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DOI:
10.1002/rsa.21126

HYPERGRAPH REGULARITY AND RANDOM SAMPLING

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Abstract. Suppose that a k-uniform hypergraph H satisfies a certain regularity instance (that is, there is a partition of H given by the hypergraph regularity lemma into a bounded number of quasirandom subhypergraphs of prescribed densities). We prove that with high probability a large enough uniform random sample of the vertex set of H also admits the same regularity instance. Here the crucial feature is that the error term measuring the quasirandomness of the subhypergraphs requires only an arbitrarily small additive correction. This has applications to combinatorial property testing. The graph case of the sampling result was proved by Alon, Fischer, Newman and Shapira.

1. Introduction

Szemerédi's regularity lemma [27] is one of the most important results in discrete mathematics and has numerous applications. Roughly speaking, it states that every graph can be partitioned into bounded number of vertex sets so that most of bipartite graphs between the parts are random-like. To be more precise, it says that for every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists an $M(\varepsilon, m)$ such that every large enough graph G admits an equipartition $V_1, \dots, V_t$ of its vertex set such that $m \leq t \leq M$ and all but at most εt^2 pairs V_i, V_j induce an ε-regular pair in G; as usual, a bipartite graph with vertex sets A, B is ε-regular if it is (ε, d)-regular for some $d \in [0, 1]$. The latter means that $|d - e(A', B')(|A'||B'|)^{-1}| \leq \varepsilon$ for all subsets $A' \subseteq A, B' \subseteq B$ with $|A'| \geq \varepsilon|A|, |B'| \geq \varepsilon|B|$.

Being ε-regular encapsulates random-like behaviour. Much of the information about the graph is captured by the vertex partition $V_1, \dots, V_t$ together with the densities $d(V_i, V_j) := e(V_i, V_j)(|V_i||V_j|)$ between the pairs $ij \in \binom{[t]}{2}$ of parts. Consequently, it turned out to be interesting in many occasions to decide whether a graph G on n vertices satisfies a particular *regularity instance* $R_\varepsilon(\mathbf{d})$ which is defined as follows.

Definition 1.1. *For given $\varepsilon > 0$, $t \in \mathbb{N} \setminus \{1\}$ and $\mathbf{d} \in [0, 1]^{\binom{[t]}{2}}$, we say that a graph G satisfies a regularity instance $R_\varepsilon(\mathbf{d})$ if G admits an equipartition $V_1, \dots, V_t$ such that V_i, V_j is an $(\varepsilon, \mathbf{d}(ij))$-regular pair for each $ij \in \binom{[t]}{2}$.*

The regularity lemma was further extended to hypergraphs in the ground-breaking work of Rödl and Frankl [12] (who proved the 3-uniform case), Rödl and Skokan [24] (who proved the r-uniform case, with the corresponding counting lemma proved by Rödl and Schacht [22]) as well as by Gowers [15, 16]. This theory was further developed in e.g. Rödl and Schacht [23], Tao [28], Allen, Böttcher, Cooley and Mycroft [1], Nagle, Rödl and Schacht [21], Allen, Davies and Skokan [2], Moshkovitz and Shapira [19]. In particular, in [23], Rödl and Schacht proved the so-called 'regular approximation lemma', which is a powerful version of the hypergraph regularity lemma and which is of central importance to our proof.

The hypergraph regularity lemma guarantees that every (large) k-uniform hypergraph admits a partition of its edge set, where most classes of the partition consist of 'regularly distributed' edges. The appropriate notion for being regularly distributed is significantly more complicated than in the graph case and it took two decades until a suitable notion was found

Date: August 11, 2022.

The research leading to these results was partially supported by the EPSRC, grant nos. EP/M009408/1 (F. Joos, D. Kühn and D. Osthus), and EP/N019504/1 (D. Kühn). The research was also partially supported by the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013) / ERC Grant 306349 (J. Kim and D. Osthus).

and a corresponding theory was established. At a high level, many concepts from graph regularity carry over to hypergraphs. In particular, one can define regularity instances for hypergraphs as a sequence of densities recorded from the cluster structure provided by the hypergraph regularity lemma. Again in many places it has turned out to be fruitful to work with the much less complex regularity instance instead of the hypergraph itself.

Another very natural way to study the properties of a given discrete structure is via random sampling. By examining a small random sample S of a combinatorial object $\mathcal{O}$, can we determine (with high probability) whether $\mathcal{O}$ has a specific property $\mathbf{P}$ or whether it is far from satisfying $\mathbf{P}$? Questions of this type are known as property testing and have been intensively studied.

Rubinfeld and Sudan [25] introduced property testing and Goldreich, Goldwasser and Ron [13] obtained results regarding k-colourability, max-cut and more general graph partitioning problems. (In fact, these results are preceded by the famous triangle removal lemma of Ruzsa and Szemerédi [26], which can be rephrased in terms of testability of triangle-freeness.) This list of problems was greatly extended, see [4, 5, 6, 7, 11, 14]. This sequence of results in property testing culminated in the result of Alon, Fischer, Newman and Shapira [5] who obtained a complete combinatorial characterization of all testable graph properties. This solved a problem posed in [13], which was regarded as one of the main open problems in the area. In fact, the combinatorial characterization says that a property is testable if and only if it can be decided whether a graph has the property solely by considering the regularity instances it satisfies. A key step in their approach is the result that one can efficiently test whether a graph satisfies a particular regularity instance.

Therefore, this suggests that is also of high importance to prove a similar result for hypergraphs. Duke and Rödl [10] proved that randomly sampled subgraphs of a dense and regular pair (V_i, V_j) almost surely span an edge. Alon, de la Vega, Kannan and Karpinski [3] as well as Mubayi and Rödl [20] then proved a stronger 'inheritance' result in the setting of uniform hypergraphs, i.e. with high probability uniform edge-distribution is inherited by random samples. Further generalizing these results, Czygrinow and Nagle [9] proved that if a hypergraph satisfies a regularity instance, then with high probability a randomly sampled hypergraph also satisfies 'the same' regularity instance. However, their result has the disadvantage that their argument gives polynomial regularity dependence which we sharpen to nearly best possible, up to an additive correction. We also reverse the inference in the result of Czygrinow and Nagle [9], that regularity instances of sampled subhypergraphs predict that of the host hypergraph.

As the precise definition of a regularity instance for hypergraphs is fairly involved, let us now only state our result for graphs; in fact, Theorem 1.2 is proved by Alon, Fischer, Newman and Shapira [5] as a crucial tool for their characterization of testable graph properties.

Theorem 1.2 ([5])**.** *For all $\varepsilon > 0$ and all $\delta > 0$ that are small in terms of ε, there exists $c > 0$ such that for all sufficiently large n and q with $n \geq q$, the following holds. Suppose (ε, t, d_t) is a regularity instance and suppose G is a graph on n vertices with vertex set V. Let $Q \in \binom{V}{q}$ be chosen uniformly at random. Then with probability at least $1 - e^{-cq}$ the following hold.*

- *If G satisfies the regularity instance (ε, t, d_t), then $G[Q]$ satisfies the regularity instance $(\varepsilon + \delta, t, d_t)$.*
- *If $G[Q]$ satisfies the regularity instance (ε, t, d_t), then G satisfies the regularity instance $(\varepsilon + \delta, t, d_t)$.*

The main result of this paper is a hypergraph version for Theorem 1.2. In another paper [17], we exploit this theorem to prove a combinatorial characterization for testable hypergraph properties. For this it is again crucial that a regularity instance can be tested with an arbitrarily small additive error. It is by no means clear that one can generalize the results of [5] to hypergraphs and obtain such a characterization of testable hypergraph properties: Austin and Tao [8] showed that for the stronger but related notion of 'repairability' the graph results do not extend to hypergraphs; see [8, 17] for a more detailed discussion.

It is beyond the scope of an introduction to present the precise statement of such a result as first several notions concerning hypergraph regularity have to be introduced. We therefore defer the statement of our main result to Section 4.

In order to prove our main result, we prove Lemma 6.1, which is a strengthening of a version of the hypergraph regularity lemma. This is a strengthening of a regular approximation lemma (Lemma 4.1) proved by Rödl and Schacht [23]. The latter is a version of the regularity lemma which of a given hypergraph H guarantees that by modifying a small proportion of the hyperedges one can obtain a hypergraph H' which has a 'high' quality regularity lemma partition, that is, with very small error terms. We believe that Lemma 6.1 is also of independent interest and will have additional applications. In particular, we apply it in [17] to derive from our testability characterization that the max cut problem is testable.

2. Concepts and tools

In this section we introduce the main concepts and tools (mainly concerning hypergraph regularity partitions) which form the basis of our approach. The constants in the hierarchies used to state our results have to be chosen from right to left. More precisely, if we claim that a result holds whenever $1/n \ll a \ll b \leq 1$ (where $n \in \mathbb{N}$ is typically the number of vertices of a hypergraph), then this means that there are non-decreasing functions $f : (0,1] \to (0,1]$ and $g : (0,1] \to (0,1]$ such that the result holds for all $0 < a, b \leq 1$ and all $n \in \mathbb{N}$ with $a \leq f(b)$ and $1/n \leq g(a)$. For a vector $\mathbf{x} = (\alpha_1, \dots, \alpha_\ell)$, we let $\mathbf{x}_* := \{\alpha_1, \dots, \alpha_\ell\}$ and write $\|\mathbf{x}\|_\infty = \max_{i \in [\ell]}\{\alpha_i\}$. We say a set E is an *i-set* if $|E| = i$. Unless stated otherwise, in the partitions considered in this paper, we allow some of the parts to be empty.

2.1. Hypergraphs.

In the following we introduce several concepts about a hypergraph H. We typically refer to $V = V(H)$ as the vertex set of H and usually let $n := |V|$. Given a hypergraph H and a set $Q \subseteq V(H)$, we denote by $H[Q]$ the hypergraph induced on H by Q. For two k-graphs G, H on the same vertex set, we often refer to $|G \triangle H|$ as the *distance* between G and H. If the vertex set of H has a partition $\{V_1, \dots, V_\ell\}$, we simply refer to H as a hypergraph on $\{V_1, \dots, V_\ell\}$.

A partition $\{V_1, \dots, V_\ell\}$ of V is an *equipartition* if $|V_i| = |V_j| \pm 1$ for all $i, j \in [\ell]$. For a partition $\{V_1, \dots, V_\ell\}$ of V and $k \in [\ell]$, we denote by $K_\ell^{(k)}(V_1, \dots, V_\ell)$ the *complete ℓ-partite k-graph* with vertex classes $V_1, \dots, V_\ell$. Let $0 \leq \lambda < 1$. If $|V_i| = (1 \pm \lambda)m$ for every $i \in [\ell]$, then an *(m, ℓ, k, λ)-graph* H on $\{V_1, \dots, V_\ell\}$ is a spanning subgraph of $K_\ell^{(k)}(V_1, \dots, V_\ell)$. For notational convenience, we consider the vertex partition $\{V_1, \dots, V_\ell\}$ as an $(m, \ell, 1, \lambda)$-graph. If $|V_i| \in \{m, m+1\}$, we drop λ and simply refer to (m, ℓ, k)-graphs. Similarly, if the value of λ is not relevant, then we say $H \subseteq K_\ell^{(k)}(V_1, \dots, V_\ell)$ is an $(m, \ell, k, *)$-graph.

Given an $(m, \ell, k, *)$-graph H on $\{V_1, \dots, V_\ell\}$, an integer $k \leq i \leq \ell$ and a set $\Lambda_i \in \binom{[\ell]}{i}$, we set $H[\Lambda_i] := H[\bigcup_{\lambda' \in \Lambda_i} V_{\lambda'}]$. If $2 \leq k \leq i \leq \ell$ and H is an $(m, \ell, k, *)$-graph, we denote by $\mathcal{K}_i(H)$ the family of all i-element subsets I of $V(H)$ for which $H[I] \cong K_i^{(k)}$, where $K_i^{(k)}$ denotes the complete k-graph on i vertices.

If $H^{(1)}$ is an $(m, \ell, 1, *)$-graph and $i \in [\ell]$, we denote by $\mathcal{K}_i(H^{(1)})$ the family of all i-element subsets I of $V(H^{(1)})$ which 'cross' the partition $\{V_1, \dots, V_\ell\}$; that is, $I \in \mathcal{K}_i(H^{(1)})$ if and only if $|I \cap V_s| \leq 1$ for all $s \in [\ell]$.

We will consider hypergraphs of different uniformity on the same vertex set. Given an $(m, \ell, k-1, \lambda)$-graph $H^{(k-1)}$ and an (m, ℓ, k, λ)-graph $H^{(k)}$ on the same vertex set, we say $H^{(k-1)}$ *underlies* $H^{(k)}$ if $H^{(k)} \subseteq \mathcal{K}_k(H^{(k-1)})$; that is, for every edge $e \in H^{(k)}$ and every $(k-1)$-subset f of e, we have $f \in H^{(k-1)}$. If we have an entire cascade of underlying hypergraphs we refer to this as a complex. More precisely, let $m \geq 1$ and $\ell \geq k \geq 1$ be integers. An (m, ℓ, k, λ)-*complex* $\mathcal{H}$ on $\{V_1, \dots, V_\ell\}$ is a collection of (m, ℓ, j, λ)-graphs $\{H^{(j)}\}_{j=1}^k$ on $\{V_1, \dots, V_\ell\}$ such that $H^{(j-1)}$ underlies $H^{(j)}$ for all $i \in [k] \setminus \{1\}$, that is, $H^{(j)} \subseteq \mathcal{K}_j(H^{(j-1)})$.

