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Optimization of Metro Vertical Alignment for Minimized Construction Costs and Traction Energy: A Dynamic Programming Approach

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Abstract: Vertical alignment significantly affects the construction costs and train traction energy use of a metro system. Existing studies on this subject mostly optimized the vertical alignment while considering train traction energy consumption or construction costs only. There is still a lack of efficient approach for the vertical alignment optimization problem which considers both construction costs and traction energy consumption. To this end, this paper proposes a two-stage optimization program involving both the design of metro vertical alignment and train speed profile, with the objective of minimizing the total costs of construction and train traction energy. An iterative approach is proposed for solving the two-stage program, in which a dynamic programming (DP) algorithm with a backward search method is designed to seek an optimal vertical alignment given an energy-efficient train speed profile. The model and algorithm approach are tested on real-world case studies on the Line 14 of Guangzhou Metro in China. The results show that compared with the existing heuristic algorithms, the DP approach performs better in computation time and solution quality. Moreover, the optimized vertical alignments outperform that designed by experienced designers in terms of total costs of construction and traction energy consumption, with an average savings rate of 6.0%.

Keywords: Metro line; Vertical alignment; Construction cost; Train movement; Dynamic programming

1 Introduction

1.1 Problem statement

Recent decades have seen continual and rapid development of metro systems in many countries (especially in many large cities in China). A constantly expanding metro network incurs considerable expenditure on construction and operation, including energy consumption. Since 2010, for example, the total investment in the construction of the Guangzhou Metro – one of China's busiest metro systems – has reached 161.8 billion Chinese Yuan (Guangzhou Statistics Bureau, 2020), and its annual energy consumption has increased by about 1.2 billion kWh in a decade. Despite numerous studies that have been conducted on various aspects of total costs reduction for a metro system, there is still room for better solutions.

The design of metro alignment is one of the technical aspects that affect the total costs for a metro system. It is aimed at finding an economical route of metro tracks between an origin and a terminal, given local soil conditions, socio-economic factors, and environmental factors such as air pollution and noise around the transit service area (Shafahi and Shahbazi, 2012, Li et al., 2016, and Ghoreishi et al., 2019). The design includes aspects of both horizontal and vertical track alignments, which are usually implemented in sequence. First, the design of horizontal alignment mostly determines locations of metro stations, and tangents, horizontal, and transition curves. On that basis, the design of vertical alignment determines the length and gradient of a slope and vertical curves that connect two adjacent slopes.

This paper deals with modeling and optimization of a metro line's vertical alignment. The vertical alignment, on the one hand, is closely related to the construction of underground infrastructure including line foundations, stations, tunnels, bridges, rights of way, utilities, and so forth (Jha et al., 2007, Bababeik and Monajjem, 2012, Li et al., 2016), which rely on huge capital investment. On the other hand, it has a significant impact on the running resistance of metro trains, and hence on train traction energy consumption (Duarte and Sotomayor, 1999). The traction energy consumption of a metro train refers to the electrical energy consumed by the traction system on the train for vehicle propulsion and on-board auxiliary equipment; it can account for approximately half of the total energy consumption in a metro system. Therefore, finding the lowest possible sum of construction and traction energy cost is important for the design of metro vertical alignment.

Metro vertical alignment optimization must adhere to geographical compliance requirements and specific geometric constraints. For underground constructions such as a metro line, it is vital to eliminate the risk of geological hazards (e.g., drift-sand and sludge layers) and bypass any existing underground facilities (e.g., oil and gas pipelines). Besides, the vertical alignment of the metro line must also satisfy specific geometric specifications, such as a maximum gradient and a minimum sloped length, to ensure train running safety. Professional engineers may decide on a feasible vertical alignment design from among a small number of candidates by using their hands-on experience. In reality, however, there could be a potentially unlimited number of feasible schemes of the vertical alignment, and the empirically-chosen one may not necessarily be optimal (Li et al., 2016, Samanta and Jha, 2011).

1.2 Literature review

Researchers have devoted great efforts to the optimization of vertical alignment in the areas of both highways and railways. The studies on the highway vertical alignment are mostly intended to minimize construction costs and energy consumption. Easa (1988) obtained an optimized vertical alignment scheme, which minimized the cost of earthwork and met the geometric specifications via an enumeration method. Fwa et al., (2002) then used a genetic algorithm (GA) to minimize the associated construction costs. Furthermore, Goktepe et al., (2005) proposed a dynamic programming (DP) approach to find an optimal vertical alignment with a minimum construction cost given a few design constraints; and later, Goktepe et al., (2009) developed a GA-based constrained curve-fitting technique considering vague soil parameters as required by the design of the highway vertical alignment. Hare et al. (2015) presented a mixed integer linear programming model, which also aims to minimize the cost of earthwork. More recently, Vandanjon et al., (2019) proposed a method thatconsidered both construction costs and energy consumption.

For optimizing railway vertical alignments, Hoang et al. (1975) proposed a model to obtain the optimal gradient and length of slopes of inter-station tracks among six types of vertical alignment. Further, Lafortune and Polis (1983) applied an "interactive computer aided analysis and design" technique to automatically produce energy-efficient vertical alignment, considering various design criteria of a metro system. Kim and Schonfeld (1997) developed a model for analyzing the train traction energy and travel time with different dipped vertical alignments designed for gravity-assisted acceleration and braking. Duarte and Sotomayor (1999) proposed a Gradient-Restoration method to determine the vertical alignment together with an optimal control policy of a train in metro systems. Xin et al. (2014) developed a simulation-based approach to optimize the metro vertical alignment within the station area for minimizing the traction energy consumption. Kim et al. (2013) compared the train traction energy among three vertical alignments connecting rail transit stations. Li et al. (2022) proposed a Gaussian pseudo-spectral method to find an optimal design of metro vertical alignment focusing on both traction energy consumption and running time deviation.

Other studies on railway vertical alignment optimization focused on the construction costs and the operating costs without considering the train traction energy. Bababeik and Monajjem (2012) developed a standard GA in a continuous search space to minimize the total construction and operating costs. Li et al. (2016) proposed a bidirectional two-dimensional distance transform (DT) algorithm to optimize the railway alignments, considering the costs of various factors including construction, operation, right-of-way, and environment. To improve computational efficiency in the optimization process, Pu et al. (2019) proposed a sequential three-dimensional DT algorithm in mountainous regions. On that basis, Song et al. (2020) considered a parallel three-dimensional DT algorithm to further boost computational efficiency. Around the same time, Costa et al. (2017) proposed an optimization method for three-dimensional alignment of the underground tunnel, which considered uncertainties associated with geology and construction processes. Ghoreishi et al. (2019) proposed a particle swarm optimization (PSO) algorithm, with the support of a geographic information system for three-dimensional optimization of railway alignments.

More recent studies have focused on optimizing the vertical alignment by minimizing concurrently the construction costs and traction energy consumption. Lai and Schonfeld (2012) proposed a GA-based method to optimize the vertical track alignment of urban rail transit. Kim et al. (2019) proposed a GA combined with a simulation model to find solutions for railway vertical alignment and train operating speed. Zhang et al. (2021) formulated a model for the 3D railway alignment optimization problem, which was solved by a stepwise and hybrid particle swarm-genetic algorithm.

Table 1 Methods for railway vertical alignment optimization

References	Construction costs	Traction energy consumption	Approaches
Hoang et al. (1975)	×	√	Enumeration
Duarte and Sotomayor (1999)	×	√	Gradient-Restoration
Fwa et al. (2002)	√	×	GA
Bababeik and Monajjem (2012)	√	×	GA
Lai and Schonfeld (2012)	√	√	GA
Li et al. (2016)	√	×	2D-DT
Kim et al. (2019)	√	√	GA
Pu et al. (2019)	√	×	3D-DT
Ghoreishi et al. (2019)	√	×	PSO

Table 1 summarizes the methods used in the relevant studies of the railway vertical alignment design. Due to nonlinear constraints and very large solution space, most of the existing studies employed heuristic algorithms to obtain a near-optimal solution. Heuristic algorithms have randomness in problem solving process and cannot guarantee the stability of their final solutions. Therefore, there is still no exact approach for tackling the vertical alignment optimization problem, while considering both the construction costs and traction energy consumption.

1.3 Aim and contributions of the study

The study presented in this paper aims to develop a two-stage optimization program involving both the design of metro vertical alignment and train speed profile, with the objective of minimizing the total costs of construction and traction energy. The main contributions of this paper are as follows:

1) Unlike most existing studies on vertical alignment design that only consider train traction energy consumption or construction costs, this study minimizes the total of the two terms to involve the impact of vertical alignment schemes on both metro construction and train operation. To explicitly characterize the interaction between the vertical alignment and train speed profile, this paper develops a two-stage program to integrate train speed profile optimization into vertical alignment design, in which the first stage determines the vertical alignment, and the second stage minimizes the train traction energy consumption by optimizing train speed profile given every vertical alignment scheme.
2) We propose an iterative approach that combines a DP algorithm and a brute force (BF) algorithm to solve the two-stage optimization program, in which a DP algorithm is designed to optimize the vertical alignment given an energy-efficient train speed profile. A two-dimensional grid system is built to transform the vertical alignment optimization problem into a multi-stage decision problem and a backward search method is proposed to seek the best vertical alignment.
3) Based on real-world case studies, the proposed DP algorithm has a better performance than the simulated annealing genetic algorithm (SA-GA) and PSO algorithm. Moreover, compared to the vertical alignment designed by experienced designers, the vertical alignment designed with the iterative approach can reduce the total costs of construction and traction energy consumption by 6.0% on average.

