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Carmesin, Johannes

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Johannes Carmesin
University of Birmingham

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Abstract

How can sparse graph theory be extended to large networks, where algorithms whose running time is estimated using the number of vertices are not good enough? I address this question by introducing 'Local Separators' of graphs. Applications include:

1. A unique decomposition theorem for graphs along their local 2-separators analogous to the 2-separator theorem;
2. an exact characterisation of graphs with no bounded subdivision of a wheel [9].

1 Introduction

One of the big challenges in Graph Theory today is to develop methods and algorithms to study sparse large networks; that is, graphs where the number of edges is about linear in the number of vertices, and the number of vertices is so large that algorithms whose running time is estimated in terms of the vertex number are not good enough. Important contributions that provide partial results towards this big aim include the following.

1. **Benjamini-Schramm limits of graphs.** Benjamini and Schramm introduced a notion of convergence of sequences of graphs that is based on neighbourhoods of vertices of bounded radius in [6]. Applications of these methods include: testing for minor closed properties [7] by Benjamini, Schramm and Shapira or the proof of recurrence of planar graph limits by Gurel-Gurevich and Nachmias [21].
2. **From Graphons to Graphexes.** Graphons have turned out to be a useful tool to study dense large networks [27, 28]. Motivated by these successes, analogues for sparse graph limits are proposed in [8, 12, 25].

3. **Graph Clustering.** The spectrum of the adjacency matrix of a graph can be used to identify large clusters, see the surveys [38] or [35].

4. **Nowhere dense classes of graphs.** In their book [29], Nešetřil and Ossana de Mendez systematically study a whole zoo of classes of sparse graphs and the relation between these classes.

5. **Refining tree-decompositions techniques.** Empirical results by Adcock, Sullivan and Mahoney suggest that some large networks do have tree-like structure [1]. In [2], these authors say that: 'Clearly, there is a need to develop Tree-Decompositions heuristics that are better-suited for the properties of realistic informatics graphs'. And they set the challenge to develop methods that combine the local and global structure of graphs using tree-decompositions methods.

Much of sparse graph theory – in particular of graph minor theory – is built upon the notion of a separator, which allows to cut graphs into smaller pieces, solve the relevant problems there and then stick together these partial solutions to global solutions. These methods include: tree-decompositions [32], the 2-separator theorem and the block-cutvertex theorem, Seymour's decomposition theorem for regular matroids [37], as well as clique sums and rank width decompositions [30]. Understanding the relevant decomposition methods properly is fundamental to recent breakthroughs such as the Graph Minor Theorem [33] or the Strong Perfect Graph Theorem [34]. As whether a given vertex set is separating depends on each vertex individually. So in the context of large networks it is unfeasible to test whether a set of vertices is separating. We believe that in order to extend such methods from sparse graphs to large networks, it is key to answer the following question. What are local separators of large networks?

Here, we answer this question. Indeed, we provide an example demonstrating that the naive definition of local separators misses key properties of separators. Then we introduce local separators of graphs that lack this defect. Our new methods have the following applications.

A) A unique decomposition theorem for graphs along their local 2-separators analogous to the 2-separator theorem;

B) an exact characterisation of graphs with no bounded subdivision of a wheel. This connects to direction (4) outlined above [9];

C) (work in progress) an analogue of the tangle-tree theorem of Robertson and Seymour, where the decomposition-tree is replaced by a general graph. This connects to direction (5).

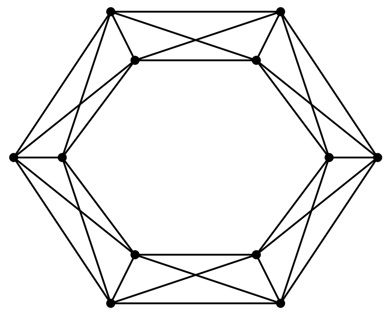

Figure 1: The graph $C_6 \boxtimes K_1$.

Example 1.1. What is the structure of the graph in Figure 1? According to the 2-separator theorem, this graph is 3-connected and hence a basic graph that cannot be decomposed further. In this paper, however, we consider finer decompositions and according to our main theorem, the structure of this graph is: a family of complete graphs K_4 glued together in a cyclic way.

Our results. The 2-separator theorem[1] (in the strong form of Cunningham and Edmonds [13]) says that every 2-connected graph has a unique minimal tree-decomposition of adhesion two all of whose torsos are 3-connected or cycles. We work with the natural extension of 'tree-decompositions' where the decomposition-tree is replaced by an arbitrary graph. We refer to them as 'graph-decompositions'.

Addressing the challenge set by Adcock, Sullivan and Mahoney, our main result is the following local strengthening of the 2-separator theorem.

Theorem 1.2. *For every $r \in \mathbb{N} \cup \{\infty\}$, every connected r-locally 2-connected graph G has a graph-decomposition of adhesion two and locality r such that all its torsos are r-locally 3-connected or cycles of length at most r.*

Moreover, the separators of this graph-decomposition are the r-local 2-separators of G that do not cross any other r-local 2-separator.

A key step in the proof of Theorem 1.2 is the following result, which seems to be of independent interest. This can be seen as a local analogue of the fact that any 2-connected graph that is not 3-connected in which any 2-separator is crossed is a cycle.

[1] See [13, Section 4] for an overview of the history of the 2-separator theorem, see also [37]. An alternative formulation of this theorem in terms of '2-sums' is given in Section 2.

Theorem 1.3. *Let $r \in \mathbb{N} \cup \{\infty\}$ and let G be a connected graph that is r-locally 2-connected. Assume that every r-local 2-separator of G is crossed by an r-local 2-separator. Then G is r-locally 3-connected or a cycle of length at most r.*

Beyond applications (A) to (C) mentioned above, this research includes the following applications.

D) An algorithmic advantage of our main theorem is that the parallel running time of the corresponding algorithm does not depend on the number of vertices of the graph but just on the local structure[2]; and we expect that our novel tool will be useful to study large networks. Indeed, a consequence of Theorem 1.2 is that one can pick the local 2-separators greedily, and all maximal graph-decompositions constructed in that way are essentially the same; in the sense that they contain the minimal graph-decomposition and additionally only have a few insignificant local 2-separators within cycles of length at most r.

E) Covers are important tools in Topology [24] and Group Theory [36, 3]. For covers of graphs, we refer the reader to the book [20] or the recent survey [26]. Recent work includes [4], [5], [15] and [17]. The universal cover of a connected graph G is always a tree and covers G. The *r-local cover*, which is obtained by relaxing all cycles not generated by cycles of length at most r, is covered by the universal cover but covers G. Our r-local 2-separator theorem lifts to the r-local cover of G, characterising the torsos of the 2-separator theorem of the cover as being the torsos of the r-local 2-separator theorem of G.

F) Local tree-decompositions are considered in [18] and [16]. Here (and in the follow-up work [11] for arbitrary local separators), we offer tools to unify such collections of local tree-decompositions to a single graph-decomposition displaying the global structure of the graph.

G) Tree-decompositions have been used to study Cayley graphs of groups and other highly symmetric objects [22, 23]. However, these tools were most helpful for infinite groups as finite groups do not look like trees (roughly speaking). The graph-decompositions we construct here are invariant under the group of automorphisms and we expect that they

[2]Indeed, we prove that for any pair of crossing local 2-separators there must be a cycle of length at most r through their vertices.

can be used as a combinatorial tool to study geometric properties of finite groups.

The remainder of this paper is structured as follows. In Section 2 we give an alternative formulation of Theorem 1.2, and start explaining basic concepts, which we continue in Section 3.

We invite all readers to look at Section 4 just after Section 3. Indeed, in there we prove a local strengthening of the block-cutvertex theorem. This is a straightforward exercise, and it is not used in the rest of the paper. However, we believe it helps to digest the rest of the paper.

In Section 5, we prove an interesting special case of our main result (the parts of the proof that are not needed in our proof of Theorem 1.2 are put into the extended online version [10]). Before proving Theorem 1.3 in Section 7, we do some preparation in Section 6.

In Section 8, we prove Theorem 8.12, which implies Theorem 2.1, a variant of Theorem 1.2. Graph-decompositions are introduced in Section 9, and we conclude this section by deducing Theorem 1.2 from Theorem 8.12. Finally, in Section 10 we discuss directions for further research. To make it easier for the reader to navigate through this paper, we added important definitions of this paper to the 'table of content', allowing readers to hyperlink to them in the pdf via the table of contents. Throughout the paper we fix a parameter $r \in \mathbb{N} \cup \{\infty\}$.

2 Constructive perspective

In this section we give an alternative formulation for Theorem 1.2 and define some basic notions for this paper.

The 2-separator theorem can be stated in the decomposition version (as we did in the Introduction) as well as as the 'constructive version'. For technical reasons we find it easier to work with the constructive version in the proofs and we will deduce the decomposition version in Section 9. We start by explaining the constructive version in this section.

We recall the classical 2-separator theorem in the constructive version in full detail. This theorem has two aspects, the existential statement (which is the easy bit), and the uniqueness statement. The *existential statement* says that every 2-connected graph G can be constructed from 3-connected graphs and cycles via 2-sums [[31],§83]. Clearly, 2-sums commute. Hence this sum is uniquely determined by the set of those summands that are basic; that is, they do not arise as a 2-sum of other summands. We refer to the set of basic summands as a *decomposition* for G. We say that one decomposition for G

is *coarser* (or smarter) than another if it has the same set of 3-connected graphs and its cycles can be build from cycles of the other decomposition (via the implicitly defined 2-sums). The *uniqueness statement* says that there is a decomposition for G with the universal property that it is coarser than any other decomposition for G.

In analogy to 2-sums, we introduce the notion of *r-local 2-sum*. This notion includes the usual 2-sums operation but additionally one is allowed to glue along edges of the same graph – as long as they have distance at least r (roughly speaking). We also introduce local 1-separators and local 2-separators and essentially[3] define that a graph is *locally 2-connected* if it has no local 1-separator; and 'locally 3-connected' is defined analogously. All these terms carry the parameter 'r' that measures how local this is (when the precise value of the parameter is not clear from the context, we shall write 'r-local' in place of just 'local'). The constructive version of Theorem 1.2 is the following.

Theorem 2.1. *Every r-locally 2-connected graph can be constructed via r-local 2-sums from r-locally 3-connected graphs and cycles of length $\leq r$.*

There is such an r-local decomposition with the universal property that it is coarser than any other r-local decomposition for G.

Remark 2.2. As for the classical 2-separator theorem, our local 2-separator theorem has two parts; the first sentence gives the existential statement and the second is the uniqueness statement. The uniqueness statement is more difficult to prove.

We continue by defining some of the basic notions for this paper rigorously. How do we define local cutvertices? Roughly, a vertex should be a local cutvertex if the ball around it gets disconnected after its removal. But which definition of ball do we take? Do we take the definition where we allow edges in the ball joining two vertices of maximum distance or not? Answer: we take both definitions, as formalised in Definition 2.3. Informally, the ball around a vertex consists of all edges and vertices on closed walks of bounded length starting at that vertex.

Definition 2.3. Given a graph G with a vertex v and an integer s, the *ball* of radius s around the vertex v is the induced subgraph of G, whose vertices are those of distance at most s from v and without all edges joining two vertices of distance precisely s. Similarly, given a half-integer $s+\frac{1}{2}$, the *ball*

[3] See Section 3 below for the complete definition.

of radius $s + \frac{1}{2}$ around the vertex v is the induced subgraph of G, whose vertices are those of distance at most s from v. We denote the ball of radius s around v by $B_s(v)$. Below we will often consider the graph $B_s(v) - v$, to which we refer as a *punctured ball*. Given a parameter $r \in \mathbb{N} \cup \{\infty\}$, a vertex v is an *r-local cutvertex* of G if it separates the ball of radius $r/2$ around v; formally: $B_{r/2}(v) - v$ is disconnected.

Lemma 2.4. *Given a parameter $r \in \mathbb{N}$ and a graph, all cycles of the subgraph $B_{r/2}(v)$ are generated by the cycles of length at most r.*

Proof. Construct a spanning tree of $B_{r/2}(v)$ rooted at v so that a vertex with distance $d \leq r/2$ in G from v has distance d in the spanning tree; this can easily be done by induction building the spanning tree layer by layer. Every fundamental cycle of this spanning tree has length at most r (if r is even, note that there is no edge between two vertices of distance $r/2$ from v. And if r is odd note that there are no vertices of distance exactly $r/2$ from v). As every cycle is generated by the fundamental cycles, the cycles of length at most r generate. $\square$

Remark 2.5. The bound r in Lemma 2.4 is sharp as can be seen by considering graphs G that are equal to cycles of length r.

Informally speaking, the '2-sums operation' on graphs can be seen as the inverse operation of cutting along 2-separators and taking torsos. In the following we will introduce a local version of the '2-sums operation' on graphs.

Definition 2.6 (Local 2-sum)**.** Given a family of weighted graphs $(G_i | i \in [n])$ and a set of weighted directed edges e_i of G_i, the *local 2-sum* of this family is the graph obtained from the disjoint union of the set of graphs $\{G_i | i \in [n]\}$ by identifying the start-vertices of the edges e_i, and the terminal vertices of the edges e_i, and then deleting all edges e_i. For this local 2-sum to be valid, it must further satisfy the following condition for each $i \in [n]$. For each $i \in [n]$, we denote by γ_i the length of the shortest path between the endvertices of the edge e_i in the graph $G_i - e_i$. By δ_i we denote the minimum of the values γ_j for $j \neq i$. We now further require that the length of the edge e_i is equal to δ_i.

Remark 2.7. We stress that the graphs G_i just form a family, so some of them may coincide, but the edges e_i form a set, so they must all be distinct. In the disjoint union of the set of graphs G_i we only have one copy for every graph, no matter how often it appears in the family.

Often, we will not explicitly specify a direction of the edges e_i but assume it is given implicitly by the context or just take an arbitrary choice.

We say that a local 2-sum is r*-local* if any pair consisting of two starting-vertices or two terminal vertices, respectively, of edges e_i and e_j that live in the same host graph $G_i = G_j$ have distance at least $r+1$.

Remark 2.8. While in this section we have been working with graphs whose edges are assigned positive integer lengths bounded by r, in the rest of the paper all graphs have no weights on their edges. This is essentially the same; indeed, to get from such a weighted graph to a genuine graph just replace each weighted edge by a path of the same length.

3 Explorer neighbourhood

In this section we define local 2-separators and explain the motivation behind our definition.

The notion of local 1-separators has been explained above. But how should one define local 2-separators? The first thing is that perhaps one only might want to consider pairs of vertices as local 2-separators if they have bounded distance between them. Indeed, otherwise if they were separating we would rather like to think about them as each being a local 1-separator. Okay, so we have a pair (v, w) of vertices of bounded distance that separates their neighbourhood. But how do we define their neighbourhood precisely? Something that looks almost right is just picking one of the vertices arbitrarily and taking a ball around them. More precisely, one could require that $B_{r/2}(v) - v - w$ is disconnected for some parameter r. However, it could be that when we swap v and w then it switches from disconnected to connected. So perhaps the next attempt would be to take $(B_{r/2}(v) \cup B_{r/2}(w)) - v - w$; just to make it symmetric in v and w. Below we will refer to this long expression as the *punctured double-ball*.

The disadvantages of this definition, although almost correct, are more subtle. The main reason is perhaps that with that definition our proofs do not seem to work, as important structural properties are simply not true. Indeed, with this definition Lemma 6.10 (Corner Lemma) does not work. This lemma is a natural generalisation of a lemma for usual separators, and we believe that any natural notion of local separators should have this property. The reason why that lemma is not true in this case is that the double-ball $B_{r/2}(v) \cup B_{r/2}(w)$ may contain cycles that are composed of a path from the ball $B_{r/2}(v)$ and from $B_{r/2}(w)$ but are not a cycle of either of these two balls, see Figure 2.