Again, if $|V_i| \in \{m, m+1\}$, then we simply drop λ and refer to such a complex as an (m, ℓ, k)-complex. If the value of λ is not relevant, then we say that $\{H^{(j)}\}_{j=1}^k$ is an $(m, \ell, k, *)$-complex. A collection of hypergraphs is a *complex* if it is an $(m, \ell, k, *)$-complex for some integers m, ℓ, k.

When m is not of primary concern, we refer to (m, ℓ, k, λ)-graphs and (m, ℓ, k, λ)-complexes simply as (ℓ, k, λ)-graphs and (ℓ, k, λ)-complexes, respectively. Again, we also omit λ if $|V_i| \in \{m, m+1\}$ and refer to (ℓ, k)-graphs and (ℓ, k)-complexes and we write the symbol '$*$' instead of λ if λ is not relevant.

Note that there is no ambiguity between an (ℓ, k, λ)-graph and an (m, ℓ, k)-graph (and similarly for complexes) as $\lambda < 1$.

Suppose $n \geq \ell \geq k$ and suppose H is an n-vertex k-graph and F is an ℓ-vertex k-graph. We define $\mathbf{Pr}(F, H)$ such that $\mathbf{Pr}(F, H)\binom{n}{\ell}$ equals the number of induced copies of F in H. For a collection $\mathcal{F}$ of ℓ-vertex k-graphs, we define $\mathbf{Pr}(\mathcal{F}, H)$ such that $\mathbf{Pr}(\mathcal{F}, H)\binom{n}{\ell}$ equals the number of induced ℓ-vertex k-graphs F in H such that $F \in \mathcal{F}$.

2.2. **Hypergraph regularity.** In this subsection we introduce ε-regularity for hypergraphs. Suppose $\ell \geq k \geq 2$ and $V_1, \dots, V_\ell$ are pairwise disjoint vertex sets. Let $H^{(k)}$ be an $(\ell, k, *)$-graph on $\{V_1, \dots, V_\ell\}$, let $\{i_1, \dots, i_k\} \in \binom{[\ell]}{k}$, and let $H^{(k-1)}$ be a $(k, k-1, *)$-graph on $\{V_{i_1}, \dots, V_{i_k}\}$. We define the *density of $H^{(k)}$ with respect to $H^{(k-1)}$* as

$$d(H^{(k)} \mid H^{(k-1)}) := \begin{cases} \frac{|H^{(k)} \cap \mathcal{K}_k(H^{(k-1)})|}{|\mathcal{K}_k(H^{(k-1)})|} & \text{if } |\mathcal{K}_k(H^{(k-1)})| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\varepsilon > 0$ and $d \geq 0$. We say $H^{(k)}$ is *(ε, d)-regular with respect to $H^{(k-1)}$* if for all $Q^{(k-1)} \subseteq H^{(k-1)}$ with

$$|\mathcal{K}_k(Q^{(k-1)})| \geq \varepsilon |\mathcal{K}_k(H^{(k-1)})|, \text{ we have } |H^{(k)} \cap \mathcal{K}_k(Q^{(k-1)})| = (d \pm \varepsilon)|\mathcal{K}_k(Q^{(k-1)})|.$$

Note that if $H^{(k)}$ is (ε, d)-regular with respect to $H^{(k-1)}$ and $H^{(k-1)} \neq \emptyset$, then we have $d(H^{(k)} \mid H^{(k-1)}) = d \pm \varepsilon$. We say $H^{(k)}$ is *ε-regular with respect to $H^{(k-1)}$* if it is (ε, d)-regular with respect to $H^{(k-1)}$ for some $d \geq 0$.

We say an $(\ell, k, *)$-graph $H^{(k)}$ on $\{V_1, \dots, V_\ell\}$ is *(ε, d)-regular with respect to an $(\ell, k-1, *)$-graph $H^{(k-1)}$* on $\{V_1, \dots, V_\ell\}$ if for every $\Lambda \in \binom{[\ell]}{k}$ $H^{(k)}$ is (ε, d)-regular with respect to the restriction $H^{(k-1)}[\Lambda]$.

Let $\mathbf{d} = (d_2, \dots, d_k) \in \mathbb{R}_{\geq 0}^{k-1}$. We say an $(\ell, k, *)$-complex $\mathcal{H} = \{H^{(j)}\}_{j=1}^k$ is *$(\varepsilon, \mathbf{d})$-regular* if $H^{(j)}$ is (ε, d_j)-regular with respect to $H^{(j-1)}$ for every $j \in [k] \setminus \{1\}$. We sometimes simply refer to a complex as being ε-regular if it is $(\varepsilon, \mathbf{d})$-regular for some vector $\mathbf{d}$.

2.3. **Partitions of hypergraphs and the regular approximation lemma.** The regular approximation lemma of Rödl and Schacht implies that for all k-graphs H, there exists a k-graph G which is very close to H and so that G has a very 'high quality' partition into ε-regular subgraphs. To state this formally we need to introduce further concepts involving partitions of hypergraphs.

Suppose $A \supseteq B$ are finite sets, $\mathscr{A}$ is a partition of A, and $\mathscr{B}$ is a partition of B. We say $\mathscr{A}$ *refines* $\mathscr{B}$ and write $\mathscr{A} \prec \mathscr{B}$ if for every $\mathcal{A} \in \mathscr{A}$ there either exists $\mathcal{B} \in \mathscr{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$ or $\mathcal{A} \subseteq A \setminus B$. The following definition concerns 'approximate' refinements. Let $\nu \geq 0$. We say that $\mathscr{A}$ *ν-refines* $\mathscr{B}$ and write $\mathscr{A} \prec_\nu \mathscr{B}$ if there exists a function $f : \mathscr{A} \to \mathscr{B} \cup \{A \setminus B\}$ such that

$$\sum_{\mathcal{A} \in \mathscr{A}} |\mathcal{A} \setminus f(\mathcal{A})| \leq \nu |A|.$$

We make the following observations.

- *$\mathscr{A} \prec \mathscr{B}$ if and only if $\mathscr{A} \prec_0 \mathscr{B}$.*
- *Suppose $\mathscr{A}, \mathscr{A}', \mathscr{A}''$ are partitions of A, A', A'' respectively and $A'' \subseteq A' \subseteq A$. If $\mathscr{A} \prec_\nu \mathscr{A}'$ and $\mathscr{A}' \prec_{\nu'} \mathscr{A}''$, then $\mathscr{A} \prec_{\nu+\nu'} \mathscr{A}''$.* (2.1)

We now introduce the concept of a polyad. Roughly speaking, given a vertex partition $\mathscr{P}^{(1)}$, an i-polyad is an i-graph which arises from a partition $\mathscr{P}^{(i)}$ of the complete partite i-graph $\mathcal{K}_i(\mathscr{P}^{(1)})$. The $(i+1)$-cliques spanned by all the i-polyads give rise to a partition $\mathscr{P}^{(i+1)}$ of $\mathcal{K}_{i+1}(\mathscr{P}^{(1)})$ (see Definition 2.1). Such a 'family of partitions' then provides a suitable framework for describing a regularity partition (see Definition 2.3).

Suppose we have a vertex partition $\mathscr{P}^{(1)} = \{V_1, \dots, V_\ell\}$ and $\ell \geq k$. For integers $k \leq \ell' \leq \ell$, we say that a hypergraph H is an *$(\ell', k, *)$-graph with respect to $\mathscr{P}^{(1)}$* if it is an $(\ell', k, *)$-graph on $\{V_i : i \in \Lambda\}$ for some $\Lambda \in \binom{[\ell]}{\ell'}$.

Recall that $\mathcal{K}_j(\mathscr{P}^{(1)})$ is the family of all crossing j-sets with respect to $\mathscr{P}^{(1)}$. Suppose that for all $i \in [k-1] \setminus \{1\}$, we have partitions $\mathscr{P}^{(i)}$ of $\mathcal{K}_i(\mathscr{P}^{(1)})$ such that each part of $\mathscr{P}^{(i)}$ is an $(i, i, *)$-graph with respect to $\mathscr{P}^{(1)}$. By definition, for each i-set $I \in \mathcal{K}_i(\mathscr{P}^{(1)})$, there exists exactly one $P^{(i)} = P^{(i)}(I) \in \mathscr{P}^{(i)}$ so that $I \in P^{(i)}$. Consider $j \in [\ell]$ and any $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$. For each $i \in [\max\{j, k-1\}]$, the *i-polyad* $\hat{P}^{(i)}(J)$ of J is defined by

$$\hat{P}^{(i)}(J) := \bigcup \left\{ P^{(i)}(I) : I \in \binom{J}{i} \right\}. \tag{2.2}$$

Thus $\hat{P}^{(i)}(J)$ is a (j, i)-graph with respect to $\mathscr{P}^{(1)}$. Moreover, let

$$\hat{\mathcal{P}}(J) := \left\{ \hat{P}^{(i)}(J) \right\}_{i=1}^{\max\{j,k-1\}}, \tag{2.3}$$

and for $j \in [k-1]$, let

$$\hat{\mathscr{P}}^{(j)} := \left\{ \hat{P}^{(j)}(J) : J \in \mathcal{K}_{j+1}(\mathscr{P}^{(1)}) \right\}. \tag{2.4}$$

We note that $\hat{\mathscr{P}}^{(1)}$ is the set consisting of all $(2,1)$-graphs with vertex classes V_s, V_t (for all distinct $s, t \in [\ell]$). In other words, each element of $\hat{\mathscr{P}}^{(1)}$ is a 1-graph with the vertex set $V_s \cup V_t$ and the edge set $V_s \cup V_t$. Also note that for any $\hat{P}^{(j)} \in \hat{\mathscr{P}}^{(j)}$, we have $\mathcal{K}_{j+1}(\hat{P}^{(j)}) \neq \emptyset$. Indeed, if $\hat{P}^{(j)} \in \hat{\mathscr{P}}^{(j)}$, it follows that there is a set $J \in \mathcal{K}_{j+1}(\mathscr{P}^{(1)})$ such that $\hat{P}^{(j)} = \hat{P}^{(j)}(J)$ and $J \in \mathcal{K}_{j+1}(\hat{P}^{(j)}(J))$.

The above definitions apply to arbitrary partitions $\mathscr{P}^{(i)}$ of $\mathcal{K}_i(\mathscr{P}^{(1)})$. However, it will be useful to consider partitions with more structure.

Definition 2.1 (Family of partitions)**.** *Suppose $k \in \mathbb{N} \setminus \{1\}$ and $\mathbf{a} = (a_1, \dots, a_{k-1}) \in \mathbb{N}^{k-1}$. We say $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}) = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ is a* family of partitions on V *if it satisfies the following for each $j \in [k-1] \setminus \{1\}$:*

(i) *$\mathscr{P}^{(1)}$ is a partition of V into $a_1 \geq k$ nonempty classes,*
(ii) *$\mathscr{P}^{(j)}$ is a partition of $\mathcal{K}_j(\mathscr{P}^{(1)})$ into nonempty j-graphs such that*
- *$\mathscr{P}^{(j)} \prec \{\mathcal{K}_j(\hat{P}^{(j-1)}) : \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ and*
- *$|\{P^{(j)} \in \mathscr{P}^{(j)} : P^{(j)} \subseteq \mathcal{K}_j(\hat{P}^{(j-1)})\}| = a_j$ for every $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$.*

We say $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ is *T-bounded* if $\|\mathbf{a}\|_\infty \leq T$. For two families of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ and $\mathscr{Q} = \mathscr{Q}(k-1, \mathbf{a}^{\mathscr{Q}})$, we say $\mathscr{P} \prec \mathscr{Q}$ if $\mathscr{P}^{(j)} \prec \mathscr{Q}^{(j)}$ for all $j \in [k-1]$. We say $\mathscr{P} \prec_\nu \mathscr{Q}$ if $\mathscr{P}^{(j)} \prec_\nu \mathscr{Q}^{(j)}$ for all $j \in [k-1]$.

As the concept of polyads is central to this paper, we emphasize the following:

Proposition 2.2. *Let $k \in \mathbb{N} \setminus \{1\}$, $\mathbf{a} \in \mathbb{N}^{(k-1)}$ and $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ be a family of partitions. Then for all $i \in [k-1]$ and $j \in [a_1]$, the following hold.*

(i) *if $i > 1$, then $\mathscr{P}^{(i)}$ is a partition of $\mathcal{K}_i(\mathscr{P}^{(1)})$ into $(i, i, *)$-graphs with respect to $\mathscr{P}^{(1)}$,*
(ii) *each $\hat{P}^{(i)} \in \hat{\mathscr{P}}^{(i)}$ is an $(i+1, i, *)$-graph with respect to $\mathscr{P}^{(1)}$,*
(iii) *for each j-set $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$, $\hat{\mathcal{P}}(J)$ as defined in (2.3) is a complex.*

We now extend the concept of ε-regularity to families of partitions.