The rest of the paper is organized as follows. Section 2 describes the problem of designing metro vertical alignment. Section 3 describes the two-stage optimization program. Then Section 4 describes the iterative approach for solving the two-stage optimization program. Section 5 demonstrates the proposed model and algorithm approach with case studies conducted on the Line 14 of Guangzhou Metro in China. Finally, Section 6 concludes the paper.

2 Problem description

This paper focuses on the vertical alignment design of a double-track metro line for a fixed horizontal alignment. As illustrated in Fig. 1, the horizontal locations of the metro stations and the horizontal curves between stations are pre-determined. The total number of metro stations is denoted by N, and $X_{\text{sta}}^{n}, n \in \{1,2,...,N\}$ denotes the horizontal location of stations. The geographical features of the metro track, such as ground surface and locations of underground facilities, are known from field surveys.

The design of track vertical alignment between a given pair of start and end stations involves the determination of a number of parameters, such as lengths and gradients of slopes and vertical curves that connect two adjacent slopes. As shown in Fig. 1, let $(X_{\text{start}}, Y_{\text{start}})$ and $(X_{\text{end}}, Y_{\text{end}})$ denote the geographical coordinates of a start point and an end point, respectively. Two adjacent slopes are connected at a vertical point of intersection (VPI); a sequence of VPIs can determine the length and gradient of the slopes. Let (X_k, Y_k) denote the geographical coordinates of the k^{th} VPI, where $k \in \{1,2,...,K\}$, with K being the total number of VPIs along the metro track.

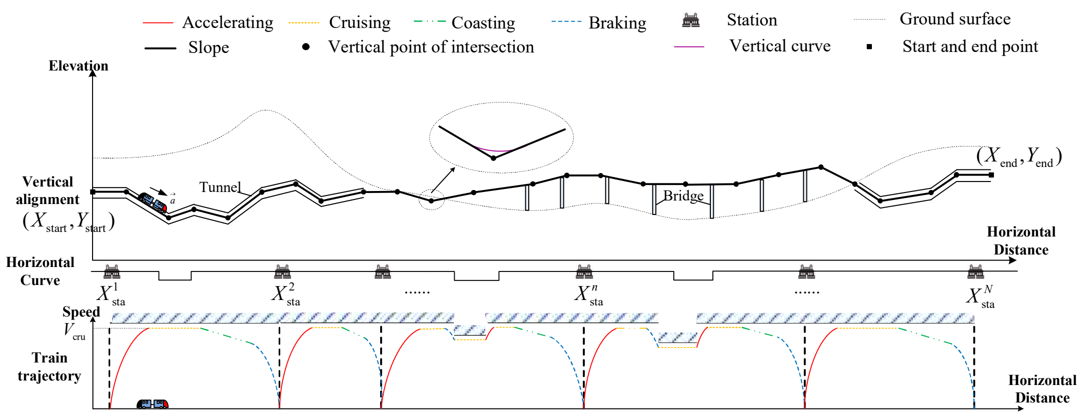


Fig. 1. The vertical alignment of a metro track.

Generally, the construction costs of a metro system include track cost C_{TRA}, earthwork cost C_{EAR}, bridge cost C_{BRI}, tunnel cost C_{TUN}, right-of-way cost C_{RIG}, and station cost C_{STA}. Specifically, the volume of cuts and fills, and the lengths of bridges and tunnels are related to the elevations of track alignment and ground surface. Besides, the area occupied by the stations and tracks is closely related to the elevation of the track alignment and to their horizontal locations, which affect the right-of-way cost and station cost. Therefore, the proper design of the vertical alignment can reduce construction costs.

Train traction energy consumption is greatly affected by the slopes along the metro line. Due to gravitational assistance, a downhill slope reduces the energy consumption of trains when they accelerate or cruise. Conversely, an uphill slope reduces the braking energy dissipation needed to decelerate trains. The traction energy consumption is an important factor in vertical alignment design. However, the traction energy consumption is also determined by train speed profile. Given a vertical alignment scheme, different train speed profiles can lead to various traction energy consumption. The optimized train speed profile should be used to evaluate the traction energy consumption. We assume that the train speed profile follows the control sequence of maximum acceleration, cruising, coasting, and maximum braking, which is proved to be the most energy-efficient train control strategy (Howlett, 1990; Howlett et al., 2009). With the energy-efficient train control sequence, the train speed profile is optimized by determining the switching points from the accelerating phase to the cruising phase and from the cruising phase to the coasting phase to minimize traction energy consumption.

The metro vertical alignment design problem aims to optimize the geographic location of VPIs under multiple constraints on geometry and geography with the fixed horizontal alignment. The objective is to minimize the total costs of construction and traction energy consumption under the optimized train speed profile.

3 Model formulation

In this section, we propose in detail an integrated two-stage optimization program that is used to determine both metro vertical alignment and train speed profile. The first-stage program optimizes metro vertical alignment by minimizing the total costs of construction and train traction energy. The second-stage program minimizes the train traction energy consumption by optimizing train speed profile given every vertical alignment scheme. The two-stage program is based on two assumptions:

Assumption 1. The radius of a vertical curve is assumed constant (see also Li et al., 2013, Song et al, 2022), as it usually remains unchanged once the design speed of the metro line is determined (Code for Design of Metro, 2013).

Assumption 2. The reuse of train regenerative braking energy is not considered (see also Lai and Schonfeld, 2012, Wang et al, 2021).

3.1 Vertical alignment optimization

3.1.1 Decision variables

The first-stage program is formulated as a vertical alignment optimization model. The core decision variables of the first-stage model are the number of VPIs and the geographical coordinates of each VPI. The gradient and length of slopes are intermediate variables of the model. Let i_k denote the gradient, and l_k the length, of the k^{th} slope. They can be further represented, respectively, by Eqs. (1) and (2):

$$i_k = \frac{Y_k - Y_{k-1}}{X_k - X_{k-1}}, \forall k \in \{1, 2, ..., K\} \tag{1}$$

$$l_k = X_k - X_{k-1}, \forall k \in \{1, 2, ..., K\} \tag{2}$$

Moreover, the elevations of a metro track are also intermediate variables. We divide the metro track by distance intervals Δs. The set of geographical elevations of the metro track can be denoted by $\{h^1_{\text{tra}}, h^2_{\text{tra}}, ..., h^s_{\text{tra}}, ..., h^S_{\text{tra}}\}$, where h^s_{tra} denotes the track elevation at horizontal location s and it can be computed with Eq. (3) as follows:

$$h^s_{\text{tra}} = i_k \cdot (s - X_k) + Y_k, \forall k \in \{1, 2, ..., K\}, \text{if } X_k \le s < X_{k+1} \tag{3}$$

3.1.2 Objective function

To consider the life cycle of the metro line, the annual total costs of construction and train traction energy is set as the overall objective function of the first-stage model, which is expressed as:

$$\min C_{\text{TOL}} = \mu \cdot C_{\text{CON}} + E(X_k, Y_k) \tag{4}$$

where C_{TOL} is the annual total costs; C_{CON} denotes the construction costs of the metro line. The ratio μ is the capital recovery factor, which is used to compute the annual construction costs. It is equal to $(1+i)^n i / [(1+i)^n - 1]$, where i denotes the interest rate per period and n represents the number of equal compounding periods in the entire economic life. $E(X_k, Y_k)$ is the second-stage objective function, which indicates the annual traction energy consumption associated with the first-stage decision variables (X_k, Y_k).

The construction costs of the metro line include the track cost C_{TRA}, bridge cost C_{BRI}, tunnel cost C_{TUN}, earthwork cost C_{EAR}, right-of-way cost C_{RIG}, and station cost C_{STA}, as expressed in Eq. (5):

$$C_{\text{CON}} = C_{\text{TRA}} + C_{\text{BRI}} + C_{\text{TUN}} + C_{\text{EAR}} + C_{\text{RIG}} + C_{\text{STA}} \tag{5}$$

According to Eq. (6), the track cost C_{TRA} is proportional to the total length and the unit track cost per meter, which is

$$C_{\text{TRA}} = \sum_{k=1}^{K} c_{\text{tra}} \cdot l_k \tag{6}$$

where c_{tra} is the unit track cost per meter.

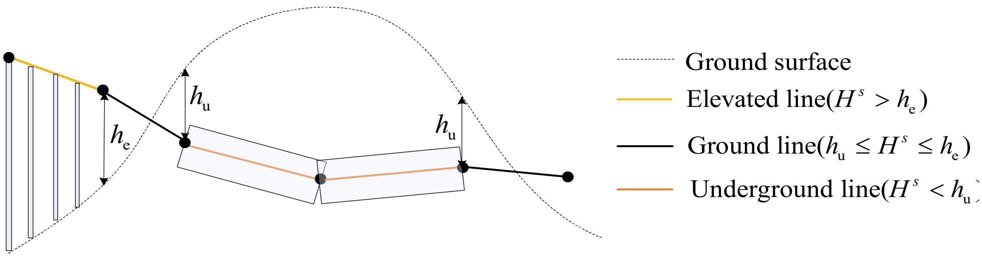


Fig. 3. The three types of metro line.