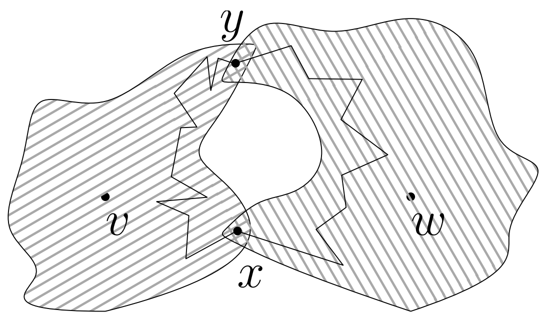

Figure 2: The balls $B_{r/2}(v)$ and $B_{r/2}(w)$ are marked by grey stripes, in rising and falling patters, respectively. Two paths between the vertices x and y, one from either ball, form a cycle that is contained in neither ball.

Informally speaking, the definition we take is similar to the double ball $B_{r/2}(v) \cup B_{r/2}(w)$ and actually agrees with it up to distance $r - d$, where d is the distance between the vertices v and w – but towards the boundary it 'gets more fuzzy'. We will call our notion of neighbourhood 'explorer-neighbourhood' and think about it as follows: imagine two explorers discovering the graph starting from the vertices v and w respectively, with the goal of separately discovering the graph and at the end combining their maps of the balls $B_{r/2}(v)$ and $B_{r/2}(w)$ into a single map. First they discover all shortest paths between the vertices v and w together and put them on the common map. We refer to the set of vertices on these paths as the *core*. Then they return to their respective starting vertices and start exploring the graph from there up to distance $r/2$. On their map they mark each vertex by the set of shortest paths to that vertex from the core (within their respective balls). There may be vertices with distance $r/2$ from the core that have distance at most $r/2$ to the vertex v but a larger distance to the vertex w. Such vertices are only discovered by the explorer based at v. There may also be vertices u discovered by both explorers. However they might not discover a common shortest path to that vertex. In this case there will be two copies of that vertex in the explorer-neighbourhood, while there is only one copy in the double ball $B_{r/2}(v) \cup B_{r/2}(w)$.

Definition 3.1 (Explorer-neighbourhood)**.** Now we give a formal definition of the *explorer-neighbourhood* of parameter r in a graph G with explorers based at the vertices v and w with distance[4] at most $\frac{r}{2}$. The *core* is the set

[4]In this paper the explorer-neighbourhood of vertices v and w of distance more than

of all vertices on shortest paths between the vertices v and w. We take a copy of the ball $B_{r/2}(v)$ where we label a vertex u with the set of shortest paths from the core to u contained in the ball $B_{r/2}(v)$. Similarly, we take a copy of the ball $B_{r/2}(w)$ where we label a vertex u with the set of shortest paths from the core to u contained in the ball $B_{r/2}(w)$. Now we take the union of these two labelled balls – with the convention that two vertices are identified if they have a common label in their sets; that is, there is a shortest path from the core to that vertex discovered by both explorers. (Note that the same vertex x of G could be in both balls but the label sets could be disjoint, see Figure 3. In this case there would be two copies of that vertex in the union. In such a case the union would not be a subgraph of the original graph). We denote the explorer neighbourhood by $\mathrm{Ex_r}(v, w)$. This completes the definition of explorer neighbourhood.

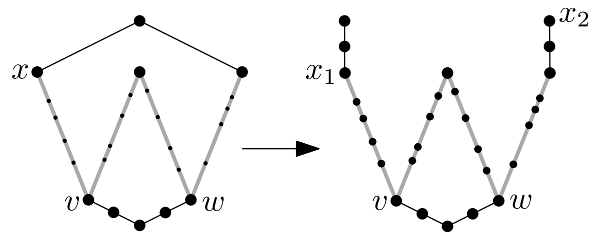

Figure 3: On the right we depicted the explorer-neighbourhood $\mathrm{Ex_r}(v, w)$ of the graph on the left. The value for $r/2$ is seven. Here the grey paths all have length equal to $(r/2) - 2$. The core is just the path of length four between v and w. The cycle of length r is still a cycle in $\mathrm{Ex_r}(v, w)$ since as a cycle it is included in both $B_{r/2}(v)$ and $B_{r/2}(w)$, see Lemma 3.4 for details. The cycle of length $r + 2$ is not contained in one of the balls $B_{r/2}(v)$ or $B_{r/2}(w)$ and hence some of its vertices get two copies in $\mathrm{Ex_r}(v, w)$. Indeed, the vertex x has distance at most r from both vertices v and w. Still it has the two copies x_1 and x_2 in the explorer-neighbourhood.

Lemma 3.2. *Given two vertices a_1 and a_2 of distance at most $r/2$, all vertices on shortest paths between a_1 and a_2 and edges incident with such vertices have unique copies in* $\mathrm{Ex_r}(a_1, a_2)$.

In particular, edges incident with vertices of the core have unique copies in $\mathrm{Ex_r}(a_1, a_2)$.

$\frac{r}{2}$ is undefined; and hence throughout the paper in statements where the explorer-neighbourhood is mentioned we have implicitly the assumption that the involved vertices have distance at most $\frac{r}{2}$.

Proof. By definition vertices on shortest paths between a_1 and a_2 have unique copies in $\mathrm{Ex_r}(a_1, a_2)$. Let e be an edge one of whose endvertices is in the core. If both endvertices of e are in the core, e has a unique copy. Otherwise, the edge e is a shortest path from the core to the other endvertex. Clearly, the edge e is in one of the balls $B_{r/2}(a_1)$ or $B_{r/2}(a_2)$. If it is in both balls, then the two copies of its endvertex must agree as they are both labelled with the edge e.

The 'In particular'-part follows immediately. □

Example 3.3. Lemma 3.2 implies that neighbours of vertices in the core have unique copies 'most of the time'. Here we give an example of a graph where neighbours of vertices in the core do not have unique copies. Let C be a cycle of length $r+1$, where r is an even number. Let a_1 and a_2 be two vertices on C of distance $\frac{r}{2}$. Then the neighbours of a_i with distance $\frac{r}{2}$ to a_{i+1} do not have unique copies in $\mathrm{Ex_r}(a_1, a_2)$; indeed, $\mathrm{Ex_r}(a_1, a_2)$ is a path of length $\frac{3r}{2}$.

Lemma 3.4. *Let o be a cycle (or more generally a closed walk) of length at most r containing vertices a_1 and a_2. Vertices of o have unique copies in $\mathrm{Ex_r}(a_1, a_2)$.*

Proof. Let o be a closed walk as in the statement of the lemma, and let x be an arbitrary vertex on o. Let S be a shortest path from x to the core in the underlying graph (not just some subballs). We will show that S is completely included in both balls $B_{r/2}(a_1)$ and $B_{r/2}(a_2)$. By symmetry, it suffices to show that S is completely included in $B_{r/2}(a_1)$.

For any pair of vertices of the set $\{a_1, a_2, x\}$, pick a shortest path between these vertices. Let o' be the closed walk obtained by concatenating these three paths. Let y be the endvertex of the path S on the core. We can pick, and we do pick, the shortest path between a_1 and a_2 so that it contains the vertex y. Hence the vertex y is on the closed walk o'. As the closed walk o also contains the vertices a_1, a_2 and x, its length is at least that of the closed walk o'; that is, the closed walk o' has length at most r.

Let o'' be the closed walk obtained by concatenating a shortest path from a_1 to x, the path S and a shortest path from the vertex y to a_1. Such a closed walk o'' can be obtained from the closed walk o' by replacing a subwalk from x via a_2 to y by the path S. As S is a shortest path between its endvertices, the length of o'' is at most that of o'; and thus at most r. Hence the closed walk o'' is completely contained within the ball $B_{r/2}(a_1)$ around a_1. Thus the shortest path S is contained in that ball. As S was chosen arbitrarily, every shortest path from x to the core is included in the ball $B_{r/2}(a_1)$. By symmetry, the same is true for 'a_2' in place of 'a_1'. Thus x has a unique copy in the explorer-neighbourhood $\mathrm{Ex_r}(a_1, a_2)$.

□

The balls $B_{r/2}(v)$ and $B_{r/2}(w)$ are embedded within the explorer neighbourhood by construction. We refer to these embedded balls as $\iota(B_{r/2}(v))'$ and $\iota(B_{r/2}(w))'$, or simply $B_{r/2}(v)'$ and $B_{r/2}(w)'$ if the embedding map ι is clear from the context.

Remark 3.5. Every cycle of $\mathrm{Ex_r}(a_1, a_2)$ of length at most r containing one of the vertices a_1 or a_2, say a_1, is a cycle of G. Indeed, it is contained in the ball of radius $r/2$ around a_1 and as such a cycle of G. This can be seen as a converse of Lemma 3.4, and we shall use this observation in various places throughout the paper.

Lemma 3.6. *Every cycle o of the explorer neighbourhood $\mathrm{Ex_r}(v, w)$ is generated from the cycles of the embedded balls $B_{r/2}(v)'$ and $B_{r/2}(w)'$.*

Proof. Each vertex of the cycle o is a vertex of $B_{r/2}(v)'$ or $B_{r/2}(w)'$. We mark it with the respective vertex v or w; and if it is in both, we mark it with both vertices v and w. For each vertex x on the cycle o marked by a vertex $y \in \{v, w\}$, we pick a shortest path from x to the core within the ball $B_{r/2}(y)'$. If a vertex is marked with v and w by the definition of explorer-neighbourhood, then we can assume, and we do assume, that we picked the same path for $y = v$ and $y = w$.

Now for each edge $e \in o$ we construct a closed walk o_e as follows. Start with e and the two paths chosen at either endvertex of e, then join their endvertices in the core by a path within the core (which is connected by construction). Since for each edge e of o, there is a mark $y \in \{v, w\}$ that is present at both endvertices of edge e, the closed walk o_e is contained in $B_{r/2}(v)'$ or $B_{r/2}(w)'$.

Our aim is to generate the cycle o from cycles of $B_{r/2}(v)'$ and $B_{r/2}(w)'$. For that we first add to o the sum of all the cycles o_e ranging over all $e \in o$ (taken over the binary field $\mathbb{F}_2$). This sum takes only non-zero entries at edges of the core. As the core is a subset of $B_{r/2}(v)' \cap B_{r/2}(w)'$, the remainder is generated from the common cycles of $B_{r/2}(v)'$ and $B_{r/2}(w)'$. □

Definition 3.7 (Local separators)**.** Given a graph G with distinct vertices v and w, we say that the set $\{v, w\}$ is an *r-local 2-separator* if the punctured explorer-neighbourhood $\mathrm{Ex_r}(v, w) - v - w$ is disconnected, and the vertices v and w have distance at most $r/2$ in the graph G.

A connected graph is *r-locally 2-connected* if it does not have an r-local cutvertex and it has a cycle of length at most r. So there are no r-locally 2-connected graphs for $r < 3$. A graph is *r-locally 2-connected* if all its components are r-locally 2-connected.

A connected r-locally 2-connected graph is *r-locally 3-connected* if it does not have an r-local 2-separator and it has at least four vertices. A graph is *r-locally 3-connected* if all its components are r-locally 3-connected.

Example 3.8. A cycle of length $r+1$ is not r-locally 2-connected. Moreover if it has more than three edges, any of its vertices together with any of its neighbours forms an r-local 2-separator.

Cycles of lengths at most r are r-locally 2-connected and not r-locally 3-connected if they have at least four edges.

Lemma 3.9. *An r-locally 3-connected graph G is r-locally 2-connected.*

Proof. Assume that the graph G is connected, has at least four vertices and contains a cycle. We are to show that if G has an r-local cutvertex, then G has an r-local 2-separator. So let x be an r-local cutvertex. If x is not contained in any cycle, x is a genuine cutvertex of the graph G. As the graph G is not a star by assumption, the vertex x has a neighbour so that x together with that neighbour is a 2-separator of G. So G has an r-local 2-separator.

So we may assume that x is contained in a cycle o. Let y be a neighbour of the vertex x on the cycle o. We claim that $\{x, y\}$ is an r-local 2-separator. Let C be a component of the punctured ball $B_{\frac{r}{2}}(x) - x$ that does not contain the vertex y. Let W be the set of edges incident with the vertex x and the other endvertex in the component C. Let z be a neighbour of x in C.

Suppose for a contradiction that the punctured explorer-neighbourhood $\mathrm{Ex_r}(x, y) - x - y$ is connected. Then in particular, $\mathrm{Ex_r}(x, y) - x$ is connected. So there is a path P in there from y to z. Then $P + xz$ is a cycle traversing the edge set W precisely once. By Lemma 2.4 and Lemma 3.6 $P + xz$ is generated by cycles of length at most r over $\mathbb{F}_2$; hence one of these cycles intersects the edge set W oddly. In particular, this cycle contains the vertex x, so it is a cycle of the ball $B_{\frac{r}{2}}(x)$. So we found a cycle of $B_{\frac{r}{2}}(x)$ that intersects the cut W oddly. This is a contradiction. Hence $\mathrm{Ex_r}(x, y) - x - y$ is disconnected; and so $\{x, y\}$ is an r-local 2-separator. □

In a sense the next lemma says that local 2-components sitting at a local 2-separator are local (in that they contain a short path between the neighbours of the two separating vertices).

Lemma 3.10 (Local 2-Connectivity Lemma)**.** *Let $\{v,w\}$ be an r-local 2-separator in an r-locally 2-connected graph G. For every connected component k of the punctured explorer-neighbourhood $\mathrm{Ex_r}(v,w)-v-w$, there is a cycle o' of length at most r containing the vertices v and w, and o' contains a vertex of the component k and o' contains an edge incident with v whose other endvertex is a vertex not in k.*

Proof. Let $k=k_1$ be an arbitrary component of the punctured explorer-neighbourhood $\mathrm{Ex_r}(v,w)-v-w$, and let k_2 be the union of all other components of the punctured explorer-neighbourhood $\mathrm{Ex_r}(v,w)-v-w$, which is nonempty as $\{v,w\}$ is a local 2-separator. If one component of $\mathrm{Ex_r}(v,w)-v-w$ had only one of the vertices v and w in its neighbourhood, then that vertex would be a local cutvertex. However, this is not possible as G is r-locally 2-connected by assumption. Hence all components of $\mathrm{Ex_r}(v,w)-v-w$ have both vertices v and w in their neighbourhood. In particular, the vertex v is adjacent to vertices of k_1 and k_2.

Let x_i be an arbitrary neighbour of the vertex v in k_i (for $i=1,2$). As the graph G is r-locally 2-connected, the vertex v is not a cutvertex of the ball $B_{r/2}(v)$. So there is a path P included in that ball from x_1 to x_2 avoiding v. Let o be the cycle obtained from P by adding the vertex v. By Lemma 2.4, the cycle o is generated from cycles of the ball $B_{r/2}(v)$ of length at most r. Consider the set $\mathcal{C}$ of these cycles that contain the vertex v. As o has precisely one edge to k_1 incident with v, there must be a cycle o' in $\mathcal{C}$ that contains an odd number of edges to k_1 incident with v. As the cycle o' has maximum degree two, it contains precisely one edge to k_1 incident with v. The other edge of o' incident with v has its other endvertex in k_2+w. This completes the proof. □

Remark 3.11. The bound r for the cycle o' in Lemma 3.10 (Local 2-Connectivity Lemma) is best possible as can be seen by considering graphs that are a single cycle of length r. The cycle o' in Lemma 3.10 is not only a cycle in $\mathrm{Ex_r}(v,w)$ but also in G by Remark 3.5.

Remark 3.12. The cycle o' of Lemma 3.10 (Local 2-Connectivity Lemma) contains an edge incident with v whose other endvertex is not in k; that is, this other endvertex is equal to the vertex w or else in a component of $\mathrm{Ex_r}(v,w)-v-w$ different from k.

Remark 3.13. The notion of the explorer-neighbourhood is crucial in the proof of Lemma 6.15 (Projection Lemma) and Lemma 6.10 (Corner Lemma) below. This is explained in detail in Remark 6.21 and Remark 6.7 below.

Remark 3.14. Above we said the explorer-neighbourhood and the double-ball 'almost lead' to the same notion of local 2-separator. This can be quantified as follows. If the punctured explorer-neighbourhood is connected, then so is the punctured double ball. If the punctured double ball of radius $r/2$ around two vertices of distance at most d is connected, then the punctured explorer-neighbourhood of radius $(r/2) + d$ is connected.

4 Intermezzo: Block-Cutvertex Graphs

The results of this section are not applied in the rest of the paper but they can be seen as a toy case for the main result. In this section we prove a generalisation of the block-cutvertex theorem allowing for r-local cutvertices, which generalise cutvertices (indeed, the r-local cutvertices for $r = \infty$ are precisely the cutvertices). See Section 2for a defintion of r-local cutvertices and Section 9 for a definition of graph-decompositions.

