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In 2018, the author, Kaluža and Kopecká showed in [3], that the best Lipschitz constant for mappings taking a given n^d-element set in the integer lattice $\mathbb{Z}^d$, with $n \in \mathbb{N}$, surjectively to the regular n times n grid $\{1, \dots, n\}^d$ may be arbitrarily large. However, there remains no known, non-trivial asymptotic bounds, either from above or below, on how this best Lipschitz constant grows with n. We approach this problem from a probabilistic point of view. More precisely, we consider the sequence space of all possible sequences in which the nth term is a configuration of n^d points inside a given finite lattice. Equipping such spaces with their natural probability measure, we establish almost sure, asymptotic upper bounds of order $\log n$ on the best Lipschitz constant of mappings taking the nth element of the set sequence, that is an n^d-element subset of the given finite lattice, surjectively to the regular n times n grid $\{1, \dots, n\}^d$.

1 Introduction

Fix a dimension $d \geq 2$. For each $n \in \mathbb{N}$, we define a mapping $F_n \colon \binom{\mathbb{Z}^d}{n^d} \to (0, \infty)$ by

$$F_n(S) = \min \left\{ \mathrm{Lip}(f) \colon f \colon S \to \{1, \dots, n\}^d \text{ surjective} \right\}, \qquad S \in \binom{\mathbb{Z}^d}{n^d}.$$

The quantity $F_n(S)$ can be thought of as a quantification of how much the set S differs from the regular $n \times n$ grid. In the 1990's Feige asked the question of whether the sequence

$$\mathbf{F}_n := \sup_{S \in \binom{\mathbb{Z}^d}{n^d}} F_n(S), \qquad n \in \mathbb{N}, \tag{1.1}$$

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is bounded. In other words, Feige's question asks whether there is some absolute constant $L > 0$ so that for any $n \in \mathbb{N}$ and any set $S \in \binom{\mathbb{Z}^d}{n^d}$ there exists a bijection $S \to [n]^d$ with Lipschitz constant at most L.

The paper [3] of the author, Kaluža and Kopecká provides a negative answer to Feige's question, proving that $\limsup_{n\to\infty} \mathbf{F}_n = \infty$. However, [3] fails to impose any non-trivial asymptotic bounds on the Feige sequence $(\mathbf{F}_n)$. This provides the motivation for the present work.

In order to prove that $\limsup_{n\to\infty} \mathbf{F}_n = \infty$, [3] uses a strategy introduced by McMullen [5] and Burago and Kleiner [1] which allows for the translation of discrete Lipschitz problems to the continuous setting. The aforementioned authors developed this strategy in order to answer a long standing open question of Gromov [4], namely whether every two separated nets in Euclidean space are bilipschitz equivalent. McMullen [5] and Burago and Kleiner [1] introduce methods of encoding measurable density functions $\rho\colon [0,1]^d \to (0,\infty)$ as separable nets in $\mathbb{R}^d$ and use these to prove that Gromov's question about separated nets is equivalent to the question of whether every bounded density $\rho\colon [0,1]^d \to (0,\infty)$ admits a bilipschitz solution $f\colon [0,1]^d \to \mathbb{R}^d$ to the pushforward equation

$$f_\sharp \rho \mathcal{L}|_{[0,1]^d} = \mathcal{L}|_{f([0,1]^d)}. \tag{1.2}$$

McMullen [5] and Burago and Kleiner [1] then resolve Gromov's question negatively by constructing ρ for which (1.2) has no bilipschitz solutions. For the negative answer to Feige's question, the authors and Kopecká constructed ρ so that (1.2) additionally has no solutions in the larger class of Lipschitz mappings $f\colon [0,1]^d \to \mathbb{R}^d$. Moreover, in a recent paper [2], the author and Kaluža find densities ρ for which (1.2) has no solutions in the class of homeomorphisms $f\colon [0,1]^d \to \mathbb{R}^d$ for which both f and f^{-1} have modulus of continuity bounded above by $\omega(t) = t \log\left(\frac{1}{t}\right)^{\varphi_0(d)}$ where $\varphi_0(d) \to 0$ as $d \to \infty$.

The latter result hints towards an asymptotic lower bound of the form

$$\mathbf{F}_n \geq (\log n)^{\varphi_0(d)} \tag{1.3}$$

on the Feige sequence $(\mathbf{F}_n)$. In [2] the author and Kaluža observe that if ρ may additionally be constructed so that (1.2) has no solutions f in the larger class of mappings where the homeomorphism condition (and the condition on f^{-1}) is dropped, i.e. the class of mappings $[0,1]^d \to \mathbb{R}^d$ with modulus of continuity bounded above by $\omega(t) = t \log\left(\frac{1}{t}\right)^{\varphi_0(d)}$, then the asymptotic bound (1.3) is valid along a subsequence of the Feige sequence $(\mathbf{F}_n)$.

In this note we identify certain types of discrete set sequences $(S_n) \in \prod_{n=1}^{\infty} \binom{\mathbb{Z}^d}{n^d}$ for which we are able to provide a non-trivial asymptotic upper bound on $(F_n(S_n))$. Furthermore, we show that these sequences (S_n) occur with high probability, in a sense to be made precise shortly. We hope that this could be a step towards establishing bounds on the Feige sequence $(\mathbf{F}_n)$. Note that the latter requires bounding the growth of $(F_n(S_n))$ for a general sequence $(S_n) \in \prod_{n=1}^{\infty} \binom{\mathbb{Z}^d}{n^d}$.

To determine the n-th Feige number $\mathbf{F}_n$, observe that it suffices to consider only sets $S \in \binom{\mathbb{Z}^d}{n^d}$ which lie inside the finite cubic grid $\left\{1, \ldots, n^d\right\}^d$ of side length n^d. Put differently, the supremum in (1.1) remains unchanged if the integer lattice $\mathbb{Z}^d$ is replaced by the finite grid $\{1,\ldots,n^d\}^d$. This holds because any set $S \in \binom{\mathbb{Z}^d}{n^d}$ may be mapped via a 1-Lipschitz, injective mapping to a subset of $\{1,\ldots,n^d\}^d$: simply take out empty hyperplanes, contract and translate. Thus, to establish asymptotic bounds on the Feige sequence $(\mathbf{F}_n)$ it suffices to provide asymptotic bounds on $F_n(S_n)$ for an arbitrary sequence (S_n) belonging to the sequence space $\prod_{n=1}^{\infty}\binom{\{1,\ldots,n^d\}^d}{n^d}$.

Restricting our attention to configurations of n^d points inside a finite cubic grid, instead of inside the entire the integer lattice, naturally invites a probabilistic approach. We can think of each possible configuration of the n^d points in the finite cubic grid as occurring with equal probability. Taking large cubic grids, such as the grid $\{1,\ldots,n^d\}^d$ discussed above, we would expect to see configurations of n^d points being very spread out with high probability, leading to F_n being uniformly bounded independent of n with high probability. Thus it makes sense to consider the problem in all smaller cubic grids of side length cn for all $c > 1$. Hence, the family of all viable sequence spaces in which it makes sense to study the probabilistic growth of $F_n(S_n)$ for a random sequence (S_n) can be described by

$$\mathcal{G}_{(c_n)} := \prod_{n=1}^{\infty}\binom{\{1,\ldots,\lfloor c_n \cdot n\rfloor\}^d}{n^d}, \tag{1.4}$$

for sequences (c_n) of real numbers with $c_n \geq 1$ for all n. Note that for all sequences (c_n) with $1 < \inf c_n \leq \sup c_n < \infty$, the methods of [3] establish that there are sequences $(S_n) \in \mathcal{G}_{(c_n)}$ for which $\limsup_{n\to\infty} F_n(S_n) = \infty$. Whilst our current methods do not allow for a non-trivial bound on the asymptotic growth of $F_n(S_n)$ for the general sequence $(S_n) \in \mathcal{G}_{(c_n)}$, we do obtain a bound which holds 'almost surely' in spaces $\mathcal{G}_{(c_n)}$ corresponding to any sequence $c_n \geq 1$ which does not converge to 1 too quickly, in a specific sense. More generally, there is less known in the case of sequences (c_n) converging to one. It is not known for which sequences $c_n \searrow 1$ the sequence space $\mathcal{G}_{(c_n)}$ contains a sequence (S_n) such that $\limsup_{n\to\infty} F_n(S_n) = \infty$.