The earthwork cost, bridge cost and tunnel cost are related to the elevation difference between the metro track and the ground surface. Let h_{gro}^{s} denote the elevation of the ground surface at the horizontal location s. The elevation difference between the metro track and the ground surface is denoted by H^s, which is computed as:

$$H^s = h_{\text{tra}}^{s} - h_{\text{gro}}^{s} \tag{7}$$

Let h_{minb} and h_{mint} denote the minimum height of a bridge and the minimum depth of a tunnel, respectively. As shown in Fig. 3, firstly, when $H^s > h_{\text{minb}}$, the elevated metro line should be built on the bridges; secondly, when $-H^s > h_{\text{mint}}$, the metro line should be constructed in tunnels; thirdly, when $-h_{\text{mint}} \leq H^s \leq h_{\text{minb}}$, the metro line is constructed on the ground with cuts and fills. With the track elevations at different horizontal locations, the numbers of bridges and tunnels, which are denoted by n_{bri} and n_{tun}, respectively, can be derived.

The bridge cost C_{BRI} is related to the bridge height and length as well as the cost of bridge abutment, which is computed by Eq. (8) as:

$$C_{\text{BRI}} = \sum_{i}^{n_{\text{bri}}} [l_{\text{bri}}^{i} \cdot c_{\text{bri}}^{i}(H_{\text{bri}}^{i}) + \left\lfloor \frac{l_{\text{bri}}^{i}}{s_{\text{maxb}}} \right\rfloor \cdot c_{\text{abu}}] \tag{8}$$

where n_{bri} is the number of bridges, l_{bri}^{i} is the length of i^{th} bridge; H_{bri}^{i} is the height of i^{th} bridge (i.e., the elevation difference between the elevated track and ground surface), c_{bri}^{i} is the unit bridge cost per meter for the i^{th} bridge, which is a function of bridge height; c_{abu} denotes the cost of one bridge abutment; s_{maxb} represents the maximum bridge length; and the number of abutment is computed as $\lfloor l_{\text{bri}}^{i} / s_{\text{maxb}} \rfloor$.

The tunnel cost C_{TUN} is largely influenced by the tunnel length and the cost of tunnel portals. It is expressed as:

$$C_{\text{TUN}} = \sum_{i}^{n_{\text{tun}}} [l_{\text{tun}}^{i} \cdot c_{\text{tun}} + 2 \cdot c_{\text{por}}] \tag{9}$$

where n_{tun} is the number of tunnels, l_{tun}^{i} is the length of the i^{th} tunnel, c_{tun} is the unit tunnel cost per meter, c_{por} denotes the cost of one tunnel portal.

The volume of cuts and fills is influenced by the area of metro track occupied and the elevation difference between the metro track and the ground surface, which can be computed with Eqs. (10) and (11) as follows:

$$V_{\text{cut}} = \sum_{s=0}^{S} \Delta s \cdot D_{\text{w}} \cdot (h_{\text{gro}}^{s} - h_{\text{tra}}^{s}), \text{if } 0 < h_{\text{tra}}^{s} - h_{\text{gro}}^{s} \le h_{\text{minb}} \tag{10}$$

$$V_{\text{fill}} = \sum_{s=0}^{S} \Delta s \cdot D_{\text{w}} \cdot (h_{\text{tra}}^{s} - h_{\text{gro}}^{s}), \text{if } -h_{\text{mint}} \le h_{\text{tra}}^{s} - h_{\text{gro}}^{s} < 0 \tag{11}$$

where V_{cut} and V_{fill} denote the volume of cuts and fills, respectively; and D_{w} represents the width of metro track alignment.

The earthwork cost C_{EAR} can be computed by summing up the product of the unit cost of cut and fill and the corresponding earthwork volume, which is represented by

$$C_{\text{EAR}} = V_{\text{cut}} \cdot c_{\text{cut}} + V_{\text{fill}} \cdot c_{\text{fill}} \tag{12}$$

where c_{cut} and c_{fill} are the unit cost of cut and fill per cubic meter, respectively.

The right-of-way cost C_{RIG} is related to the right-of-way area occupied by the metro track and the unit right-of-way cost, which is computed by

$$C_{\text{RIG}} = \sum_{s=0}^{S} \Delta s \times D_{\text{w}} \cdot c_{\text{right}}^{s} (h_{\text{tra}}^{s}) \tag{13}$$

where c_{right}^{s} denotes the unit right-of-way cost at horizontal location s, which is a function of with regard to the metro track elevation.

The station cost C_{STA} is determined by the station area and the unit construction cost of the station. This is represented by Eq. (14) as follows:

$$C_{\text{STA}} = \sum_{n=1}^{N} M_{\text{sta}} \cdot c_{\text{sta}}^{n} (h_{\text{tra}}^{s}) \tag{14}$$

where M_{sta} denotes the area of metro station, and c_{sta}^{n} the unit construction cost of the n^{th} metro station, which is related to the elevation of the metro track in the station area.

3.1.3 Constraints

The VPIs between any two slopes must satisfy various geometric constraints and geographical requirements. The geometric specifications on the metro design, which is referred as to "Code for Design of Metro (CDM)" in China, specifies the requirements of metro alignment design. The geometric specifications and geographical requirements should be treated as constraints when minimizing the objective function.

(1) Geometric constraints

1) The minimum slope length within station area

According to the CDM, the slope length within station area must be no less than the platform length of metro station, as shown in Fig. 4(a). This constraint can be represented by formula (15)

$$\begin{cases} X_{k+1} - X_{\text{sta}}^{n} \ge L_{\text{P}} / 2 \\ X_{\text{sta}}^{n} - X_{k} \ge L_{\text{P}} / 2 \end{cases}, \forall n \in \{1, 2, ...N\}, k \in \{1, 2, ...K\}, \text{if } X_k < X_{\text{sta}}^{n} < X_{k+1} \quad (15)$$

where N is the total number of stations; X_{sta}^{n} denotes the X coordinate of the central point of the n^{th} station; L_{P} denotes the platform length of stations.

2) The maximum and minimum slope gradient within station area

Considering the requirements of good drainage and safe parking, the gradient of any slope i_k within a station area must be selected from an integer set Q (see Fig. 4(a)). The constraint is represented by

$$i_k = \frac{Y_{k+1} - Y_k}{X_{k+1} - X_k} \in Q, \forall n \in \{1, 2, ...N\}, k \in \{1, 2, ..., K\}, \text{if } X_k < X_{\text{sta}}^{n} < X_{k+1} \quad (16)$$

3) The minimum slope length beyond station area

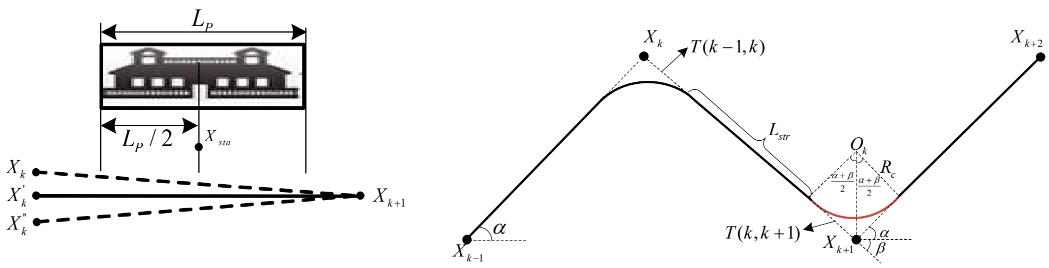
The length of any slope outside the station area must be no less than the train length. The train never runs over more than two slopes at one time for driving safety and passenger comfort. This constraint is represented by

$$L_{\text{T}} \le X_{k+1} - X_k, \forall n \in \{1, 2, ...N\}, k \in \{1, 2, ..., K\}, \text{if } X_{\text{sta}}^{n} < X_k \text{ and } X_{k+1} < X_{\text{sta}}^{n+1} \quad (17)$$

4) The maximum and minimum slope gradient beyond station area

The gradient of any slope beyond the station area should not exceed $i_{\max}$, subject to the traction performance of train motors. Meanwhile, the gradient should also be no less than $i_{\min}$ to facilitate drainage in tunnels. The constraint is represented by

$$i_{\min} \le \frac{Y_{k+1} - Y_k}{X_{k+1} - X_k} \le i_{\max}, \forall n \in (1, N), k \in \{1, 2, ...K\}, \text{if } X_{\text{sta}}^{n} < X_k \text{ and } X_{k+1} < X_{\text{sta}}^{n+1} \quad (18)$$



(a) The slope within station area (b) The minimum length of straight line

Fig. 4. Illustration of constraints 1), 2) and 5).

5) The minimum length of straight lines between adjacent vertical curves

According to the CDM, when the gradient difference between two adjacent slopes reaches 2‰, a vertical curve must be set at VPI to reduce the vibrations of train movement, as shown in Fig. 4(b). To further avoid the superposition of train vibrations, the length of the straight line between any two vertical curves should be no less than a minimum value. The constraint can be expressed as

$$X_{k+1} - X_k \geq T(k-1,k) + T(k,k+1) + L_{\text{str}}^{\min}, \forall k \in \{1,2,...K\} \tag{19}$$

where $L_{\text{str}}^{\min}$ is the minimum length of the straight line; $T(k-1,k)$ denotes the tangent length of the vertical curve between the (k-1)th and the kth slopes, and $T(k,k+1)$ denotes that between the kth and the (k+1)th slopes.