Definition 2.3 (Equitable family of partitions)**.** *Let $k \in \mathbb{N} \setminus \{1\}$. Suppose $\eta > 0$ and $\mathbf{a} = (a_1, \dots, a_{k-1}) \in \mathbb{N}^{k-1}$. Let V be a vertex set of size n. We say a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ on V is* $(\eta, \varepsilon, \mathbf{a}, \lambda)$-equitable *if it satisfies the following:*

(i) $a_1 \geq \eta^{-1}$,
(ii) $\mathscr{P}^{(1)} = \{V_i : i \in [a_1]\}$ *satisfies* $|V_i| = (1 \pm \lambda)n/a_1$ *for all* $i \in [a_1]$, *and*
(iii) *if* $k \geq 3$, *then for every k-set* $K \in \mathcal{K}_k(\mathscr{P}^{(1)})$ *the collection* $\hat{\mathcal{P}}(K) = \{\hat{P}^{(j)}(K)\}_{j=1}^{k-1}$ *is an* $(\varepsilon, \mathbf{d})$*-regular* $(k, k-1, *)$*-complex, where* $\mathbf{d} = (1/a_2, \dots, 1/a_{k-1})$.

As before we drop λ if $|V_i| \in \{\lfloor n/a_1 \rfloor, \lfloor n/a_1 \rfloor + 1\}$ and say $\mathscr{P}$ is $(\eta, \varepsilon, \mathbf{a})$-equitable. Note that for any $\lambda \leq 1/3$, every $(\eta, \varepsilon, \mathbf{a}, \lambda)$-equitable family of partitions $\mathscr{P}$ satisfies

$$\left| \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{P}^{(1)}) \right| \leq k^2 \eta \binom{n}{k}. \tag{2.5}$$

We next introduce the concept of perfect ε-regularity with respect to a family of partitions.

Definition 2.4 (Perfectly regular)**.** *Suppose $\varepsilon > 0$ and $k \in \mathbb{N} \setminus \{1\}$. Let $H^{(k)}$ be a k-graph with vertex set V and let $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ be a family of partitions on V. We say $H^{(k)}$ is perfectly ε-regular with respect to $\mathscr{P}$ if for every $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ the graph $H^{(k)}$ is ε-regular with respect to $\hat{P}^{(k-1)}$.*

Having introduced the necessary notation, we are now ready to state the regular approximation lemma due to Rödl and Schacht. It states that for every k-graph H, there is a k-graph G that is close to H and that has very good regularity properties.

Theorem 2.5 (Regular approximation lemma [23])**.** *Let $k \in \mathbb{N} \setminus \{1\}$. For all $\eta, \nu > 0$ and every function $\varepsilon : \mathbb{N}^{k-1} \to (0, 1]$, there are integers $t_0 := t_{2.5}(\eta, \nu, \varepsilon)$ and $n_0 := n_{2.5}(\eta, \nu, \varepsilon)$ so that the following holds:*

For every k-graph H on at least $n \geq n_0$ vertices, there exists a k-graph G on $V(H)$ and a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ on $V(H)$ so that

(i) $\mathscr{P}$ *is* $(\eta, \varepsilon(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}})$*-equitable and* t_0*-bounded,*
(ii) G *is perfectly* $\varepsilon(\mathbf{a}^{\mathscr{P}})$*-regular with respect to* $\mathscr{P}$, *and*
(iii) $|G \triangle H| \leq \nu \binom{n}{k}$.

The crucial point here is that in applications we may apply Theorem 2.5 with a function ε such that $\varepsilon(\mathbf{a}^{\mathscr{P}}) \ll \|\mathbf{a}^{\mathscr{P}}\|_\infty^{-1}$. This is in contrast to other versions (see e.g. [16, 24, 28]) where (roughly speaking) in (iii) we have $G = H$ but in (ii) we have an error parameter ε' which may be large compared to $\|\mathbf{a}^{\mathscr{P}}\|_\infty^{-1}$.

2.4. **The address space.** Later on, we will need to explicitly refer to the densities arising, for example, in Theorem 2.5(ii). For this (and other reasons) it is convenient to consider the 'address space'. Roughly speaking the address space consists of a collection of vectors where each vector identifies a polyad.

For $a, s \in \mathbb{N}$, we recursively define $[a]^s$ by $[a]^s := [a]^{s-1} \times [a]$ and $[a]^1 := [a]$. To define the address space, let us write $\binom{[a_1]}{\ell}_< := \{(\alpha_1, \dots, \alpha_\ell) \in [a_1]^\ell : \alpha_1 < \dots < \alpha_\ell\}$.

Suppose $k', \ell, p \in \mathbb{N}$, $\ell \geq k'$, and $p \geq \max\{k'-1, 1\}$, and $\mathbf{a} = (a_1, \dots, a_p) \in \mathbb{N}^p$. We define

$$\hat{A}(\ell, k'-1, \mathbf{a}) := \binom{[a_1]}{\ell}_< \times \prod_{j=2}^{k'-1} [a_j]^{\binom{\ell}{j}}$$

to be the (ℓ, k')-*address space*. Observe that $\hat{A}(1, 0, \mathbf{a}) = [a_1]$ and $\hat{A}(2, 1, \mathbf{a}) = \binom{[a_1]}{2}_<$. Recall that for a vector $\mathbf{x}$, the set $\mathbf{x}_*$ was defined at the beginning of Section 2. Note that if $k' > 1$, then each $\hat{\mathbf{x}} \in \hat{A}(\ell, k'-1, \mathbf{a})$ can be written as $\hat{\mathbf{x}} = (\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k'-1)})$, where $\mathbf{x}^{(1)} \in \binom{[a_1]}{\ell}_<$ and $\mathbf{x}^{(j)} \in [a_j]^{\binom{\ell}{j}}$ for each $j \in [k'-1] \setminus \{1\}$. Thus each entry of the vector $\mathbf{x}^{(j)}$ corresponds to (i.e. is indexed by) a subset of $\binom{[\ell]}{j}$. We order the elements of both $\binom{[\ell]}{j}$ and $\binom{\mathbf{x}_*^{(1)}}{j}$ lexicographically and consider the bijection $g : \binom{\mathbf{x}_*^{(1)}}{j} \to \binom{[\ell]}{j}$ which preserves this ordering. For each $\Lambda \in \binom{\mathbf{x}_*^{(1)}}{j}$ and $j \in [k'-1]$, we denote by $\mathbf{x}_\Lambda^{(j)}$ the entry of $\mathbf{x}^{(j)}$ which corresponds to the set $g(\Lambda)$.

2.4.1. *Basic properties of the address space.* Let $k \in \mathbb{N} \setminus \{1\}$ and let V be a vertex set of size n. Let $\mathscr{P}(k-1, \mathbf{a})$ be a family of partitions on V. For each crossing ℓ-set $L \in \mathcal{K}_\ell(\mathscr{P}^{(1)})$, the address space allows us to identify (and thus refer to) the set of polyads 'supporting' L. We will achieve this by defining a suitable operator $\hat{\mathbf{x}}(L)$ which maps L to the address space.

To do this, write $\mathscr{P}^{(1)} = \{V_i : i \in [a_1]\}$. Recall from Definition 2.1(ii) that for $j \in [k-1] \setminus \{1\}$, we partition $\mathcal{K}_j(\hat{P}^{(j-1)})$ of every $(j-1)$-polyad $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$ into a_j nonempty parts in such a way that $\mathscr{P}^{(j)}$ is the collection of all these parts. Thus, there is a labelling $\phi^{(j)} : \mathscr{P}^{(j)} \to [a_j]$ such that for every polyad $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$, the restriction of $\phi^{(j)}$ to $\{P^{(j)} \in \mathscr{P}^{(j)} : P^{(j)} \subseteq \mathcal{K}_j(\hat{P}^{(j-1)})\}$ is injective. The set $\mathbf{\Phi} := \{\phi^{(2)}, \ldots, \phi^{(k-1)}\}$ is called an $\mathbf{a}$-*labelling* of $\mathscr{P}(k-1, \mathbf{a})$. For a given set $L \in \mathcal{K}_\ell(\mathscr{P}^{(1)})$, we denote $\mathrm{cl}(L) := \{i : V_i \cap L \neq \emptyset\}$.

Consider any $\ell \in [a_1]$. Let $j' := \min\{k-1, \ell-1\}$ and let $j'' := \max\{j', 1\}$. For every ℓ-set $L \in \mathcal{K}_\ell(\mathscr{P}^{(1)})$ we define an integer vector $\hat{\mathbf{x}}(L) = (\mathbf{x}^{(1)}(L), \ldots, \mathbf{x}^{(j'')}(L))$ by

- $\mathbf{x}^{(1)}(L) := (\alpha_1, \ldots, \alpha_\ell)$, *where* $\alpha_1 < \ldots < \alpha_\ell$ *and* $L \cap V_{\alpha_i} = \{v_{\alpha_i}\}$,
- *and for* $i \in [j'] \setminus \{1\}$ *we set*

$$\mathbf{x}^{(i)}(L) := \Big(\phi^{(i)}(P^{(i)}) : \{v_\lambda : \lambda \in \Lambda\} \in P^{(i)}, P^{(i)} \in \mathscr{P}^{(i)}\Big)_{\Lambda \in \binom{\mathrm{cl}(L)}{i}}. \tag{2.6}$$

Here, we order $\binom{\mathrm{cl}(L)}{i}$ lexicographically. In particular, $\mathbf{x}^{(i)}(L)$ is a vector of length $\binom{\ell}{i}$.

By definition, $\hat{\mathbf{x}}(L) \in \hat{A}(\ell, j', \mathbf{a})$ for every $L \in \mathcal{K}_\ell(\mathscr{P}^{(1)})$ with ℓ, j' as above. Our next aim is to define an operator $\hat{\mathbf{x}}(\cdot)$ which maps the set $\hat{\mathscr{P}}^{(j-1)}$ of $(j-1)$-polyads injectively into the address space $\hat{A}(j, j-1, \mathbf{a})$ (see (2.7)). We will then extend this further into a bijection between elements of the address spaces and their corresponding hypergraphs. However, before we can define $\hat{\mathbf{x}}(\cdot)$, we need to introduce some more notation.

Suppose $j \in [k'-1]$. For $\hat{\mathbf{x}} \in \hat{A}(\ell, k'-1, \mathbf{a})$ and $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$ with $\mathrm{cl}(J) \subseteq \mathbf{x}_*^{(1)}$, we define $\mathbf{x}_J^{(j)} := \mathbf{x}_{\mathrm{cl}(J)}^{(j)}$. Thus from now on, we may refer to the entries of $\mathbf{x}^{(j)}$ either by an index set $\Lambda \in \binom{\mathbf{x}_*^{(1)}}{j}$ or by a set $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$.

Next we introduce a relation on the elements of (possibly different) address spaces. Consider $\hat{\mathbf{x}} = (\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(k'-1)}) \in \hat{A}(\ell, k'-1, \mathbf{a})$ with $\ell' \leq \ell$ and $k'' \leq k'$. We define $\hat{\mathbf{y}} \leq_{\ell', k''-1} \hat{\mathbf{x}}$ if

- $\hat{\mathbf{y}} = (\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \ldots, \mathbf{y}^{(k''-1)}) \in \hat{A}(\ell', k''-1, \mathbf{a})$,
- $\mathbf{y}_*^{(1)} \subseteq \mathbf{x}_*^{(1)}$ and
- $\mathbf{x}_\Lambda^{(j)} = \mathbf{y}_\Lambda^{(j)}$ for any $\Lambda \in \binom{\mathbf{y}_*^{(1)}}{j}$ and $j \in [k''-1] \setminus \{1\}$.

Thus any $\hat{\mathbf{y}} \in \hat{A}(\ell', k''-1, \mathbf{a})$ with $\hat{\mathbf{y}} \leq_{\ell', k''-1} \hat{\mathbf{x}}$ can be viewed as the restriction of $\hat{\mathbf{x}}$ to an ℓ'-subset of the ℓ-set $\mathbf{x}_*^{(1)}$. Hence for $\hat{\mathbf{x}} \in \hat{A}(\ell, k'-1, \mathbf{a})$, there are exactly $\binom{\ell}{\ell'}$ distinct integer vectors $\hat{\mathbf{y}} \in \hat{A}(\ell', k''-1, \mathbf{a})$ such that $\hat{\mathbf{y}} \leq_{\ell', k''-1} \hat{\mathbf{x}}$. Also it is easy to check the following properties.

Proposition 2.6. *Suppose* $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ *is a family of partitions,* $i \in [a_1]$ *and* $i' := \min\{i, k\}$.

(i) *Whenever* $I \in \mathcal{K}_i(\mathscr{P}^{(1)})$ *and* $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$ *with* $I \subseteq J$*, then* $\hat{\mathbf{x}}(I) \leq_{i, i'-1} \hat{\mathbf{x}}(J)$.

(ii) *If* $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$ *and* $\hat{\mathbf{y}} \leq_{i, i'-1} \hat{\mathbf{x}}(J)$*, then there exists a unique* $I \in \binom{J}{i}$ *such that* $\hat{\mathbf{y}} = \hat{\mathbf{x}}(I)$.

