . \tag{138}
$$

STEP 2: CONSTRUCTION OF M. To see that $M_n \leqslant M^n$ for some $M > 0$, we argue as follows. Consider the following pair of power series in an abstract variable t:

$$
\widehat{p}(t) := \sum_{n=0}^{\infty} M_n t^n \quad \text{and} \quad Q(t) := \sum_{m=0}^{\infty} B^m t^m . \tag{139}
$$

Notice that $\widehat{p}(0) = M_0 = 0$ and that $Q(t)$ is convergent. We will show that $\widehat{p}(t)$ is also convergent. The key is the observation that they satisfy the following equation:

$$
\widehat{p}(t) = A t Q(t) Q\big(\widehat{p}(t)\big) = A t Q(t) \sum_{m=0}^{\infty} B^m \widehat{p}(t)^m . \tag{140}
$$

This equation was found by trial and error, and it is straightforward to verify directly by substituting the power series $\widehat{p}(t)$ and $Q(t)$ and comparing the coefficients of t^{n+1} using formula (138).

Now, consider the following holomorphic function in two complex variables (t, p):

$$
F(t, p) := -p + A t Q(t) Q(p)
$$

It has the following properties:

$$
F(0,0) = 0 \qquad \text{and} \qquad \left. \frac{\partial F}{\partial p} \right|_{(t,p)=(0,0)} = -1 \neq 0 .
$$

By the Holomorphic Implicit Function Theorem, there exists a unique holomorphic function $p(t)$ near $t = 0$ such that $p(t) = 0$ and $F\big(t, p(t)\big) = 0$. Thus, $\widehat{p}(t)$ must be the convergent Taylor series expansion at $t = 0$ for $p(t)$, so its coefficients grow at most exponentially: there is a constant $M > 0$ such that $M_n \leqslant M^n$. ∎

§ B.4. Exact Perturbation Theory

Now we show that the convergent Borel transform $\widehat{\varphi}(x, \xi)$ of the formal solution admits an analytic continuation along a ray in the Borel ξ-plane and furthermore its Laplace transform is well-defined. First, we prove the following lemma.

B.6. Lemma. *Let $\mathsf{X}_0 \subset \mathbb{C}_x^d$ be a domain. Let $\mathsf{S} := \{\hbar \mid \mathrm{Re}(1/\hbar) > 1/R\} \subset \mathbb{C}_\hbar$ be the Borel disc of some diameter $R > 0$. Recall that the opening of S is $\mathsf{A}_+ := (-\pi/2, +\pi/2)$. Let $G = G(x, \hbar, w)$ be a holomorphic function on $\mathsf{X}_0 \times \mathsf{S} \times \mathbb{C}_w$ which admits a Gevrey asymptotic expansion*

$$G(x, \hbar, w) \simeq \widehat{G}(x, \hbar, w) \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta}_+ , \tag{141}$$

uniformly for all $x \in \mathsf{X}_0$ and locally uniformly for all $w \in \mathbb{C}_w$. Then there is a Borel disc $\mathsf{S}_0 := \{\hbar \mid \mathrm{Re}(1/\hbar) > 1/R_0\} \subset \mathsf{S}$ of possibly smaller diameter $R_0 \in (0, R]$ such that there is a unique holomorphic function $g = g(x, \hbar)$ on $\mathsf{X}_0 \times \mathsf{S}_0$ which admits a uniform Gevrey asymptotic expansion

$$g(x, \hbar) \simeq \widehat{g}(x, \hbar) \quad \text{as } \hbar \to 0 \text{ along } \overline{\Theta}_+, \text{ unif. } \forall x \in \mathsf{X}_0 , \tag{142}$$

and such that $g(x, \hbar) = \hbar G\big(x, \hbar, g(x, \hbar)\big) = 0$ for all $(x, \hbar) \in \mathsf{X}_0 \times \mathsf{S}_0$. Furthermore, g is the uniform Borel resummation of $\widehat{g}$: for all $(x, \hbar) \in \mathsf{X}_0 \times \mathsf{S}_0$,