It seems to us that the most natural generalisation of the block-cutvertex theorem to this context is the following.

Theorem 4.1. *Given $r \in \mathbb{N} \cup \{\infty\}$, every connected graph has a graph-decomposition of adhesion one and locality r such that all its bags are r-locally 2-connected or single edges.*

Remark 4.2. The strengthening of Theorem 4.1 with 'bags are r-locally 2-connected' replaced by 'bags are r-locally 2-connected subgraphs' is not true. An example is given in Figure 8.

As a preparation for the proof of Theorem 4.1, we investigate the operation of locally cutting vertices, defined as follows.

Given a parameter $r \geq 1$ and a graph G with a vertex v, the graph obtained from G by *r-locally cutting* the vertex v is defined as follows. Let X be the set of connected components of the ball of radius r around v with v removed; formally X is the set of components of the graph $B_{r/2}(v) - v$. Define a new graph from G by replacing the vertex v by one new vertex for each element of the set X, where the vertex labelled with $x \in X$ inherits the incidences with those edges incident with v that are incident with a vertex of the connected component X. We refer to the new vertices as the *slices of v*. This completes the construction of the r-local cutting of G.

Observation 4.3. *Let G' be obtained from G by r-locally cutting a vertex v into a set X of new vertices. Then in the graph G', no vertex $x \in X$ is an r-local cutvertex.* $\square$

The next lemma says that r-local cuttings commute.

Lemma 4.4. *Given a graph G with vertices v and w, first r-locally cutting v and then w results in the same graph as first locally cutting w and then v.*

Proof. Consider the graph G' obtained from G by r-locally cutting the vertex v. We denote the ball of radius $r/2$ around the vertex w in the graph G by $B_{r/2}(w)$, and by $B'_{r/2}(w)$ we denote the ball of radius $r/2$ around the vertex w in the graph G'.

In the graphs G and G', the vertex w has the same neighbours. Indeed, if v and w are not adjacent, this is immediate. Otherwise w is adjacent with a unique slice of v in G', and in the following we will suppress a bijection between the vertex v and this particular slice of v – in order to simplify notation. With this notation at hand, we next prove the following.

Sublemma 4.5. *Two neighbours x and y of w are in the same connected component of $B_{r/2}(w) - w$ if and only if they are in the same connected component of $B'_{r/2}(w) - w$.*

Proof. If x and y are in the same connected component of $B'_{r/2}(w) - w$, they are joined by a path in that graph and this path is also is a path (or a walk) in the graph $B_{r/2}(w) - w$.

Hence conversely assume that x and y are vertices of the same connected component of the ball $B_{r/2}(w) - w$. Let P be a path between these two vertices in the graph $B_{r/2}(w) - w$. Then this path P together with the vertex w forms a cycle, which we denote by o. By Lemma 2.4, the cycle o is generated by cycles of length at most r in the graph $B_{r/2}(w) - w$.

If one of these cycles does not include the vertex v, then it is also a cycle in the graph $B'_{r/2}(w)$. Otherwise, such a cycle is also a cycle completely contained with in the ball $B_{r/2}(v)$ around v in G. In particular this cycle witnesses that the two neighbours on that cycle adjacent to v are in the same connected component of $B_{r/2}(v) - v$. Thus these two neighbours are neighbours of the same slice of the vertex v in G'. Hence this cycle is also a cycle in G' and hence in the ball $B'_{r/2}(w)$. To summarise, all those cycles of length at most r that generate o are cycles in $B'_{r/2}(w)$. In the ball $B'_{r/2}(w)$ they generate (the edge set of) o. So o is an eulerian subgraph in $B'_{r/2}(w)$, and so a cycle as it cannot have a vertex of degree strictly more than two and it is connected. In particular the vertices x and y are in the same connected component of the punctured ball $B'_{r/2}(w) - w$. $\square$

It is a direct consequence of Sublemma 4.5 that cutting locally commutes. $\square$

Lemma 4.6. *Let G be a connected graph. Let G' be obtained from G by r-locally cutting all r-local cutvertices of G. Then G' is r-locally 2-connected.*

Proof. First we remark that the graph G' is well-defined by Lemma 4.4. Let $v_1, v_2, ..., v_n$ be an enumeration of the vertices of G. Here we stress that we include vertices in this enumeration that are not r-local cutvertices; and cutting them does not change the graph at all. We may assume by Lemma 4.4 that we obtain G' from G by first cutting v_1, then v_2, etc., so that in the final step we cut the vertex v_n. By Observation 4.3, all slices of the vertex v_n are not r-local cutvertices. As cutting locally commutes by Lemma 4.4, we can argue the same for any other ordering of the vertices of G. Hence no vertex of the graph G' is an r-local cutvertex. □

Proof of Theorem 4.1. Let $r \in \mathbb{N} \cup \{\infty\}$ be a parameter. Let G be a connected graph. We construct the graph H from G by r-locally cutting all r-local cutvertices of G. By Lemma 4.4 this is well-defined, and the graph H is r-locally 2-connected by Lemma 4.6.

Let S be the set of r-local cutvertices of G. Let B be the set of connected components of the graph H. We define a bipartite graph with bipartition (B, S), where we add one edge between an r-local cutvertex s of G to a connected component k of the graph H for every slice of s that is contained in k. The map associated to that edge map the singleton subgraph s to its corresponding slice. We set $G_s = s$ and $G_b = b$ for $s \in S$ or $b \in B$, respectively.

This defines a graph-decomposition of adhesion one and locality r all of whose bags are r-locally 2-connected; compare Section 9 for definitions. It is straightforward to check that the underlying graph of that graph-decomposition is the graph G. □

5 The existential statement of the local 2-separator theorem

In this section, we prove the lemmas necessary to deduce the existential statement of the local 2-separator theorem; that is, the first sentence of Theorem 2.1.

Definition 5.1 (Local cutting)**.** Given a graph G with an r-local 2-separator $\{v_0, v_1\}$, the graph obtained from G by *r-locally cutting* $\{v_0, v_1\}$ is defined as follows. Let X be the set of connected components of the punctured explorer-neighbourhood $\mathrm{Ex_r}(v_0, v_1) - v_0 - v_1$. We now replace in the graph

G the vertices v_0 and v_1 each by one copy for every element of X. Here a copy of v_i labelled by some $x \in X$ inherits an edge from v_i if the other endvertex of that edge is a vertex of the component x. We refer to the newly added vertices as the *slices* of the vertices v_1 or v_2, respectively. We additionally add a weighted edge between any two slices for the same $x \in X$. Its weight is given by the minimum length of a path between v_0 and v_1 in the explorer-neighbourhood $\mathrm{Ex}_\mathrm{r}(v_0, v_1)$ with the component x removed. It follows that all but one of these weights are always the same. We refer to these additional edges as *torso edges*. If the vertices v_0 and v_1 share an edge e in G, we add a new connected component consisting of the edge e and one edge in parallel to e. This other edge is a torso edge and its length is the minimum length of a path between v_0 and v_1 in the explorer-neighbourhood $\mathrm{Ex}_\mathrm{r}(v_0, v_1)$ minus e. Finally, we replace each torso edge by a path of the same length; we refer to such paths as *torso-paths*[5]. This completes the definition of local cutting, see Figure 4 for an example.

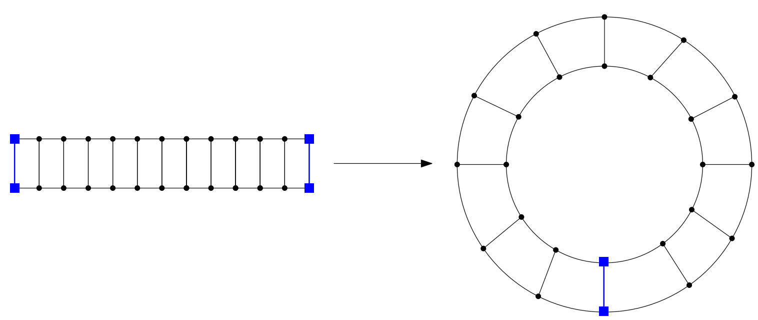

Figure 4: The graph on the left is obtained from the graph on the right by locally splitting at the local 2-separator given by the blue edge. This blue edge gets two copies on the left, one for each local component. In Section 9 we shall investigate the inverse operation of local cutting.

Remark 5.2. All edges incident with the vertices v_0 or v_1 are inherited by a unique slice except for possibly an edge between v_0 and v_1, which is in this artificial component of size two. If an edge between v_0 and v_1 in G is a shortest path between these vertices, its length is the length of all torso edges.

[5]This technical step reduces technicalities elsewhere; indeed, the explorer-neighbourhood is not defined for weighted graphs, and doing so would lead to technicalities.

Lemma 5.3. *Slices of the same vertex have distance at least $r + 1$.*

And slices for the same component have distance at most $r/2$ in a graph obtained from an r-locally 2-connected graph by local cutting.

Proof. Firstly, suppose for a contradiction that there was a path P of length at most r joining two slices of a vertex v. Pick P so that no interior vertex is a slice of v. Then all vertices on the path P have distance at most $r/2$ from v in the original graph. So the path P projects to a closed walk completely contained in the ball $B_{\frac{r}{2}}(v)$. Hence all interior vertices of P are in the same local component in any explorer-neighbourhood around v and some other vertex w. So we get a contradiction to the assumption that P joins two different slices of the vertex v.

Secondly, let $\{a_1, a_2\}$ be an r-local 2-separator of a graph G, and let a_1' and a_2' be slices for a local component k in the graph G' obtained from G by local cutting. As G is r-locally 2-connected, by Lemma 3.10 (Local 2-Connectivity Lemma) there is a cycle of length at most r of G through the vertices a_1 and a_2 that contains a vertex of the local component k. This cycle is a cycle of G by Remark 3.5. In the graph G' there is a cycle o' obtained from o by possibly replacing one of its subpaths from a_1 to a_2 by a torso path. The cycle o' has the same length as o and contains the vertices a_1' and a_2', as it contains their neighbours from k. So a_1' and a_2' have distance at most $r/2$. □

Lemma 5.4. *Let G' be obtained from a graph G by r-locally cutting a local 2-separator $\{v, w\}$. Then G can be obtained from G' by r-local sums.*

Proof. The family of graphs for the local sum consists of copies of the graph G', one copy for each component of the punctured explorer-neighbourhood $\mathrm{Ex_r}(v, w) - v - w$, together with the artificial component of size two if vw is an edge of G. We move from G' to a weighted graph by replacing the torso paths by the torso edges of the cutting. Now we take an r-local 2-sum where the gluing edges are the torso edges.

It follows directly from the definitions of local cutting and local sums that the graph G is equal to the graph obtained from G' by applying the local 2-sum as described above. This local sum is r-local by Lemma 5.3. □

Lemma 5.5. *Let G' be a graph obtained from an r-locally 2-connected graph G by r-locally cutting a local 2-separator. Then the graph G' is r-locally 2-connected.*

Proof. By Lemma 3.10 (Local 2-Connectivity Lemma), every connected component of the graph G' contains a cycle of length at most r. So it remains to show that there are no r-local cutvertices. Let v be an arbitrary vertex of the graph G'. We distinguish two cases.

Case 1: the vertex v is a slice. We denote the local 2-separator of G at which we cut by $\{a, b\}$. We may assume, and we do assume, that the vertex v is a slice of the vertex a. Let X denote the set of components of the punctured explorer-neighbourhood $\mathrm{Ex_r}(a, b) - a - b$. Recall that the ball $B_{r/2}(a)$ of radius $r/2$ around a in the graph G is a naturally embedded subgraph of the explorer-neighbourhood $\mathrm{Ex_r}(a, b)$ (and we shall suppress this natural embedding). For each component $x \in X$, we let H_x be the intersection of the punctured ball $B_{r/2}(a) - a$ with the component x. Note that b has distance at most $r/2$ from a in G by Lemma 3.10 (Local 2-Connectivity Lemma). The punctured ball $B_{r/2}(a) - a$ is obtained by taking the union of the graphs H_x and adding the vertex b together with its incident edges. As this punctured ball is connected by assumption, all graphs H_x must have the vertex b in their neighbourhood and all graphs $H_x + b$ must be connected. The punctured ball $B_{r/2}(v) - v$ around v in G' is $H_y + b$, where y is the component of $\mathrm{Ex_r}(a, b) - a - b$ that belongs to the slice v. So $B_{r/2}(v) - v$ is connected. Thus the vertex v is not an r-local cutvertex. This completes Case 1.

Case 2: the vertex v is not a slice. Then the vertex v is a vertex of the graph G.

Suppose for a contradiction that the punctured ball $B_{r/2}(v) - v$ around v of radius $r/2$ in the graph G' is disconnected. Let w_1' and w_2' be two arbitrary neighbours of v in G' in different components of that punctured ball. Let w_1 and w_2 be the vertices of the graph G from which the vertices w_1' and w_2' are slices of or that are equal to them, respectively. Then the vertices w_1 and w_2 are adjacent to the vertex v in the graph G by the definition of local cutting. As the punctured ball $B_{r/2}(v) - v$ of radius $r/2$ around the vertex v in the graph G is connected by assumption, there is a path P within that punctured ball from w_1 to w_2. This path together with the vertex v is a cycle o within that ball. So by Lemma 2.4 this cycle is generated by cycles within that ball of length at most r.

Let W' be the set of neighbours of the vertex v in the graph G' in the component of the punctured ball containing the vertex w_1'. Let W be the set of vertices of the graph G that are equal to vertices in W' or that have slices in the set W'. By $E(W)$ we denote the set of edges in the graph G from v to a vertex in W.

By construction the cycle o contains precisely one edge from the set $E(W)$. Hence there must be a cycle $\hat{o}$ of G from the generating set that contains an odd number of edges from $E(W)$. As $\hat{o}$ has maximum degree two, it contains precisely one edge from the set $E(W)$.

We denote the local 2-separator of G at which we locally cut by $\{a, b\}$.

Case 2A: the cycle $\hat{o}$ does not contain any of the vertices a or b. Then $\hat{o} - v$ is a path in the graph G' from a vertex of W' to a neighbour of the vertex v outside W'. This is a contradiction to the assumption that the punctured ball is disconnected. This completes this case.

Case 2B: the cycle $\hat{o}$ contains one of the vertices a or b. If $\hat{o}$ was a cycle of the graph G', then we would get the desired contradiction as the path $\hat{o} - v$ would join two vertices in different components of the punctured ball around v in G'. So assume that this is not the case. Then $\hat{o}$ contains both vertices a and b. Let Q be the a-b-subpath of $\hat{o}$ containing v. Then Q plus a torso edge between two slices of a and b is a cycle of G' whose length is no longer than the length of $\hat{o}$. Denote this cycle by o'. Then $o' - v$ joins two vertices in different components of the punctured ball around v in G', a contradiction. This completes Case 2, and hence the whole proof. □

Remark 5.6. In [10] we give an alternative proof of the first sentence of Theorem 2.1 that only relies on lemmas of the paper proved up to this point. We encourage the reader to look at this proof next.

6 Properties of local 2-separators

In this section we prove some lemmas that are used in our proof of Theorem 1.3 and Theorem 1.2.

A *cut* is the set of edges between a bipartition of the vertex set. The bipartition classes are referred to as the *sides* of the cut.

Lemma 6.1. *Let Y be a cut in a graph G. Then the endvertices of a path P are on the same side of Y if and only if P intersects Y evenly.*

Proof: by induction on the length of the path P. □

Definition 6.2 (Crossing)**.** Given an r-local 2-separator $\{v, w\}$ and a pair of vertices $\{a, b\}$ of the explorer-neighbourhood $\mathrm{Ex_r}(v, w)$, we say that $\{a, b\}$ *pre-crosses* $\{v, w\}$ if a and b are in different components of the punctured explorer-neighbourhood $\mathrm{Ex_r}(v, w) - v - w$. And $\{a, b\}$ *crosses* the r-local 2-separator $\{v, w\}$ if it pre-crosses it and there is a cycle of length at most r

in $\mathrm{Ex_r}(v,w)$ through a and b in the explorer-neighbourhood; note that this cycle contains v and w[6].