Now we are ready to introduce the promised bijection between the elements of address spaces and their corresponding hypergraphs.

Consider $j \in [k] \setminus \{1\}$. Recall that for every j-set $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$, we have $\hat{\mathbf{x}}(J) \in \hat{A}(j, j-1, \mathbf{a})$. Moreover, recall that $\mathcal{K}_j(\hat{P}^{(j-1)}) \neq \emptyset$ for any $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$, and note that $\hat{\mathbf{x}}(J) = \hat{\mathbf{x}}(J')$

for all $J, J' \in \mathcal{K}_j(\hat{P}^{(j-1)})$ and all $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$. Hence, for each $\hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$ we can define

$$\hat{\mathbf{x}}(\hat{P}^{(j-1)}) := \hat{\mathbf{x}}(J) \text{ for some } J \in \mathcal{K}_j(\hat{P}^{(j-1)}). \tag{2.7}$$

Let

$$\begin{aligned}\hat{A}(j, j-1, \mathbf{a})_{\neq\emptyset} &:= \{\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}) : \exists \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)} \text{ such that } \hat{\mathbf{x}}(\hat{P}^{(j-1)}) = \hat{\mathbf{x}}\} \\ &= \{\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}) : \exists J \in \mathcal{K}_j(\mathscr{P}^{(1)}) \text{ such that } \hat{\mathbf{x}} = \hat{\mathbf{x}}(J)\}, \\ \hat{A}(j, j-1, \mathbf{a})_{\emptyset} &:= \hat{A}(j, j-1, \mathbf{a}) \setminus \hat{A}(j, j-1, \mathbf{a})_{\neq\emptyset}.\end{aligned}$$

Clearly (2.7) gives rise to a bijection between $\hat{\mathscr{P}}^{(j-1)}$ and $\hat{A}(j, j-1, \mathbf{a})_{\neq\emptyset}$. Thus for each $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})_{\neq\emptyset}$, we can define the polyad $\hat{P}^{(j-1)}(\hat{\mathbf{x}})$ of $\hat{\mathbf{x}}$ by

$$\hat{P}^{(j-1)}(\hat{\mathbf{x}}) := \hat{P}^{(j-1)} \text{ such that } \hat{P}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)} \text{ with } \hat{\mathbf{x}} = \hat{\mathbf{x}}(\hat{P}^{(j-1)}). \tag{2.8}$$

Note that for any $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$, we have $\hat{P}^{(j-1)}(\hat{\mathbf{x}}(J)) = \hat{P}^{(j-1)}(J)$.

We will frequently make use of an explicit description of a polyad in terms of the partition classes it contains (see (2.12)). For this, we proceed as follows. For each $b \in [a_1]$, let $P^{(1)}(b, b) := V_b$. For each $j \in [k-1] \setminus \{1\}$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a})_{\neq\emptyset} \times [a_j]$, we let

$$P^{(j)}(\hat{\mathbf{x}}, b) := P^{(j)} \in \mathscr{P}^{(j)} \text{ such that } \phi^{(j)}(P^{(j)}) = b \text{ and } P^{(j)} \subseteq \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}})). \tag{2.9}$$

Using Definition 2.1(ii), we conclude that so far $P^{(j)}(\hat{\mathbf{x}}, b)$ is well-defined for each $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a})_{\neq\emptyset} \times [a_j]$ and all $j \in [k-1] \setminus \{1\}$.

For convenience we now extend the domain of the above definitions to cover the 'trivial' cases. For $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a})_{\emptyset} \times [a_j]$, we let

$$P^{(j)}(\hat{\mathbf{x}}, b) := \emptyset. \tag{2.10}$$

We also let $P^{(1)}(a, b) := \emptyset$ for all $a, b \in [a_1]$ with $a \neq b$. For all $j \in [k-1]$ and $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})_{\emptyset}$, we define

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) := \bigcup_{\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{x}}} P^{(j)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\mathbf{y}^{(1)}_*}). \tag{2.11}$$

To summarize, given a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ and an $\mathbf{a}$-labelling $\mathbf{\Phi}$, for each $j \in [k-1]$, this defines $P^{(j)}(\hat{\mathbf{x}}, b)$ for $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$ and $b \in [a_j]$ and $\hat{P}^{(j)}(\hat{\mathbf{x}})$ for all $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$. For later reference, we collect the relevant properties of these objects below. For each $j \in [k-1] \setminus \{1\}$, it will be convenient to extend the domain of the $\mathbf{a}$-labelling $\phi^{(j)}$ of $\mathscr{P}^{(j)}$ to all j-sets $J \in \mathcal{K}_j(\mathscr{P}^{(1)})$ by setting $\phi^{(j)}(J) := \phi^{(j)}(P^{(j)})$, where $P^{(j)} \in \mathscr{P}^{(j)}$ is the unique j-graph that contains J.

Proposition 2.7. *For a given family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ and an $\mathbf{a}$-labelling $\mathbf{\Phi}$, the following hold for all $j \in [k-1]$.*

(i) $\hat{P}^{(j)}(\cdot) : \hat{A}(j+1, j, \mathbf{a})_{\neq\emptyset} \to \hat{\mathscr{P}}^{(j)}$ *is a bijection.*
(ii) *For $j \geq 2$, the restriction of $P^{(j)}(\cdot, \cdot)$ onto $\hat{A}(j, j-1, \mathbf{a})_{\neq\emptyset} \times [a_j]$ is a bijection onto $\mathscr{P}^{(j)}$.*
(iii) *$\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})_{\neq\emptyset}$ if and only if $\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) \neq \emptyset$.*
(iv) *Each $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$ satisfies*

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) = \bigcup_{\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{x}}} P^{(j)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\mathbf{y}^{(1)}_*}). \tag{2.12}$$

(v) *$\{P^{(j)}(\hat{\mathbf{x}}, b) : \hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}), b \in [a_j]\}$ forms a partition of $\mathcal{K}_j(\mathscr{P}^{(1)})$.*
(vi) *$\{\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})\}$ forms a partition of $\mathcal{K}_{j+1}(\mathscr{P}^{(1)})$.*
(vii) *$\{P^{(j+1)}(\hat{\mathbf{x}}, b) : \hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a}), b \in [a_{j+1}]\} \prec \{\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})\}$.*
(viii) *If $\mathscr{P}(k-1, \mathbf{a})$ is T-bounded, then $|\hat{\mathscr{P}}^{(j)}| \leq |\hat{A}(j+1, j, \mathbf{a})| \leq T^{2^{j+1}-1}$ and $|\mathscr{P}^{(j)}| \leq T^{2^j}$.*

(ix) *If* $\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) \neq \emptyset$ *for all* $\hat{\mathbf{x}} \in \hat{A}(j+1,j,\mathbf{a})$, *then* $\hat{P}^{(j)}(\cdot) : \hat{A}(j+1,j,\mathbf{a}) \to \hat{\mathscr{P}}^{(j)}$ *is a bijection and, if in addition* $j < k-1$, *then* $P^{(j+1)}(\cdot,\cdot) : \hat{A}(j+1,j,\mathbf{a}) \times [a_{j+1}] \to \mathscr{P}^{(j+1)}$ *is also a bijection.*

(x) $\hat{A}(j,j-1,\mathbf{a})_\emptyset = \emptyset$ *for all* $j \in [2]$ *and thus* $\hat{P}^{(1)}(\cdot)$ *and* $P^{(2)}(\cdot,\cdot)$ *are always bijections.*

(xi) *If* $\mathscr{P} \prec \mathscr{Q}(k-1,\mathbf{a}^{\mathscr{Q}})$, *then* $\{\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j+1,j,\mathbf{a})\} \prec \{\mathcal{K}_{j+1}(\hat{Q}^{(j)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{Q}})\}$.

We postpone the proof of Proposition 2.7 to Section 4.

We remark that the counting lemma (see Lemma 5.4) will enable us to restrict our attention to families of partitions as in Proposition 2.7(ix). This is formalized in Lemma 5.5.

For $j \in [k-1]$, $\ell \geq j+1$ and for each $\hat{\mathbf{x}} \in \hat{A}(\ell,j,\mathbf{a})$, we define the *polyad of* $\hat{\mathbf{x}}$ by

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) := \bigcup_{\hat{\mathbf{y}} \leq_{j+1,j} \hat{\mathbf{x}}} \hat{P}^{(j)}(\hat{\mathbf{y}}) \overset{(2.12)}{=} \bigcup_{\hat{\mathbf{z}} \leq_{j,j-1} \hat{\mathbf{x}}} P^{(j)}(\hat{\mathbf{z}}, \mathbf{x}^{(j)}_{\mathbf{z}^{(1)}_*}). \tag{2.13}$$

(Note that this generalizes the definition made in (2.8) for the case $\ell = j+1$.) The following fact follows easily from the definition.

Proposition 2.8. *Let* $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a})$ *be a family of partitions. Let* $j \in [k-1]$ *and* $\ell \geq j+1$. *Then for every* $L \in \mathcal{K}_\ell(\mathscr{P}^{(1)})$, *there exists a unique* $\hat{\mathbf{x}} \in \hat{A}(\ell,j,\mathbf{a})$ *such that* $L \in \mathcal{K}_\ell(\hat{P}^{(j)}(\hat{\mathbf{x}}))$.

Note that (2.9) and (2.13) together imply that, for all $j \in [k-1]$ and $\hat{\mathbf{x}} \in \hat{A}(j+1,j,\mathbf{a})$,

$$\hat{\mathcal{P}}(\hat{\mathbf{x}}) := \left\{ \bigcup_{\hat{\mathbf{y}} \leq_{j+1,i} \hat{\mathbf{x}}} \hat{P}^{(i)}(\hat{\mathbf{y}}) \right\}_{i \in [j]} \tag{2.14}$$

is a $(j+1,j,*)$-complex. Moreover, using Proposition 2.7(iii) it is easy to check that for each $\hat{\mathbf{x}} \in \hat{A}(j+1,j,\mathbf{a})$ with $\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) \neq \emptyset$, we have (for $\hat{\mathcal{P}}(J)$ as defined in (2.3))

$$\hat{\mathcal{P}}(\hat{\mathbf{x}}) = \hat{\mathcal{P}}(J) \text{ for some } J \in \mathcal{K}_{j+1}(\mathscr{P}^{(1)}). \tag{2.15}$$

2.4.2. *Density functions of address spaces.* For $k \in \mathbb{N} \setminus \{1\}$ and $\mathbf{a} \in \mathbb{N}^{k-1}$, we say a function $d_{\mathbf{a},k} : \hat{A}(k,k-1,\mathbf{a}) \to [0,1]$ is a *density function* of $\hat{A}(k,k-1,\mathbf{a})$. For two density functions $d^1_{\mathbf{a},k}$ and $d^2_{\mathbf{a},k}$, we define the *distance* between $d^1_{\mathbf{a},k}$ and $d^2_{\mathbf{a},k}$ by

$$\operatorname{dist}(d^1_{\mathbf{a},k}, d^2_{\mathbf{a},k}) := k! \prod_{i=1}^{k-1} a_i^{-\binom{k}{i}} \sum_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} |d^1_{\mathbf{a},k}(\hat{\mathbf{x}}) - d^2_{\mathbf{a},k}(\hat{\mathbf{x}})|.$$

Since $|\hat{A}(k,k-1,\mathbf{a})| = \binom{a_1}{k} \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}$, we always have that $\operatorname{dist}(d^1_{\mathbf{a},k}, d^2_{\mathbf{a},k}) \leq 1$. Suppose we are given a density function $d_{\mathbf{a},k}$, a real $\varepsilon > 0$, and a k-graph $H^{(k)}$. We say a family of partitions $\mathscr{P} = \mathscr{P}(k-1,\mathbf{a})$ on $V(H^{(k)})$ is an $(\varepsilon, d_{\mathbf{a},k})$-*partition* of $H^{(k)}$ if for every $\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})$ the k-graph $H^{(k)}$ is $(\varepsilon, d_{\mathbf{a},k}(\hat{\mathbf{x}}))$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{x}})$. If $\mathscr{P}$ is also $(1/a_1, \varepsilon, \mathbf{a})$-equitable (as specified in Definition 2.3), we say $\mathscr{P}$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-*equitable partition* of $H^{(k)}$. Note that

if $\hat{P}^{(k-1)}(\cdot) : \hat{A}(k,k-1,\mathbf{a}) \to \hat{\mathscr{P}}^{(k-1)}$ *is a bijection, then* $H^{(k)}$ *is perfectly* ε*-regular with respect to* $\mathscr{P}$ *if and only if there exists a density function* $d_{\mathbf{a},k}$ *such that* $\mathscr{P}$ *is an* $(\varepsilon, d_{\mathbf{a},k})$*-partition of* $H^{(k)}$. (2.16)

2.5. **Regularity instances.** A regularity instance R encodes an address space, an associated density function and a regularity parameter. Roughly speaking, a regularity instance can be thought of as encoding a weighted 'reduced multihypergraph' obtained from an application of the regularity lemma for hypergraphs. To formalize this, let $\varepsilon_{2.9}(\cdot,\cdot) : \mathbb{N} \times \mathbb{N} \to (0,1]$ be a function which satisfies the following.