$$g(x, \hbar) = \mathcal{S}\big[\widehat{g}\big](x, \hbar) . \tag{143}$$

Proof. First, uniqueness of g follows from the asymptotic property (142). Indeed, suppose g' is another such function. Then the difference $g - g'$ is a holomorphic function on $\mathsf{X}_0 \times \mathsf{S}_0$ which is uniformly Gevrey asymptotic to 0 as $\hbar \to 0$ along the closed arc $\overline{\Theta}_+$ of opening angle π. By Nevanlinna's Theorem ([Nev18, pp.44-45] and [Sok80]; see also [Nik20, Theorem B.11]), there can only be one holomorphic function on S_0 (namely, the constant function 0) which is Gevrey asymptotic to 0 as $\hbar \to 0$ along $\overline{\Theta}_+$. Thus, $g - g'$ must be identically zero.

To construct g, start by expanding G as a uniformly convergent power series in w:

$$G(x, \hbar, w) = \sum_{m=0}^{\infty} A_m(x, \hbar) w^m . \tag{144}$$

Step 1: The Borel Transform. Let $a_m = a_m(x)$ be the $\hbar$-leading-order part of A_m and let $\alpha_m(x, \xi) := \mathfrak{B}\big[A_m\big](x, \xi)$. By the assumption (141), there is some $\varepsilon > 0$ such that α_m is a holomorphic function on $\mathsf{X}_0 \times \Xi$, where

$$\Xi := \{\xi \mid \mathrm{dist}(\xi, \mathbb{R}_+) < \varepsilon\} ,$$

with uniformly at most exponential growth at infinity in ξ (cf. Part A.6), and

$$A_m(x, \hbar) = a_m(x) + \mathfrak{L}\big[\alpha_m\big](x, \hbar) \tag{145}$$

for all $(x, \hbar) \in \mathsf{X}_0 \times \mathsf{S}$ provided that the diameter R is sufficiently small.

Dividing through by $\hbar$ and applying the analytic Borel transform to the equation

$w = \hbar G(x, \hbar, w)$, we obtain the following nonlinear ordinary differential equation:

$$\partial_\xi \sigma = \alpha_0 + \sum_{m=1}^\infty \left(a_m \sigma^{*m} + \alpha_m * \sigma^{*m} \right) , \tag{146}$$

where the unknown variables w and σ are related by $\sigma = \mathfrak{B}[w]$ and $w = \mathfrak{L}[\sigma]$. A solution of (146) with initial condition $\sigma(x, 0) = a_0(x)$ is equivalently the solution of the integral equation

$$\sigma = a_0 + \int_0^\xi \left[\alpha_0 + \sum_{m=1}^\infty \left(a_m \sigma^{*m} + \alpha_m * \sigma^{*m} \right) \right] \mathrm{d}t . \tag{147}$$

Step 2: Method of Successive Approximations. We solve this integral equation using the method of successive approximations. To this end, define a sequence of holomorphic functions $\{\sigma_n\}_{n=0}^\infty$ on $\mathsf{X}_0 \times \Xi$ as follows:

$$\sigma_0 := a_0, \qquad \sigma_1 := \int_0^\xi \left[\alpha_0 + a_1 \sigma_0 \right] \mathrm{d}t , \tag{148}$$

and for $n \geqslant 2$ by

$$\sigma_n := \int_0^\xi \sum_{m=1}^n \left(a_m \sum_{i_1+\cdots+i_m=n-m} \sigma_{i_1} * \cdots * \sigma_{i_m} + \alpha_m * \sum_{i_1+\cdots+i_m=n-m-1} \sigma_{i_1} * \cdots * \sigma_{i_m} \right) \mathrm{d}t \tag{149}$$

Main Technical Claim. *The infinite series*

$$\sigma(x, \xi) := \sum_{n=0}^\infty \sigma_n(x, \xi) \tag{150}$$

converges uniformly for all $(x, \xi) \in \mathsf{X}_0 \times \Xi$ and defines a holomorphic solution of the integral equation (147) *with uniformly at-most-exponential growth at infinity in ξ; that is, there are constants $D, K > 0$ such that*

$$\left| \sigma(x, \xi) \right| \leqslant D e^{K|\xi|} \qquad \forall (x, \xi) \in \mathsf{X}_0 \times \Xi . \tag{151}$$

Furthermore, the formal Borel transform

$$\widehat{\sigma}(x, \xi) = \widehat{\mathfrak{B}}[\widehat{g}](x, \xi) = \sum_{n=0}^\infty \tfrac{1}{n!} g_{n+1}(x) \xi^n \tag{152}$$

of the formal solution $\widehat{g}(x, \hbar)$ is the Taylor series expansion of σ at $\xi = 0$.