The notion of 'almost surely' refers to the uniform probability measure on the sequence space $\mathcal{G}_{(c_n)}$. In the present work we only consider one type of probability space, namely that given by a product of finite sets equipped with the uniform probability measure. If X is a finite set, we consider the uniform probability measure on $(X, 2^X)$ defined by

$$\mathbb{P}_X(A) = \frac{|A|}{|X|}, \qquad A \subseteq X, \tag{1.5}$$

where $|{-}|$ denotes the cardinality. Secondly, if $Y = \prod_{n=1}^{\infty} X_n$, where each X_n is a finite set, we define the uniform probability measure on Y by

$$\mathbb{P}_Y := \prod_{n=1}^{\infty} \mathbb{P}_{X_n}. \tag{1.6}$$

Since it will always be clear from the context which probability space we are working in, we will always just write $\mathbb{P}$ (without a subscript) to denote the uniform probability measure.

We are now ready to state the main result[1] of this note:

Theorem 1.1. *Let $d \in \mathbb{N}$ with $d \geq 4$ and (c_n) be a sequence of positive real numbers $c_n \geq \left(1 + \frac{2^{d+5}}{\log n}\right)^{1/d}$. For each $n \in \mathbb{N}$, let the mapping $F_n \colon \binom{\mathbb{Z}^d}{n^d} \to (0, \infty)$ be defined by*

$$F_n(S) = \min\left\{\mathrm{Lip}(f) \colon f \colon S \to \{1, \dots, n\}^d \text{ surjective}\right\}, \qquad S \in \binom{\mathbb{Z}^d}{n^d}.$$

Then a random sequence in the probability space $\mathcal{G}_{(c_n)}$, defined by (1.4), (1.5) and (1.6), satisfies

$$\mathbb{P}\left[\limsup_{n\to\infty} F_n(S_n)(\log n)^{-1} < \infty\right] = 1.$$

In addition, we identify certain real sequences (c_n) for which Theorem 1.1 may be improved. For these sequences, we establish a stronger, almost sure, asymptotic upper bound on $F_n(S_n)$ of order $(\log n)^{\alpha/d}$, which we emphasise is dependent on the dimension d. Moreover, the inequality (1.7) in the following statement indicates, that such sequences $c_n \geq 1$ may be chosen, in a sense, asymptotically equivalent to any given sequence $e_n \geq 1$.

Theorem 1.2. *Let $d \in \mathbb{N}$ with $d \geq 2$ and (e_n) be a sequence of positive real numbers $e_n \geq 1$. Then there exists a sequence (c_n) of positive real numbers c_n satisfying*

$$|c_n - e_n| \leq \frac{2^{d+8}}{(\log n)^{1/d}}, \tag{1.7}$$

such that the following statement holds: For each $n \in \mathbb{N}$, let the mapping $F_n \colon \binom{\mathbb{Z}^d}{n^d} \to (0, \infty)$ be defined by

$$F_n(S) = \min\left\{\mathrm{Lip}(f) \colon f \colon S \to \{1, \dots, n\}^d \text{ surjective}\right\}, \qquad S \in \binom{\mathbb{Z}^d}{n^d}.$$

Then a random sequence in the probability space $\mathcal{G}_{(c_n)}$, defined by (1.4), (1.5) and (1.6), satisfies

$$\mathbb{P}\left[\limsup_{n\to\infty} F_n(S_n)(\log n)^{\frac{-\alpha}{d}} < \infty\right] = 1$$

for all $\alpha > 3$.

Roughly speaking, Theorem 1.2 tentatively supports the conjecture of an upper bound of order say $(\log n)^{4/d}$ on the Feige sequence $(\mathbf{F}_n)$. This is interesting because it coincides in form with the conjectured lower bound of order $(\log n)^{\varphi(d)}$, where $\lim_{d\to\infty} \varphi(d) = 0$, coming from the completely independent results of [2].

[1] We actually prove a slightly more general, but more technical statement which covers dimensions 2 and 3; see Theorem 5.1.

2 Preliminaries, Convention and Notation.

Let us quickly summarise some basic notation which may not be completely standard. The dimension d of the Euclidean space $\mathbb{R}^d$ in which we work will be considered fixed throughout the whole paper. Thus, all objects defined in the paper should be thought of as having a suppressed subindex d; for example $F_n = F_{n,d}$ and $\mathcal{G}_{(c_n)} = \mathcal{G}_{(c_n),d}$. For a set A and $k \in \mathbb{N}$ we let $\binom{A}{k}$ denote the set of all subsets of A with precisely k elements. Given $t \geq 0$ we write $\lfloor t \rfloor$ for the integer part of t and $[t]$ for the set of integers $\{1, 2, \dots, \lfloor t \rfloor\}$. The symbol $\mathcal{L}$ will denote the Lebesgue measure. Since powers of 2 arise frequently in the calculations we take, for convenience, the logarithm function log with base 2. We will also write $\exp(x)$ to denote 2^x.

Given a measure μ on $[0,1]^d$, we call a collection $\mathcal{T}$ of subsets of $[0,1]^d$ a μ-partition of $[0,1]^d$ if $\mu\left([0,1]^d \setminus \bigcup \mathcal{T}\right) = 0$ and $\mu(T \cap T') = 0$ for all $T, T' \in \mathcal{T}$ with $T \neq T'$. For each $k \in \mathbb{N}$ we let

$$\mathcal{T}_k = \left\{ \prod_{i=1}^{d} \left(\frac{p_i}{k}, \frac{p_i + 1}{k} \right] : p_1, \dots, p_d \in \{0, 1, \dots, k-1\} \right\}, \tag{2.1}$$

Note that each $\mathcal{T}_k$ is, in particular, an $\mathcal{L}$-partition of $[0,1]^d$.

The next lemma is our main mechanism for relating measures to the question of best Lipschitz constants for mappings of finite sets.