- $\varepsilon_{2.9}(\cdot,k)$ is a decreasing function for any fixed $k \in \mathbb{N}$ with $\lim_{t\to\infty} \varepsilon_{2.9}(t,k) = 0$,

- $\varepsilon_{2.9}(t, \cdot)$ is a decreasing function for any fixed $t \in \mathbb{N}$,
- $\varepsilon_{2.9}(t, k) < t^{-4^k}\varepsilon_{5.4}(1/t, 1/t, k-1, k)/4$, where $\varepsilon_{5.4}$ is defined in Lemma 5.4.

Definition 2.9 (Regularity instance)**.** *A regularity instance $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$ is a triple, where $\mathbf{a} = (a_1, \dots, a_{k-1}) \in \mathbb{N}^{k-1}$ with $0 < \varepsilon \leq \varepsilon_{2.9}(\|\mathbf{a}\|_\infty, k)$, and $d_{\mathbf{a},k}$ is a density function of $\hat{A}(k, k-1, \mathbf{a})$. A k-graph H satisfies the regularity instance R if there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ such that $\mathscr{P}$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of H. The* complexity *of R is $1/\varepsilon$.*

Since $\varepsilon_{2.9}$ depends only on $\|\mathbf{a}\|_\infty$ and k, it follows that for given r and fixed k, the number of vectors $\mathbf{a}$ which could belong to a regularity instance R with complexity r is bounded by a function of r.

We will often make use of the fact that if we apply the regular approximation lemma (Theorem 2.5) to a k-graph H to obtain G and $\mathscr{P}$, then $\mathbf{a}^{\mathscr{P}}$ together with the densities of G with respect to the polyads in $\hat{\mathscr{P}}^{(k-1)}$ naturally give rise to a regularity instance R where G satisfies R and H is close to satisfying R.

3. Main result: Sampling regularity instances

Let us now turn to the statement of our main result thereby extending Theorem 1.2 to k-graphs. It states that not too small random samples of vertex subsets satisfy with high probability essentially the same regularity instance; that is, only an additive error term is needed.

Theorem 3.1. *Suppose $0 < 1/n < 1/q \ll c \ll \delta \ll \varepsilon_0 \leq 1$ and $k \in \mathbb{N} \setminus \{1\}$. Suppose $R = (2\varepsilon_0/3, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance. Suppose H is a k-graph on vertex set V with $|V| = n$. Let $Q \in \binom{V}{q}$ be chosen uniformly at random. Then with probability at least $1 - e^{-cq}$ the following hold.*

(Q1)$_{3.1}$ *If there exists an $(\varepsilon_0, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_1$ of H, then there exists an $(\varepsilon_0 + \delta, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_2$ of $H[Q]$.*

(Q2)$_{3.1}$ *If there exists an $(\varepsilon_0, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_2$ of $H[Q]$, then there exists an $(\varepsilon_0 + \delta, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_1$ of H.*

We use Theorem 3.1 to completely characterize all testable hypergraph properties in the companion paper [17]. Similar in the graph setting, regular instances are the key objects and, roughly speaking, a property is testable if regularity instances determine the property.

Now we illustrate a few key points in our approach. Roughly speaking, Theorem 3.1 states the following.

> *Suppose H is a k-graph and Q a random subset of $V(H)$. Then with high probability, the following hold (where $\delta \ll \varepsilon_0$).*
>
> - *If $\mathscr{O}_1$ is an ε_0-equitable partition of H with density function $d_{\mathbf{a},k}$, then there is an $(\varepsilon_0 + \delta)$-equitable partition of $H[Q]$ with the same density function $d_{\mathbf{a},k}$.*
> - *If $\mathscr{O}_2$ is an ε_0-equitable partition of $H[Q]$ with density function $d_{\mathbf{a},k}$, then there is an $(\varepsilon_0 + \delta)$-equitable partition of H with the same density function $d_{\mathbf{a},k}$.*

The crucial point here is that the transfer between H and $H[Q]$ incurs only an additive increase in the regularity parameter ε_0. In fact, this additive increase can then be eliminated by slightly adjusting H (or $H[Q]$).

The key ingredient in the proof of Theorem 3.1 is Lemma 7.1. Roughly speaking, Lemma 7.1 states the following.

Suppose the following hold (where $\varepsilon \ll \delta \ll \varepsilon_0$).

- *H_1 is a k-graph on vertex set V_1 and $\mathscr{Q}_1$ is an ε-equitable partition of H_1 with density function $d_{\mathbf{a}^{\mathscr{Q}},k}$.*
- *H_2 is a k-graph on vertex set V_2 and $\mathscr{Q}_2$ is an ε-equitable partition of H_2 with the same density function $d_{\mathbf{a}^{\mathscr{Q}},k}$.*
- *$\mathscr{O}_1$ is an ε_0-equitable partition of H_1 with density function $d_{\mathbf{a}^{\mathscr{O}},k}$.*

Then there is an $(\varepsilon_0 + \delta)$-equitable partition $\mathscr{O}_2$ of H_2, also with density function $d_{\mathbf{a}^{\mathscr{O}},k}$.

One may think of this results as follows; if two k-graphs both satisfy *some* 'high quality' regularity partition with the same parameters, then *all* 'low quality' regularity partitions from one k-graph are also regularity partitions of the other k-graph to the expense of only a small additive increase in the regularity parameter.

To prove Theorem 3.1, we will apply Lemma 7.1 with H playing the role of H_1 and with the random sample $H[Q]$ playing the role of H_2 (and vice versa). In turn, our strengthening of the regular approximation lemma (Lemma 6.1) will be one of the main tools in the proof of Lemma 7.1; see the beginning of Section 7 for a more detailed sketch.

4. More concepts and tools

Here we collect some further results that we need later in our proofs but which are not needed to understand our main theorem.

4.1. A stronger hypergraph regularity lemma.

We next state Lemma 4.1 which is a generalization of the regular approximation lemma which was also proved by Rödl and Schacht (see Lemma 25 in [23]). Lemma 4.1 has two additional features in comparison to Theorem 2.5. Firstly, we can prescribe a family of partitions $\mathscr{Q}$ and obtain a refinement $\mathscr{P}$ of $\mathscr{Q}$, and secondly, we are not only given one k-graph H but a collection of k-graphs H_i that partitions the complete k-graph. Thus we may view Lemma 4.1 as a 'partition version' of Theorem 2.5. We will use it in the proof of Lemma 6.1.

Lemma 4.1 (Rödl and Schacht [23]). *For all $o, s \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{1\}$, all $\eta, \nu > 0$, and every function $\varepsilon : \mathbb{N}^{k-1} \to (0, 1]$, there are $\mu = \mu_{4.1}(k, o, s, \eta, \nu, \varepsilon) > 0$ and $t = t_{4.1}(k, o, s, \eta, \nu, \varepsilon) \in \mathbb{N}$ and $n_0 = n_{4.1}(k, o, s, \eta, \nu, \varepsilon) \in \mathbb{N}$ such that the following hold. Suppose*

(O1)$_{4.1}$ *V is a set and $|V| = n \geq n_0$,*
(O2)$_{4.1}$ *$\mathscr{Q} = \mathscr{Q}(k, \mathbf{a}^{\mathscr{Q}})$ is a $(1/a_1^{\mathscr{Q}}, \mu, \mathbf{a}^{\mathscr{Q}})$-equitable o-bounded family of partitions on V,*
(O3)$_{4.1}$ *$\mathscr{H}^{(k)} = \{H_1^{(k)}, \ldots, H_s^{(k)}\}$ is a partition of $\binom{V}{k}$ so that $\mathscr{H}^{(k)} \prec \mathscr{Q}^{(k)}$.*

Then there exist a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ and a partition $\mathscr{G}^{(k)} = \{G_1^{(k)}, \ldots, G_s^{(k)}\}$ of $\binom{V}{k}$ satisfying the following for every $i \in [s]$ and $j \in [k-1]$.

(P1)$_{4.1}$ *$\mathscr{P}$ is a t-bounded $(\eta, \varepsilon(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}})$-equitable family of partitions, and $a_j^{\mathscr{Q}}$ divides $a_j^{\mathscr{P}}$,*
(P2)$_{4.1}$ *$\mathscr{P} \prec \{\mathscr{Q}^{(j)}\}_{j=1}^{k-1}$,*
(P3)$_{4.1}$ *$G_i^{(k)}$ is perfectly $\varepsilon(\mathbf{a}^{\mathscr{P}})$-regular with respect to $\mathscr{P}$,*
(P4)$_{4.1}$ *$\sum_{i=1}^{s} |G_i^{(k)} \triangle H_i^{(k)}| \leq \nu \binom{n}{k}$, and*
(P5)$_{4.1}$ *$\mathscr{G}^{(k)} \prec \mathscr{Q}^{(k)}$ and if $H_i^{(k)} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$, then $G_i^{(k)} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$.*

In Lemma 4.1 we may assume without loss of generality that $1/\mu, t, n_0$ are non-decreasing in k, o, s and non-increasing in η, ν.

4.2. **The proof of Proposition 2.7.**

Proof of Proposition 2.7. Observe that (i) and (ii) hold by definition. Note that $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})_{\neq\emptyset}$ if and only if $\hat{\mathbf{x}} = \hat{\mathbf{x}}(J)$ for some $J \in \mathcal{K}_{j+1}(\mathscr{P}^{(1)})$ if and only if there exists a set $J \in \mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}}))$. Thus (iii) holds. To show (iv), by (2.11), we may assume $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})_{\neq\emptyset}$. Thus we know that $\mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}}))$ contains at least one $(j+1)$-set J and $\hat{\mathbf{x}} = \hat{\mathbf{x}}(J)$. By (2.2), we have

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) = \hat{P}^{(j)}(J) = \bigcup_{I \in \binom{J}{j}} P^{(j)}(I).$$

By Proposition 2.6(ii), we know that $\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{x}}$ if and only if $\hat{\mathbf{y}} = \hat{\mathbf{x}}(I)$ for some $I \in \binom{J}{j}$. Consider any j-set $I \subseteq J$. Recall that $\hat{P}^{(j-1)}(\hat{\mathbf{x}}(I)) = \hat{P}^{(j-1)}(I)$, and thus $I \in \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}}(I)))$. Together with (2.9) this implies that $P^{(j)}(I) = P^{(j)}(\hat{\mathbf{x}}(I), \phi^{(j)}(I))$, where $P^{(j)}(I)$ is the unique part of $\mathscr{P}^{(j)}$ that contains I. Since $\phi^{(j)}(I) = \phi^{(j)}(P^{(j)}(I)) = \mathbf{x}^{(j)}(J)_I$ holds by (2.6), we have

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) = \bigcup_{I \in \binom{J}{j}} P^{(j)}(I) = \bigcup_{I \in \binom{J}{j}} P^{(j)}(\hat{\mathbf{x}}(I), \mathbf{x}^{(j)}(J)_I) = \bigcup_{\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{x}}} P^{(j)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\mathbf{y}^{(1)}_*}).$$

This shows that (iv) holds. It is easy to see that (i), (ii), (iii) and Definition 2.1(ii) together imply (v), (vi) and (vii). If $\mathscr{P}(k-1, \mathbf{a})$ is T-bounded, (i) implies that

$$|\hat{\mathscr{P}}^{(j)}| \leq |\hat{A}(j+1, j, \mathbf{a})| \leq \prod_{i=1}^{j} a_i^{\binom{j+1}{i}} \leq \prod_{i=1}^{j} T^{\binom{j+1}{i}} \leq T^{2^{j+1}-1}.$$

Thus for $j \in [k-1] \setminus \{1\}$, we have $|\mathscr{P}^{(j)}| \leq a_j |\hat{\mathscr{P}}^{(j-1)}| \leq T^{2^j}$. Also $|\mathscr{P}^{(1)}| = a_1 \leq T$, thus we have (viii). Statement (ix) follows from (i), (ii) and (iii). Property (x) is trivial from the definitions.

Finally we show (xi). Suppose $J \in \mathcal{K}_{j+1}(\hat{P}^{(j)}(\hat{\mathbf{x}})) \cap \mathcal{K}_{j+1}(\hat{Q}^{(j)}(\hat{\mathbf{y}}))$ for some $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$ and $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{Q}})$. Then (iii) implies that $\hat{P}^{(j)}(\hat{\mathbf{x}}) = \hat{P}^{(j)}(J)$ and $\hat{Q}^{(j)}(\hat{\mathbf{y}}) = \hat{Q}^{(j)}(J)$. Since $\mathscr{P} \prec \mathscr{Q}$, we have $P^{(j)}(I) \subseteq Q^{(j)}(I)$. Thus

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) \overset{(2.2)}{=} \bigcup_{I \in \binom{J}{j}} P^{(j)}(I) \subseteq \bigcup_{I \in \binom{J}{j}} Q^{(j)}(I) \overset{(2.2)}{=} \hat{Q}^{(j)}(\hat{\mathbf{y}}).$$

Thus we have $\mathcal{K}_{j+1}(\hat{P}^{(j)}) \subseteq \mathcal{K}_{j+1}(\hat{Q}^{(j)}(J))$. This implies (xi). □

4.3. **Constructing families of partitions using the address space.**

On several occasions we will construct $P^{(j)}(\hat{\mathbf{x}}, b)$ and $\hat{P}^{(j)}(\hat{\mathbf{x}})$ first and then show that they actually give rise to a family of partitions for which we can use the properties listed in Proposition 2.7. The following lemma, which can easily be proved by induction, provides a criterion to show that this is indeed the case.