As illustrated in Fig. 4(b), the relation between the tangent length, the radius of the vertical curve and the coordinates of VPI can be represented by Eq. (20):

$$\begin{aligned} T(k-1,k) &= R_{\text{c}} \times \tan[\frac{1}{2}\cdot(\alpha+\beta)] \approx \frac{1}{2}\cdot R_{\text{c}}\cdot\tan(\alpha+\beta) \\ &\approx \frac{1}{2}\cdot R_{\text{c}}\cdot\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\times\tan\beta} \approx \frac{1}{2}\cdot R_{\text{c}}\times(\tan\alpha+\tan\beta) \\ &\approx \frac{1}{2}\cdot R_{\text{c}}\cdot[\tan\alpha-\tan(\pi-\beta)] \\ &= \frac{1}{2}\cdot R_{\text{c}}\cdot\left|\frac{Y_{k+1}-Y_k}{X_{k+1}-X_k}-\frac{Y_k-Y_{k-1}}{X_k-X_{k-1}}\right| \end{aligned} \tag{20}$$

(2) Geographic restrictions

For safety purposes, the metro vertical alignment design must avoid the drift-sand and sludge layers that are not suitable for tunnel construction and must bypass underground facilities (such as oil pipelines or gas pipelines).

Let J denote the set of the points of the restricted spaces. For any restricted point $j \in J$, the minimal distance $dis(j)$ between the metro track and the restricted point should be larger than an acceptable value ε (see Fig. 5). This can be expressed by formula (21). The minimum distance dis(j) between the metro track and the restricted point can be computed with the Algebraic Method, which is the formula for finding the shortest distance from a point to a line.

$$dis(j) > \varepsilon, \forall j \in J \tag{21}$$

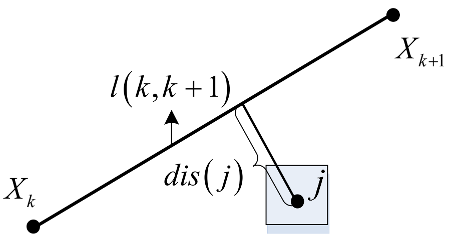


Fig. 5. Explanation of the constraints of geographic restrictions.

3.2 Energy-efficient train control

Given the decisions from the first stage, the second-stage program determines the optimal train speed profile by considering the annual total traction energy cost in both uphill and downhill directions, which is expressed in Eq. (22). The model is developed based on an optimal train control strategy composed of maximum acceleration, cruising, coasting, and maximum braking. The train speed profile is optimized by searching the switching points from the accelerating phase to the cruising phase a_j^1 and from the cruising phase to the coasting phase a_j^2 (Scheepmaker and Goverde, 2015), in which *j* denotes the index of interstation track of a metro line.

$$E(X_k, Y_k) = \tau \cdot N_{\mathrm{T}} \cdot \sum_{j=1}^{N-1} [\min E_{\mathrm{TRA}}^{\mathrm{u}}(j) + \min E_{\mathrm{TRA}}^{\mathrm{d}}(j)] \tag{22}$$

where the ratio τ denotes the unit cost of electricity; N_{T} represents the average number of one-direction train trips per year; $E_{\mathrm{TRA}}^{\mathrm{u}}(j)$ denotes the traction energy consumption of one train trip in the uphill direction of the j^{th} interstation track. $E_{\mathrm{TRA}}^{\mathrm{d}}(j)$ denotes the traction energy consumption of one train trip in the downhill direction of the j^{th} interstation track.

The traction energy consumption in the uphill direction of the j^{th} interstation track $E_{\mathrm{TRA}}^{\mathrm{u}}(j)$ in Eq. (22) can be computed by multiplying the train traction force over horizontal distance *s*, which is expressed as:

$$E_{\mathrm{TRA}}^{\mathrm{u}}(j) = \frac{1}{\eta} \cdot \int_0^S F_{\mathrm{tra}}^s ds \tag{23}$$

where *S* is the total length of the j^{th} interstation track; F_{tra}^s denotes the train traction force at location *s*, and η the conversion factor from electricity to kinetic energy.

The second-stage model is subject to the following constraints. First, to provide a sufficient operating service and meet the passenger demand, the total running time should not exceed the maximum allowable running time *T*.

$$\int_0^S \frac{s}{v_s} ds \le T \tag{24}$$

Then, according to Newton's second law, the following three constraints should be obeyed for train movements.

$$\frac{dv(s)}{dt(s)} = \frac{F_{\mathrm{tra}}^s - F_{\mathrm{bra}}^s - w_0(v_s) - w_{\mathrm{g}}(s) - w_{\mathrm{c}}(s)}{M \cdot (1+\delta)} \tag{25}$$

$$w_0(v_s) = (\varphi_0 + \varphi_1 \cdot v_s + \varphi_2 \cdot v_s^2) \cdot M \cdot g \tag{26}$$

$$w_{\mathrm{g}}(s) = (\sum_{k=1}^{K} i_k^s \cdot \frac{L_{\mathrm{g}}^{k,s}}{L_{\mathrm{T}}}) \cdot M \cdot g \tag{27}$$

$$w_{\mathrm{c}}(s) = (\frac{600}{R_s} \cdot \frac{L_{\mathrm{c}}^s}{L_{\mathrm{T}}}) \cdot M \cdot g \tag{28}$$

where F_{bra}^s denotes the train braking force at location *s*, δ denotes the rotational inertia of the train,

M denotes the train mass; v_s represents the train speed at location s; φ_0, φ_1 and φ_2, are coefficients of the Davis Equation (Davis, 1926), which are provided by rolling stock manufacturers; i_k^s and $L_{\mathrm{g}}^{k,s}$ are, respectively, the gradient and length of k^{th} slope upon which the train is at location s; L_{T} denotes the train length; R_s and L_{c}^s denote the radius and length of horizontal curve the train covers at location s respectively, which are known from the pre-determined horizontal alignment of metro track.

Train traction and braking force at any location are given by Eqs. (29) and (30),:

$$F_{\mathrm{tra}}^s = \begin{cases} F_{\mathrm{tra}}^{\max}(v_s), & \text{if } 0 \le s \le a_j^1 \\ \max[0, w_0(v_s) + w_{\mathrm{g}}(s) + w_{\mathrm{c}}(s)], & \text{if } a_j^1 < s \le a_j^2 \\ 0, & \text{if } a_j^2 < s \le S \end{cases} \tag{29}$$

$$F_{\mathrm{bra}}^s = \begin{cases} 0, & \text{if } 0 \le s \le a_j^1 \\ \min[0, -w_0(v_s) - w_{\mathrm{g}}(s) - w_{\mathrm{c}}(s)], & \text{if } a_j^1 < s \le a_j^2 \\ 0, & \text{if } a_j^2 < s \le a_j^3 \\ F_{\mathrm{bra}}^{\max}(v_s), & \text{if } a_j^3 < s \le S \end{cases} \tag{30}$$

where $F_{\mathrm{tra}}^{\max}(v_s)$ and $F_{\mathrm{bra}}^{\max}(v_s)$ denote the maximum traction and braking forces of a train at a speed of v_s, respectively; a_j^3 is the switching point from the coasting phase to the braking phase.

Considering the riding comfort for passengers (especially standing ones), train speed acceleration is limited within a reasonable range.

$$\underline{a} \le \frac{dv(s)}{dt(s)} \le \bar{a} \tag{31}$$

Train speed at the origin and terminal of each inter-station track must be 0 km/h, as represented by Eq. (32):

$$\begin{cases} v_0 = 0 \\ v_S = 0 \end{cases} \tag{32}$$

Finally, for operational safety, the train speed must not exceed the speed limits at any position, which is expressed as:

$$v_s \le v_s^{\mathrm{limit}} \tag{33}$$

where v_s^{limit} denotes the limit speed at location s.

The traction energy consumption in the j^{th} interstation track in the downhill direction $E_{\mathrm{TRA}}^{\mathrm{d}}(j)$ can also be optimized by using this method. For simplicity, we do not present the detailed formulation.

4 Solution approach

In this section, we design an iterative approach for solving the two-stage program involving both the optimization of metro vertical alignment and train speed profile. We first introduce the structure of the iterative approach. Then, we propose a DP algorithm with a backward method to optimize the vertical alignment given a train speed profile.

4.1 Iterative approach

The iterative approach combines the BF algorithm and the DP algorithm, whose structure is shown in Fig. 6. The DP algorithm with a backward search method solves the first-stage program on the vertical alignment optimization given a train speed profile. Given a vertical alignment scheme, the BF algorithm solves the second-stage program to find the train speed profile leading to minimal traction energy consumption. The BF algorithm searches for the optimal switching points of the train among the accelerating phase, cruising phase and coasting phase, following the control sequence of maximum acceleration, cruising, coasting, and maximum braking. The specific steps of the BF algorithm are provided in Zhou et al (2018). With the iterative approach, the interaction of the vertical alignment and train speed profile can be considered.