We say that a pair $\{a,b\}$ of (distinct) vertices of G *crosses* a local 2-separator $\{v,w\}$ of G if there exist copies a' and b' of a and b in the explorer-neighbourhood $\mathrm{Ex_r}(v,w)$, respectively, so that $\{a',b'\}$ crosses $\{v,w\}$.

Remark 6.3. If $\{a,b\}$ is a local separator in an r-locally 2-connected graph, then the existence of a cycle o of length at most r through a and b is guaranteed by Lemma 3.10 (Local 2-Connectivity Lemma). Hence 'crossing' essentially means 'pre-crossing' plus the crossing vertices are 'near' to the local separator. Phrasing being 'near' in terms of this cycle seems particularly natural in view of Lemma 6.5 (Alternating Cycle Lemma) and Corollary 6.6 below.

Definition 6.4 (Alternating cycle)**.** Given two disjoint sets A_1 and A_2, we say that a cyclic ordering *alternates* between A_1 and A_2 if it has even length and each element of the cyclic ordering in A_i has its two neighbours in A_{i+1} (for $i\in\mathbb{F}_2$).

A *pre-alternating cycle* is a cycle o together with two local 2-separators $\{a_1,a_2\}$ and $\{b_1,b_2\}$ such that the order in which these four vertices appear on the cycle o alternates between the two local separators (i.e., it is $a_1b_1a_2b_2$ or its reverse $a_1b_2a_2b_1$). An *alternating cycle* is a pre-alternating cycle o such that the two neighbours of a_i on o are in different components of $\mathrm{Ex_r}(a_1,a_2)-a_1-a_2$ for $i=1,2$, and analoguously the two neighbours of b_i on o are in different components of $\mathrm{Ex_r}(b_1,b_2)-b_1-b_2$ for $i=1,2$. Below sometimes it will be more convenient to refer to this situation by saying that the cycle o *alternates between* the local 2-separators $\{a_1,a_2\}$ and $\{b_1,b_2\}$, see Figure 5.

The 'nearness' condition in the definition of 'crossing' is equivalent to the following stronger property.

Lemma 6.5 (Alternating Cycle Lemma)**.** *Let $\{a_1,a_2\}$ and $\{b_1,b_2\}$ be r-local 2-separators in an r-locally 2-connected graph G. The following are equivalent:*

1. *$\{a_1,a_2\}$ crosses $\{b_1,b_2\}$;*

[6] We remark that this definition is a little subtle, as there may well be vertices a and b of G that lie on a common cycle of G of length at most r and that are in different components of

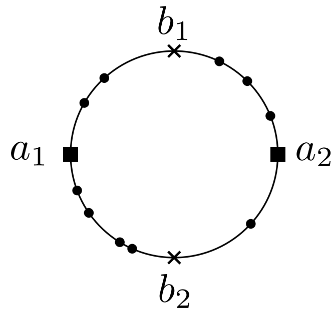

Figure 5: An alternating cycle. The vertices a_1 and a_2 of the first local 2-separator are indicated by boxes, the vertices b_1 and b_2 of the second local 2-separator are indicated by crosses. The cyclic order of the cycle induced on these four vertices alternates between the two local separators.

2. *there is a cycle of G of length at most r alternating between $\{a_1, a_2\}$ and $\{b_1, b_2\}$;*

3. *every cycle of length at most r of G containing a_1 and a_2 alternates between $\{a_1, a_2\}$ and $\{b_1, b_2\}$;*

4. *every cycle of length at most r of G containing a_1 and a_2 pre-alternates between $\{a_1, a_2\}$ and $\{b_1, b_2\}$ and the neighbours of b_1 on o are in different components of* $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$.

Proof. To see that (3) implies (2) note that by r-locally 2-connectivity there is a cycle of length at most r containing a_1 and a_2 by Lemma 3.10 (Local 2-Connectivity Lemma); and it is a cycle of G by Remark 3.5.

To see that (2) implies (1), let o be a cycle alternating between $\{a_1, a_2\}$ and $\{b_1, b_2\}$. By Lemma 3.4, o is a cycle of $\mathrm{Ex_r}(b_1, b_2)$ and all its vertices have unique copies in $\mathrm{Ex_r}(b_1, b_2)$. For simplicity, we suppress a map between the vertices a_i and their unique copies in $\mathrm{Ex_r}(b_1, b_2)$. As o is alternating, the vertices a_1 and a_2 are in different components of the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$.

Next we show that (1) implies (4). For this assume that the vertices a_1 and a_2 have copies a_1' and a_2', respectively, in different components of the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$ such that there is a cycle o of $\mathrm{Ex_r}(b_1, b_2)$ of length at most r containing a_1' and a_2'. So the cycle o has to intersect the 2-separator $\{b_1, b_2\}$ of $\mathrm{Ex_r}(b_1, b_2)$ in both its vertices. Thus by Lemma 3.4, all vertices on o have unique copies in $\mathrm{Ex_r}(b_1, b_2)$. For simplicity, we suppress a map between the vertex a_i and its unique copy a_i' in $\mathrm{Ex_r}(b_1, b_2)$. We have shown that in G there is an a_1-a_2-path P of length at most $r/2$ containing one of the vertices b_i, say b_1.

Now let u be an arbitrary cycle of length at most r containing a_1 and a_2. Let Q be a subpath of G of that cycle from a_1 to a_2 of length at most $r/2$. The closed walk PQ has length at most r. As it contains the vertex b_1, by Remark 3.5 it is a closed walk in $\mathrm{Ex_r}(b_1, b_2)$. Hence the path Q traverses the separator $\{b_1, b_2\}$. So it contains a vertex b_i. So by Remark 3.5, the cycle u is a cycle of $\mathrm{Ex_r}(b_1, b_2)$. It follows that the cycle u pre-alternates between the local 2-separators $\{a_1, a_2\}$ and $\{b_1, b_2\}$. As a_1 and a_2 are in different components of $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$, this completes the proof that (1) implies (4).

Finally, we show how (4) implies (3). Let u be a cycle length at most r of G containing a_1 and a_2. By (4) it pre-alternates between $\{a_1, a_2\}$ and $\{b_1, b_2\}$ and the neighbours of b_1 on o are in different components of $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$. By Lemma 3.4, the vertices b_1 and b_2 have unique copies in $\mathrm{Ex_r}(a_1, a_2)$, and we suppress a map between the vertex b_i and its unique copy in $\mathrm{Ex_r}(a_1, a_2)$. It suffices to show that the vertices b_1 and b_2 are in different components of $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$.

Suppose for a contradiction that they are in the same component. By Lemma 3.10 (Local 2-Connectivity Lemma), there is a cycle c of length at most r in $\mathrm{Ex_r}(a_1, a_2)$ containing a_1 and a_2 such that one of its a_1-a_2-subpaths does not contain a vertex b_i. By Remark 3.5, c is a cycle of G. Let R be an a_1-a_2-subpath of c of length at most $r/2$. Then PR is a closed walk of G of length at most r. As it contains the vertex b_1, it is a cycle of $\mathrm{Ex_r}(b_1, b_2)$. So R contains a vertex b_i. So by Remark 3.5, c is a cycle of $\mathrm{Ex_r}(b_1, b_2)$. As the vertices a_1 and a_2 are in different components of $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$, every subpath of c has to contain a vertex b_i, a contradiction to the choice of c. Hence the vertices b_1 and b_2 are in different components of $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$. So every cycle u alternates between $\{a_1, a_2\}$ and $\{b_1, b_2\}$. □

The next lemma essentially says that crossing is a symmetric relation on r-local 2-separators.

Corollary 6.6. *Let G be an r-locally 2-connected graph with two r-local 2-separators $\{a_1, a_2\}$ and $\{b_1, b_2\}$. If $\{a_1, a_2\}$ crosses $\{b_1, b_2\}$, then $\{b_1, b_2\}$ crosses $\{a_1, a_2\}$.*

Furthermore the punctured explorer-neighbourhood $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$ *has precisely two components.*

Proof. For the first part assume that $\{a_1, a_2\}$ crosses $\{b_1, b_2\}$. This is equivalent to the symmetric condition (2) in Lemma 6.5 (Alternating Cycle Lemma), so it follows that $\{b_1, b_2\}$ crosses $\{a_1, a_2\}$.

To see the 'Furthermore'-part, by condition (3) of Lemma 6.5 (Alternating Cycle Lemma) every cycle c of length at most r in $\mathrm{Ex_r}(a_1, a_2)$ containing a_1 and a_2 has to contain a vertex b_i on each of its a_1-a_2-subpaths. By the contrapositive of Lemma 3.10 (Local 2-Connectivity Lemma), we conclude that every component of $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$ contains a vertex b_i. So $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$ has at most two components, and it has exactly two as $\{a_1, a_2\}$ is a local separator. $\square$

Remark 6.7. A key-feature of separators in graphs is the 'Corner Property[7]'. In the classic version for 2-separators, the Corner-Lemma says that if two 2-separators $\{a_1, a_2\}$ and $\{b_1, b_2\}$ cross, then $\{a_1, b_1\}$ is a 2-separator – under certain non-triviality conditions. We shall prove that this property also holds for our local 2-separators. This is in fact a central lemma of the paper and the notion of 'explorer-neighbourhood' is key to this lemma.

Intuitively speaking, the reason why this lemma is true is the following. As the Corner-Lemma is true for separators in the classical version, the only reason why a local version could break is essentially if one of the involved vertices, say a_1, would explore a new path around the local separator $\{b_1, b_2\}$ to the other component of the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$ that was not known to b_1 or b_2. If we used 'double balls' instead of our local notion of 'explorer-neighbourhoods', this could well happen, see Figure 6 below for an example. The intuition now is that a_1 may well 'explore' a new path to the other local component but when the explorers compare their maps, they have given the things different names and so the explorers do not realise that between them they know a path around. Hence they believe that the corner $\{a_1, b_1\}$ is separating. That is how we think about locally separating: the explorers cannot prove that there is a way round with their local information.

This is somewhat similar to the following situation. Imagine you are running on a graph and at any point in time you can only see your neighbours. If the graph is a cycle, you cannot tell its length – and you even cannot distinguish it from the 2-way-infinite path $\mathbb{Z}$.

Now we start setting up some notation for Lemma 6.10 below.

Definition 6.8 (Corner-Setting)**.** Given an r-local separator $\{a_1, a_2\}$ crossing an r-local 2-separator $\{b_1, b_2\}$, by Corollary 6.6 and Lemma 6.5 (Al-

[7] In modern terms, it just says that the connectivity function is submodular.

ternating Cycle Lemma), $\{b_1, b_2\}$ crosses $\{a_1, a_2\}$ and there is a cycle o of length at most r alternating between these two local separators. A *person of type one* is a vertex x of $V(o) - a_1 - a_2 - b_1 - b_2$ that has a copy in the same component of $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$ as b_1 and a copy in the same component of $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$ as a_1. A *person of type two* is a neighbour x of b_1 outside the cycle o that has a copy in the same component of $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$ as a_1. A *person of type three* is a neighbour x of a_1 outside the cycle o that has a copy in the same component of $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$ as b_1. A *person* is a person of type one, two or three. We say that a person *lives in the corner* $\{a_1, b_1\}$ if there exists a person.

Remark 6.9. A person of type one has unique copies in $\mathrm{Ex_r}(a_1, a_2)$, $\mathrm{Ex_r}(b_1, b_2)$ and $\mathrm{Ex_r}(a_1, b_1)$ by Lemma 3.4. For a person x of type two, the edge xb_1 has unique copies in $\mathrm{Ex_r}(b_1, b_2)$ and $\mathrm{Ex_r}(a_1, b_1)$ by Lemma 3.2. Hence we define the copy of a person of type two in these neighbourhoods to be the unique endvertex of that edge that is a copy of x. Hence to simplify notation, below we suppress a bijection between a person and its unique copies in the explorer-neighbourhoods where copies are unique.

Lemma 6.10 (Corner Lemma)**.** *Assume G is r-locally 2-connected and assume the corner-setting. Assume a person x lives in the corner $\{a_1, b_1\}$, then $\{a_1, b_1\}$ is an r-local 2-separator.*

Moreover, x is in a different component of $\mathrm{Ex_r}(a_1, b_1) - a_1 - b_1$ *than (copies of) a_2 and b_2.*

Proof. By Lemma 3.4, the vertices a_1, a_2, b_1 and b_2 have unique copies in $\mathrm{Ex_r}(a_1, b_1)$; hence for this proof we suppress a bijection between these vertices and their copies in $\mathrm{Ex_r}(a_1, b_1)$. We start by showing the following.

Sublemma 6.11. *The vertices b_1 and b_2 are in different components of the graph* $\mathrm{Ex_r}(a_1, b_1) - a_1 - a_2$*.*

Proof. Suppose not for a contradiction. Then there is a path P of the graph $\mathrm{Ex_r}(a_1, b_1) - a_1 - a_2$ from b_1 to b_2.

Let W be the neighbourhood of the set $\{a_1, a_2\}$ in the punctured explorer-neighbourhood $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$ in the component containing b_1. By Lemma 3.2 neighbours of a_1 or a_2 have unique copies in the explorer-neighbourhood $\mathrm{Ex_r}(a_1, a_2)$; hence there is a bijection between the neighbours of a_1 and a_2 in G and the explorer-neighbourhood. To simplify notation we suppress this map in our notation. And we will simply consider W as a vertex set of the graph G, as well. Let $E(W)$ be the set of edges of the graph G from $\{a_1, a_2\}$ to W.

Recall that by the corner-setting, there is a cycle of length at most r alternating between the local separators $\{a_1, a_2\}$ and $\{b_1, b_2\}$. Thus it has a subpath Q from b_1 to b_2 of length at most $r/2$; this subpath contains precisely one of the vertices a_1 and a_2. This alternating cycle is also a cycle of the explorer-neighbourhood $\mathrm{Ex_r}(a_1, a_2)$ and the path Q has to intersect the set $E(W)$ oddly[8] as it connects vertices in different components. Vertices of o have unique copies in $\mathrm{Ex_r}(a_1, b_1)$ by Lemma 3.4. We suppress a bijection between vertices of o and their unique copies in $\mathrm{Ex_r}(a_1, b_1)$. So in $\mathrm{Ex_r}(a_1, b_1)$, Q is a path from b_1 to b_2.

Let u be the closed walk of the explorer-neighbourhood $\mathrm{Ex_r}(a_1, b_1)$ obtained by concatenating P and Q. The path P considered as a walk of the graph G contains no vertex a_i and thus does not intersect the edge set $E(W)$. Thus the closed walk u, considered as an edge set of the graph G intersects the edge set $E(W)$ in an odd number of edges.

By Lemma 3.6 the closed walk u is generated by cycles of G included in the balls $B_{r/2}(a_1)$ and $B_{r/2}(b_1)$. These cycles are in turn by Lemma 2.4 generated by cycles of length at most r included in these balls. Hence one of the generating cycles has to intersect the edge set $E(W)$ oddly. Call such a cycle u'. So the cycle u' contains the vertex a_1 or a_2. As it has bounded length, it is a cycle of the explorer-neighbourhood $\mathrm{Ex_r}(a_1, a_2)$. This is a contradiction as in the explorer-neighbourhood $\mathrm{Ex_r}(a_1, a_2)$ the cycle u' and the cut $E(W)$ cannot intersect oddly. Thus the vertices b_1 and b_2 must be in different components of the graph $\mathrm{Ex_r}(a_1, b_1) - a_1 - a_2$. □

By exchanging the roles of the 'a_i' and 'b_i' in Sublemma 6.11 one obtains the following.

Sublemma 6.12. *The vertices a_1 and a_2 are in different components of the graph $\mathrm{Ex_r}(a_1, b_1) - b_1 - b_2$.*

Proof. The proof is analogous to that of Sublemma 6.11. □

By $C(a, i)$ we denote the component of $\mathrm{Ex_r}(a_1, b_1) - a_1 - a_2$ containing the vertex b_i. By $C(b, i)$ we denote the component of $\mathrm{Ex_r}(a_1, b_1) - b_1 - b_2$ containing the vertex a_i.