The assertion of Lemma B.6 follows from this claim by defining

$$g(x, \hbar) := \mathfrak{L}[\sigma](x, \hbar) = \int_0^{+\infty} e^{-\xi/\hbar} \sigma(x, \xi) \, \mathrm{d}\xi . \tag{153}$$

Indeed, the exponential estimate (151) implies that the Laplace transform of σ is uniformly convergent for all $(x, \hbar) \in \mathsf{X}_0 \times \mathsf{S}_0$ where $\mathsf{S}_0 = \left\{ \hbar \mid \mathrm{Re}(1/\hbar) > 1/R_0 \right\}$ as long as $R_0 < K^{-1}$. We now turn to the proof of the Main Technical Claim.

Step 3: Solution Check. First, assuming that the infinite series σ is uniformly convergent for all $(x, \xi) \in X_0 \times \Xi$, we check that it satisfies (147) by direct substitution. Start by removing σ_0 and σ_1 via the following manipulation:

$$
a_0 + \int_0^\xi \left[\alpha_0 + \sum_{m=1}^\infty a_m \left(\sum_{n=0}^\infty \sigma_n \right)^{*m} + \sum_{m=1}^\infty \alpha_m * \left(\sum_{n=0}^\infty \sigma_n \right)^{*m} \right] \mathrm{d}t
$$

$$
= a_0 + \int_0^\xi \left[\alpha_0 + a_1 \sigma_0 + a_1 \sum_{n=1}^\infty \sigma_n + \sum_{m=2}^\infty a_m \left(\sum_{n=0}^\infty \sigma_n \right)^{*m} + \sum_{m=1}^\infty \alpha_m * \left(\sum_{n=0}^\infty \sigma_n \right)^{*m} \right] \mathrm{d}t
$$

$$
= \sigma_0 + \sigma_1 + \int_0^\xi \left[a_1 \sum_{n=1}^\infty \sigma_n + \sum_{m=2}^\infty a_m \left(\sum_{n=0}^\infty \sigma_n \right)^{*m} + \sum_{m=1}^\infty \alpha_m * \left(\sum_{n=0}^\infty \sigma_n \right)^{*m} \right] \mathrm{d}t .
$$

The goal is to show that the the integral in the above expression is $\sum_{n \geqslant 2} \sigma_n$. Focus now on the expression inside the integral and use the formula

$$
\left(\sum_{n=0}^\infty \sigma_n \right)^{*m} = \sum_{n=0}^\infty \sum_{i_1 + \cdots + i_m = n} \sigma_{i_1} * \cdots * \sigma_{i_m} . \tag{154}
$$

Then we manipulate it as follows:

$$
a_1 \sum_{n=1}^\infty \sigma_n + \sum_{m=2}^\infty a_m \left(\sum_{n=0}^\infty \sigma_n \right)^{*m} + \sum_{m=1}^\infty \alpha_m * \left(\sum_{n=0}^\infty \sigma_n \right)^{*m}
$$

$$
= a_1 \sum_{n=1}^\infty \sigma_n + \sum_{m=2}^\infty a_m \sum_{n=0}^\infty \sum_{i_1 + \cdots + i_m = n} \sigma_{i_1} * \cdots * \sigma_{i_m} + \sum_{m=1}^\infty \alpha_m * \sum_{n=0}^\infty \sum_{i_1 + \cdots + i_m = n} \sigma_{i_1} * \cdots * \sigma_{i_m} .
$$

Now we shift the summation index n up by 1 in the first green sum, by m in the second, and by $m + 1$ in the third:

$$
= a_1 \sum_{n=2}^\infty \sigma_{n-1} + \sum_{m=2}^\infty a_m \sum_{n=m}^\infty \sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n - m}} \sigma_{i_1} * \cdots * \sigma_{i_m} + \sum_{m=1}^\infty \alpha_m * \sum_{n=m+1}^\infty \sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n - m - 1}} \sigma_{i_1} * \cdots * \sigma_{i_m} .
$$