Lemma 2.1. *Let μ, ν be Borel probability measures on the unit cube $[0,1]^d$. Let $n \in \mathbb{N}$, $\mathcal{T}$ be a finite μ-partition of $[0,1]^d$, $c > 1$ and*

$$X \subseteq \frac{1}{cn}\mathbb{Z}^d \cap \bigcup \mathcal{T}, \qquad Y \subseteq \frac{1}{n}\mathbb{Z}^d \cap [0,1]^d$$

be finite sets such that

$$\mu(T) \geq \frac{1}{n^d} \left| X \cap T \right|, \qquad \textit{for every } T \in \mathcal{T}\textit{, and} \tag{2.2}$$

$$\nu(E) \leq \frac{1}{n^d} \left| \left\{ y \in Y \colon d_\infty(y, E) \leq \frac{1}{n} \right\} \right| \qquad \textit{for every } \nu\textit{-measurable } E \subseteq [0,1]^d, \tag{2.3}$$

where d_∞ denotes the distance induced by the norm $\|-\|_\infty$. Let $f\colon [0,1]^d \to [0,1]^d$ be a Lipschitz mapping with $f_\sharp \mu = \nu$. Then there exist a constant $\Lambda = \Lambda(d) > 0$ and an injective mapping $g\colon X \to Y$ with

$$\operatorname{Lip}(g) \leq \Lambda \max\{1, \operatorname{Lip}(f)\}\, c \left(n \cdot \max_{T \in \mathcal{T}} \operatorname{diam} T + 1 \right).$$

Proof. For a point $x \in X$ we denote by $T(x)$ a choice of set $T \in \mathcal{T}$ which contains x. We further define a set valued mapping $R\colon X \to 2^Y$ by

$$R(x) = \left\{ y \in Y \colon d_\infty(y, f(T(x))) \leq \frac{1}{n} \right\}.$$

In what follows we obtain an injective mapping $g\colon X \to Y$ with the property that

$$g(x) \in R(x), \qquad x \in X. \tag{2.4}$$

We may then complete the proof in the following way. For points $x, x' \in X$ we observe that

$$\left\|g(x') - g(x)\right\|_2 \leq \left\|g(x') - f(x')\right\|_2 + \left\|f(x') - f(x)\right\|_2 + \|f(x) - g(x)\|_2$$

Now, from condition (2.4) we have

$$\begin{aligned}
\|g - f\|_\infty &\leq \max_{T \in \mathcal{T}} \operatorname{diam} f(T) + \frac{\sqrt{d}}{n} \\
&\leq \operatorname{Lip}(f)\sqrt{d} \max_{T \in \mathcal{T}} \operatorname{diam} T + \frac{\sqrt{d}}{n} \\
&\leq \sqrt{d} \max\{1, \operatorname{Lip}(f)\} \left(\max_{T \in \mathcal{T}} \operatorname{diam} T + \frac{1}{n}\right).
\end{aligned}$$

Hence, using $\|x' - x\|_2 \geq \frac{1}{cn}$, we obtain

$$\begin{aligned}
\left\|g(x') - g(x)\right\|_2 &\leq 2\sqrt{d} \max\{1, \operatorname{Lip}(f)\} \left(\max_{T \in \mathcal{T}} \operatorname{diam} T + \frac{1}{n}\right) + \operatorname{Lip}(f) \left\|x' - x\right\|_2 \\
&\leq 3\sqrt{d} \max\{1, \operatorname{Lip}(f)\}\, c \left(n \cdot \max_{T \in \mathcal{T}} \operatorname{diam} T + 1\right) \left\|x' - x\right\|_2.
\end{aligned}$$

It only remains to verify the existence of the mapping g. To do this we will adopt a similar strategy to that employed in [5, Theorem 4.1]. By Hall's Marriage Theorem it suffices to verify that $|A| \leq |R(A)|$ for any set $A \subseteq X$.

Let $A \subseteq X$, $T_1, \ldots, T_p$ be an enumeration of $\{T(x) \colon x \in A\}$ and $E := \bigcup_{j \in [p]} f(T_j)$. Then

$$\left\{y \in Y \colon d_\infty(y, E) \leq \frac{1}{n}\right\} = R(A).$$

Therefore, by (2.3),

$$\nu\left(\bigcup_{j \in [p]} f(T_j)\right) \leq \frac{1}{n^d} \cdot |R(A)|. \tag{2.5}$$

On the other hand, using $f_\sharp \mu = \nu$ and (2.2) we may derive

$$\nu\left(\bigcup_{j \in [p]} f(T_j)\right) \quad \geq \quad \sum_{j \in [p]} \mu(T_j) \quad \geq \quad \frac{1}{n^d} \sum_{j \in [p]} |X \cap T_j| \quad \geq \quad \frac{1}{n^d} \quad \cdot \quad |A|. \tag{2.6}$$

Combining (2.5) and (2.6) we get

$$|A| \leq |R(A)|,$$

as required. $\square$

3 Well-distributed sets.

In this section we derive an upper bound on the best Lipschitz constant $F_n(S)$ for sets $S \in \binom{[cn]^d}{n^d}$ which are 'well-distributed' in the sense that the points are quite evenly spread relative to the grid partition coming from $\mathcal{T}_m$.

Lemma 3.1. *Let $m, n, l \in \mathbb{N}$ with $m = 2^l \leq n$, $0 < \theta < a < b < \frac{1}{\theta}$, $c > 1$ and $S \subseteq \mathbb{Z}^d \cap [0, cn]^d$ be a finite set with $|S| = n^d$ and*

$$\frac{an^d}{m^d} \leq |S \cap (cn \cdot T)| \leq \frac{bn^d}{m^d}$$

for all $T \in \mathcal{T}_m$, where $\mathcal{T}_m$ is defined by (2.1). Then there exists a bijection $g\colon S \to [n]^d$ and constants $\Lambda := \Lambda(d)$, $\Delta := \Delta(d, \theta)$ with

$$\mathrm{Lip}(g) \leq \Lambda 2^{\log n - l(1-\Delta(b-a))}.$$

Let us begin working towards a proof of Lemma 3.1. The bound will be established by applying Lemma 2.1 in the case that ν is the Lebesgue measure on $[0,1]^d$ and μ has the form $\mu = \rho\mathcal{L}$, where ρ is of the form considered in the next lemma.

Lemma 3.2. *Let $l \in \mathbb{N}$, $\theta \in (0,1)$, $\mathcal{T}_k$ be defined by (2.1) for each $k \in \mathbb{N}$, and $\rho\colon [0,1]^d \to (0, \infty)$ be a function such that $\rho|_T$ is constant for each $T \in \mathcal{T}_{2^l}$, $\int_{[0,1]^d} \rho \, d\mathcal{L} = 1$ and $\theta \leq \min \rho \leq \max \rho \leq \frac{1}{\theta}$. Then there exists a Lipschitz mapping $f\colon [0,1]^d \to \mathbb{R}^d$ and a constant $\Delta = \Delta(d, \theta) > 0$ such that*

$$f_\sharp \rho\mathcal{L} = \mathcal{L}|_{[0,1]^d},$$

and

$$\mathrm{Lip}(f) \leq (1 + \Delta(\max \rho - \min \rho))^l.$$

The proof of Lemma 3.2 is due to Rivieré and Ye [6]. However, there the argument is used to prove a more general statement and Lemma 3.2 is not stated or proved explicitly. The proof is based on the following lemma.

Lemma 3.3 ([6, Lemma 1])**.** *Let $D = [0,1]^d$, $A = [0,1]^{d-1} \times \left[0, \frac{1}{2}\right]$ and $B = [0,1]^{d-1} \times \left[\frac{1}{2}, 1\right]$. Let $\alpha, \beta \geq 0$ be such that $\alpha + \beta = 1$ and let $\eta > 0$ be such that $\eta \leq \alpha \leq 1 - \eta$. Then there exists a Lipschitz homeomorphism $\Phi\colon [0,1]^d \to [0,1]^d$ and a constant $\Delta = \Delta(\eta)$ such that*

(i) $\Phi|_{\partial[0,1]^d} = \mathrm{id}_{\partial[0,1]^d}$,

(ii) $\mathrm{Jac}(\Phi)(x) = \begin{cases} 2\alpha & \text{if } x \in A, \\ 2\beta & \text{if } x \in B, \end{cases}$ *for a.e.* $x \in [0,1]^d$,

(iii) $\mathrm{Lip}(\Phi - \mathrm{id}) \leq \Delta\,|1 - 2\alpha|$.