By assumption there is vertex x of G that is a person living in the corner $\{a_1, b_1\}$. If x is a person of type one, then it is contained in both components $C(a, 1)$ and $C(b, 1)$, as the vertices b_1 and a_1 are in the respective components. If x is a person of type two, then it is contained in the component $C(a, 1)$ as $b_1 \in C(a, 1)$. Next we verify that it is in $C(b, 1)$. By definition

[8] In fact it intersects this set just once but we will not need that strengthening.

$a_1 \in C(b,1)$. As x is in the sam component of $\mathrm{Ex_r}(b_1,b_2)-b_1-b_2$ as a_1, pick a path joining them in there and add the edges b_1x and the subpath b_1oa_1 of o that avoids b_2. This forms a cycle. By Lemma 3.6 and Lemma 2.4, this cycle is generated by cycles of length at most r. From these cycles take those that contain the vertex b_1. The sum of these cycles with the vertex b_1 removed gives a closed walk in $B_{r/2}(b_1)$ from x to the neighbour of b_1 on b_1oa_1 (which is equal to a_1 or a person of type one). This walk witnesses that $x \in C(b,1)$. Like for persons of type two, we show for persons of type three that they are contained in $C(a,1)$ and $C(b,1)$. To summarise, in every case the intersection of $C(a,1)$ and $C(b,1)$ contains the person x.

In particular, the intersection of $C(a,1)$ and $C(b,1)$ is nonempty and includes a component of $\mathrm{Ex_r}(a_1,b_1)-a_1-a_2-b_1-b_2$. Denote such a component containing the person x by k. As the component k is included in $C(a,1)$, it does not contain any neighbour of the vertex b_2 by Sublemma 6.11. Similarly, k does not contain any neighbour of the vertex a_2 by Sublemma 6.12. Hence k is also a component of $\mathrm{Ex_r}(a_1,b_1)-a_1-b_1$. As k does not contain the vertex a_2, the punctured explorer-neighbourhood $\mathrm{Ex_r}(a_1,b_1)-a_1-b_1$ is disconnected. Thus $\{a_1,b_1\}$ is an r-local 2-separator.

The 'Moreover'-part is clear by construction. □

Remark 6.13. In the next paragraph we define 'contacts'. This definition is technical and results in quite a few technicalities later on. In contrast to this, the intuition behind contacts is rather simple: given a graph G' obtained from a graph G by locally cutting at $\{a_1,a_2\}$. We would like to move back and forth between local separators of G and G'. Specifically, if $\{b_1,b_2\}$ is disjoint from $\{a_1,a_2\}$, then $\{b_1,b_2\}$ is a local separator of G if and only if it is a local separator of G'; compare Lemma 6.15 (Projection Lemma) and Lemma 8.4 (Lifting Lemma) below. If the sets $\{b_1,b_2\}$ and $\{a_1,a_2\}$ are not disjoint, we still want to move back and forth between local separators, and contacts are our way to formalise it; they associate to every vertex b_i of G a canonical vertex of G' relative to $\{b_1,b_2\}$. (Imagine you phone a 'helpline' and they redirect your call to the 'next contact'. Don't worry: unlike for many helplines, here this happens only once.) The proofs below would simplify a lot if one (carelessly) assumed that $\{b_1,b_2\}$ and $\{a_1,a_2\}$ were disjoint.

Definition 6.14 (Contacts)**.** Given a graph G' obtained from G by locally cutting a local 2-separator, by definition there is a bijection between the edges of G and the edges of G' that are not on torso paths. To simplify notation, we suppress this bijection from our notation. Let b be a vertex of G and let b' be a vertex of G' that is equal to b or a slice thereof. Let e be an edge of G that is incident with the vertex b. Then the edge e is incident with the vertex b' in G' or else the vertex b' must be a slice, and thus is incident with a unique edge on a torso path. The *contact* of the edge e at the vertex b' is the edge e itself if e is incident with b' in G' or else the contact is the unique edge on a torso path incident with b'.

Given a local 2-separator $\{b_1, b_2\}$ in a graph G and two edges e and f each with one endvertex in $\{b_1, b_2\}$, we say that e and f are *separated* by $\{b_1, b_2\}$ if the edges e and f have endvertices in different components of the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$.

For a vertex x' of G' that is not an interior vertex of a torso-path, there is a unique vertex of G that is equal to x' or such that x' is a slice of that vertex. We denote this vertex by x.

Lemma 6.15 (Projection Lemma)**.** *Assume G is r-locally 2-connected. For any r-local 2-separator $\{b'_1, b'_2\}$ of G' such that the b'_i are not interior vertices of torso-paths, the set $\{b_1, b_2\}$ is an r-local 2-separator of G.*

More specifically, edges e and f each with one endvertex in $\{b_1, b_2\}$ are separated by $\{b_1, b_2\}$ in G if their contacts are separated by $\{b'_1, b'_2\}$ in G'.

Proof. In this proof we will distinguish between the vertices of G and G' by adding a dash to the vertices of the graph G'; for example we write b'_1 when we consider b_1 as a vertex of G' and b_1 when we consider it as a vertex of the graph G (whenever b'_1 is not an interior vertex of a torso path). We prove the 'More specifically'-part, as this suffices.

Let e and f be edges incident with precisely one of b_1 or b_2 such that their contacts are separated by $\{b'_1, b'_2\}$ in G'.

Sublemma 6.16. *There is at most one edge on a torso path incident with vertices of $\{b'_1, b'_2\}$.*

Proof. Let $\{a_1, a_2\}$ be a local 2-separator of G such that G' is obtained from G by locally cutting at $\{a_1, a_2\}$. Suppose for a contradiction that there are two edges on torso paths incident with vertices of $\{b'_1, b'_2\}$. As each vertex is incident with at most one torso edge, both b'_1 and b'_2 must be slices.

As $\{b'_1, b'_2\}$ is an r-local 2-separator, the vertices b'_1 and b'_2 have distance at most $r/2$. So b'_1 and b'_2 cannot be slices of the same vertex by Lemma 5.3. By symmetry assume that b'_1 is a slice of a_1 and b'_2 is a slice of a_2. As there are at least two torso edges, b'_1 and b'_2 must be slices for different components of $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$. As $\{a_1, a_2\}$ is an r-local 2-separator, the vertices a_1 and a_2 have distance at most $r/2$. It follows from the definition of local cutting that slices of a_1 and a_2 have the same distance as a_1 and a_2, so this distance is upper-bounded by $r/2$.

Let a_1' be the slice of the vertex a_1 for the same component as the slice b_2' of a_2. So the distance from a_1' to b_2' is at most $r/2$. So the distance between the distinct slices a_1' and b_1' of a_1 is at most r. This is a contradiction to Lemma 5.3. Hence there is at most one torso edge incident with vertices of $\{b_1', b_2'\}$. $\square$

Let k' be the component of the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1', b_2') - b_1' - b_2'$ in G' that contains an endvertex of the contact for e[9]. Let W be the set of edges of G' with one endvertex in $\{b_1', b_2'\}$ and the other endvertex in k'. By Sublemma 6.16 and by exchanging the roles of the edges e and f if necessary, we may assume, and we do assume, that no edge on a torso path incident with a vertex of $\{b_1', b_2'\}$ has its other endvertex in the component k'. Hence the edge set W is also an edge set of the graph G. By Lemma 3.2, edges of W have unique copies in $\mathrm{Ex_r}(b_1, b_2)$. For simplicity we suppress a bijection between W and its uniquely defined copy in $\mathrm{Ex_r}(b_1, b_2)$.

Sublemma 6.17. *There is a path Q from b_1 to b_2 contained in* $\mathrm{Ex_r}(b_1, b_2)$ *that contains an even number of edges from W.*

Proof. As G' is r-locally 2-connected by Lemma 5.5, by Lemma 3.10 (Local 2-Connectivity Lemma) there is a cycle o' of length at most r included in $\mathrm{Ex_r}(b_1', b_2')$ containing the vertices b_1' and b_2'. By Lemma 5.3, the cycle o' can contain edges of at most one torso path (note that if a cycle contains edges from a torso path, it must include the whole torso path). Hence there is a path Q' from b_1' to b_2' included in o' that does not contain any edges of torso paths. As Q' has both its endvertices on the same side of the cut W, it intersects that cut evenly. The edges of Q' form a path Q in the graph G from b_1 to b_2. By definition of local splitting, there is a cycle o of G of the same length as o' that includes the path Q. So by Lemma 3.4, Q is a path in $\mathrm{Ex_r}(b_1, b_2)$. $\square$

Suppose for a contradiction that the edges e and f are not separated by $\{b_1, b_2\}$ in G; that is, they are incident with vertices of the same component of the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$.

Sublemma 6.18. *There is a cycle o of the explorer-neighbourhood* $\mathrm{Ex_r}(b_1, b_2)$ *in G that intersects the set W oddly.*

[9]This is well-defined as the contact for e is an edge that has exactly one endvertex in $\{b_1', b_2'\}$, and the other endvertex defines the component k' uniquely.

Proof. By assumption, there is a path P included in the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$ between the endvertices of the edges e and f outside $\{b_1, b_2\}$. Now we extend the path P to a walk by adding the edges e and f at the endvertices of P. The endvertices of this extended walk are in the set $\{b_1, b_2\}$. This walk intersects the edge set W precisely in the edge e. Either this walk is a cycle, or it is a path whose endvertices are b_1 and b_2. While we are done immediately in the first case, in the second case we concatenate this path ePf with a path Q as in Sublemma 6.17. This way we obtain a closed walk, which includes the desired cycle o. □

Sublemma 6.19. *There is a cycle o_1 of G contained in the explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2)$ of length bounded by r that intersects the set W oddly.*

Proof. Let o be a cycle as in Sublemma 6.18. By Lemma 3.6, the cycle o is generated from cycles that are included within the balls of radius $r/2$ around the vertices b_1 and b_2. These cycles, in turn by Lemma 2.4, are generated by cycles within the respective balls of length bounded by r. To summarise: the cycle o is generated over the finite field $\mathbb{F}_2$ by cycles of G contained in the explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2)$ of length bounded by r. As the cycle o intersects the set W oddly, one of the cycles in the generating set has to intersect the set W oddly. We pick such a cycle for o_1. □

Sublemma 6.20. *There is a cycle o' of G' included in the explorer-neighbourhood $\mathrm{Ex_r}(b'_1, b'_2)$ of length bounded by r that intersects the set W oddly.*

Proof. Let o_1 be a cycle as in Sublemma 6.19. First assume the cycle o_1 does not traverse the local 2-separator $\{a_1, a_2\}$ (here we say that a cycle *traverses* a local 2-separator if the cycle traverses[10] this set as a separator of the explorer-neighbourhood [of parameter r]). Then the cycle o_1 of G is a cycle of G'. So we can take $o' = o_1$ and are done. Hence we may assume, and we do assume, that the cycle o_1 traverses the local 2-separator $\{a_1, a_2\}$. We remark that as the cycle o_1 is a cycle of the graph G – not just of the explorer-neighbourhood – it cannot contain two copies of a vertex of the graph G.

One of the subpaths of o_1 from a_1 to a_2 contains an even number of edges of W, the other one an odd number of edges of W. Let P be the subpath of o_1 from a_1 to a_2 that contains an odd number of edges of W. Then the

[10]Given a cycle o and separator S in a graph G, a *traversal* of o is a minimal subpath of o such that the vertex just after and just before that subpath are in different components of $G \setminus S$. A cycle traverses a 2-separator in exactly two subpaths, each being a single vertex, or not at all.

edges of P form a path of the graph G' from a slice of a_1 to a slice of a_2. We obtain the cycle o' from P by adding a torso path (which is added by the definition of local cutting between the two slices a_1' and a_2'), which is does not intersect W by the choice of the component k'. $\square$

The edge set W is a cut of the explorer-neighbourhood $\mathrm{Ex_r}(b_1', b_2')$, and the cycle o', which is given by Sublemma 6.20, is a cycle of that graph. So they must intersect evenly (as all cuts and cycles do). This is a direct contradiction to Sublemma 6.20. Hence the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$ in G is disconnected, and so $\{b_1, b_2\}$ is an r-local separator of the graph G. More specifically, the edges e and f are separated by $\{b_1, b_2\}$. $\square$

Remark 6.21. Here we come back to Remark 3.13. If the punctured double ball around two vertices is disconnected, then so is the punctured explorer-neighbourhood around them. We say that an r-local separator $\{v, w\}$ is *mundane* if the punctured double-ball around v and w is disconnected. Lemma 6.15 (Projection Lemma) says that the class of local 2-separators has the corner property, a property which plays a central role in the structural theory of genuine separators. The subclass of mudane local 2-separators does not have this property, which can be seen from Figure 6 as follows. If one obtained G' from the depicted graph by locally cutting at the mudane local separator $\{b_1, b_2\}$, the corner $\{a_1, b_1\}$ becomes a mudane local separator, which does not come from a mudane local separator of G.

Example 6.22. While Figure 6, already shows that a notion of local 2-separators based on punctured double balls would not have all the desired properties, here we give a second more advanced example that demonstrates that even more properties fail for the class of mudane local separators.

We start by giving a formal definition of the graph G depicted in Figure 7. Given a parameter $r \geq 6$, let M be the graph obtained from a cycle o of length $2r$ by adding a paths of length two between two antipodal vertices of o (that is, any two vertices of o of distance precisely r). We obtain G from M by picking an edge e of o arbitrarily and subdividing it four times. Informally speaking, the graph G is a subdivided Moebius strip, see Figure 7.

Now we give names to four vertices of G. Denote the two endvertices of the edge e by a_2 and b_2. Denote the antipodal vertex of the vertex a_2 on o by a_1, and let b_1 be the subdivision vertex of the edge e that has distance two from a_2 (and consequently distance three from b_2).

We observe (and prove below) that:

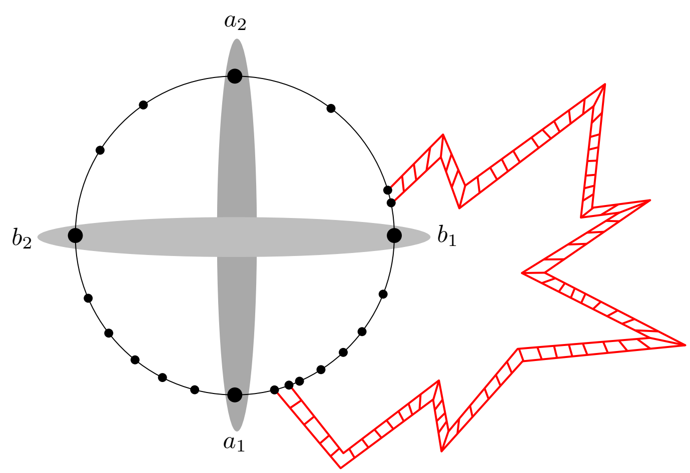

Figure 6: Two crossing local separators. They are highlighted in grey and denoted by $\{a_1, a_2\}$ and $\{b_1, b_2\}$. The long red strip joins a neighbour of a_1 with a neighbour of b_1. Its length is long enough so that the punctured double-balls $(B_{r/2}(a_1) \cup B_{r/2}(a_2)) - a_1 - a_2$ and $(B_{r/2}(b_1) \cup B_{r/2}(b_2)) - b_1 - b_2$ are disconnected but so short that the punctured double-ball $(B_{r/2}(a_1) \cup B_{r/2}(b_1)) - a_1 - b_1$ is connected.

(1) $B_{\frac{r}{2}}(a_1) \cup B_{\frac{r}{2}}(b_1) - a_1 - b_1$ is disconnected;

(2) $B_{\frac{r}{2}}(b_1) \cup B_{\frac{r}{2}}(b_2) - b_1 - b_2$ is disconnected ALTHOUGH;

(3) $B_{\frac{r}{2}}(a_1) \cup B_{\frac{r}{2}}(a_2) - a_1 - a_2$ is connected AND;

(4) $B_{\frac{r}{2}}(a_1) \cup B_{\frac{r}{2}}(b_2) - a_1 - b_2$ is connected;

To see (1) note that the ball $B_{\frac{r}{2}}(b_1)$ is included in the ball $B_{\frac{r}{2}}(a_1)$ and $B_{\frac{r}{2}}(a_1) - a_1 - b_1$ is disconnected. Condition (2) follows from the fact that $\{b_1, b_2\}$ is a global 2-separator. To see (4), just note that the union of these two balls includes the whole graph G; and similarly to see (3) note that the union of the two relevant balls is equal to the graph G with a single vertex removed.