Notice that all terms in the second green sum with $n < m$ are zero, so we can start the summation over n from $n = 2$ (which is the lowest possible value of m) without altering the result. Similarly, all terms in the third green sum with $n < m + 1$ are zero, so we can start from $n = 2$. The first green sum is left unaltered. Thus:

$$
= a_1 \sum_{n=2}^\infty \sigma_{n-1} + \sum_{m=2}^\infty a_m \sum_{n=2}^\infty \sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n - m}} \sigma_{i_1} * \cdots * \sigma_{i_m} + \sum_{m=1}^\infty \alpha_m * \sum_{n=2}^\infty \sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n - m - 1}} \sigma_{i_1} * \cdots * \sigma_{i_m} .
$$

The advantage of this way of expressing the sums is that we can now interchange the summations over m and n in the second and third green sums:

$$
= a_1 \sum_{n=2}^\infty \sigma_{n-1} + \sum_{n=2}^\infty \sum_{m=2}^\infty a_m \sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n - m}} \sigma_{i_1} * \cdots * \sigma_{i_m} + \sum_{n=2}^\infty \sum_{m=1}^\infty \alpha_m * \sum_{\substack{i_1, \ldots, i_m \geqslant 0 \\ i_1 + \cdots + i_m = n - m - 1}} \sigma_{i_1} * \cdots * \sigma_{i_m}
$$

$$= \sum_{n=2}^{\infty} \left\{ a_1 \sigma_{n-1} + \sum_{m=2}^{\infty} a_m \sum_{\substack{i_1,\ldots,i_m \geqslant 0 \\ i_1 + \cdots + i_m = n-m}} \sigma_{i_1} * \cdots * \sigma_{i_m} + \sum_{m=1}^{\infty} \alpha_m * \sum_{\substack{i_1,\ldots,i_m \geqslant 0 \\ i_1 + \cdots + i_m = n-m-1}} \sigma_{i_1} * \cdots * \sigma_{i_m} \right\}.$$

Notice that the term $a_1 \sigma_{n-1}$ fits well into the first sum over m to give the $m = 1$ addend. Thus:

$$= \sum_{n=2}^{\infty} \left\{ \sum_{m=1}^{\infty} a_m \sum_{\substack{i_1,\ldots,i_m \geqslant 0 \\ i_1 + \cdots + i_m = n-m}} \sigma_{i_1} * \cdots * \sigma_{i_m} + \sum_{m=1}^{\infty} \alpha_m * \sum_{\substack{i_1,\ldots,i_m \geqslant 0 \\ i_1 + \cdots + i_m = n-m-1}} \sigma_{i_1} * \cdots * \sigma_{i_m} \right\}$$

$$= \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \left\{ a_m \sum_{\substack{i_1,\ldots,i_m \geqslant 0 \\ i_1 + \cdots + i_m = n-m}} \sigma_{i_1} * \cdots * \sigma_{i_m} + \alpha_m * \sum_{\substack{i_1,\ldots,i_m \geqslant 0 \\ i_1 + \cdots + i_m = n-m-1}} \sigma_{i_1} * \cdots * \sigma_{i_m} \right\}.$$

Finally, notice that that both sums are empty for $m > n$, so:

$$= \sum_{n=2}^{\infty} \sum_{m=1}^{n} \left\{ a_m \sum_{\substack{i_1,\ldots,i_m \geqslant 0 \\ i_1 + \cdots + i_m = n-m}} \sigma_{i_1} * \cdots * \sigma_{i_m} + \alpha_m * \sum_{\substack{i_1,\ldots,i_m \geqslant 0 \\ i_1 + \cdots + i_m = n-m-1}} \sigma_{i_1} * \cdots * \sigma_{i_m} \right\}.$$

The sum over m is precisely the expression inside the integral in (149) defining σ_n. This shows that σ satisfies the integral equation (147).

Step 4: Convergence. Now we show that σ is a uniformly convergent infinite series on $\mathsf{X}_0 \times \Xi$ and therefore defines a holomorphic function. In the process, we also establish the estimate (151).