As shown in Fig. 6, initially, the train speed profile is optimized by the BF algorithm given a flat track. During the iterations, the vertical alignment is solved by the DP algorithm given the optimized train speed profile. Then, the train speed profile is optimized given the vertical alignment scheme and the optimized profile will be forwarded to the first-stage model for the following vertical alignment optimization. The vertical alignment and train speed profile are iteratively optimized until the difference between the train speed profiles in two consecutive iterations is rather small. Here we set the terminal condition that the speed difference at every location of the train inter-station run is less than 1 km/h.

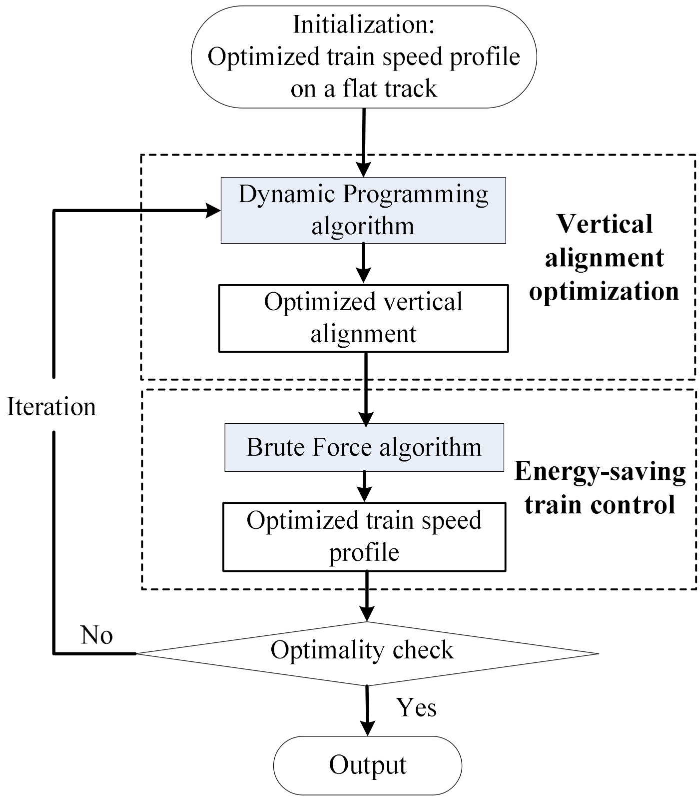


Fig. 6. Procedure of the iterative approach.

4.2 Dynamic programming algorithm

4.2.1 The framework of the DP algorithm

It is very challenging to find the exact solution to the first-stage model on vertical alignment optimization. The complexity of the proposed model stems from two aspects. First, the model has a nonlinear objective function due to the construction costs. For example, the bridge cost is nonlinearly related to the track elevation. The track elevation affects both the length of the bridges and the unit bridge cost per meter, as the unit bridge cost is not a fixed value but varies depending on the bridge height. In other words, the objective function involves a product of two decision variables. Moreover, the track elevation has a nonlinear impact on the number of bridge abutments, which makes the model even harder to solve. On the other hand, the solution space of the proposed model is fairly large for a real-world metro line. The decision variables of the model are the geographical location of VPIs of the metro line. In a continuous search area, the feasible location of a VPI is numerous. For a real-world metro line with hundreds of VPIs, the solution space of the model can be extremely large, which makes it quite difficult to solve.

In this section, we design a DP algorithm with a backward search method to solve the first-stage model on vertical alignment optimization. The DP algorithm has two advantages in solving this model. First, the DP algorithm is a powerful approach to handle nonlinear programming models (Coaker, 1965), which provides an alternative to solve for an exact solution. Moreover, the proposed model can be transformed into a multi-stage decision model for determining the coordinates of VPIs, which fit the DP algorithm well. The best solution of the model can be efficiently obtained by solving and recording the solutions of each sub-model.

To build the DP framework, a two-dimensional grid is first developed to discretize the search space. In the two-dimensional grid, the width and height of a cell are determined beforehand. The VPIs are restricted to be located at the vertexes of the cells, as shown in Fig. 7. The set of all vertexes of the cells is denoted by $\Re$.

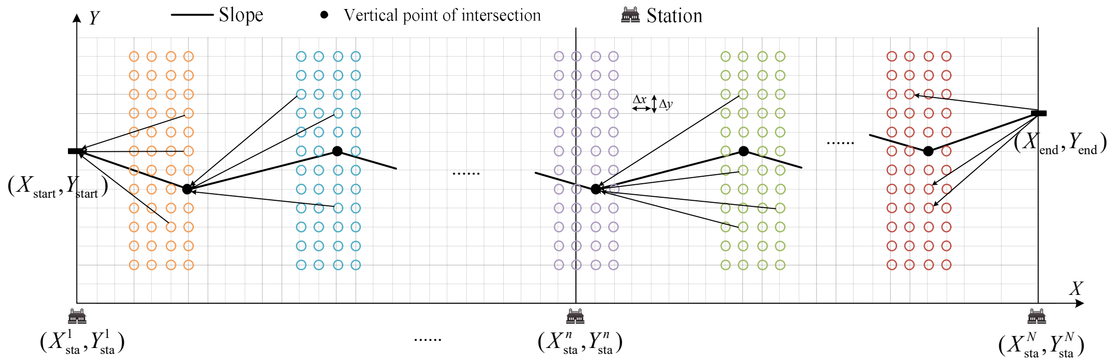


Fig. 7. A two-dimensional grid for the vertical alignment optimization

We then transform the vertical alignment optimization problem into a multi-stage decision problem based on the decisions of the VPI locations. The framework of the DP algorithm could thus be reformulated in terms of stage, state, decision and cost function.

Stage: The vertical alignment can be divided into K+1 parts by the K VPIs. The vertical alignment optimization problem can thus be transformed into a multi-stage decision problem with K+1 stages. In the DP algorithm, let k denote the index of the stage.

State: The state in the DP algorithm is the location of the VPIs. At stage k, the state variable is represented by the geographic coordinates $(X_k, Y_k) \in \Re$ of the k^{th} VPIs.

Decision: The decision in the DP algorithm is the action of selecting the next VPI (X_{k+1}, Y_{k+1}) based on the location of the current VPI (X_k, Y_k). Let $\mathrm{A}_{(X_k, Y_k)}$ denote the action space of the state variable (X_k, Y_k). Each action $a_k = (X_k, Y_k) \to (X_{k+1}, Y_{k+1}) \in \mathrm{A}_{(X_k, Y_k)}$ will connect the point (X_k, Y_k) to a point (X_{k+1}, Y_{k+1}) by a slope, and the action must satisfy all the constraints in Section 3.3. The final metro vertical alignment will be determined by the best set of actions $\P = \{a_0^*, \ldots, a_k^*, \ldots, a_K^*\}$.

Cost function: The metro vertical alignment optimization problem can be transformed into the following a multi-stage decision problem:

$$\min C_{\text{TOL}} = \min_{\P} \sum_{k=0}^{K} J_k \tag{34}$$

where J_k indicates the costs associated with the action a_k, which includes the construction costs and the train traction energy consumption.

According to the Bellman principle of optimality (Bellman, 1966), the recurrence equation of a backward dynamic program is expressed as:

$$C_{\text{TOL}}^{*}(X_{k-1}, Y_{k-1}) = \min_{a_k^*}[J_k + C_{\text{TOL}}^{*}(X_k, Y_k)] \tag{35}$$

where $C_{\text{TOL}}^{*}(X_{k-1}, Y_{k-1})$ denotes the minimum cumulative costs from the state (X_{k-1}, Y_{k-1}) to the state $(X_{\text{end}}, Y_{\text{end}})$, and $C_{\text{TOL}}^{*}(X_k, Y_k)$ denotes that from (X_k, Y_k) to $(X_{\text{end}}, Y_{\text{end}})$.

4.2.2 Backward search method

To solve the best action set of the first-stage model on vertical alignment optimization, a backward search method is proposed to compute the minimum cumulative costs from the state $(X_{\text{start}}, Y_{\text{start}})$ to the state $(X_{\text{end}}, Y_{\text{end}})$. As shown in Fig. 7, the backward search method starts at $(X_{\text{end}}, Y_{\text{end}})$ and ends at $(X_{\text{start}}, Y_{\text{start}})$.

Let $\square_k$ denote the set of all current states at stage *k*. The search process is depicted in Fig. 8. Let $Pred(X_k, Y_k)$ denote the predecessor state of the state (X_k, Y_k) that causes the minimum cumulative costs to the state $(X_{\text{end}}, Y_{\text{end}})$.

Step 0. Initialization: Set stage *k*=0, and $Pred(X,Y) = \varnothing, \forall (X,Y) \in \Re$. Set $C_{\text{TOL}}^{*}(X_{\text{end}}, Y_{\text{end}}) = 0$ and $C_{\text{TOL}}^{*}(X,Y) = \infty$ for all states (X,Y) in $\Re$ except $(X_{\text{end}}, Y_{\text{end}})$. Add the state $(X_{\text{end}}, Y_{\text{end}})$ to the set of current states $\square_k$.

Step 1. For each current state $(X,Y) \in \square_k$, compute the corresponding action space $\text{A}_{(X,Y)}$.