Similarly as Figure 6, this example shows that the corner property does not in general hold for the class of mudane local separators. Also the following rather obscure thing happens. Let G' be the graph obtained from G by r-locally cutting at the local 2-separator $\{a_1, b_1\}$. Then $B_{\frac{r}{2}}(a_1) \cup B_{\frac{r}{2}}(a_2) - a_1 - a_2$ is disconnected in the graph G', while (3) above says that this property does not hold in G. It seems like $\{a_1, a_2\}$ should be an r-local 2-separator in any sensible r-local decomposition for G.

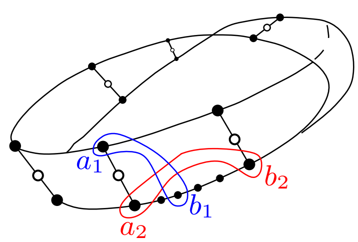

Figure 7: A subdivided Moebius strip.

7 When all local 2-separators are crossed...

In this section we prove Theorem 1.3, which later will be used in the proof of Theorem 2.1.

Proof of Theorem 1.3. Let $r \in \mathbb{N} \cup \{\infty\}$ and let G be a connected graph that is r-locally 2-connected. Assume that every r-local 2-separator of G

is crossed by an r-local 2-separator. We are to show that G is r-locally 3-connected or a cycle of length at most r.

Let $\{a_1, a_2\}$ be an r-local 2-separator, and let $\{b_1, b_2\}$ be an r-local 2-separator that crosses it. By Lemma 6.5 (Alternating Cycle Lemma), there is a cycle o of length at most r alternating between these two local separators. Our aim is to show that the graph G is equal to the cycle o. Suppose not for a contradiction. Then there is a vertex outside the cycle o. As the graph G is connected, there is a vertex that is outside o and adjacent to a vertex on the cycle o. Pick such a vertex and call it x_2, and denote one of its neighbours on the cycle o by x_1. We distinguish two cases.

Case 1: there is no vertex y on the cycle o such that $\{x_1, y\}$ is an r-local 2-separator.

By $\mathcal{S}$ we denote the set of r-local 2-separators with both their vertices on the cycle o. Given $\{c_1, c_2\} \in \mathcal{S}$, by Lemma 3.4 all vertices on the cycle o have a unique copy in the explorer-neighbourhood $\mathrm{Ex_r}(c_1, c_2)$. For these vertices we suppress a bijection between them and their unique copies in $\mathrm{Ex_r}(c_1, c_2)$ to simplify notation. By $\Gamma(c_1, c_2)$ we denote the set of vertices on the cycle o in the component of $o - c_1 - c_2$ that contains the vertex x_1. The *size* of a local separator $\{c_1, c_2\}$ in $\mathcal{S}$ is $|\Gamma(c_1, c_2)|$. The set $\mathcal{S}$ is nonempty as $\{a_1, a_2\} \in \mathcal{S}$. Pick a local separator $\{w_1, w_2\} \in \mathcal{S}$ of minimal size.

Sublemma 7.1. *In Case 1, no r-local 2-separator crosses $\{w_1, w_2\}$.*

Proof. Suppose for a contradiction there is an r-local 2-separator $\{v_1, v_2\}$ that crosses $\{w_1, w_2\}$. By Lemma 6.5 (Alternating Cycle Lemma) there is a cycle o' alternating between $\{v_1, v_2\}$ and $\{w_1, w_2\}$. Hence by Lemma 3.4 the vertices v_1, v_2, w_1 and w_2 have unique copies in $\mathrm{Ex_r}(v_1, v_2)$; and so in the following we will suppress a bijection between them and their copies in $\mathrm{Ex_r}(v_1, v_2)$ from our notation. Let P' be a subpath of o' between w_1 and w_2 of length at most $r/2$. Let P be a subpath of o between w_1 and w_2 of length at most $r/2$. Let o'' be the closed walk obtained by concatenating P and P'. The path P' must contain one of the vertices v_1 or v_2, say v_1. Hence o'' is a closed walk through v_1 of length at most r; so it is contained within $B_{r/2}(v_1)$. So the path P is a path of the ball $B_{r/2}(v_1)$. As w_1 and w_2 are in different components of the punctured explorer-neighbourhood $\mathrm{Ex_r}(v_1, v_2) - v_1 - v_2$, the path P must contain the vertex v_1 or v_2. Thus the cycle o contains the vertex v_1 or v_2. By condition (2) of Lemma 6.5 (Alternating Cycle Lemma), the cycle o alternates between the local separators $\{v_1, v_2\}$ and $\{w_1, w_2\}$.

By Corollary 6.6 each of the punctured explorer-neighbourhoods $\mathrm{Ex_r}(v_1, v_2) -$

$v_1 - v_2$ and $\mathrm{Ex_r}(w_1, w_2) - w_1 - w_2$ has precisely two components. Hence there is a corner $\{v_i, w_j\}$ with $i, j \in \{1, 2\}$ such that the vertex x_1 is a person of type one living in the corner $\{v_i, w_j\}$. Hence by Lemma 6.10 (Corner Lemma), $\{v_i, w_j\}$ is an r-local 2-separator. It is in the set $\mathcal{S}$. We claim that its size is strictly smaller than that of $\{w_1, w_2\}$. Indeed, by the 'Moreover'-part of Lemma 6.10 (Corner Lemma), $\Gamma(v_i, w_j)$ only contains those vertices of o on the path between v_i and w_j containing x_1. As the vertices x_1 and v_i are in the same component of the punctured explorer-neighbourhood $\mathrm{Ex_r}(w_1, w_2) - w_1 - w_2$, all vertices of $\Gamma(v_i, w_j)$ are also in $\Gamma(w_1, w_2)$ but that set additionally contains the vertex v_i. This is the desired contradiction. Thus no r-local 2-separator crosses $\{w_1, w_2\}$. □

Sublemma 7.1 contradicts the assumptions of the theorem. Hence the graph G is a cycle. Having finished Case 1, it remains to treat the following (which will be somewhat similar).

Case 2: not Case 1; that is, there is a vertex y on the cycle o such that $\{x_1, y\}$ is an r-local 2-separator.

By $\mathcal{S}$ we denote the set of r-local 2-separators with both their vertices on the cycle o and one of these vertices is equal to the vertex x_1. Given $\{c_1, c_2\} \in \mathcal{S}$, by Lemma 3.4 all vertices on the cycle o have a unique copy in the explorer-neighbourhood $\mathrm{Ex_r}(c_1, c_2)$. For these vertices we suppress a bijection between them and their unique copies in $\mathrm{Ex_r}(c_1, c_2)$ to simplify notation. Moreover, the edge x_1x_2 of G has a unique copy in $\mathrm{Ex_r}(c_1, c_2)$ by Lemma 3.2. We suppress a bijection between the vertex x_2 of G and the endvertex of the edge x_1x_2 in $\mathrm{Ex_r}(c_1, c_2)$ that is a copy of x_2.

By $\Gamma(c_1, c_2)$ we denote the set of vertices on the cycle o in the component of the punctured explorer-neighbourhood $\mathrm{Ex_r}(c_1, c_2) - c_1 - c_2$ that contains the vertex x_2. The *size* of a local separator $\{c_1, c_2\}$ in $\mathcal{S}$ is $|\Gamma(c_1, c_2)|$. The set $\mathcal{S}$ is nonempty by the assumption of Case 2. Pick a local separator $\{w_1, w_2\} \in \mathcal{S}$ of minimal size.

Arguing the same as in the proof of Sublemma 7.1 but referring to the fact that 'x_2 is a person of type two' instead of 'x_1 is a person of type one', one proves the following[11].

Sublemma 7.2. *In Case 2, no r-local 2-separator crosses $\{w_1, w_2\}$.* □

This completes all the cases. Hence in all cases, there is an r-local 2-separator that is not crossed by any other r-local 2-separator. This is a

[11]By local 2-connectivity, if there is no person of type one but a person of type two, there is also a person of type three. So we need not to rely on persons of type three in our arguments.

contradiction to the assumptions of this theorem. Hence the graph G must be equal to the cycle o. $\square$

8 The uniqueness statement of the local 2-separator theorem

In this section we prove Theorem 2.1. Our first goal is to prove Lemma 8.4 below, which can be seen as the 'inverse' of Lemma 6.15 (Projection Lemma).

Let G' be a graph obtained from a graph G by r-locally cutting an r-local 2-separator $\{a_1, a_2\}$. Let $\{b_1, b_2\}$ be an r-local 2-separator of G.

Lemma 8.1. *Assume G is r-locally 2-connected. Assume $\{b_1, b_2\}$ is not crossed by $\{a_1, a_2\}$. Then there is a cycle o' of the graph G' of length at most r that contains vertices b'_i that are equal to b_i or a slice of b_i (for $i = 1, 2$).*

Moreover, for any $i \in \{1, 2\}$ with $b_i \notin \{a_1, a_2\}$ the two edges of o' incident with b'_i are separated by $\{b'_1, b'_2\}$ – or else the edge $b'_1 b'_2$ is on o'.

We remark that the neighbours of b'_i are also neighbours of b_i (unless the edge joining them is on a torso path), and they have unique copies in $\mathrm{Ex_r}(b_1, b_2)$ by Lemma 3.2; in this sense the 'Moreover'-part is unambiguously defined.

Proof of Lemma 8.1. As the graph G is r-locally 2-connected, we can apply Lemma 3.10 (Local 2-Connectivity Lemma) to deduce that there is a cycle o of length at most r in G through the vertices b_1 and b_2 and such that interior vertices of different subpaths of o between b_1 and b_2 are in different components of the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$ – or else the cycle o contain the edge $b_1 b_2$.

If the cycle o does not contain any vertex a_i, it is a cycle of the graph G' and we are done. So we may assume, and we do assume, that a vertex a_i, say a_1, is on the cycle o. So the cycle o is a cycle of the explorer-neighbourhood $\mathrm{Ex_r}(a_1, a_2)$. In there, $\{a_1, a_2\}$ is a genuine 2-separator. There are three cases: if o does not contain any of its vertices, then o is a cycle of the graph G' and we are done. If o contains precisely one vertex, all other vertices must be in a single local component and so o is a cycle of the graph G' and we are done. Hence it remains to consider the third case that both vertices a_1 and a_2 are on the cycle o. As the local separator $\{a_1, a_2\}$ does not cross $\{b_1, b_2\}$, the vertices a_1 and a_2 must be in the same component of the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$. In particular, the cycle o does not pre-alternate between the local separators $\{a_1, a_2\}$ and $\{b_1, b_2\}$. So there is a subpath P of o between the a_i containing no vertex b_j as an interior vertex. We obtain o' from o by replacing the path P by a torso path to obtain a cycle of the graph G'. By the definition of the length of the torso paths the length of o' is at most that of o. And o' contains the vertices b_i or slices thereof. This completes the proof except for the 'Moreover'-part.

To see the 'Moreover'-part, pick b_i under the constraint that it is not in $\{a_1, a_2\}$. The two incident edges of the vertex b'_i on o' are not edges on torso paths. By the construction of the cycle o', the two neighbours of b'_i on o' are separated by $\{b'_1, b'_2\}$. $\square$

Definition 8.2 (Lift)**.** A *lift* of a local 2-separator $\{b_1, b_2\}$ of G is a set $\{b'_1, b'_2\}$ of G' such that each vertex b'_i is equal to b_i or a slice thereof, and such that there is a cycle o' of length at most r containing b'_1 and b'_2.

Remark 8.3. Under the circumstances of Lemma 8.1, it can be shown that for each local 2-separator $\{b_1, b_2\}$ a lift is *uniquely* defined – unless $\{b_1, b_2\}$ is identical to $\{a_1, a_2\}$ (note that $\{a_1, a_2\}$ does not cross itself). Indeed, if no b_i is in the set $\{a_1, a_2\}$, this is clear. Otherwise there can be at most one vertex b_i that is in $\{a_1, a_2\}$, say it is b_1. A vertex b'_1 in a lift has distance at most $r/2$ from $b_2 = b'_2$. As any other slice x' of b_1 has distance at least $r+1$ from b'_1 by Lemma 5.3, the vertex b_2 has a too large distance from that vertex. So $\{x', b_2\}$ cannot be a lift. Thus we will in the following always refer to 'the' lift.

Lemma 8.4 (Lifting Lemma)**.** *Assume G is r-locally 2-connected. Let $\{b_1, b_2\}$ be an r-local 2-separator of G. Assume $\{b_1, b_2\}$ is not crossed by $\{a_1, a_2\}$ and they are not identical. Then the lift $\{b'_1, b'_2\}$ of $\{b_1, b_2\}$ is an r-local 2-separator of G'.*

More specifically, if edges e and f with precisely one endvertex in $\{b_1, b_2\}$ are separated by $\{b_1, b_2\}$ in G, then their contacts are separated by $\{b'_1, b'_2\}$ in G'.

Proof. The proof strategy is somewhat similar to that of Lemma 6.15 (Projection Lemma). Like in the proof of Lemma 6.15, in this proof we will distinguish between the vertices of G and G' by adding a dash to the vertices of the graph G'.

Let e and f be edges incident with precisely one of b_1 or b_2 that are separated by $\{b_1, b_2\}$ in G.

Sublemma 8.5. *There is a single component k of $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$ containing an endvertex of every edge incident with precisely one of b_1 or b_2 whose contact is an edge of a torso path.*

Proof. By assumption not both vertices b_1 and b_2 are in the set $\{a_1, a_2\}$. As we are done otherwise, we assume that precisely one of the vertices b_i is in $\{a_1, a_2\}$. By symmetry, we assume that $b_1 = a_1$.

Next, we determine the component of $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$ belonging to the slice b_1' of a_1. As $\{b_1, b_2\}$ is an r-local 2-separator, by Lemma 3.10 (Local 2-Connectivity Lemma), there is a cycle o of length at most r through b_1 and b_2. By Lemma 3.4, the vertex b_2 has a unique copy in $\mathrm{Ex_r}(a_1, a_2)$. So there is a unique slice of the vertex a_1 that has distance at most $r/2$ from the vertex $b_2 = b_2'$. This is the slice for the component of $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$ containing b_2. Denote that component by k_2. Hence the vertex b_1' is the slice of the vertex a_1 for the component k_2.

Next we define the component k. As $\{a_1, a_2\}$ is an r-local 2-separator, by Lemma 3.10 (Local 2-Connectivity Lemma), there is a cycle of length at most r through a_1 and a_2. By Lemma 3.4, the vertex a_2 has a unique copy in $\mathrm{Ex_r}(b_1, b_2)$. Let k be the component of $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$ that contains the vertex a_2.

Now let e be an edge of G incident with precisely one of b_1 or b_2 whose contact is an edge of a torso path. Then e is incident with the vertex $b_1 = a_1$. Let x be the endvertex of the edge e aside from b_1. As the contact for e is an edge on a torso path, the vertex x is outside the component k_2 of $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$. As G is r-locally 2-connected, the punctured ball $B_{r/2}(a_1) - a_1$ is connected. So there is a path P from x to a_2 within that ball. As the vertex b_2 is in a different connected component of $\mathrm{Ex_r}(a_1, a_2) - a_1 - a_2$ than x, it is also in a different connected component of $B_{r/2}(a_1) - a_1 - a_2$ than x. So the path P does not contain the vertex b_2. Thus P is a path from x to a_2 included in the component k. Note that P is a path of $\mathrm{Ex_r}(b_1, b_2)$; indeed, it is within the ball of radius $r/2$ around b_1 (which is equal to a_1 here). So the path P witnesses that the vertex x is in the component k of $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$.

As the edge e was arbitrary, the component k contains an endvertex of every edge incident with precisely one of b_1 or b_2 whose contact is a torso edge. $\square$

Let k_1 be the component of the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$ containing an endvertex of the edge e. By exchanging the roles of e and f if necessary, we assume that the component k_1 is different from the component k in Sublemma 8.5. Let W be the set of edges of G with one endvertex in $\{b_1, b_2\}$ and the other endvertex in k_1. By Sublemma 8.5, the edge set W does not contain an edge of a torso path and hence is an edge set of the graph G' consisting of edges with precisely one endvertex in the set $\{b'_1, b'_2\}$.