Let $B, C, L > 0$ be such that for all $(x, \xi) \in \mathsf{X}_0 \times \Xi$ and all $m \in \mathbb{N}$,

$$\left| a_m(x) \right| \leqslant CB^m \qquad \text{and} \qquad \left| \alpha_m(x, \xi) \right| \leqslant CB^m e^{L|\xi|}. \tag{155}$$

We claim that there are $D, M > 0$ such that for all $(x, \xi) \in \mathsf{X}_0 \times \Xi$ and $n \in \mathbb{N}$,

$$\left| \sigma_n(x, \xi) \right| \leqslant DM^n \frac{|\xi|^n}{n!} e^{L|\xi|}. \tag{156}$$

If we achieve (156), then the uniform convergence and the exponential estimate (151) both follow at once because

$$\left| \sigma(x, \xi) \right| \leqslant \sum_{n=0}^{\infty} \left| \sigma_n(x, \xi) \right| \leqslant \sum_{n=0}^{\infty} DM^n \frac{|\xi|^n}{n!} e^{L|\xi|} \leqslant De^{(M+L)|\xi|}.$$

To demonstrate (156), we proceed in two steps. First, we construct a sequence of positive real numbers $\{M_n\}_{n=0}^{\infty}$ such that for all n and all $(x, \xi) \in \mathsf{X}_0 \times \Xi$,

$$\left| \sigma_n(x, \xi) \right| \leqslant M_n \frac{|\xi|^n}{n!} e^{L|\xi|}. \tag{157}$$

We will then show that there are constants D, M such that $M_n \leqslant DM^n$ for all n.

Step 4.1: Construction of $\{M_n\}$. We can take $M_0 := C$ and $M_1 := C(1 + BM_0)$ because $\sigma_0 = a_0$ and

$$|\sigma_1| \leqslant \left| \int_0^\xi \left(|\alpha_0| + |a_1||\sigma_0| \right) \right| \mathrm{d}t \,| \leqslant C(1 + BM_0) \int_0^{|\xi|} e^{Lr} \, \mathrm{d}r \leqslant C(1 + BM_0)|\xi|e^{L|\xi|} \; ,$$

where in the final step we used Lemma A.10. Let us assume now that we have constructed $M_0, \ldots, M_{n-1}$ such that $|\sigma_i| \leqslant M_i \frac{|\xi|^i}{i!} e^{L|\xi|}$ for all $i = 0, \ldots, n-1$. Then we use formula (149) together with Lemma A.10 and Lemma A.11 in order to derive an estimate for σ_n:

$$
\begin{aligned}
|\sigma_n| &\leqslant \left| \int_0^\xi \sum_{m=1}^n \left(|a_m| \sum_{i_1 + \cdots + i_m = n-m} |\sigma_{i_1} * \cdots * \sigma_{i_m}| + \sum_{i_1 + \cdots + i_m = n-m-1} |\alpha_m * \sigma_{i_1} * \cdots * \sigma_{i_m}| \right) \right| \mathrm{d}t \,| \\
&\leqslant \sum_{m=1}^n \left(CB^m \sum_{i_1 + \cdots + i_m = n-m} M_{i_1} \cdots M_{i_m} + CB^m \sum_{i_1 + \cdots + i_m = n-m-1} M_{i_1} \cdots M_{i_m} \right) \left| \int_0^\xi \frac{|t|^{n-1}}{(n-1)!} e^{L|t|} \right| \mathrm{d}t \,| \\
&\leqslant \sum_{m=1}^n CB^m \left(\sum_{i_1 + \cdots + i_m = n-m} M_{i_1} \cdots M_{i_m} + \sum_{i_1 + \cdots + i_m = n-m-1} M_{i_1} \cdots M_{i_m} \right) \frac{|\xi|^n}{n!} e^{L|\xi|} \; .
\end{aligned}
$$

Thus, this expression allows us to define the constant M_n for $n \geqslant 2$. In fact, a quick glance at this formula reveals that it can be extended to $n = 0, 1$ by defining

$$M_n := \sum_{m=0}^n CB^m \left(\sum_{i_1 + \cdots + i_m = n-m} M_{i_1} \cdots M_{i_m} + \sum_{i_1 + \cdots + i_m = n-m-1} M_{i_1} \cdots M_{i_m} \right) \; . \tag{158}$$

Indeed, if $m = 0$, then the two sums inside the brackets are nonzero only when $n = 0$, so we recover $M_0 = C$. Likewise, if $n = 1$, then the only the terms $m = 0$ and $m = 1$ are nonzero in this formula, and they are respectively $0 + C$ and $CBM_0 + 0$, so we recover M_1 defined previously.