Since we only require the argument of Rivieré and Ye [6] for a particular special case, the following restricted version of the argument is more convenient for the reader.

Proof of Lemma 3.2. For each $k = (k_1, \dots, k_d) \in (\{0\} \cup [2^i - 1])^d$ and each $i \in \mathbb{N}$ we let

$$C(k,i) := \prod_{1 \le j \le d} \left[\frac{k_j}{2^i}, \frac{k_j + 1}{2^i} \right],$$

define a labelling of the cubes in $\mathcal{T}_m$. We define a homeomorphism $\Phi_i \colon [0,1]^d \to [0,1]^d$ by prescribing it on each cube $C(k,i)$.

Fix $k = (k_1, \dots, k_n) \in (\{0\} \cup [2^i - 1])^d$. For each $p \in [d]$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{0,1\}^d$ let

$$A_i^p(\varepsilon) = \prod_{1=j}^{p-1} \left[\frac{k_j}{2^i}, \frac{k_j+1}{2^i} \right] \times \left[\frac{k_p}{2^i}, \frac{2k_p+1}{2^{i+1}} \right] \times \prod_{j=p+1}^{d} \left[\frac{k_j}{2^i} + \frac{\varepsilon_j}{2^{i+1}}, \frac{k_j}{2^i} + \frac{\varepsilon_j + 1}{2^{i+1}} \right],$$

$$B_i^p(\varepsilon) = \prod_{1=j}^{p-1} \left[\frac{k_j}{2^i}, \frac{k_j+1}{2^i} \right] \times \left[\frac{2k_p+1}{2^{i+1}}, \frac{k_p+1}{2^i} \right] \times \prod_{j=p+1}^{d} \left[\frac{k_j}{2^i} + \frac{\varepsilon_j}{2^{i+1}}, \frac{k_j}{2^i} + \frac{\varepsilon_j + 1}{2^{i+1}} \right],$$

$$\alpha_i^p(\varepsilon) = \frac{\int_{A_i^p(\varepsilon)} \rho \, d\mathcal{L}}{\int_{A_i^p(\varepsilon) \cup B_i^p(\varepsilon)} \rho \, d\mathcal{L}}, \qquad \beta_i^p(\varepsilon) = \frac{\int_{B_i^p(\varepsilon)} \rho \, d\mathcal{L}}{\int_{A_i^p(\varepsilon) \cup B_i^p(\varepsilon)} \rho \, d\mathcal{L}}.$$

Note that, for each fixed p, the sets $A_i^p(\varepsilon)$, $B_i^p(\varepsilon)$ indexed by $\varepsilon \in \{0,1\}^d$ form a partition of $C(k,i)$. More precisely, these sets have pairwise disjoint interiors their union is $C(k,i)$.

For each $j \in [d]$ we define a homeomorphism $\Phi_i^j \colon [0,1]^d \to [0,1]^d$ as follows: For each $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{0,1\}^d$ define $\Phi_i^j|_{A_i^j(\varepsilon) \cup B_i^j(\varepsilon)}$ as the homeomorphism given by the conclusion of Lemma 3.3 applied with $D = A_i^j(\varepsilon) \cup B_i^j(\varepsilon)$, $A = A_i^j(\varepsilon)$, $B = B_i^j(\varepsilon)$, $\alpha = \alpha_i^j(\varepsilon)$ and $\beta = \beta_i^j(\varepsilon)$. Note that here we have to use Lemma 3.3 in combination with suitable affine transformations. Further, the parameter η in Lemma 3.3 may be taken as $\eta = \frac{\theta^2}{1+\theta^2}$ and we have

$$|1 - 2\alpha_i^p(\varepsilon)| \le \frac{1}{2\theta} (\max \rho - \min \rho), \qquad p \in [d], \, \varepsilon \in \{0,1\}^d.$$

Hence, this application of Lemma 3.3 provides a Lipschitz homeomorphism

$$\Phi_i^j \colon A_i^j(\varepsilon) \cup B_i^j(\varepsilon) \to A_i^j(\varepsilon) \cup B_i^j(\varepsilon)$$

with properties (i)–(iii), where (iii) translates to

$$\operatorname{Lip}(\Phi_i^j - \mathrm{id}) \le \Delta (\max \rho - \min \rho), \qquad j \in [d],$$

for a constant $\Delta = \Delta(\theta) > 0$. Due to property (i), we can glue all of these homeomorphisms together to obtain a homeomorphism $\Phi_i^j \colon [0,1]^d \to [0,1]^d$ with the same Lipschitz constant. Finally set

$$\Phi_i := \Phi_i^d \circ \Phi_i^{d-1} \circ \ldots \circ \Phi_i^1,$$
$$f_q := \Phi_1 \circ \Phi_2 \circ \ldots \Phi_q \text{ for } q \in [l], \qquad \text{and } f := f_l.$$

By induction, it is readily verified that $\operatorname{Jac}(f_q)(x) = \frac{1}{\mathcal{L}(T)} \int_T \rho \, d\mathcal{L}$ for a.e. $x \in T$ and $T \in \mathcal{T}_{2^q}$. Hence $\operatorname{Jac}(f)(x) = \rho(x)$ for a.e. $x \in [0,1]^d$ and accordingly $f_\sharp \rho\mathcal{L} = \mathcal{L}|_{[0,1]^d}$. Moreover, we have

$$\begin{aligned}\operatorname{Lip}(f) \leq \prod_{q=1}^{l} \operatorname{Lip}(\Phi_q) &\leq \prod_{q=1}^{l} \prod_{j=1}^{d} \operatorname{Lip}(\Phi_q^j) \\ &\leq (1 + \Delta(\max \rho - \min \rho))^{ld} \leq (1 + \Delta(\max \rho - \min \rho))^{l},\end{aligned}$$

where in the final expression the constant $\Delta(\theta)$ becomes $\Delta(d, \theta)$. $\square$

Proof of Lemma 3.1. Define $\rho \colon [0,1]^d \to (0, \infty)$ in $L^\infty([0,1]^d)$ by

$$\rho|_T \equiv \frac{m^d}{n^d} \cdot |S \cap (cn \cdot T)|\,, \qquad T \in \mathcal{T}_m.$$

Thus, ρ is constant on each $T \in \mathcal{T}_m$ and $a \leq \rho \leq b$. By Lemma 3.2 there exists a Lipschitz mapping $f \colon [0,1]^d \to [0,1]^d$ and a constant $\Delta = \Delta(d, \theta)$ such that $f_\sharp \rho\mathcal{L} = \mathcal{L}|_{[0,1]^d}$ and $\operatorname{Lip}(f) \leq (1 + \Delta(b - a))^l$. We may now apply Lemma 2.1 to $\mu = \rho\mathcal{L}$, $\nu = \mathcal{L}|_{[0,1]^d}$, n, $\mathcal{T} = \mathcal{T}_m$, c, $X = \frac{1}{cn} S$, $Y = \frac{1}{n}[n]^d$ and f to get a bijective mapping $\widetilde{g} \colon \frac{1}{cn} \cdot S \to \frac{1}{n}[n]^d$ and a constant $\Lambda = \Lambda(d)$ with

$$\begin{aligned}\operatorname{Lip}(\widetilde{g}) &\leq \Lambda \max\{\operatorname{Lip}(f), 1\}\, c\frac{n}{m} \leq \Lambda c 2^{\log n - l}(1 + \Delta(b - a))^l \\ &= \Lambda c \exp\left(\log n - l(1 - \log(1 + \Delta(b - a)))\right) \leq \Lambda c \exp\left(\log n - l(1 - \Delta(b - a))\right).\end{aligned}$$

Finally, we define $g \colon S \to [n]^d$ by

$$g(x) = n \cdot \widetilde{g}\left(\frac{x}{cn}\right), \qquad x \in S.$$

$\square$

4 Random sequences.

In this section we show that for $C > 1$ and large n, a random set $S \in \binom{[C^{1/d} n]^d}{n^d}$ is well-distributed in the sense of Section 3 with high probability.