Step 2. For each current state $(X,Y) \in \square_k$ and for each $a = (X,Y) \to (X', Y') \in \text{A}_{(X,Y)}$, compute the next state (X', Y') and the stage cost $J_k(X,Y,a)$ between the current and the next state. If the next state is not the same as the state $(X_{\text{start}}, Y_{\text{start}})$, add the next state to the set $\square_{k+1}$.

Step 3. Update the minimum cumulative costs and predecessor state:

For each next state $(X', Y') \in \square_{k+1}$, if $J_k(X,Y,a) + C_{\text{TOL}}^{*}(X,Y) < C_{\text{TOL}}^{*}(X', Y'), \forall (X,Y) \in \text{¡}_k$, then update the minimum cumulative costs and predecessor state using Eq. (36).

$$\begin{cases} C_{\text{TOL}}^{*}(X', Y') = J_k(X,Y,a) + C_{\text{TOL}}^{*}(X,Y) \\ Pred(X', Y') = (X,Y) \end{cases} \tag{36}$$

Step 4. If $\square_{k+1} = \varnothing$, set $\square_k = \square_{k+1}$ and *k*=*k*+1, then go to Step 1; otherwise output the cumulative costs and output the predecessor states that belong to $\Re$.

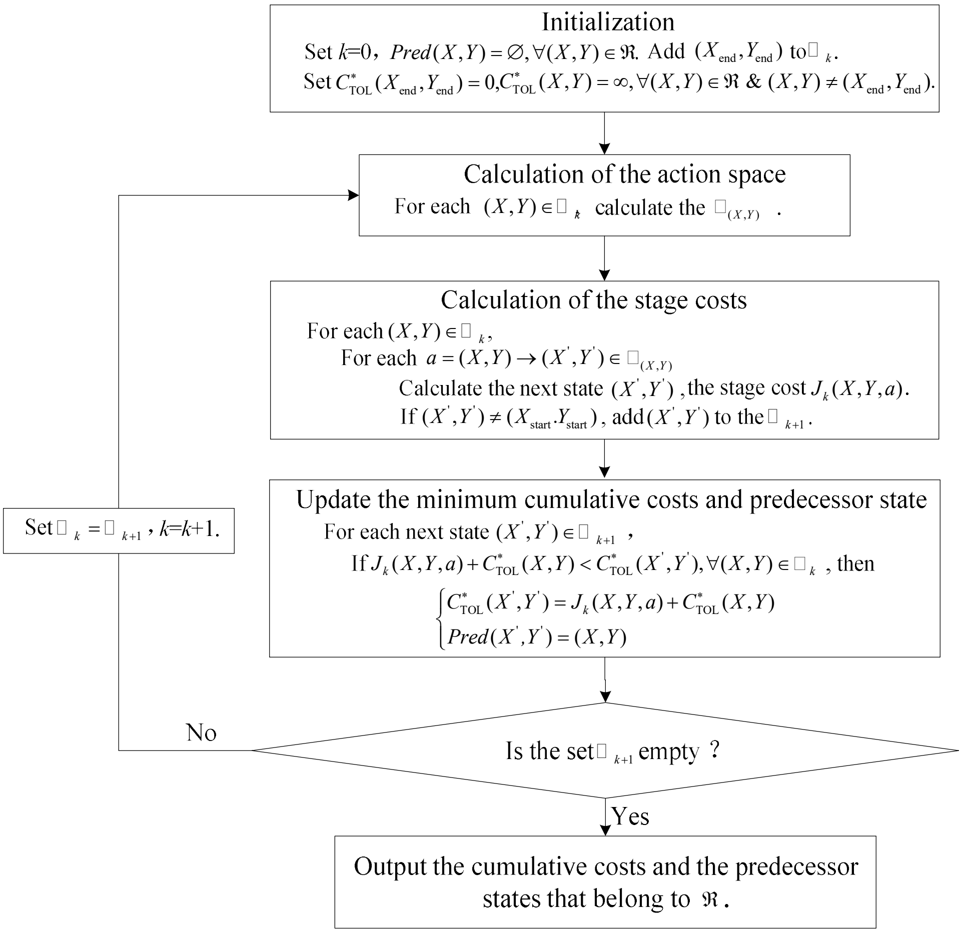


Fig. 8. Flowchart of backward dynamic programming search.

From the backward search process recorded by the predecessor states, all the states can be obtained from $(X_{\text{start}}, Y_{\text{start}})$ to $(X_{\text{end}}, Y_{\text{end}})$, with the minimum cumulative costs. The best vertical alignment in the two-dimensional grid can thus be found.

5 Case studies

The case studies are based on the Guangzhou Metro Line 14 in China. There are seven stations along the line, which runs from the southwest across the city center to the northeast. Its length is 27.15 km. The trains running on Line 14 consist of four motorized vehicles and two trailers, in which each motorized vehicle is equipped with four motors. The parameters of the trains are listed in Table 2. The traction and braking characteristics of a motor of the metro train are shown in Fig. 9.

The parameters associated with the constraints are presented in Table 3. Table 4 lists the parameters of the unit costs in the objective function. The case studies are conducted on a desktop

PC with an Intel Core i5-9500F CPU (2.5 GHz), 16GB of RAM, and a Microsoft Windows 10 Pro 64-Bit operating system.

Table 2 Parameters of Train A

Parameter	Value
Train mass (M)	293.4 t
Train length (L_T)	120 m
Number of motors (n_m)	16
Coefficients of the Davis Equation (φ_0 / φ_1 / φ_2)	1.8214 / 0.030612 / 0.000251
Conversion factor from electricity to kinetic energy (η)	0.9
Rotational inertia (δ)	0.38

Table 3 Parameters of model constraints

Description	Value
The integer set of the gradient of slope within station area (Q)	{-2‰, 0‰, 2‰}
The platform length of metro station (L_P)	150 m
Minimum gradient of slope beyond station area (i_{min})	3‰
Maximum gradient of slope beyond station area (i_{max})	30‰
The radius of the vertical curve (R_c)	5000 m
Minimum length of straight line between adjacent vertical curves (L_{str}^{min})	20 m
Minimum difference between track and restricted space (ε)	1 m
The average number of one-direction train trips per year (N_T)	67525
The maximum bridge length (s_{maxb})	20 m
Station area (M_{sta})	3000 m^2
Number of compounding years of metro economic life (n)	60 years
Interest rate per year (i)	5 %
The width of metro track alignment (D_w)	20 m
The minimum bridge height (h_{minb})	10 m
The minimum depth of tunnel (h_{mint})	13 m

Table 4 Unit costs

Items	Value	Items	Value
Electricity (τ)	1.0 CNY/kWh	Metro track (c_{tra})	20000 CNY/m
Tunnel (c_{tun})	80000 CNY/m	One tunnel portal (c_{por})	500000 CNY
Cutting earthwork (c_{cut})	35 CNY/m^3	One abutment of bridge (c_{abu})	40000 CNY
Filling earthwork (c_{fill})	40 CNY/m^3	(H_{bri} ≤10m) Bridge (c_{bri})	42000 CNY/m
Underground station (c_{sta})	40000 CNY/m^2	(10m< H_{bri} ≤15m) Bridge (c_{bri})	54600 CNY/m
Elevated station (c_{sta})	20000 CNY/m^2	(15m< H_{bri} ≤20m) Bridge (c_{bri})	67200 CNY/m

Ground station (c_{sta})	10000 CNY/m^2	(H_{bri} >20 m) Bridge (c_{bri})	79800 CNY/m
(s ≤14km) Right-of-way cost of the underground line (c_{right})	2000 CNY/m^2	(s >14km) Right-of-way cost of the underground line (c_{right})	10000 CNY/m^2
(s ≤14km) Right-of-way cost of the ground line (c_{right})	30000 CNY/m^2	(s >14km) Right-of-way cost of the ground line (c_{right})	20000 CNY/m^2
(s ≤14km) Right-of-way cost of the elevated line (c_{right})	13000 CNY/m^2	(s >14km) Right-of-way cost of the elevated line (c_{right})	3000 CNY/m^2

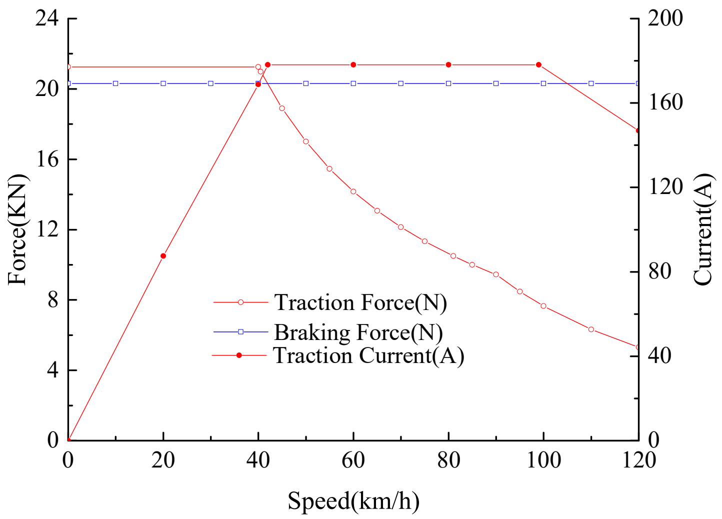


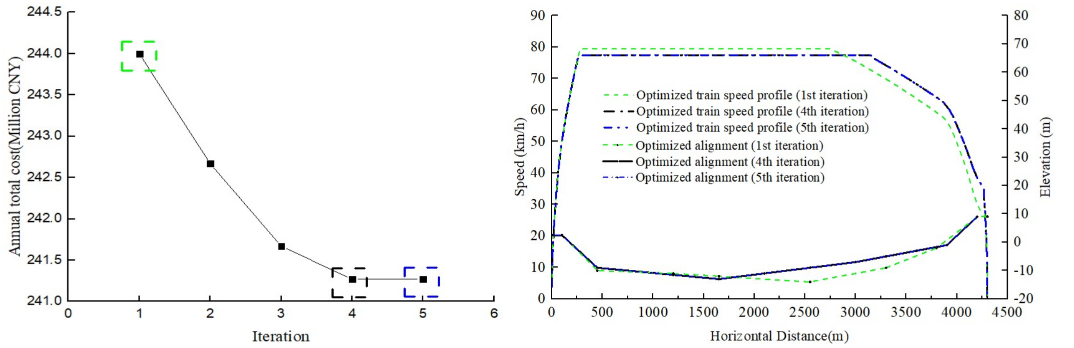
Fig. 9. Traction and braking performance of a train motor.