Step 4.2: Bounding M_n. To see that $M_n \leqslant DM^n$ for some $D, M > 0$, consider the following power series in an abstract variable t:

$$\widehat{p}(t) := \sum_{n=0}^\infty M_n t^n \qquad \text{and} \qquad Q(t) := \sum_{m=0}^\infty CB^m t^m \; . \tag{159}$$

Notice that $Q(t)$ is a convergent and $Q(0) = C = M_0$. We will show that $\widehat{p}(t)$ is also a convergent power series. The key observation is that $\widehat{p}$ satisfies the following functional equation:

$$\widehat{p}(t) = (1 + t)Q\big(t\widehat{p}(t)\big) \; . \tag{160}$$

This equation was found by trial and error, and it is straightforward to verify by direct substitution of the expansions (159) and comparing the coefficients of t^n using the defining formula (158) for M_n. Explicitly, the righthand side of (160) expands as follows:

$$(1 + t) \sum_{m=0}^\infty CB^m \left(t \sum_{n=0}^\infty M_n t^n \right)^m$$

$$= (1+t) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} CB^m \sum_{i_1+\cdots+i_m=n} M_{i_1} \cdots M_{i_m} t^{n+m}$$

$$= (1+t) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} CB^m \sum_{i_1+\cdots+i_m=n-m} M_{i_1} \cdots M_{i_m} t^{n}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} CB^m \left(\sum_{i_1+\cdots+i_m=n-m} M_{i_1} \cdots M_{i_m} t^{n} + \sum_{i_1+\cdots+i_m=n-m} M_{i_1} \cdots M_{i_m} t^{n+1} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} CB^m \left(\sum_{i_1+\cdots+i_m=n-m} M_{i_1} \cdots M_{i_m} + \sum_{i_1+\cdots+i_m=n-m-1} M_{i_1} \cdots M_{i_m} \right) t^{n} .$$

Now, consider the following holomorphic function in two variables (t,p):

$$F(t,p) := -p + (1+t)Q(tp) . \tag{161}$$

It has the following properties:

$$F(0,C) = 0 \qquad \text{and} \qquad \left. \frac{\partial P}{\partial p} \right|_{(t,p)=(0,C)} = -1 \neq 0 .$$

By the Holomorphic Implicit Function Theorem, there exists a unique holomorphic function $p(t)$ near $t = 0$ such that $p(0) = C$ and $F\big(t,p(t)\big) = 0$. Therefore, $\widehat{p}(t)$ must be the convergent Taylor series expansion of $p(t)$ at $t = 0$, so its coefficients grow at most exponentially: there are constants $D, M > 0$ such that $M_n \leqslant DM^n$. This completes the proof of the Main Technical Claim and hence of Lemma B.6. ∎

At last, we are able to collect all our work in order to finish the proof of the Gevrey Asymptotic Implicit Function Theorem (Theorem 1.1) with $N = 1$.

Proof of Theorem 1.1 ($N = 1$). By the Formal Implicit Function Theorem (Proposition B.1), there is a subdomain $\mathsf{X}_0 \subset \mathsf{X}$ containing x_0 such that the equation $F(x,\hbar,z) = 0$ has a unique formal solution $\widehat{f}$ satisfying $f_0(x_0) = z_0$. Let f_0, f_1 be its leading- and next-to-leading-order parts in $\hbar$. As in Lemma B.2, we change variables as $z = f_0 + \hbar f_1 + \hbar w$ to transform the equation $F(x,\hbar,z) = 0$ into $w = \hbar G(x,\hbar,w)$. By Lemma B.6, this equation has a unique holomorphic solution $w = g(x,\hbar)$ on $\mathsf{X}_0 \times \mathsf{S}_0$ for some sectorial subdomain $\mathsf{S}_0 \subset \mathsf{S}$ still with opening Θ and admitting a uniform Gevrey asymptotic expansion as $\hbar \to 0$ along $\overline{\Theta}$. Finally, we define $f := f_0 + \hbar f_1 + \hbar g$ which is readily seen to have all the desired properties. ∎

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