Calculating probabilities in the space $\binom{[C^{1/d} n]^d}{n^d}$ will inevitably lead to expressions involving large binomial coefficients. To estimate these numbers, we will use the following standard lemma which follows easily from Stirling's approximation of the factorial.

In what follows H denotes the binary entropy function

$$H(t) = -t \log t - (1 - t) \log(1 - t), \qquad t \in [0,1].$$

Later on we will use certain important properties of the binary entropy function H, namely that it is strictly convex, differentiable and that its derivative is given by

$$H'(t) = -\log\left(\frac{t}{1-t}\right), \qquad t \in (0,1).$$

Lemma 4.1. *There is an absolute constant $\Lambda > 0$ such that*

$$\begin{cases} \Lambda^{-1}\sqrt{\frac{p}{2\pi q(p-q)}} \cdot 2^{pH\left(\frac{q}{p}\right)} \le \binom{p}{q} \le \Lambda\sqrt{\frac{p}{2\pi q(p-q)}} \cdot 2^{pH\left(\frac{q}{p}\right)} & \textit{if } q \in [p-1]\setminus\{0\}\,, \\ \Lambda^{-1}2^{pH\left(\frac{q}{p}\right)} \le \binom{p}{q} \le \Lambda 2^{pH\left(\frac{q}{p}\right)} & \textit{if } q \in \{0,p\}\,. \end{cases}, \qquad p \in \mathbb{N}.$$

Note that the inequalities of Lemma 4.1 for the case $q \in \{0,p\}$ are trivial, because $\binom{p}{0} = \binom{p}{p} = 1$. We write them here because we wish to treat the case $q \in \{0,p\}$ together with the case $q \in [p-1]$ later on.

Proof of Lemma 4.1. By Stirling's Approximation of $n!$ (see for example [?]), the quantities

$$\alpha := \min_{n\in\mathbb{N}} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} > 0,$$

and

$$\beta := \max_{n\in\mathbb{N}} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} < \infty,$$

are absolute constants. Let $p \in \mathbb{N}$ and $q \in [p-1]\setminus\{0\}$. Then,

$$\binom{p}{q} = \frac{p!}{q!(p-q)!} \le \frac{\beta}{\alpha^2}\cdot\sqrt{\frac{p}{2\pi q(p-q)}}\cdot\frac{p^p}{q^q(p-q)^{p-q}} = \frac{\beta}{\alpha^2}\cdot\sqrt{\frac{p}{2\pi q(p-q)}}\cdot 2^{pH\left(\frac{q}{p}\right)},$$

and similarly

$$\binom{p}{q} \ge \frac{\alpha}{\beta^2}\cdot\sqrt{\frac{p}{2\pi q(p-q)}}\cdot 2^{pH\left(\frac{q}{p}\right)}.$$

Now let $p \in \mathbb{N}$ and $q \in \{0,p\}$. Then, $H\left(\frac{q}{p}\right) = 0$ and

$$\binom{p}{q} = 1 = 2^{pH\left(\frac{q}{p}\right)}.$$

Therefore, we may take $\Lambda = \frac{\max\{\beta,\beta^2\}}{\min\{\alpha,\alpha^2\}}$. ☐

Lemma 4.2. *Let $I \subseteq \mathbb{R}$ be an open interval, $f\colon I \to (0,\infty)$ be a differentiable, concave and strictly increasing function and let $s,t \in I$ with $s < t$ and let $\lambda \in (0,1)$. Then*

$$f((1-\lambda)s + \lambda t) \le \frac{f'(s)}{f'(t)}\cdot\left((1-\lambda)f(s) + \lambda f(t)\right).$$

Proof. If the inequality holds for the function $g(u) := f(u) - f(s)$ in place of f then it also holds for f. This is readily verified using the concavity and positivity of f. Thus, we may assume that $f(s) = 0$. This allows us to write

$$\frac{f((1-\lambda)s+\lambda t)}{(1-\lambda)f(s)+\lambda f(t)} = \frac{\int_s^{(1-\lambda)s+\lambda t} f'(u)\,du}{\lambda\int_s^t f'(u)\,du} \le \frac{f'(s)}{f'(t)}.$$

☐

Lemma 4.3. *Let $\delta \in [0, \frac{1}{2})$, $N \in \mathbb{N}$, $M > 1$, $\frac{1}{2} < a < 1 - \delta$, $1 + 2\delta < b < 2$, X be a finite set with $|X| > bN$ and $Y \subseteq X$ be a set with*

$$\frac{(1-\delta)\,|X|}{M} \le |Y| \le \frac{(1+\delta)\,|X|}{M}. \tag{4.1}$$

Then, there is an absolute constant $\Gamma > 0$ such that a random set $S \in \binom{X}{N}$ satisfies

$$\mathbb{P}\left[|S \cap Y| \le \frac{aN}{M}\right] \le \Gamma \cdot \sqrt{\frac{|X| - N}{|X|\left(1 - \frac{2}{M}\right) - N}} \cdot \frac{N^{3/2}}{M} \cdot \exp\left(-\frac{(1-(a+\delta))^2 N(|X| - N)}{\Gamma M(|X| - (a+\delta)N)}\right) \tag{4.2}$$

and

$$\mathbb{P}\left[|S \cap Y| \ge \frac{bN}{M}\right] \le \Gamma \cdot \sqrt{\frac{|X| - N}{|X|\left(1 - \frac{2}{M}\right) - N}} \cdot N^{3/2} \cdot \exp\left(-\frac{(b - 2\delta - 1)^2 N(|X| - (b-2\delta)N)}{\Gamma M(|X| - N)}\right). \tag{4.3}$$

Proof. The probabilities considered in (4.2) and (4.3) are bounded above by

$$\binom{|X|}{N}^{-1} \sum_k \binom{|Y|}{k}\binom{|X| - |Y|}{N - k},$$

where the sum is taken over $0 \le k \le \frac{aN}{M}$ for (4.2) and over $\frac{bN}{M} \le k \le \min\{N, |Y|\}$ for (4.3). Our first aim is to establish an upper bound for the quantity

$$\binom{|X|}{N}^{-1} \binom{|Y|}{k}\binom{|X| - |Y|}{N - k} \tag{4.4}$$

for $0 \le k \le \min\{N, |Y|\}$.