Lemma 4.2. *Suppose $k \in \mathbb{N} \setminus \{1\}$ and $\mathbf{a} \in \mathbb{N}^{k-1}$. Suppose $\mathscr{P}^{(1)} = \{V_1, \ldots, V_{a_1}\}$ is a partition of a vertex set V. Suppose that for each $j \in [k-1] \setminus \{1\}$ and each $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}) \times [a_j]$, we are given a j-graph $P'^{(j)}(\hat{\mathbf{x}}, b)$, and for each $j \in [k] \setminus \{1\}$ and $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$, we are given a $(j-1)$-graph $\hat{P'}^{(j-1)}(\hat{\mathbf{x}})$. Let*

$$P'^{(1)}(b, b) := V_b \text{ for all } b \in [a_1], \text{ and}$$
$$\mathscr{P}^{(j)} := \{P'^{(j)}(\hat{\mathbf{x}}, b) : (\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}) \times [a_j]\} \text{ for all } j \in [k-1] \setminus \{1\}.$$

Suppose the following conditions hold:

- (FP1) *$P'^{(1)}(b, b) \neq \emptyset$ for each $b \in [a_1]$; moreover for each $j \in [k-1] \setminus \{1\}$ and each $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}) \times [a_j]$, we have $P'^{(j)}(\hat{\mathbf{x}}, b) \neq \emptyset$.*
- (FP2) *For each $j \in [k-1] \setminus \{1\}$ and $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$, the set $\{P'^{(j)}(\hat{\mathbf{x}}, b) : b \in [a_j]\}$ has size a_j and forms a partition of $\mathcal{K}_j(\hat{P'}^{(j-1)}(\hat{\mathbf{x}}))$.*

(FP3) *For each* $j \in [k-1]$ *and* $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$, *we have*

$$\hat{P'}^{(j)}(\hat{\mathbf{x}}) = \bigcup_{\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{x}}} P'^{(j)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\mathbf{y}^{(1)}_*}).$$

Then the $\mathbf{a}$*-labelling* $\mathbf{\Phi} = \{\phi^{(i)}\}_{i=2}^{k-1}$ *given by* $\phi^{(i)}(P'^{(i)}(\hat{\mathbf{x}}, b)) = b$ *for each* $(\hat{\mathbf{x}}, b) \in \hat{A}(i, i-1, \mathbf{a}) \times [a_i]$ *is well-defined and satisfies the following:*

(FQ1) $\mathscr{P} = \{\mathscr{P}^{(i)}\}_{i=1}^{k-1}$ *is a family of partitions on* V.

(FQ2) *The maps* $P^{(j)}(\cdot,\cdot)$ *and* $\hat{P}^{(j)}(\cdot)$ *defined in* (2.8)–(2.11) *for* $\mathscr{P}$, $\mathbf{\Phi}$ *satisfy that for each* $j \in [k-1] \setminus \{1\}$ *and* $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}) \times [a_j]$, *we have*

$$P^{(j)}(\hat{\mathbf{x}}, b) = P'^{(j)}(\hat{\mathbf{x}}, b),$$

and for each $j \in [k-1]$ *and* $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$ *we have*

$$\hat{P}^{(j)}(\hat{\mathbf{x}}) = \hat{P'}^{(j)}(\hat{\mathbf{x}}).$$

5. Hypergraph regularity: counting lemmas and approximation

In this section we present several results about hypergraph regularity. The first few results are simple observations which follow either from the definition of ε-regularity or can be easily proved by standard probabilistic arguments. We omit the proofs. In Section 5.2 we then derive an induced version of the 'counting lemma' that is suitable for our needs (see Lemma 5.8).

In Section 5.3 we make two simple observations on refinements of partitions and in Section 5.4 we consider small perturbations of a given family of partitions.

5.1. Simple hypergraph regularity results.

We will use the following results which follow easily from the definition of hypergraph regularity (see Section 2.2).

Lemma 5.1. *Suppose* $m \in \mathbb{N}$, $0 < \varepsilon \leq \alpha^2 < 1$ *and* $d \in [0,1]$. *Suppose* $H^{(k)}$ *is an* $(m, k, k, 1/2)$*-graph which is* (ε, d)*-regular with respect to an* $(m, k, k-1, 1/2)$*-graph* $H^{(k-1)}$. *Suppose* $Q^{(k-1)} \subseteq H^{(k-1)}$ *and* $H'^{(k)} \subseteq H^{(k)}$ *such that* $|\mathcal{K}_k(Q^{(k-1)})| \geq \alpha |\mathcal{K}_k(H^{(k-1)})|$ *and* $H'^{(k)}$ *is* (ε, d')*-regular with respect to* $H^{(k-1)}$ *for some* $d' \leq d$. *Then*

(i) $\mathcal{K}_k(H^{(k-1)}) \setminus H^{(k)}$ *is* $(\varepsilon, 1-d)$*-regular with respect to* $H^{(k-1)}$,

(ii) $H^{(k)}$ *is* $(\varepsilon/\alpha, d)$*-regular with respect to* $Q^{(k-1)}$, *and*

(iii) $H^{(k)} \setminus H'^{(k)}$ *is* $(2\varepsilon, d-d')$*-regular with respect to* $H^{(k-1)}$.

Lemma 5.2 (Union lemma)**.** *Suppose* $0 < \varepsilon \ll 1/k, 1/s$. *Suppose that* $H_1^{(k)}, \dots, H_s^{(k)}$ *are edge-disjoint* $(k, k, *)$*-graphs such that each* $H_i^{(k)}$ *is* ε*-regular with respect to a* $(k, k-1, *)$*-graph* $H^{(k-1)}$. *Then* $\bigcup_{i=1}^s H_i^{(k)}$ *is* $s\varepsilon$*-regular with respect to* $H^{(k-1)}$.

We will also use the following observation (see for example [23]), which can be easily proved using Chernoff's inequality.

Lemma 5.3 (Slicing lemma [23])**.** *Suppose* $0 < 1/m \ll d, \varepsilon, p_0, 1/s$ *and* $d \geq 2\varepsilon$. *Suppose that*

- $H^{(k)}$ *is an* (ε, d)*-regular* k*-graph with respect to a* $(k-1)$*-graph* $H^{(k-1)}$,
- $|\mathcal{K}_k(H^{(k-1)})| \geq m^k / \log m$,
- $p_1, \dots, p_s \geq p_0$ *and* $\sum_{i=1}^s p_i \leq 1$.

Then there exists a partition $\{H_0^{(k)}, H_1^{(k)}, \dots, H_s^{(k)}\}$ *of* $H^{(k)}$ *such that* $H_i^{(k)}$ *is* $(3\varepsilon, p_i d)$*-regular with respect to* $H^{(k-1)}$ *for every* $i \in [s]$, *and* $H_0^{(k)}$ *is* $(3\varepsilon, (1 - \sum p_i)d)$*-regular with respect to* $H^{(k-1)}$.

5.2. **Counting lemmas.** Kohayakawa, Rödl and Skokan proved the following 'counting lemma' (Theorem 6.5 in [18]), which asserts that the number of copies of a given $K_\ell^{(k)}$ in an $(\varepsilon, \mathbf{d})$-regular complex is close to what one could expect in a corresponding random complex. We will deduce several versions of this which suit our needs.

Lemma 5.4 (Counting lemma [18])**.** *For all $\gamma, d_0 > 0$ and $k, \ell \in \mathbb{N}\setminus\{1\}$ with $k \leq \ell$, there exist $\varepsilon_0 := \varepsilon_{5.4}(\gamma, d_0, k, \ell) \leq 1$ and $m_0 := n_{5.4}(\gamma, d_0, k, \ell)$ such that the following holds: Suppose $0 \leq \lambda < 1/4$. Suppose $0 < \varepsilon \leq \varepsilon_0$ and $m_0 \leq m$ and $\mathbf{d} = (d_2, \dots, d_k) \in \mathbb{R}^{k-1}$ such that $d_j \geq d_0$ for every $j \in [k] \setminus \{1\}$. Suppose that $\mathcal{H} = \{H^{(j)}\}_{j=1}^k$ is an $(\varepsilon, \mathbf{d})$-regular (m, ℓ, k, λ)-complex, and $H^{(1)} = \{V_1, \dots, V_\ell\}$ with $m_i = |V_i|$ for every $i \in [\ell]$. Then*

$$|\mathcal{K}_\ell(H^{(k)})| = (1 \pm \gamma) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \cdot \prod_{i=1}^{\ell} m_i.$$

Recall that equitable families of partitions were defined in Section 2.3. Based on the counting lemma, it is easy to show that for an equitable family of partitions $\mathscr{P}$ and an **a**-labelling $\mathbf{\Phi}$, the maps $\hat{P}^{(j-1)}(\cdot) : \hat{A}(j, j-1, \mathbf{a}) \to \hat{\mathscr{P}}^{(j-1)}$ and $P^{(j)}(\cdot, \cdot) : \hat{A}(j, j-1, \mathbf{a}) \times [a_j] \to \mathscr{P}^{(j)}$ defined in Section 2.4 are bijections. We will frequently make use of this fact in subsequent sections, often without referring to Lemma 5.5 explicitly.

Lemma 5.5. *Suppose that $k, t \in \mathbb{N} \setminus \{1\}$, $0 \leq \lambda < 1/4$ and $\varepsilon/3 \leq \varepsilon_{2.9}(t, k)$ and $\mathbf{a} = (a_1, \dots, a_{k-1}) \in [t]^{k-1}$ and $|V| = n$ with $1/n \ll 1/t, 1/k$. If $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ is a $(1/a_1, \varepsilon, \mathbf{a}, \lambda)$-equitable family of partitions on V, and $\mathscr{P}$ with an $\mathbf{a}$-labelling $\mathbf{\Phi}$ defines maps $\hat{P}^{(j-1)}(\cdot)$ and $P^{(j-1)}(\cdot, \cdot)$, then the following hold.*

- (i) *For each $j \in [k-1]$, $\hat{P}^{(j)}(\cdot) : \hat{A}(j+1, j, \mathbf{a}) \to \hat{\mathscr{P}}^{(j)}$ is a bijection and if $j > 1$, then $P^{(j)}(\cdot, \cdot) : \hat{A}(j, j-1, \mathbf{a}) \times [a_j] \to \mathscr{P}^{(j)}$ is also a bijection. In particular, $\hat{A}(j, j-1, \mathbf{a}) = \hat{A}(j, j-1, \mathbf{a})_{\neq\emptyset}$.*
- (ii) *For each $j \in [k-1]\setminus\{1\}$ and $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a})$, $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ is an $(\varepsilon, (1/a_2, \dots, 1/a_j))$-regular $(j+1, j, \lambda)$-complex.*

Note that in Lemma 5.4 the graphs $H^{(k)}[\Lambda]$ for $\Lambda \in \binom{\ell}{k}$ are all (ε, d_k)-regular with respect to $H^{(k-1)}[\Lambda]$. In view of Lemma 5.3, we obtain the following corollary, which allows for varying densities at the k-th 'level'.

Corollary 5.6. *For all $\gamma, d_0 > 0$ and $k, \ell \in \mathbb{N}\setminus\{1\}$ with $k \leq \ell$, there exist $\varepsilon_0 := \varepsilon_{5.6}(\gamma, d_0, k, \ell)$ and $m_0 := n_{5.6}(\gamma, d_0, k, \ell)$ such that the following holds: Suppose $0 \leq \lambda < 1/4$. Suppose $d' \geq d_0$, $0 < \varepsilon \leq \varepsilon_0$ and $m \geq m_0$ for each $i \in [\ell]$, and $\mathbf{d} = (d_2, \dots, d_{k-1}) \in \mathbb{R}^{k-2}$ such that $d_j \geq d_0$ for each $j \in [k-1] \setminus \{1\}$. Suppose $\mathcal{H} = \{H^{(j)}\}_{j=1}^k$ is an (m, ℓ, k, λ)-complex, $H^{(1)} = \{V_1, \dots, V_\ell\}$ with $m_i = |V_i|$ for every $i \in [\ell]$, and for every $\Lambda \in \binom{\ell}{k}$, the complex $\mathcal{H}[\Lambda]$ is $(\varepsilon, (d_2, \dots, d_{k-1}, p_\Lambda))$-regular, where p_Λ is a multiple of d'. Then*

$$|\mathcal{K}_\ell(H^{(k)})| = (1 \pm \gamma) \prod_{\Lambda \in \binom{\ell}{k}} p_\Lambda \cdot \prod_{j=2}^{k-1} d_j^{\binom{\ell}{j}} \cdot \prod_{i=1}^{\ell} m_i.$$

Note that in the above lemma, some p_Λ are allowed to be zero.