5.1 Performance of the solution approach

5.1.1 The convergence of the iterative approach

In this subsection, we test the convergence of the proposed iterative approach in optimizing the vertical alignment of Guangzhou Metro Line 14. The convergence process of the iterative approach is illustrated in Fig. 10(a). The iterative approach achieves convergence after 5 iterations. By the first optimization, the annual total cost is reduced to 244.00 million. With the iterations, the annual total cost is further decreased to 241.27 million CNY.

Taking the first inter-station track as an example, we give the optimized train speed profile and the vertical alignment in the first and last two iterations (see Fig. 10(b)). It can be seen that the optimized train speed profiles at the last two iterations are overlapped, which indicates the convergence of the iterative approach in solving the two-stage program.



(a) Costs changes in successive iterations (b) The optimized speed profile and vertical alignment

Fig. 10. The convergence of the iterative approach.

5.1.2 The efficiency of the DP algorithm

We further analyze the efficiency of the DP algorithm in solving the first-stage model on vertical alignment optimization, for a given train speed profile with a maximum cruising speed of 78 km/h. The performance of the DP algorithm depends on the cell size in the grid. Smaller cell size in the grid can improve computation accuracy and thus solutions, but also increases the number of decision variables and computation time. Therefore, different cell sizes are set, in which the width of cells Δx varies from 150 m to 50 m, and the height of cells Δy varies from 1.5 m to 0.5 m.

The computation results are presented in Table 5. With the decrease of the cell width and height, the search space is enlarged, and thus the computation time is increased greatly from 0.1 hours to 14 hours. The annual total costs are also decreased from 259.09 million to 252.78 million CNY. When the width and height of the cell are set as Δx=50 m and Δy=0.5 m, the annual total costs are 252.78 million CNY, but the computation time is too long at 14 hours. When the cell size is increased to Δx=100m and Δy=1m, the annual total costs are increased to 255.19 million CNY by only 0.94%, which is also a satisfactory solution, but the computation time can be significantly shortened to 2 hours. In a nutshell, the quality of the solutions obtained by the DP algorithm can be enhanced if the accuracy of the two-dimensional grid is improved.

Table 5 The performance of the DP algorithm under different cell sizes in the grid

Cell width Δx (m)	Cell height Δy (m)	Annual total costs (million CNY)	Computation Time (h)
50	0.5	252.78	14.0
50	1.0	252.81	10.0
50	1.5	254.00	6.0
100	0.5	254.86	4.0
100	1.0	255.19	2.0
100	1.5	255.70	1.5

150	0.5	257.62	0.5
150	1.0	258.16	0.2
150	1.5	259.09	0.1

For comparison, two heuristic algorithms in the existing studies are applied, which include the SA-GA (Sun et al., 2020) and the PSO algorithm (Shafahi and Bagherian, 2013). Unlike the DP algorithm using the two-dimensional grid, the two heuristic algorithms optimize the geographical locations of the VPIs in a continuous search space. The parameters are set as follows. In the SA-GA, the population size, the crossover rate, and the mutation rate are set as 250, 0.8, and 0.15, respectively. In the PSO, the population size, the decreasing inertia weight, the maximum velocity, and the two accelerator coefficients are set as 250, 0.6, 0.3, 2 and 2, respectively. The two heuristic algorithms are tested using the same computation time as the DP algorithm.

The convergence process of the SA-GA and PSO is illustrated in Fig. 11. With the computation time increases, the solution quality of the SA-GA and the PSO is increased, and the annual total costs is decreased from 382.63 million to 253.22 million CNY, and from 324.79 million to 255.43 million CNY, respectively. The SA-GA can reach a better solution than the PSO after about 50 iterations. Fig. 12 illustrates the vertical alignments optimized by the DP, SA-GA and PSO. Compared to the vertical alignment optimized by the PSO, the alignment optimized by the SA-GA is closer to that optimized by the DP algorithm.

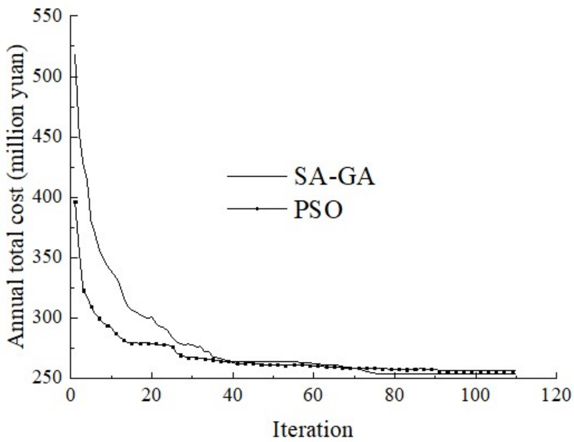


Fig. 11. Annual total costs changes in successive iterations of the SA-GA and the PSO.

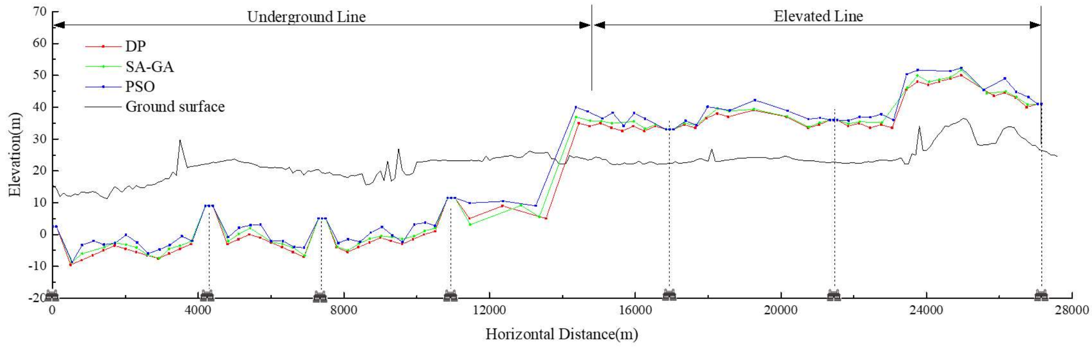


Fig. 12. The vertical alignments optimized by the DP, SA-GA and PSO.

As shown in Table 6, the DP algorithm outperforms the two heuristic algorithms, i.e., SA-GA and PSO, in the case study. Given the same computation time, the annual total costs associated with the vertical alignment produced by the DP algorithm are less than that by the heuristic algorithms. Within two hours, the DP algorithm can produce much better solutions as the heuristic algorithms cannot search for high-quality solutions in such a short period of time. After more than 10 hours of computation, the DP algorithm also showed better performance as it should find the optimal solution in a high-accuracy two-dimensional grid. When the computation time is 4 or 6 hours, the annual total costs computed by the DP algorithm is slightly greater than that by the SA-GA; the gap between the two algorithms is 0.34% and 0.17%, respectively, due to the accuracy loss when splitting the grid during the implementation of the DP algorithm.

Table 6 Comparison between the DP algorithm and heuristic algorithms

Computation Time (h)	Annual total cost (Million CNY)		Saving rate	Annual total cost (Million CNY)		Saving rate
	DP	SA-GA		DP	PSO	
14.0	252.78	253.22	0.17%	252.78	255.43	1.04%
10.0	252.81	253.22	0.16%	252.81	256.38	1.39%
6.0	254.00	253.58	-0.17%	254.00	259.55	2.14%
4.0	254.86	253.99	-0.34%	254.86	262.26	2.82%
2.0	255.19	262.49	2.78%	255.19	266.99	4.42%
1.5	255.70	268.38	4.72%	255.70	269.74	5.21%
0.5	257.62	307.97	16.35%	257.62	280.2	8.06%
0.2	258.16	361.39	28.56%	258.16	297.6	13.25%
0.1	259.09	382.63	32.29%	259.09	324.79	20.23%

5.2 The optimized vertical alignment

In this subsection, we compare the experienced designers' manual vertical alignment and the optimized vertical alignments with different optimization objectives to illustrate the effectiveness of the two-stage optimization program. The maximum allowable train running time is set as 1474s for 6 inter-station tracks. The width and height of the cell in the DP algorithm are set as Δx=50 m and Δy=0.5 m. Hereafter, we refer to the optimized alignment with the minimal construction costs as "construction cost-oriented alignment", the optimized alignment with the minimal traction energy consumption as "traction energy-oriented alignment", and the optimized alignment with minimizing both the traction energy consumption and construction costs by the two-stage program as "total cost-oriented alignment".