Fix $0 \le k \le \min\{N, |Y|\}$ and define

$$V_k := \begin{cases} \sqrt{\frac{|Y|}{k(|Y|-k)}} & \text{if } k \notin \{0, |Y|\}, \\ 1 & \text{if } k \in \{0, |Y|\}, \end{cases}$$

$$W_k := \begin{cases} \sqrt{\frac{|X|-|Y|}{(N-k)(|X|-|Y|-N+k)}} & \text{if } N - k \notin \{0, |X| - |Y|\}, \\ 1 & \text{if } N - k \in \{0, |X| - |Y|\}. \end{cases}$$

Then, we may use Lemma 4.1 to bound the product in (4.4) above by

$$\Lambda \cdot \left(\sqrt{\frac{|X|}{N(|X| - N)}}\right)^{-1} V_k W_k \cdot \exp\left(-|Y|\,\gamma_k\right), \tag{4.5}$$

where $\Lambda > 0$ is an absolute constant and

$$\gamma_k := \frac{|X|}{|Y|} H\left(\frac{N}{|X|}\right) - H\left(\frac{k}{|Y|}\right) - \left(\frac{|X|}{|Y|} - 1\right) H\left(\frac{N-k}{|X|-|Y|}\right). \tag{4.6}$$

The product $\left(\sqrt{\frac{|X|}{N(|X|-N)}}\right)^{-1} V_k W_k$ in (4.5) may be bounded above by

$$\sqrt{\frac{|Y|}{|Y|-1}} \cdot \sqrt{\frac{N}{N-(N-1)}} \cdot \sqrt{\frac{|X|-N}{|X|-N-|Y|}} \quad \leq \quad \sqrt{2}\sqrt{N}\sqrt{\frac{|X|-N}{|X|-N-|Y|}}.$$

Therefore, we obtain an absolute constant $\Gamma > 0$ such that

$$\binom{|X|}{N}^{-1} \binom{|Y|}{k} \binom{|X|-|Y|}{N-k} \leq \Gamma \sqrt{\frac{N(|X|-N)}{|X|-N-|Y|}} \cdot \exp\left(-|Y|\gamma_k\right). \tag{4.7}$$

Our task is now to establish a lower bound on γ_k. To this end, we rewrite the formula (4.6) for γ_k as

$$\begin{aligned}
\gamma_k &= H\left(\frac{N}{|X|}\right) - H\left(\frac{k}{|Y|}\right) - \left(\frac{|X|}{|Y|} - 1\right)\left(H\left(\frac{N-k}{|X|-|Y|}\right) - H\left(\frac{N}{|X|}\right)\right) \\
&= H\left(\frac{N}{|X|}\right) - H\left(\frac{k}{|Y|}\right) - \left(\frac{N}{|X|} - \frac{k}{|Y|}\right) H'(\xi_k)
\end{aligned}$$

for some ξ_k lying in the interval with endpoints $\frac{N}{|X|}$ and $\frac{N-k}{|X|-|Y|}$. From the bounds on $|Y|$, a and b given by the hypothesis of the lemma, we have

$$\frac{k}{|Y|} < \frac{N}{|X|} \leq \xi_k \leq \frac{N-k}{|X|-|Y|} \qquad \text{if } 0 \leq k \leq \frac{aN}{M}, \tag{4.8}$$

$$\frac{N-k}{|X|-|Y|} \leq \xi_k \leq \frac{N}{|X|} < \frac{k}{|Y|} \qquad \text{if } \frac{bN}{M} \leq k \leq \min\{N, |Y|\}. \tag{4.9}$$

Assume first, that $0 \leq k \leq \frac{aN}{M}$. Then (4.8), together with that fact that H is strictly concave, allows us to write

$$\begin{aligned}
\gamma_k = \int_{\frac{k}{|Y|}}^{\frac{N}{|X|}} H'(t) - H'(\xi_k)\, dt &\geq \int_{\frac{(a+\delta)N}{|X|}}^{\frac{N}{|X|}} H'(t) - H'\left(\frac{N}{|X|}\right) dt \\
&\geq H\left(\frac{N}{|X|}\right) - H\left(\frac{((a+\delta)N}{|X|}\right) - \frac{(1-(a+\delta))N}{|X|} H'\left(\frac{N}{|X|}\right) \\
&= -\frac{N}{|X|}\log\left(\frac{N}{|X|}\right) - \left(1 - \frac{N}{|X|}\right)\log\left(1 - \frac{N}{|X|}\right) + \frac{(a+\delta)N}{|X|}\log\left(\frac{(a+\delta)N}{|X|}\right) \\
&\quad + \left(1 - \frac{(a+\delta)N}{|X|}\right)\log\left(1 - \frac{(a+\delta)N}{|X|}\right) + \frac{(1-(a+\delta))N}{|X|}\log\left(\frac{\frac{N}{|X|}}{1-\frac{N}{|X|}}\right) \\
&= \left(1 - \frac{(a+\delta)N}{|X|}\right)\log\left(\frac{|X|-(a+\delta)N}{|X|-N}\right) + \frac{(a+\delta)N}{|X|}\log(a+\delta).
\end{aligned}$$

Finally, we apply Lemma 4.2 to $I = (0\,\infty)$, $f = \log$, $s = a$ and $t = \frac{|X|-(a+\delta)N}{|X|-N}$ in order to bound the latter expression below by

$$\frac{\log'\left(\frac{|X|-(a+\delta)N}{|X|-N}\right)}{\log'(a+\delta)}\log\left(1+\frac{(1-(a+\delta))^2N}{|X|-N}\right) \geq \frac{(1-(a+\delta))^2N(|X|-N)}{2\,|X|\,(|X|-(a+\delta)N)}, \tag{4.10}$$

where the latter inequality is derived by applying the inequality $\log(1+x) \geq \frac{x}{1+x}$. Similarly, if $\frac{bN}{M} \leq k \leq \min\{N, |Y|\}$, we use (4.9) and the strict concavity of H to derive

$$\begin{aligned}
\gamma_k = \int_{\frac{N}{|X|}}^{\frac{k}{|Y|}} H'(\xi_k) - H'(t)\,dt &\geq \int_{\frac{N}{|X|}}^{\frac{(b-2\delta)N}{|X|}} H'\left(\frac{N}{|X|}\right) - H'(t)\,dt \\
&\geq \frac{(b-2\delta-1)N}{|X|}H'\left(\frac{N}{|X|}\right) - \left(H\left(\frac{(b-2\delta)N}{|X|}\right) - H\left(\frac{N}{|X|}\right)\right) \\
&\geq \frac{-(b-2\delta-1)N}{|X|}\log\left(\frac{\frac{N}{|X|}}{1-\frac{N}{|X|}}\right) + \frac{(b-2\delta)N}{|X|}\log\left(\frac{(b-2\delta)N}{|X|}\right) \\
&\quad + \left(1-\frac{(b-2\delta)N}{|X|}\right)\log\left(1-\frac{(b-2\delta)N}{|X|}\right) \\
&\quad - \frac{N}{|X|}\log\left(\frac{N}{|X|}\right) - \left(1-\frac{N}{|X|}\right)\log\left(1-\frac{N}{|X|}\right) \\
&= \frac{(b-2\delta)N}{|X|}\log(b-2\delta) + \left(1-\frac{(b-2\delta)N}{|X|}\right)\log\left(\frac{|X|-(b-2\delta)N}{|X|-N}\right) \\
&\geq \frac{\log'(b-2\delta)}{\log'\left(\frac{|X|-(b-2\delta)N}{|X|-N}\right)}\cdot\log\left(1+\frac{(b-2\delta-1)^2N}{|X|-N}\right) \\
&\geq \frac{(b-2\delta-1)^2N(|X|-(b-2\delta)N)}{2\,|X|\,(|X|-N)}.
\end{aligned} \tag{4.11}$$