Let F be an ℓ-vertex k-graph and $\mathcal{H} = \{H^{(j)}\}_{j=1}^k$ be a complex with $H^{(1)} = \{V_1, \dots, V_\ell\}$. For a bijection $\sigma : V(F) \to [\ell]$, we say an induced copy F' of F in $H^{(k)}$ is *σ-induced* if for each $v \in V(F)$ the vertex of F' corresponding to v lies in $V_{\sigma(v)}$. Let $IC_\sigma(F, \mathcal{H})$ be the number of σ-induced copies F' of F in $H^{(k)}$ such that F' is contained in an element of $\mathcal{K}_\ell(H^{(k-1)})$.

Lemma 5.7 (Induced counting lemma for many clusters)**.** *Suppose $0 < 1/m \ll \varepsilon \ll \gamma, d_0, 1/k, 1/\ell$ with $k \in \mathbb{N} \setminus \{1\}$ and suppose that*

- *F is an ℓ-vertex k-graph,*
- *$d_0 \leq d_j \leq 1 - d_0$ for every $j \in [k-1] \setminus \{1\}$,*

- $\mathcal{H} = \{H^{(j)}\}_{j=1}^{k}$ *is an* (m, ℓ, k)*-complex with* $H^{(1)} = \{V_1, \dots, V_\ell\}$,
- *for each* $\Lambda \in \binom{[\ell]}{k}$*, the complex* $\mathcal{H}[\Lambda]$ *is an* $(\varepsilon, (d_2, \dots, d_{k-1}, p_\Lambda))$*-regular* (m, k, k)*-complex, and*
- $\sigma : V(F) \to [\ell]$ *is a bijection.*

Then

$$IC_\sigma(F, \mathcal{H}) = \left(\prod_{e \in F} p_{\sigma(e)} \prod_{e \notin F, |e| = k} (1 - p_{\sigma(e)}) \pm \gamma \right) \prod_{j=2}^{k-1} d_j^{\binom{\ell}{j}} \cdot m^\ell.$$

Proof. We select $q \in \mathbb{N}$ such that $1/m \ll \varepsilon \ll 1/q \ll \gamma, d_0, 1/k, 1/\ell$ and define $\overline{H}^{(k)} := \mathcal{K}_k(H^{(k-1)}) \setminus H^{(k)}$. We also define an (m, ℓ, k)-graph H' on $\{V_1, \dots, V_\ell\}$ so that for each $e \in \binom{V(F)}{k}$, we have

$$H'[\sigma(e)] := \begin{cases} H^{(k)}[\sigma(e)] & \text{if } e \in F, \\ \overline{H}^{(k)}[\sigma(e)] & \text{otherwise,} \end{cases}$$

and let $H' := \bigcup_{e \in \binom{V(F)}{k}} H'[\sigma(e)]$. Note that $H^{(k-1)}$ underlies H'. Observe that there is a bijection between the set of all σ-induced copies F' of $F^{(k)}$ in $H^{(k)}$ such that F' is contained in an element of $\mathcal{K}_\ell(H^{(k-1)})$ and the set of copies of $K_\ell^{(k)}$ in H'. For $e \in \binom{V(F)}{k}$, we define

$$p'_{\sigma(e)} := \begin{cases} p_{\sigma(e)} & \text{if } e \in F, \\ 1 - p_{\sigma(e)} & \text{otherwise.} \end{cases}$$

By Lemma 5.1(i), for each $\Lambda \in \binom{[\ell]}{k}$, the set $\{H^{(j)}[\Lambda]\}_{j=1}^{k-1} \cup \{H'[\Lambda]\}$ is an $(\varepsilon, (d_2, \dots, d_{k-1}, p'_\Lambda))$-regular (m, k, k)-complex. It suffices to show that

$$|\mathcal{K}_\ell(H')| = \left(\prod_{\Lambda \in \binom{\ell}{k}} p'_\Lambda \pm \gamma \right) \cdot \prod_{j=2}^{k-1} d_j^{\binom{\ell}{j}} \cdot m^\ell. \tag{5.1}$$

We apply the slicing lemma (Lemma 5.3) to find for each $\Lambda \in \binom{[\ell]}{k}$ a subgraph $H'_1[\Lambda]$ of $H'[\Lambda]$ which is $(3\varepsilon, \lfloor q p'_\Lambda \rfloor / q)$-regular with respect to $H^{(k-1)}[\Lambda]$. Similarly, for each $\Lambda \in \binom{[\ell]}{k}$, we apply Lemma 5.3 to the graph $\mathcal{K}_k(H^{(k-1)}[\Lambda]) \setminus H'[\Lambda]$. In combination with the union lemma (Lemma 5.2) this gives a supergraph $H'_2[\Lambda]$ of $H'[\Lambda]$ which is $(6\varepsilon, \lceil q p'_\Lambda \rceil / q)$-regular with respect to $H^{(k-1)}[\Lambda]$.

Let $H'_i := \bigcup_{\Lambda \in \binom{[\ell]}{k}} H'_i[\Lambda]$ for each $i \in [2]$. Observe that $|\mathcal{K}_\ell(H'_1)| \le |\mathcal{K}_\ell(H')| \le |\mathcal{K}_\ell(H'_2)|$. By Corollary 5.6 with $\gamma/2, 1/q, 1/q$ playing the roles of γ, d', d_0, respectively,

$$|\mathcal{K}_\ell(H'_1)| \ge \left(1 - \frac{\gamma}{2}\right) \prod_{\Lambda \in \binom{\ell}{k}} \lfloor q p'_\Lambda \rfloor / q \cdot \prod_{j=2}^{k-1} d_j^{\binom{\ell}{j}} \cdot m^\ell \quad \text{and}$$

$$|\mathcal{K}_\ell(H'_2)| \le \left(1 + \frac{\gamma}{2}\right) \prod_{\Lambda \in \binom{\ell}{k}} \lceil q p'_\Lambda \rceil / q \cdot \prod_{j=2}^{k-1} d_j^{\binom{\ell}{j}} \cdot m^\ell.$$

Note that for each $\Lambda \in \binom{\ell}{k}$, we have $p'_\Lambda - 1/q \le \lfloor q p'_\Lambda \rfloor / q$ and $\lceil q p'_\Lambda \rceil / q \le p'_\Lambda + 1/q$. Thus we obtain (5.1) as required. □

The previous lemma counts σ-induced copies of a k-graph F. However, ultimately, we want to count all induced copies of F. Let us introduce the necessary notation for this step.

Suppose $k, \ell \in \mathbb{N} \setminus \{1\}$ such that $\ell \ge k$ and suppose $\mathbf{a} \in \mathbb{N}^{k-1}$. Suppose that $d_{\mathbf{a},k} : \hat{A}(k, k-1, \mathbf{a}) \to [0, 1]$ is a density function. Suppose F is a k-graph on ℓ vertices. Let $\mathrm{Bij}(A, B)$ be the set of all bijections from A to B. Suppose $\hat{\mathbf{x}} \in \hat{A}(\ell, k-1, \mathbf{a})$ and $\sigma \in \mathrm{Bij}(V(F), \mathbf{x}_*^{(1)})$ is a bijection. Let $A(F)$ be the size of the automorphism group of F. We now define three functions in terms of the parameters above that will estimate the number of induced copies of F in certain parts of an ε-regular k-graph. Let

$$\begin{aligned} IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}, \sigma) &:= \prod_{\substack{\hat{\mathbf{y}} \leq_{k,k-1} \hat{\mathbf{x}}, \\ \mathbf{y}_*^{(1)} \in \sigma(F)}} d_{\mathbf{a},k}(\hat{\mathbf{y}}) \prod_{\substack{\hat{\mathbf{y}} \leq_{k,k-1} \hat{\mathbf{x}}, \\ \mathbf{y}_*^{(1)} \notin \sigma(F)}} (1 - d_{\mathbf{a},k}(\hat{\mathbf{y}})) \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}}, \\ IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}) &:= \frac{1}{A(F)} \sum_{\sigma \in \mathrm{Bij}(V(F), \mathbf{x}_*^{(1)})} IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}, \sigma), \\ IC(F, d_{\mathbf{a},k}) &:= \binom{a_1}{\ell}^{-1} \sum_{\hat{\mathbf{x}} \in \hat{A}(\ell, k-1, \mathbf{a})} IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}). \end{aligned}$$

We will now show that for a k-graph H satisfying a suitable regularity instance $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$, the value $IC(F, d_{\mathbf{a},k})$ is a very accurate estimate for $\mathbf{Pr}(F, H)$ (recall the latter was introduced in Section 2.1). The same is true if F is replaced by a finite family of k-graphs (see Corollary 5.9).

Lemma 5.8 (Induced counting lemma for general hypergraphs)**.** *Suppose $0 < 1/n \ll \varepsilon \ll 1/t, 1/a_1 \ll \gamma, 1/k, 1/\ell$ with $2 \leq k \leq \ell$. Suppose F is an ℓ-vertex k-graph and $\mathbf{a} \in [t]^{k-1}$. Suppose H is an n-vertex k-graph satisfying a regularity instance $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$. Then*

$$\mathbf{Pr}(F, H) = IC(F, d_{\mathbf{a},k}) \pm \gamma.$$

Proof. Since H satisfies the regularity instance R, there exists a $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ of H (as defined in Section 2.4.2). Let $\mathscr{P}^{(1)} = \{V_1, \dots, V_{a_1}\}$ and $m := \lfloor n/a_1 \rfloor$. We say an induced copy F' of F in H is *crossing-induced* if $V(F') \in \mathcal{K}_\ell(\mathscr{P}^{(1)})$ and *non-crossing-induced* otherwise. Then by (2.5),

$$\text{there are at most } \tfrac{\gamma}{3}\tbinom{n}{\ell} \text{ non-crossing-induced copies of } F. \tag{5.2}$$

The strategy of the proof is as follows. We only consider crossing-induced copies of F, as the number of non-crossing-induced copies is negligible. For each $\hat{\mathbf{x}} \in \hat{A}(\ell, k-1, \mathbf{a})$, we fix some bijection σ between $V(F)$ and $\mathbf{x}_*^{(1)}$. By Lemma 5.7, we can accurately estimate the number of σ-induced copies of F. By summing over all choices for $\hat{\mathbf{x}}$ and σ and taking in account which copies we counted multiple times, we can estimate the number of crossing-induced copies of F in H.

For each $\hat{\mathbf{x}} \in \hat{A}(\ell, k-1, \mathbf{a})$, we consider the $(k-1)$-polyad $\hat{P}^{(k-1)}(\hat{\mathbf{x}}) = \bigcup_{\hat{\mathbf{y}} \leq_{k,k-1} \hat{\mathbf{x}}} \hat{P}^{(k-1)}(\hat{\mathbf{y}})$ as defined in (2.13). By Proposition 2.8, for every crossing-induced copy F' of F in H, there is a unique $\hat{\mathbf{x}} \in \hat{A}(\ell, k-1, \mathbf{a})$ such that F' is contained in some element of $\mathcal{K}_\ell(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))$.

Consider any $\hat{\mathbf{x}} \in \hat{A}(\ell, k-1, \mathbf{a})$ and a bijection $\sigma : V(F) \to \mathbf{x}_*^{(1)}$. Let

$$\mathcal{H}'(\hat{\mathbf{x}}) := \left\{ \bigcup_{\hat{\mathbf{z}} \leq_{k,i} \hat{\mathbf{x}}} \hat{P}^{(i)}(\hat{\mathbf{z}}) \right\}_{i \in [k-1]} \quad \text{and} \quad \mathcal{H}(\hat{\mathbf{x}}) := \mathcal{H}'(\hat{\mathbf{x}}) \cup \{H \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))\}.$$

Hence $\mathcal{H}(\hat{\mathbf{x}})$ is an (ℓ, k)-complex and $\mathcal{H}'(\hat{\mathbf{x}})$ is an $(\ell, k-1)$-complex. Note that $\mathcal{H}'(\hat{\mathbf{x}}) = \bigcup_{\hat{\mathbf{y}} \leq_{k,k-1} \hat{\mathbf{x}}} \hat{\mathcal{P}}(\hat{\mathbf{y}})$, where $\hat{\mathcal{P}}(\hat{\mathbf{y}})$ is as defined in (2.14).

Lemma 5.5 implies that each $\hat{\mathcal{P}}(\hat{\mathbf{y}}) = \mathcal{H}'(\hat{\mathbf{x}})[\mathbf{y}_*^{(1)}]$ is $(\varepsilon, (1/a_2, \dots, 1/a_{k-1}))$-regular (if $k \geq 3$). Furthermore, since $\mathscr{P}$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of H, for each $e \in \binom{V(F)}{k}$, the k-graph $H[\sigma(e)]$ is $(\varepsilon, d_{\mathbf{a},k}(\hat{\mathbf{y}}))$-regular with respect to $\hat{P}^{(k-1)}(\hat{\mathbf{y}}) = \hat{P}^{(k-1)}(\hat{\mathbf{x}})[\bigcup_{i \in \mathbf{y}_*^{(1)}} V_i]$, where $\hat{\mathbf{y}}$ is the unique vector satisfying $\hat{\mathbf{y}} \leq_{k,k-1} \hat{\mathbf{x}}$ and $\mathbf{y}_*^{(1)} = \sigma(e)$.