With the different vertical alignments, the results of the annual construction costs (ACC), the annual traction energy cost (ATEC) and the annual total costs (ATC) is shown in Table 7. Table 8 shows details of the construction costs with the different vertical alignments.

Table 7 Costs comparison among different vertical alignments (Million CNY)

	ACC	ATEC	ATC
Manual alignment	226.32	30.42	256.74
Construction cost-oriented alignment	214.10(-5.4%)	31.27(+2.8%)	245.37(-4.4%)
Traction energy-oriented alignment	433.56(+91.6%)	27.55(-9.4%)	461.11(+79.6%)
Total cost-oriented alignment	213.15(-5.8%)	28.12(-7.6%)	241.27(-6.0%)

Table 8 Construction costs comparison among different vertical alignments (Million CNY)

	C_{TRA}	C_{BRI}	C_{TUN}	C_{EAR}	C_{RIG}	C_{STA}	ACC
Manual Alignment	20.02	49.65	40.17	8.14	89.98	18.36	226.32
Construction cost-oriented alignment	20.02	44.88	35.74	6.70	92.62	14.13	214.10
Traction energy-oriented alignment	20.02	10.10	19.23	59.23	312.77	12.21	433.56
Total cost-oriented alignment	20.02	46.15	34.51	6.70	92.04	13.73	213.15

According to Table 7, compared with the manual alignment, the construction cost-oriented alignment decreases the construction costs by 5.4% and the traction energy-oriented alignment decreases the traction energy cost by 9.4%. The two alignments provide the minimal cost of construction and traction energy consumption, respectively. However, the construction cost-oriented alignment increases the traction energy cost by 2.8%, and the traction energy-oriented alignment significantly increases the construction costs by 91.6%. Based on that, the annual total costs are reduced by 4.4% in the construction cost-oriented alignment, whereas they are increased by 79.6% in the traction energy-oriented alignment. This shows that the optimization of the vertical alignment cannot neglect the construction cost.

Besides, the total cost-oriented alignment reduces both the construction costs and the traction energy cost by 5.8% and 7.6%, respectively. The total costs can thus be reduced by 6.0%, which is larger than the decrease rate of 4.4% in the construction cost-oriented alignment. Compared with the construction cost-oriented alignment, the total cost-oriented alignment can reduce the traction energy cost yet with a very slight increase of the construction cost. This result illustrates the good performance of the two-stage program in reducing construction costs and traction energy.

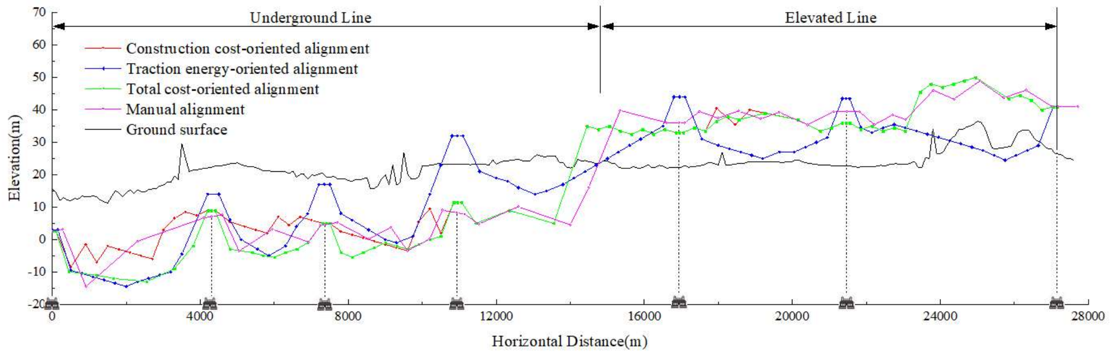


Fig. 13. The manual and optimized vertical alignments with different optimization objectives.

The manual and optimized alignments are shown in Fig. 13. It can be seen that, except for the traction energy-oriented alignment, the other alignments are built underground when the horizontal distance is less than 14000 m and turn to be elevated when their distance is longer than 14000 m. The reason is that the right-of-way cost is fully considered in the construction cost-oriented alignment and the total cost-oriented alignment. The metro vertical alignment should be built underground as the unit right-of-way cost of an elevated or ground metro line is high when the horizontal distance is between 0 and 14000 m (as shown in Table 4). Similarly, when the horizontal distance is between 14000 and 27150 m, the unit right-of-way cost of an elevated line is lower than an underground or ground line, and a bridge is preferable when the horizontal distance exceeds 14000 m.

A steep downhill slope in the train running direction contributes to reducing the required traction energy during the train's acceleration process. For the underground line, as the unit construction costs of the tunnel is unrelated to the depth of the tunnel (Patino-Ramirez et al., 2020), it is proper to set steep slopes near the station area for traction energy saving. As shown in Fig. 13, the gradient near the station area of the traction energy-oriented alignment is steep. However, the steep downhill slope increases construction costs when the metro line is built on the bridge. As shown in Fig. 13, to reduce construction cost, the gradient near the station area of the total cost-oriented alignment is flat when it is built on the bridge.

Based on the analyses, two broad conclusions on metro vertical alignment design can be summarized as follows: 1) for an underground metro line, steep downhill slopes near a station in the train running direction can assist trains in accelerating and reducing the traction energy consumption; 2) for an elevated metro line, a flat vertical alignment rather than steep slopes reduces life-cycle cost, as the construction costs reduction exceeds the increment in traction energy cost.

5.3 Sensitivity analysis on the maximum allowable running time

Given the vertical alignment, the maximum allowable train running time has a significant influence on the train traction energy. In this section, we analyze the impact of the maximum allowable train running time on the energy performance of the optimized vertical alignment. Firstly, we optimize the vertical alignment by the two-stage optimization program based on a preset maximum train running time (i.e., 1474s). Then, given the optimized vertical alignment, we further optimize train speed profile by the second-stage model under different maximum allowable running times.

Table 10 gives the ATEC of the manual and optimized alignment for various train speed profiles limited by different train running times. With the preset train running time, the optimized vertical alignment decreases the ATEC by 9.43%, compared to the manually designed vertical alignment. When the train running time decreases by 8% and increases by 8% and 16% from the preset one, the energy-saving rates of the optimized vertical alignments are 4.61%, 6.38%, and 6.41%, respectively. We can see that the energy performance of the optimized vertical alignment is better than that of the manually designed vertical alignment for different train running times.

Table 10 Annual traction energy cost comparison under different running times

Maximum allowable running time (s)	ATEC (Million CNY)		Energy-saving rate (%)
	Manual	Optimized	
1474	**30.42**	**27.55**	**9.43**
1354 (-8%)	41.44	39.53	4.61
1594 (+8%)	27.59	25.83	6.38
1714 (+16%)	25.42	23.79	6.41

6 Conclusions

Metro vertical alignment design is a complex engineering problem since it has many nonlinear constraints and a very large solution space. Existing studies on this subject mostly employed heuristic algorithms to obtain a near-optimal solution, and the stability of the solution would not be guaranteed. This paper proposes an iterative approach for solving the vertical alignment optimization problem, considering both the construction costs and traction energy consumption. The following are the main contributions of this paper:

1. An integrated two-stage optimization program is proposed involving both metro vertical alignment and train speed profile, which considers a wide variety of constraints in terms of both geographical and geometrical requirements around the metro line area. It ensures the practical applicability of the resulting vertical alignment to minimize the costs of construction and traction energy.
2. An iterative approach is proposed to solve the two-stage program. A dynamic programming (DP) algorithm is designed based on a two-dimensional grid system, where the vertical alignment optimization problem is transformed into a multi-stage decision problem, given the vertical point of intersection (VPI) locations. In that way, a backward search method is proposed to search for the best vertical alignment given an energy-efficient train speed profile.
3. The DP algorithm was tested with real data from Guangzhou Metro Line 14, and the results were compared with those of two heuristic algorithms. The results demonstrated that the DP algorithm generally provides better solution quality within the same computation time. The optimized vertical alignment based on our proposed model and the iterative approach could help achieve a 6.0% reduction in the total costs of construction and traction energy.
4. From our study, we can reach a few broad conclusions about metro vertical alignment, which may serve as a useful reference for its practical design: 1) for an underground metro line, steep downhill slopes near a station in the train running direction can assist trains in accelerating and reducing the traction energy consumption; 2) for an elevated metro line, a flat vertical alignment rather than steep slopes leads to a lower life-cycle cost, as the saving on construction costs exceeds the increment on traction energy cost.

The work presented in this paper focused only on the vertical alignment of metro tracks; however, the metro vertical alignment interacts with its horizontal alignment, which would also affect the construction costs and traction energy consumption. Therefore, joint optimization of the horizontal and vertical alignment for the metro tracks is well worth future research efforts.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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