Sublemma 8.6. *There is a path Q' from b'_1 to b'_2 contained in* $\mathrm{Ex_r}(b'_1, b'_2)$ *that contains an even number of edges from W.*

Proof. As G is r-locally 2-connected, by Lemma 3.10 (Local 2-Connectivity Lemma) there is a cycle o included in $\mathrm{Ex_r}(b_1, b_2)$ containing the vertices b_1 and b_2 of length at most r, and o contains vertices in two different components of $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$ or an edge from b_1 to b_2. In the second case, we immediately set Q to be the edge from b_1 to b_2. In the first case we define Q as follows. As $\{a_1, a_2\}$ does not cross $\{b_1, b_2\}$ by condition (4) in Lemma 6.5 (Alternating Cycle Lemma), one of the two subpaths of o from b_1 to b_2 has no vertex a_i as interior vertex. Pick such a path and call it Q. As $\{a_1, a_2\} \neq \{b_1, b_2\}$, one of the vertices b'_i is equal to b_i. Hence the cycle o is in the ball of radius $r/2$ around that vertex, and so in the explorer-neighbourhood $\mathrm{Ex_r}(b'_1, b'_2)$. Thus the subpath Q is within $\mathrm{Ex_r}(b'_1, b'_2)$.

As Q has both its endvertices on the same side of the cut W, it intersects that cut evenly. The edges of Q form a path Q' in the graph G' from b'_1 to b'_2, which is also a path in $\mathrm{Ex_r}(b'_1, b'_2)$ as shown above. □

Suppose for a contradiction that the contacts for the edges e and f are not separated by $\{b'_1, b'_2\}$ in G'; that is, they are incident with vertices of the same component of the punctured explorer-neighbourhood $\mathrm{Ex_r}(b'_1, b'_2) - b'_1 - b'_2$.

Sublemma 8.7. *There is a cycle o' of the explorer-neighbourhood* $\mathrm{Ex_r}(b'_1, b'_2)$ *in G' that intersects the set W oddly.*

Proof. By assumption, there is a path P' included in the punctured explorer-neighbourhood between the endvertices of the contacts for the edges e and f outside $\{b'_1, b'_2\}$. Now we extend the path P' to a walk by adding the contacts for the edges e and f at the two ends. This walk intersects the set W precisely in the edge[12] e. If we added the same vertex to both ends, we obtained the desired cycle o'.

Otherwise we obtain a path between the vertices b'_1 and b'_2 in the explorer-neighbourhood that intersects the set W oddly. Concatenating this path with a path Q' as in Sublemma 8.6 yields a closed walk intersecting oddly. This closed walk is an edge-disjoint union of cycles and one of those cycles intersects W oddly. □

[12] Sublemma 8.5 implies that one of the edges e and f is its own contact. By fixing the roles of e and f above in the definition of the set W, we ensured that the edge e is its own contact.

Sublemma 8.8. *There is a cycle o'_1 of G' contained in the explorer-neighbourhood $\mathrm{Ex}_\mathrm{r}(b'_1, b'_2)$ of length bounded by r that intersects the set W oddly.*

Proof. Let o' be a cycle as in Sublemma 8.7. By Lemma 3.6, the cycle o' is generated from cycles of that are included within the balls of radius $r/2$ around the vertices b'_1 and b'_2. These cycles, in turn by Lemma 2.4, are generated by cycles within the respective balls of length bounded by r. To summarise: the cycle o' is generated over the finite field $\mathbb{F}_2$ by cycles of G' contained in the explorer-neighbourhood $\mathrm{Ex}_\mathrm{r}(b'_1, b'_2)$ of length bounded by r. As the cycle o' intersects the set W oddly, one of the cycles in the generating set has to intersect the set W oddly. We pick such a cycle for o'_1. $\square$

Sublemma 8.9. *There is a cycle o of G of length bounded by r that intersects the set W oddly. Moreover, o is a cycle of $\mathrm{Ex}_\mathrm{r}(b_1, b_2)$.*

Proof. Let o'_1 be a cycle as in Sublemma 8.8. We remark that as the cycle o'_1 is a cycle of the graph G' – not just of the explorer-neighbourhood – it cannot contain two copies of a vertex of the graph G'. As the cycle o'_1 has length at most r, by Lemma 5.3 it contains at most one slice of any vertex of G. In particular, the cycle o'_1 contains at most one torso path.

First assume the cycle o'_1 does not use any edges on torso paths. Then the edges of the cycle o'_1 of G' form a cycle o of G. So we are done in this case.

Hence we may assume, and we do assume, that the cycle o'_1 contains a unique torso path. We denote by Q' the subpath of o'_1 obtained by removing this torso path. Then Q' bijects to an edge set Q of the graph G. As the torso path is not in W by construction, we deduce that Q contains an odd number of edges of W.

By the definition of local cutting, there is a path P of[13] G between a_1 and a_2 associated to the torso path of o'_1 such that the cycle o'_1 with the torso path replaced by P is a cycle of the graph G of the same length as o'_1. We denote that cycle by o.

To see the 'Moreover'-Part note that the cycle o contains at least one edge of W and so must contain a vertex b_i and so is in the ball of radius $r/2$ around that vertex and thus is embedded in the explorer-neighbourhood $\mathrm{Ex}_\mathrm{r}(b_1, b_2)$. From now on we consider this embedding of o in $\mathrm{Ex}_\mathrm{r}(b_1, b_2)$ (note that this is unambiguous as if o contained both vertices b_i, then the embedding would be unique by Lemma 3.4).

[13] Formally, the path P lives in the explorer-neighbourhood $\mathrm{Ex}_\mathrm{r}(a_1, a_2)$ and in there it is included in a cycle of length at most r and by Remark 3.5, this cycle is a cycle of G, and so P is a path of G.

As Q contains an odd number of edges of W, it remains to show that the path P contains an even number of edges from the set W. Denote by a_1' and a_2' the copies of the vertices a_1 and a_2 on o, respectively. The existence of o implies that $\{a_1, a_2\}$ does not pre-cross $\{b_1, b_2\}$; that is, the copies a_1' and a_2' are in the same component of $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$. So they are on the same side of the cut W. So by Lemma 6.1, P contains an even number of edges of W. Thus the cycle o, which is composed of the paths P and Q contains an odd number of edges of W. The 'Moreover'-Part has been shown in the text, so this completes the proof. □

The edge set W is a cut of the explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2)$, and a cycle o as in Sublemma 8.9 is a cycle of that graph. So they must intersect evenly (as all cuts and cycles do). This is a direct contradiction to Sublemma 8.9. Hence the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1', b_2') - b_1' - b_2'$ in G' is disconnected, and so $\{b_1', b_2'\}$ is an r-local separator of the graph G'. More specifically, the contacts for e and f are separated by $\{b_1', b_2'\}$. □

Lemma 8.10. *Let G be an r-locally 2-connected graph. Let $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$ be r-local 2-separators so that $\{a_1, a_2\}$ crosses neither $\{b_1, b_2\}$ nor $\{c_1, c_2\}$. Construct G' from G by r-locally cutting $\{a_1, a_2\}$.*

Then the lifts of $\{b_1, b_2\}$ and $\{c_1, c_2\}$ cross in G' if and only if $\{b_1, b_2\}$ and $\{c_1, c_2\}$ cross in G.

Proof. Assume $\{b_1, b_2\}$ and $\{c_1, c_2\}$ cross in G. Then by condition (2) of Lemma 6.5 (Alternating Cycle Lemma) there is a cycle o of length at most r alternating between $\{b_1, b_2\}$ and $\{c_1, c_2\}$ traversing the local separator $\{b_1, b_2\}$ twice (that is, the subpaths of o between b_1 and b_2 have interior vertices in different components of the punctured explorer-neighbourhood $\mathrm{Ex_r}(b_1, b_2) - b_1 - b_2$).

If the cycle o does not traverse the local separator $\{a_1, a_2\}$ twice, then o is a cycle of the graph G' (by the argument already given in the proof of Lemma 8.1). Then by Lemma 8.4 (Lifting Lemma), o traverses the lift of $\{b_1, b_2\}$ twice. Thus the lifts of $\{b_1, b_2\}$ and $\{c_1, c_2\}$ cross in G'.

Hence we may assume, and we do assume, that the cycle o traverses the local separator $\{a_1, a_2\}$ twice. As $\{a_1, a_2\}$ crosses neither $\{b_1, b_2\}$ nor $\{c_1, c_2\}$, one a_1-a_2-subpath of o must contain both vertices b_1 and b_2. As the cycle o alternates, this path must also contain one vertex c_i between b_1 and b_2 and it must contain the second vertex c_i as $\{a_1, a_2\}$ does not cross $\{c_1, c_2\}$ (in the form of condition (4) of Lemma 6.5 (Alternating Cycle Lemma)). So there is an a_1-a_2-subpath P of o containing none of the vertices b_i or c_j as interior vertices. Let o' be the cycle obtained from o by replacing the path P by a torso path. The cycle o' contains a lift of $\{b_1, b_2\}$ by construction. As the lift is unique by Remark 8.3, it must contain the lift of $\{b_1, b_2\}$. The cycle o' traverses the lift of $\{b_1, b_2\}$ twice by the 'More specifically'-part of Lemma 8.4 (Lifting Lemma). Similarly, o' traverses the lift of $\{c_1, c_2\}$ twice. So it alternates between the lifts of $\{b_1, b_2\}$ and $\{c_1, c_2\}$. Thus the lifts of $\{b_1, b_2\}$ and $\{c_1, c_2\}$ cross by Lemma 6.5 (Alternating Cycle Lemma).

Next assume that the lifts $\{b_1', b_2'\}$ and $\{c_1', c_2'\}$ of $\{b_1, b_2\}$ and $\{c_1, c_2\}$ cross. Then by condition (2) of Lemma 6.5 (Alternating Cycle Lemma) there is a cycle o' of length at most r alternating between $\{b_1', b_2'\}$ and $\{c_1', c_2'\}$ traversing $\{b_1', b_2'\}$ twice. Note that o' contains at most one torso path by Lemma 5.3. Let o be the cycle obtained from the cycle o' by replacing torso paths by paths of G of the same length. By Lemma 6.15 (Projection Lemma) the cycle o traverses the local separator $\{b_1, b_2\}$ twice. As the cycle o alternates between the local separators $\{b_1, b_2\}$ and $\{c_1, c_2\}$, these two local separators cross by Lemma 6.5 (Alternating Cycle Lemma). $\square$

Sometimes we will omit the term 'lift' and simply consider local separators of G as local separators of a graph G' obtained by cutting. The next lemma says that r-local cuttings along non-crossing 2-separators commute.

Lemma 8.11. *Let G be an r-locally 2-connected graph with distinct non-crossing r-local 2-separators $\{a_1, a_2\}$ and $\{b_1, b_2\}$. Then the graphs obtained from r-locally cutting these two local separators in either order are identical.*

Proof. Let G' be the graph obtained from G by r-locally cutting $\{a_1, a_2\}$. Let G'' be the graph obtained from G' by r-locally cutting (the lift of) $\{b_1, b_2\}$. Let G_2 be the graph obtained from G by first cutting $\{b_1, b_2\}$ and then (the lift of) $\{a_1, a_2\}$. Let $\{b_1', b_2'\}$ be the lift of $\{b_1, b_2\}$ in G'. By Lemma 8.4 (Lifting Lemma), the slices of $\{b_1, b_2\}$ in G'' and G_2 are identical. By symmetry, the same is true for slices of $\{a_1, a_2\}$. Vertices that are not slices are identical by construction. This defines a bijection between the vertices of the graphs G'' and G_2. It is straightforward to check that this bijection between the vertices extends to a bijection between the edges. $\square$

Given a set $\mathcal{S}$ of r-local 2-separators of a graph G that pairwise do not cross, we say that a graph G' is obtained from G by *r-locally cutting $\mathcal{S}$* if G' is obtained from G by the following procedure. Pick a linear ordering of the set $\mathcal{S}$. Then starting with the graph G we cut along the local separators of $\mathcal{S}$ in that linear order. By Lemma 8.4 (Lifting Lemma) and Lemma 8.10 this is well-defined. By Lemma 8.11, changing the linear ordering does not affect the graph obtained by cutting. Hence in the following we shall speak of *the* graph obtained from G by cutting along $\mathcal{S}$.

Given a graph G, by $\mathcal{N}$ we denote the set of all r-local 2-separators of G that are not crossed by any r-local 2-separator.

Theorem 8.12. *Assume G is r-locally 2-connected, and let G' be the graph obtained from G by r-locally cutting $\mathcal{N}$. Then every connected component of G' is r-locally 3-connected or a cycle of length at most r.*

Let G'' be a graph obtained from G by r-local cuttings such that all connected components are r-locally 3-connected or cycles of length at most r. Then in the construction of G'' one has to cut at any local separator X or one of its lifts for all $X \in \mathcal{N}$.

Proof. Fix a linear ordering of the set $\mathcal{N}$ and let G_i be the graph obtained from G by r-locally cutting the first i elements of $\mathcal{N}$. By Lemma 5.5 applied recursively each graph G_i is r-locally 2-connected. The graph G' is the last graph G_i.

Suppose for a contradiction that some connected component of the graph G' is neither r-locally 3-connected nor a cycle of length at most r. Then by Theorem 1.3, the graph G' has an r-local 2-separator $\{v, w\}$ that is not crossed by any other r-local 2-separator of G'. By Lemma 6.15 (Projection Lemma) applied recursively, $\{v, w\}$ is also an r-local 2-separator of the graph G. By the construction of the graph G', the local separator $\{v, w\}$ is not in the set $\mathcal{N}$. Thus there is some r-local 2-separator $\{a, b\}$ of G that crosses $\{v, w\}$. By the choice of the set $\mathcal{N}$, the local 2-separators $\{v, w\}$ and $\{a, b\}$ are not crossed by any local separator of $\mathcal{N}$. Hence we can apply Lemma 8.4 (Lifting Lemma) recursively to deduce that $\{a, b\}$ is a local separator of the graph G'. Applying Lemma 8.10 recursively yields that $\{v, w\}$ and $\{a, b\}$ are crossing in the graph G'. This is a contradiction to the fact that $\{v, w\}$ ia not crossed in G'. Thus every connected component of the graph G' is r-locally 3-connected or a cycle of length at most r.

Now let G'' be a graph obtained from G by r-local cuttings such that all connected components are r-locally 3-connected or cycles of length at most r. Let (G_i) be a sequence of graphs starting with $G_0 = G$ and ending with $G_n = G''$ such that G_{i+1} is obtained from G_i by r-local cutting. By Lemma 5.5 applied recursively each graph G_i is r-locally 2-connected.

Suppose for a contraction there is an r-local 2-separator $\{v, w\}$ of the set $\mathcal{N}$ such that neither it nor any of its lifts is identical to a local 2-separator at which we locally cut to obtain G_{i+1} from G_i.

We will show by induction that $\{v, w\}$ does not cross any local separator of any graph G_i. Assume we have shown it does not cross any local separator of a graph G_j. By Lemma 6.15 (Projection Lemma), all local 2-separators of G_{j+1} are lifts of local 2-separators of G_j. Applying this recursively yields that all local 2-separators at which we locally cut are lifts of local 2-separators of the graph G. So by Lemma 8.10 $\{v, w\}$ does not cross any local separator of the graphs G_{j+1}. This completes the induction step.

Thus the above is also true for the last graph $G_n = G''$, all of whose connected components are r-locally 3-connected or cycles of size at most r by assumption. Such graphs do not have an r-local 2-separator that is not crossed. Hence $\{v, w\}$ cannot exist. This is a contraction. So for any local separator of $\mathcal{N}$, it or one of its lifts has to appear in the construction of G''. $\square$

Proof of Theorem 2.1. This theorem is a direct consequence of Theorem 8.12. $\square$

9 Graph-Decompositions

The purpose of this section is to define graph-decompositions and explain why they can be understood as a generalisation of tree-decompositions with the decomposition-tree replaced by a general graph. See also [14].