Finally, we substitute the lower bounds (4.10) and (4.11) for γ_k into (4.7), to acquire upper bounds on the product in (4.4) in the cases $0 \leq k \leq \frac{aN}{M}$ and $\frac{bN}{M} \leq k \leq \min\{N, |Y|\}$ respectively. Moreover, in both cases these upper bounds are independent of k. Thus, by summing the relevant upper bounds over $0 \leq k \leq \frac{aN}{M}$ and $\frac{bN}{M} \leq k \leq \min\{N, |Y|\}$ respectively and additionally applying the bounds on $|Y|$ from (4.1), we establish (4.2) and (4.3). In case of possible future relevance, we point out that the factor $N^{3/2}$ in (4.3) may be replaced by $N^{1/2}\cdot\min\{N, |Y|\}$. This comes from keeping the term $\min\{N, |Y|\}$ when summing over $\frac{bN}{M} \leq k \leq \min\{N, |Y|\}$, rather than bounding it above by N, as we do, for simplicity, to get (4.3). $\square$

Lemma 4.4. *Let $d, n, m \in \mathbb{N}$ and $C, q \in \mathbb{R}$ with*

$$\frac{n}{2(\log n)^q} \leq m \leq \frac{2n}{(\log n)^q}, \qquad C \geq 1 + \frac{2^{d+7}}{\log n}, \qquad \begin{cases} q \geq 1 & \text{if } \frac{C^{1/d}n}{m} \notin \mathbb{Z} \\ q > 0 & \text{if } \frac{C^{1/d}n}{m} \in \mathbb{Z} \end{cases}. \tag{4.12}$$

Let $\mathcal{T}_m$ be defined by (2.1). Then there exists a constant $\Gamma = \Gamma(d) > 0$ such that a random set $S \in \binom{[C^{1/d}n]^d}{n^d}$ satisfies

$$\mathbb{P}\left[\exists T \in \mathcal{T}_m \text{ s.t. } \left|S \cap \left(C^{1/d}n \cdot T\right)\right| \leq \frac{\left(1 - \frac{\Gamma}{\log n}\right) n^d}{m^d}\right] \\ \leq \Gamma n^\Gamma \exp\left(-\frac{(\log n)^{qd-2}}{\Gamma}\right) \tag{4.13}$$

and

$$\mathbb{P}\left[\exists T \in \mathcal{T}_m \text{ s.t. } \left|S \cap \left(C^{1/d}n \cdot T\right)\right| \geq \frac{\left(1 + \frac{\Gamma}{\log n}\right) n^d}{m^d}\right] \\ \leq \Gamma n^\Gamma \exp\left(-\frac{(\log n)^{qd-2}}{\Gamma}\right). \tag{4.14}$$

Proof. In the present proof, Γ will always denote a (large) constant which may depend only on d and whose value is allowed to increase in each occurence. So, to give an example of the use of this convention, we would write the inequality $\Gamma n^d + dn \leq \Gamma n^d$ for $n \in \mathbb{N}$ instead of writing $\Gamma n^d + dn \leq (\Gamma + d)n^d$. Moreover, we point out that it suffices to verify the conclusions (4.13) and (4.14) of the lemma with an additional assumption that n is larger than some threshold depending only on d. The finitely many remaining $n \in \mathbb{N}$ can then be treated by adjusting the constant $\Gamma = \Gamma(d)$ if necessary. Therefore, in the present proof, every inequality involving n should be read with an additional condition that n is sufficiently large, where the sufficiently large condition depends only on d.

Fix $T \in \mathcal{T}_m$ and let $X := [C^{1/d}n]^d$ and $Y := \left(C^{1/d}n \cdot T\right) \cap X$. Then,

$$Cn^d\left(1 - \frac{2^d}{n}\right) \leq |X| \leq Cn^d, \quad \text{and}$$
$$\frac{Cn^d}{m^d}\left(1 - \frac{2^d m}{n}\right) \leq |Y| \leq \frac{Cn^d}{m^d}\left(1 + \frac{2^d m}{n}\right).$$

These inequalities imply

$$\frac{\left(1 - \frac{2^d m}{n}\right)|X|}{m^d} \leq |Y| \leq \frac{\left(1 + \frac{2^{d+2} m}{n}\right)|X|}{m^d}.$$

In the special case that $\frac{C^{1/d}n}{m} \in \mathbb{Z}$, we note that $|Y| = \frac{Cn^d}{m^d} = \frac{|X|}{m^d}$.

Set $N = n^d$, $M := m^d$ and $\delta := \begin{cases} \frac{2^{d+2}m}{n} & \text{if } \frac{C^{1/d}n}{m} \notin \mathbb{Z} \\ 0 & \text{if } \frac{C^{1/d}n}{m} \in \mathbb{Z} \end{cases}$, so that (4.1) is satisfied and

$$\frac{2^{d+1}}{(\log n)^q} \leq \delta \leq \frac{2^{d+3}}{(\log n)^q}, \qquad \text{if } \frac{C^{1/d}n}{m} \notin \mathbb{Z}. \tag{4.15}$$

We apply Lemma 4.3 to δ, N, M, $a := 1 - \frac{2^{d+5}}{\log n}$, $b = 1 + \frac{2^{d+5}}{\log n}$, X and Y. After applying the bounds or substituting the values for the parameters, the probability inequalities (4.2) and (4.3) given by Lemma 4.3 become

$$\mathbb{P}\left[\left|S \cap \left(C^{1/d} n \cdot T\right)\right| \leq \frac{\left(1 - \frac{2^{d+5}}{\log n}\right) n^d}{m^d}\right] \leq \Gamma n^{d/2} (\log n)^{qd} \exp\left(-\frac{(\log n)^{qd-2}}{\Gamma}\right), \quad (4.16)$$

and

$$\mathbb{P}\left[\left|S \cap \left(C^{1/d} n \cdot T\right)\right| \geq \frac{\left(1 + \frac{2^{d+5}}{\log n}\right) n^d}{m^d}\right] \leq \Gamma n^{3d/2} \exp\left(-\frac{(\log n)^{qd-2}}{\Gamma}\right). \quad (4.17)$$

To aid in the verification of (4.16) and (4.17) we list the following utilised bounds on terms from (4.2) and (4.3):

$$\frac{|X| - N}{|X|\left(1 - \frac{2}{M}\right) - N} \leq 2, \quad (4.18)$$

$$\frac{|X| - N}{|X| - (a+\delta)N} \geq \frac{1}{2}, \quad (4.19)$$

$$\frac{|X| - (b - 2\delta)N}{|X| - N} \geq \frac{1}{2}, \quad (4.20)$$

We presently explain how to verify each of the bounds (4.18)–(4.20): For (4.18), first note that

$$\frac{|X| - N}{|X|\left(1 - \frac{2}{M}\right) - N} \leq \frac{(C-1)}{C\left(1 - \frac{2^d}{n}\right)\left(1 - \frac{2}{M}\right) - 1} \leq \frac{C-1}{(C-1) - C\left(\frac{2^d}{n} + \frac{2}{M}\right)},$$

and then observe that

$$\frac{2^d}{n} + \frac{2}{M} < \frac{1}{\log n} < \frac{C-1}{2C}.$$

For (4.19), observe that

$$\frac{|X| - N}{|X| - (a+\delta)N} \geq \frac{C\left(1 - \frac{2^d}{n}\right) - 1}{C - (a+\delta)},$$

and the inequality $\frac{C\left(1 - \frac{2^d}{n}\right) - 1}{C - (a+\delta)} \geq \frac{1}{2}$ is equivalent to

$$C \geq \frac{1 - \frac{(a+\delta)}{2}}{\frac{1}{2} - \frac{2^d}{n}} = 1 + \frac{\frac{2^{d+4}}{\log n} - \frac{\delta}{2} + \frac{2^d}{n}}{\frac{1}{2} - \frac{2^d}{n}},$$

which evidently holds, in light of (4.12) and (4.15). The verification of (4.20) can be done similarly to that of (4.19).