Definition 9.1. First we need some preparation. Given a graph G, a graph F and a family $\mathcal{F}$ of subgraph-embeddings of the graph F in G, the graph obtained from G by *identifying along* $\mathcal{F}$ is the graph obtained from G by identifying all elements of $\mathcal{F}$. In this paper, all embeddings of the graph F will be vertex-disjoint, although we do not make this part of the definition. Formally, the vertex set of this new graph is the vertex set of G modulo the equivalence relation generated by the relation where two vertices v_1 and v_2 are related if there are graphs $F_1, F_2 \in \mathcal{F}$ with $v_i \in F_i$ (for $i = 1, 2$) such that after applying the isomorphisms to F the vertices v_1 and v_2 are equal to the same vertex of F. The edges of the quotient-graph are the edges of G, where the endvertices are the equivalence classes of the original endvertices of G – with the following exception: if two vertices v and w in F are joined by edge, then in the quotient-graph we keep only one copy of all the edges of G between the copies of v and w in the graphs $F' \in \mathcal{F}$. This completes the definition of gluing. Examples of gluing are given in Figure 8 and Figure 9.

Remark 9.2. Even if the graph G has no loops or parallel edges, graphs obtained from G by identifying along a family may have loops or parallel edges, see Figure 9.

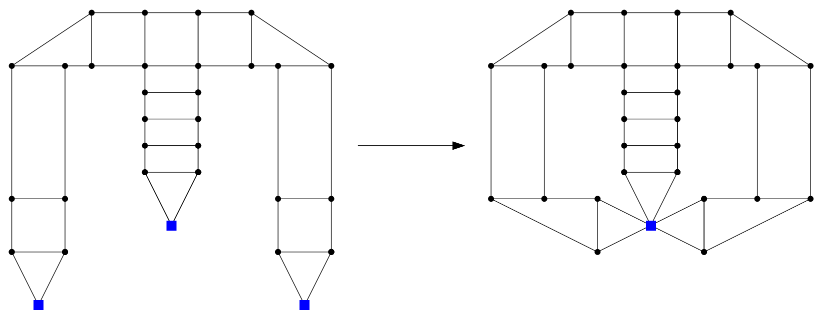

Figure 8: An example of a gluing. Here the graph F consists of a single vertex and the family $\mathcal{F}$ has three members, which are marked in blue. The graph G before the gluing is depicted on the left. The graph after the gluing is depicted on the right.

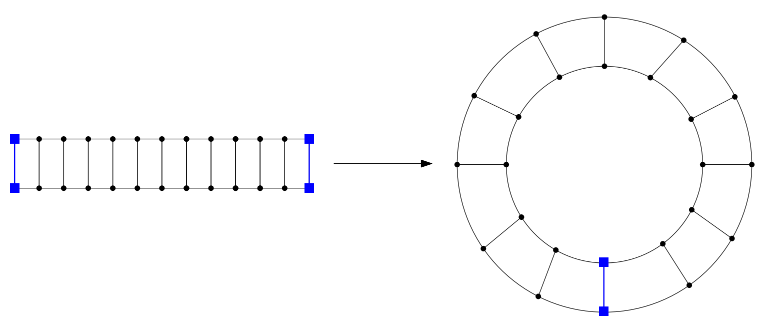

Figure 9: An example of a gluing. Here the graph F consists of two vertices joined by an edge and the family $\mathcal{F}$ has two members, which are marked in blue. The graph G before the gluing is depicted on the left. The graph after the gluing is depicted on the right. If the graph F consisted just of the two vertices without the edge, the gluing would be the graph obtained by the graph on the right by adding an edge in parallel to the blue edge. (This figure is the same as Figure 4.)

Definition 9.3 (Graph-decomposition)**.** A *graph-decomposition* consists of a bipartite graph (B, S) with bipartition classes B and S, where the elements of B are referred to as 'bags-nodes' and the elements of S are referred to as 'separating-nodes'. This bipartite graph is referred to as the 'decomposition graph'. For each node x of the decomposition graph, there is a graph G_x associated to x. Moreover for every edge e of the decomposition graph from a separating-node s to a bag-node b, there is a map ι_e that maps the associated graph G_s to a subgraph of the associated graph G_b. We refer to G_s with $s \in S$ as a *local separator* and to G_b with $b \in B$ as a *bag*.

The *underlying graph* of a graph-decomposition $(G_x|x \in V(B, S))$ is constructed from the disjoint union of the bags G_b with $b \in B$ by identifying along all the families given by the copies of the graphs G_s for $s \in S$. Formally, for each separating-node $s \in S$, its family is $(\iota_e(G_s)))$, where the index ranges over the edges of (B, S) incident with s.

Remark 9.4. For some applications, it may be attractive to not construct the underlying graph all at once, as we do here, but rather 'step by step' by doing the gluing in a certain order. We expect that one essentially gets the same graph in the natural cases, independently of the ordering. As gluing is the reverse operation to local splitting and for local splitting we have proved commutativity (compare Lemma 8.11), for the graph-decompositions related to Theorem 1.2 this is indeed the case. However, we require the technical tool of contacts to formalise this. We shall not use this in this paper.

Now we perform the identification for all these families separately. We remark that different orderings in which we perform these identification result in the same graph.

The *width* of a graph-decomposition $(G_x|x \in V(B, S))$ is the maximal vertex number of a bag G_b with $b \in B$ take away one[14]. The *adhesion* of a graph-decomposition $(G_x|x \in V(B, S))$ is the maximum vertex number of a local separator G_s with $s \in S$. This completes the definition of graph-decompositions and related concepts.

Example 9.5. Essentially, tree-decompositions are examples of graph-decompositions. Indeed, given a tree-decomposition, one obtains a new tree-decomposition by subdividing every edge once and associating to each new vertex the separator associated to that edge (this separator is given by taking the intersection of the two bags at the endvertices of that edge).

[14]This convention to take away one is common in the literature. Consequently, trees have tree-width one.

This defines a graph-decomposition whose decomposition-graph is the decomposition-tree of this newly constructed tree-decomposition. Its bags are the original bags of that tree-decomposition and its local separators are separators of the old tree-decomposition; that is, the bags associated to the new vertices of the new tree-decomposition.

The notions of width and adhesion as defined above for graph-decompositions whose decomposition graphs are trees coincide with the standard notions for tree-decompositions when interpreted as graph-decompositions in the way explained above.

Example 9.6. The graph on the right of Figure 8 has a graph-decomposition with only one bag which is given by the graph on the left, and only one local separator, which is given by the blue vertex. Its decomposition-graph is the bipartite graph consisting of three edges in parallel.

Example 9.7. The graph on the right of Figure 9 has a graph-decomposition with only one bag which is given by the graph on the left, and only one local separator, which consists of an edge. Its decomposition-graph is the bipartite graph consisting of two edges in parallel.

Definition 9.8. This definition is slightly technical. For this definition consider graph-decompositions such that for every separating-node s there are no two distinct embedding maps ι_f that map vertices of s to the same vertex or endvertices of the same edge (informally, this means that the images of s under the maps ι_f are sufficiently far away from each other. For sets s we are interested in here – those that come from r-local 2-separators – this will always be the case).

Given a graph-decomposition of a graph G and an edge f of the decomposition-graph incident with a separating-node s and a bag-node b, the *local cut* associated to f consists of those edges of G that have precisely one endvertex in $\iota_f(G_s)$ and the other endvertex in the bag $\iota_f(G_b)$. We say that a cycle *traverses the local cut associated to the edge f oddly* if it contains an odd number of edges from that local cut.

Definition 9.9. A graph-decomposition has *locality* r if every cycle traversing a local cut corresponding to an edge of the decomposition-graph oddly[15] has length larger than r.

[15]An alternative definition would be to replace 'traversing oddly' by 'traversing effectively zero', taking additionally orientations of traversals into account. For simplicity we just make the definition with 'oddly'.

Example 9.10. In graph-decompositions whose decomposition graph is a tree all cycles traverse evenly and hence their locality is infinite.

Example 9.11. The locality of the graph-decomposition described in Figure 9 is 11, as the shortest cycle traversing oddly has length 12. The locality of the graph-decomposition described in Figure 8 is 9.

There is a correspondence between nested sets of separations and tree-decompositions. A corresponding fact for graph-decompositions is also true. Here we only need the following special case of this correspondence.

Lemma 9.12. *Let G be a graph with a set S of non-crossing r-local 2-separators. Let B be the set of connected components of the graph obtained from G by r-locally cutting along S.*

Then there is a decomposition graph with bipartition (B, S) of a graph-decomposition of G of adhesion two and locality r.

Proof. For every local component of a local separator G_S with $s \in S$ we have an edge e in the graph (B, S) from s to the unique bag-node $b \in B$ such that G_b contains the slices of $\iota_e(G_s)$ for that local component. The map ι_e maps G_s to this copy of G_s in G_b. So (B, S) is the decomposition graph of a graph-decomposition of G. It has adhesion two as all elements G_s with $s \in S$ have size two. It has locality r as all local separators G_s with $s \in S$ are r-local. $\square$

A *torso* of a bag G_b of a graph-decomposition is obtained from G_b by joining for every map ι from a local separator G_s any two vertices in the image of ι by an edge whenever sb is an edge of the decomposition graph.

Proof of Theorem 1.2. Combine Lemma 9.12 and Theorem 8.12. $\square$

10 Outlook

Having finished the proof of the local 2-separator theorem, we outline possible ways in which this theorem could be extended. We continue our investigation of local separators in [11] by proving a local version of the tangle tree theorem. Another direction, might be to prove a matroidal analogue of our local 2-separator theorem.

Question 10.1. *Can you prove a local 2-separator theorem for (representable) matroids that is reminiscent of Theorem 1.2?*

A natural next step would be to prove a local version of the Grohe-Decomposition-Theorem, which gives a decomposition of a 3-connected graph into 'quasi 4-connected components' [19]. We hope that with the methods of this paper one should be able to prove the following.

It is natural to try to extend the notion of local 2-separators to local 3-separators using explorer-neighbourhoods around three vertices; this is work in progress [11]. As for local 2-separators the intuition for this comes from separators of the r-local cover. I think that understanding local separators through separators of the r-local cover is an exciting area for future research. Given a parameter $s \in \mathbb{N} \cup \{\infty\}$, we say that a graph is *s-locally quasi 4-connected* if for every s-local 3-separator all but one of its components (that is, components of the punctured explorer-neighbourhood) contain at most one vertex.

Conjecture 10.2. *For every parameter $r \in \mathbb{N} \cup \{\infty\}$ there is a parameter s such that every s-locally 3-connected graph G has a graph-decomposition of locality at least r and adhesion at most 3 such that all its torsos are minors of G that are either r-locally quasi-4-connected or a complete graph of order at most 4.*

Remark 10.3. We do not conjecture any relationship between r and s, and even $r = s$ may be possible.

References

[1] Aaron B Adcock, Blair D Sullivan, and Michael W Mahoney. Tree-like structure in large social and information networks. In *2013 IEEE 13th International Conference on Data Mining*, pages 1–10. IEEE, 2013.

[2] Aaron B Adcock, Blair D Sullivan, and Michael W Mahoney. Tree decompositions and social graphs. *Internet Mathematics*, 12(5):315–361, 2016.

[3] Hyman Bass. Covering theory for graphs of groups. *Journal of pure and applied algebra*, 89(1-2):3–47, 1993.

[4] Itai Benjamini and David Ellis. On the structure of graphs which are locally indistinguishable from a lattice. In *Forum of Mathematics, Sigma*, volume 4. Cambridge University Press, 2016.

[5] Itai Benjamini and David Ellis. On the structure of random graphs with constant r-balls. *arXiv preprint: arXiv:1802.02002*, 2018.

[6] Itai Benjamini and Oded Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6:no. 23, 13, 2001.

[7] Itai Benjamini, Oded Schramm, and Asaf Shapira. Every minor-closed property of sparse graphs is testable. *Advances in mathematics*, 223(6):2200–2218, 2010.

[8] Christian Borgs, Jennifer T Chayes, Henry Cohn, and Nina Holden. Sparse exchangeable graphs and their limits via graphon processes. *The Journal of Machine Learning Research*, 18(1):7740–7810, 2017.

[9] Johannes Carmesin. Characterising graphs with no subdivision of a wheel of bounded diameter. Preprint, available at `http://web.mat.bham.ac.uk/J.Carmesin/`.

[10] Johannes Carmesin. Local 2-separators. Preprint, available at `http://web.mat.bham.ac.uk/J.Carmesin/`.

[11] Johannes Carmesin. Local k-separators. In preparation.

[12] François Caron and Emily B Fox. Sparse graphs using exchangeable random measures. *Journal of the Royal Statistical Society. Series B, Statistical Methodology*, 79(5):1295, 2017.

[13] William H Cunningham and Jack Edmonds. A combinatorial decomposition theory. *Canadian Journal of Mathematics*, 32(3):734–765, 1980.

[14] Reinhard Diestel and Daniela Kühn. Graph minor hierarchies. *Discrete Applied Mathematics*, 145(2):167–182, 2005.

[15] Jiří Fiala, Pavel Klavík, Jan Kratochvíl, and Roman Nedela. 3-connected reduction for regular graph covers. *European Journal of Combinatorics*, 73:170–210, 2018.

[16] Markus Frick and Martin Grohe. Deciding first-order properties of locally tree-decomposable structures. *Journal of the ACM (JACM)*, 48(6):1184–1206, 2001.

[17] Agelos Georgakopoulos. On covers of graphs by cayley graphs. *European Journal of Combinatorics*, 64:57–65, 2017.

[18] Martin Grohe. Local tree-width, excluded minors, and approximation algorithms. *Combinatorica*, 23(4):613–632, 2003.

[19] Martin Grohe. Quasi-4-connected components. *ICALP*, 2016.

[20] Jonathan L Gross and Thomas W Tucker. *Topological graph theory*. Courier Corporation, 2001.

[21] Ori Gurel-Gurevich and Asaf Nachmias. Recurrence of planar graph limits. *Annals of Mathematics*, pages 761–781, 2013.

[22] Matthias Hamann. The classification of finite and locally finite connected-homogeneous digraphs. *Combinatorica*, 37(2):183–222, 2017.

[23] Matthias Hamann and Fabian Hundertmark. The classification of connected-homogeneous digraphs with more than one end. *Transactions of the American Mathematical Society*, 365(1):531–553, 2013.

[24] Allen Hatcher. *Algebraic Topology*. Cambridge Univ. Press, 2002.

[25] Tue Herlau, Mikkel N Schmidt, and Morten Mørup. Completely random measures for modelling block-structured sparse networks. In *Advances in Neural Information Processing Systems*, pages 4260–4268, 2016.

[26] Jin Ho Kwak and Roman Nedela. Graphs and their coverings. *Lecture Notes Series*, 17, 2007.

[27] László Lovász. *Large networks and graph limits*, volume 60. American Mathematical Soc., 2012.

[28] László Lovász and Balázs Szegedy. Arbeitsgemeinschaft mit aktuellem thema: Limits of discrete structures.

[29] Jaroslav Nešetřil and Patrice Ossona de Mendez. *Sparsity: graphs, structures, and algorithms*. Springer, 2012.

[30] Sang-il Oum. Rank-width and vertex-minors. *J. Combin. Theory, Series B*, 95(1):79–100, 2005.

[31] James Oxley. *Matroid Theory* (2nd edition). Oxford University Press, 2011.

[32] Neil Robertson and Paul D Seymour. Graph minors. X. Obstructions to tree-decompositions. *J. Combin. Theory, Series B*, 52:153–190, 1991.

[33] Neil Robertson and Paul D Seymour. Graph minors. XX. Wagner's conjecture. *J. Combin. Theory Ser. B*, 92(2):325–357, 2004.

[34] Neil Robertson, Robin Thomas, Maria Chudnovsky, and Paul D Seymour. The strong perfect graph theorem. *Annals of mathematics*, 164(1):51–229, 2006.

[35] Satu Elisa Schaeffer. Graph clustering. *Computer science review*, 1(1):27–64, 2007.

[36] Jean-Pierre Serre. *Trees*. Springer-Verlag, 1980. (Translated from the French by John Stillwell.).

[37] Paul D Seymour. Decomposition of regular matroids. *J. Combin. Theory, Series B*, 28(3):305–359, 1980.

[38] Ulrike Von Luxburg. A tutorial on spectral clustering. *Statistics and computing*, 17(4):395–416, 2007.