Having established (4.16) and (4.17), we obtain (4.13) and (4.14) by summing (4.16) and (4.17) over $T \in \mathcal{T}_m$ and applying the bound $|\mathcal{T}_m| = m^d \leq \frac{2^d n^d}{(\log n)^{qd}}$. $\square$

5 Proof of Main Results.

To finish this note, we give a proof of Theorems 1.1 and 1.2. In the case of Theorem 1.1, we will actually prove a slightly more general statement which includes dimensions $d = 2$ and $d = 3$. Since this makes the formulation of the statement somewhat more cumbersome, we decided to state Theorem 1.1 in the introduction only for dimensions $d \geq 4$.

Theorem 5.1. *[Slightly stronger than Theorem 1.1] Let $d \in \mathbb{N}$ with $d \geq 2$, (c_n) be a sequence of real numbers $c_n \geq \left(1 + \frac{2^{d+7}}{\log n}\right)^{1/d}$ and $q \in \mathbb{R}$ with $q \geq 1$ and $q > 3/d$. For each $n \in \mathbb{N}$, let the mapping $F_n \colon \binom{\mathbb{Z}^d}{n^d} \to (0, \infty)$ be defined by*

$$F_n(S) = \min \left\{ \mathrm{Lip}(f) \colon f \colon S \to \{1, \ldots, n\}^d \text{ surjective} \right\}, \qquad S \in \binom{\mathbb{Z}^d}{n^d}.$$

Then a random sequence in the probability space $\mathcal{G}_{(c_n)}$, defined by (1.4), (1.5) and (1.6), satisfies

$$\mathbb{P}\left[\limsup_{n \to \infty} F_n(S_n)(\log n)^{-q} < \infty \right] = 1.$$

For the reader's convenience, we repeat the statement of Theorem 1.2:

Theorem 1.2. *Let $d \in \mathbb{N}$ with $d \geq 2$ and (e_n) be a sequence of positive real numbers $e_n \geq 1$. Then there exists a sequence (c_n) of positive real numbers c_n satisfying*

$$|c_n - e_n| \leq \frac{2^{d+8}}{(\log n)^{1/d}}, \tag{1.7}$$

such that the following statement holds: For each $n \in \mathbb{N}$, let the mapping $F_n \colon \binom{\mathbb{Z}^d}{n^d} \to (0, \infty)$ be defined by

$$F_n(S) = \min \left\{ \mathrm{Lip}(f) \colon f \colon S \to \{1, \ldots, n\}^d \text{ surjective} \right\}, \qquad S \in \binom{\mathbb{Z}^d}{n^d}.$$

Then a random sequence in the probability space $\mathcal{G}_{(c_n)}$, defined by (1.4), (1.5) and (1.6), satisfies

$$\mathbb{P}\left[\limsup_{n \to \infty} F_n(S_n)(\log n)^{\frac{-\alpha}{d}} < \infty \right] = 1$$

for all $\alpha > 3$.

Proof of both Theorems 5.1 and 1.2. In this proof we will adopt the same convention with the constant Γ as used in the proof of Lemma 4.4, see the start of the proof of Lemma 4.4 for an explanation. We provide an argument which simultaneously provides a proof of both Theorems 5.1 and 1.2. The objects q and (c_n) should be understood differently, depending on which statement the reader wishes to verify.

Fix $\alpha > 3$. For the proof of Theorem 1.2, we take $q = \frac{\alpha}{d}$. For the proof of Theorem 5.1, q is given by the statement of the theorem.

Set $l_n := \lfloor \log n - q \log\log n \rfloor$ and $m_n := 2^{l_n}$ for all $n \in \mathbb{N}$ starting at a certain threshold so that all expressions make sense. For the finitely many remaining n we define m_n in the same way, but set $l_n = 1$.

For the proof of Theorem 5.1, (c_n) is already defined by the statement of the theorem. For the proof of Theorem 1.2, we define (c_n) as follows: Note that $\frac{n}{m_n} \geq (\log n)^q$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, every interval I of length at least $(\log n)^{-q}$ contains a number $t \in I$ such that $\frac{tn}{m_n} \in \mathbb{Z}$. This allows us to choose a sequence (c_n) with

$$c_n \in \left[e_n + \frac{2^{d+7}}{(\log n)^{1/d}}, e_n + \frac{2^{d+7}+1}{(\log n)^{1/d}} \right] \quad \text{and} \quad \frac{c_n n}{m_n} \in \mathbb{Z}$$

for each $n \in \mathbb{N}$. Observe that this sequence (c_n) satisfies the condition (1.7) of Theorem 1.2.

From this point on, the proof reads identically for both Theorems 5.1 and 1.2. Let $\mathcal{S} = (S_n) \in \mathcal{G}_{(c_n)}$ be a random sequence. It suffices to show that there is a constant $\Upsilon = \Upsilon(d) > 0$ such that

$$\mathbb{P}\left[F_n(S_n) > \Upsilon(\log n)^q\right] \leq \Upsilon n^{-2} \tag{5.1}$$

for all $n \in \mathbb{N}$. Indeed, this will establish that

$$\sum_{n=1}^{\infty} \mathbb{P}\left[F_n(S_n) > \Upsilon(\log n)^q\right] < \infty.$$

The assertion of the theorem is then verified by applying the Borel-Cantelli Lemma.

We set $C_n := c_n^d$. The conditions of Lemma 4.4 are satisfied for d, n, $m = m_n$, $C = C_n$ and q. Applying Lemma 4.4, we deduce that there is a constant $\Gamma = \Gamma(d) > 0$ such that

$$\begin{aligned} \mathbb{P}\left[\frac{\left(1 - \frac{\Gamma}{\log n}\right) n^d}{m_n^d} \leq \left| S \cap \left(C_n^{1/d} n \cdot T \right) \right| \leq \frac{\left(1 + \frac{\Gamma}{\log n}\right) n^d}{m_n^d} \quad \text{for all } T \in \mathcal{T}_m \right] \\ \geq 1 - \Gamma n^{\Gamma} \exp\left(- \frac{(\log n)^{qd-2}}{\Gamma} \right) \geq 1 - \Gamma n^{-2}, \end{aligned} \tag{5.2}$$

for all $n \in \mathbb{N}$. Let $\Lambda = \Lambda(d) > 0$ and $\Delta = \Delta(d, \frac{1}{2})$ be the constants given by the conclusion of Lemma 3.1. Then, combining (5.2) and Lemma 3.1, we conclude that

$$\mathbb{P}\left[F_n(S_n) > \Lambda \exp\left(\log n - l_n \left(1 - \frac{2\Delta\Gamma}{\log n} \right) \right) \right] \leq \Gamma n^{-2},$$

for all $n \in \mathbb{N}$. To finish the proof, it only remains to observe

$$\begin{aligned} \exp\left(\log n - l_n \left(1 - \frac{2\Delta\Gamma}{\log n} \right) \right) &\leq \exp\left(\log n - (\log n - q \log\log n - 1) \cdot \left(1 - \frac{2\Delta\Gamma}{\log n} \right) \right) \\ &= \exp\left(2\Delta\Gamma + (q \log\log n + 1) \left(1 - \frac{2\Delta\Gamma}{\log n} \right) \right) \leq \Upsilon(\log n)^q, \end{aligned}$$

for some constant $\Upsilon = \Upsilon(d) > 0$. ∎

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