Thus, by applying Lemma 5.7 with $\mathcal{H}(\hat{\mathbf{x}}), a_i^{-1}, \gamma/(3\ell!), d_{\mathbf{a},k}(\hat{\mathbf{y}})$ playing the roles of $\mathcal{H}, d_i, \gamma, p_{\mathbf{y}_*^{(1)}}$, we conclude that (with $IC_\sigma(F, \mathcal{H}(\hat{\mathbf{x}}))$ defined as in Lemma 5.7)

$$\begin{aligned} IC_\sigma(F, \mathcal{H}(\hat{\mathbf{x}})) &= \left(\prod_{\substack{\hat{\mathbf{y}} \leq_{k,k-1} \hat{\mathbf{x}}, \\ \mathbf{y}_*^{(1)} \in \sigma(F)}} d_{\mathbf{a},k}(\hat{\mathbf{y}}) \prod_{\substack{\hat{\mathbf{y}} \leq_{k,k-1} \hat{\mathbf{x}}, \\ \mathbf{y}_*^{(1)} \notin \sigma(F)}} (1 - d_{\mathbf{a},k}(\hat{\mathbf{y}})) \pm \frac{\gamma}{3\ell!} \right) \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}} \cdot m^\ell \\ &= \left(IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}, \sigma) \pm \frac{\gamma}{3\ell!} \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}} \right) m^\ell. \end{aligned}$$

Next we want to estimate the number of all crossing-induced copies of F in H which lie in some element of $\mathcal{K}_\ell(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))$. Observe that we count every copy of F exactly $A(F)$ times if we sum over all possible bijections σ. Therefore, the number of crossing-induced copies of F in H which lie in some element of $\mathcal{K}_\ell(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))$ is

$$\begin{aligned} \frac{1}{A(F)} \sum_\sigma IC_\sigma(F, \mathcal{H}(\hat{\mathbf{x}})) &= \frac{1}{A(F)} \sum_\sigma \left(IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}, \sigma) \pm \frac{\gamma}{3\ell!} \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}} \right) m^\ell \\ &= \left(IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}) \pm \frac{\gamma}{3} \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}} \right) m^\ell. \end{aligned}$$

Note that $|\hat{A}(\ell, k-1, \mathbf{a})| = \binom{a_1}{\ell} \prod_{j=2}^{k-1} a_j^{\binom{\ell}{j}}$ and $\binom{a_1}{\ell} m^\ell = (1 \pm \gamma/10)\binom{n}{\ell}$, because $1/a_1 \ll \gamma, 1/\ell$. Hence the number of crossing-induced copies of F in H is

$$\begin{aligned} &\sum_{\hat{\mathbf{x}} \in \hat{A}(\ell,k-1,\mathbf{a})} \left(IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}) \pm \frac{\gamma}{3} \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}} \right) m^\ell \\ &\quad = \left(\sum_{\hat{\mathbf{x}} \in \hat{A}(\ell,k-1,\mathbf{a})} IC(F, d_{\mathbf{a},k}, \hat{\mathbf{x}}) \right) m^\ell \pm \frac{\gamma}{3} \prod_{j=2}^{k-1} a_j^{-\binom{\ell}{j}} |\hat{A}(\ell, k-1, \mathbf{a})| m^\ell \\ &\quad = (IC(F, d_{\mathbf{a},k}) \pm \gamma/2) \binom{n}{\ell}. \end{aligned}$$

This together with (5.2) implies the desired statement. □

In the previous lemma we counted the number of induced copies of a single k-graph F in H. It is not difficult to extend this approach to a finite family of k-graphs. For a finite family $\mathcal{F}$ of k-graphs, we define

$$IC(\mathcal{F}, d_{\mathbf{a},k}) := \sum_{F \in \mathcal{F}} IC(F, d_{\mathbf{a},k}). \tag{5.3}$$

Corollary 5.9. *Suppose $0 < 1/n \ll \varepsilon \ll 1/t, 1/a_1 \ll \gamma, 1/k, 1/\ell$ with $2 \leq k \leq \ell$. Let $\mathcal{F}$ be a collection of k-graphs on ℓ vertices. Suppose H is an n-vertex k-graph satisfying a regularity instance $R = (\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$ where $\mathbf{a} \in [t]^{k-1}$. Then*

$$\mathbf{Pr}(\mathcal{F}, H) = IC(\mathcal{F}, d_{\mathbf{a},k}) \pm \gamma.$$

Proof. For each $F \in \mathcal{F}$, we apply Lemma 5.8 with $\gamma/2^{\binom{\ell}{k}}$ playing the role of γ. As $|\mathcal{F}| \leq 2^{\binom{\ell}{k}}$, this completes the proof. □

5.3. **Refining a partition.** In this subsection we make a simple observation regarding refinements of a given partition. This shows that we can refine a family of partitions without significantly affecting the regularity parameters.

Lemma 5.10. *Suppose* $0 < 1/n \ll \varepsilon \ll 1/t, 1/k$ *with* $k, t \in \mathbb{N} \setminus \{1\}$, $0 < \eta < 1$, *and* $\mathbf{a} \in \mathbb{N}^{k-1}$. *Suppose* $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ *is an* $(\eta, \varepsilon, \mathbf{a})$*-equitable family of partitions on* V *with* $|V| = n$. *Suppose* $\mathbf{b} \in [t]^{k-1}$ *and* $a_i \mid b_i$ *for all* $i \in [k-1]$. *Then there exists a family of partitions* $\mathscr{Q} = \mathscr{Q}(k-1, \mathbf{b})$ *on* V *which is* $(\eta, \varepsilon^{1/3}, \mathbf{b})$*-equitable and* $\mathscr{Q} \prec \mathscr{P}$.

It is easy to prove this by induction on k via an appropriate application of the slicing lemma (Lemma 5.3). We omit the details.

5.4. **Small perturbations of partitions.** Here we consider the effect of small changes in a partition on the resulting parameters. In particular, the next lemma implies that for any family of partitions $\mathscr{P}$, every family of partitions that is close to $\mathscr{P}$ in distance is a family of partitions with almost the same parameters. This is proved in [17].

Lemma 5.11. *Suppose* $k \in \mathbb{N} \setminus \{1\}$, $0 < 1/n \ll \nu \ll \varepsilon$, *and* $0 \leq \lambda \leq 1/4$. *Suppose* $R = (\varepsilon/3, \mathbf{a}, d_{\mathbf{a},k})$ *is a regularity instance. Suppose* V *is a vertex set of size* n *and suppose* $G^{(k)}, H^{(k)}$ *are* k*-graphs on* V *with* $|G^{(k)} \triangle H^{(k)}| \leq \nu\binom{n}{k}$. *Suppose* $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ *is a* $(1/a_1, \varepsilon, \mathbf{a}, \lambda)$*-equitable family of partitions on* V *which is an* $(\varepsilon, d_{\mathbf{a},k})$*-partition of* $H^{(k)}$. *Suppose* $\mathscr{Q} = \mathscr{Q}(k-1, \mathbf{a})$ *is a family of partitions on* V *such that for any* $j \in [k-1]$, $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$, *and* $b \in [a_j]$, *we have*

$$|P^{(j)}(\hat{\mathbf{x}}, b) \triangle Q^{(j)}(\hat{\mathbf{x}}, b)| \leq \nu\binom{n}{j}. \tag{5.4}$$

Then $\mathscr{Q}$ *is a* $(1/a_1, \varepsilon+\nu^{1/6}, \mathbf{a}, \lambda+\nu^{1/6})$*-equitable family of partitions which is an* $(\varepsilon+\nu^{1/6}, d_{\mathbf{a},k})$*-partition of* $G^{(k)}$.

The following lemma shows that for every equitable family of partitions $\mathscr{P}$ whose vertex partition $\mathscr{P}^{(1)}$ is an almost equipartition, there is an equitable family of partitions with almost the same parameters whose vertex partition is an equipartition.

Lemma 5.12. *Suppose* $0 < 1/n \ll \lambda \ll \varepsilon \leq 1$, $k \in \mathbb{N} \setminus \{1\}$, *and* $R = (\varepsilon/3, \mathbf{a}, d_{\mathbf{a},k})$ *is a regularity instance. Suppose* $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ *is a* $(1/a_1, \varepsilon, \mathbf{a}, \lambda)$*-equitable family of partitions and an* $(\varepsilon, d_{\mathbf{a},k})$*-partition of an* n*-vertex* k*-graph* $H^{(k)}$. *Then there exists a family of partitions* $\mathscr{Q} = \mathscr{Q}(k-1, \mathbf{a})$ *which is an* $(\varepsilon + \lambda^{1/10}, \mathbf{a}, d_{\mathbf{a},k})$*-equitable partition of* $H^{(k)}$.

Proof. Let $m := \lfloor n/a_1 \rfloor$. We write $\mathscr{P}^{(1)} = \{V_1, \dots, V_{a_1}\}$. Since $\mathscr{P}$ is a $(1/a_1, \varepsilon, \mathbf{a}, \lambda)$-equitable family of partitions, we have $|V_i| = (1 \pm \lambda)m$ for all $i \in [a_1]$, and Lemma 5.5 implies that for each $j \in [k-1]$, the function $\hat{P}^{(j)}(\cdot) : \hat{A}(j+1, j, \mathbf{a}) \to \hat{\mathscr{P}}^{(j)}$ is a bijection and for each $j \in [k-1] \setminus \{1\}$, the function $P^{(j)}(\cdot, \cdot) : \hat{A}(j, j-1, \mathbf{a}) \times [a_j] \to \mathscr{P}^{(j)}$ is also a bijection. Next, we fix the size of the parts in the new equitable partition $\mathscr{Q}$ of $V := V_1 \cup \dots \cup V_{a_1}$. For each $i \in [a_1]$, let $m_i := \lfloor (n+i-1)/a_1 \rfloor$. Thus $m_i \in \{m, m+1\}$. Choose $U_i' \subseteq V_i$ of size $\max\{|V_i|, m_i\}$ and let $U_0' := \bigcup_{i \in [a_1]} V_i \setminus U_i'$. We partition U_0' into $U_1'', \dots, U_{a_1}''$ in an arbitrary manner such that $|U_i''| = m_i - |U_i'|$. For each $i \in [a_1]$, let

$$U_i := U_i' \cup U_i'' \text{ and } \mathscr{Q}^{(1)} := \{U_1, \dots, U_{a_1}\}.$$

Moreover, let $Q^{(1)}(b, b) := U_b$ for each $b \in [a_1]$, and $Q^{(1)}(b, b') := \emptyset$ for all distinct $b, b' \in [a_1]$. For each $i \in [a_1]$, we have

$$|U_i \triangle V_i| \leq |(1 \pm \lambda)m - m_i| \leq \lambda m + 1.$$

For each $\hat{\mathbf{x}} = (\alpha_1, \alpha_2) \in \hat{A}(2, 1, \mathbf{a})$, let $\hat{Q}^{(1)}(\hat{\mathbf{x}}) := U_{\alpha_1} \cup U_{\alpha_2}$. Note that $\{\mathcal{K}_2(\hat{Q}^{(1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(2, 1, \mathbf{a})\}$ forms a partition of $\mathcal{K}_2(\mathscr{Q}^{(1)})$.

Now, we inductively construct $\mathscr{Q}^{(2)}, \dots, \mathscr{Q}^{(k-1)}$ in this order. Assume that for some $j \in [k] \setminus \{1\}$, we have already defined $\{\mathscr{Q}^{(i)}\}_{i=1}^{j-1}$ with $\mathscr{Q}^{(i)} = \{Q^{(i)}(\hat{\mathbf{x}}, b) : \hat{\mathbf{x}} \in \hat{A}(i, i-1, \mathbf{a}), b \in [a_i]\}$ and $\hat{\mathscr{Q}}^{(i)} = \{\hat{Q}^{(i)}(\hat{\mathbf{x}}) : \hat{\mathbf{x}} \in \hat{A}(i+1, i, \mathbf{a})\}$ for each $i \in [j-1]$ such that the following hold.

(Q1)$_{j-1}$ For each $i \in [j-1]$, $\hat{\mathbf{x}} \in \hat{A}(i, i-1, \mathbf{a})$ and $b \in [a_i]$, we have $|P^{(i)}(\hat{\mathbf{x}}, b) \triangle Q^{(i)}(\hat{\mathbf{x}}, b)| \leq 2^i i! \lambda n^i$.

(Q2)$_{j-1}$ For each $i \in [j-1] \setminus \{1\}$ and $\hat{\mathbf{x}} \in \hat{A}(i, i-1, \mathbf{a})$, the collection $\{Q^{(i)}(\hat{\mathbf{x}}, b) : b \in [a_i]\}$ forms a partition of $\mathcal{K}_i(\hat{Q}^{(i-1)}(\hat{\mathbf{x}}))$.

(Q3)$_{j-1}$ For each $i \in [j-1]$ and $\hat{\mathbf{x}} \in \hat{A}(i+1, i, \mathbf{a})$, we have $\hat{Q}^{(i)}(\hat{\mathbf{x}}) = \bigcup_{\hat{\mathbf{y}} \leq_{i,i-1} \hat{\mathbf{x}}} Q^{(i)}(\hat{\mathbf{y}}, \mathbf{x}^{(i)}_{\mathbf{y}^{(1)}_*})$.

Note that $\mathscr{Q}^{(1)}$ satisfies (Q1)$_1$–(Q3)$_1$. Suppose first that $j \leq k-1$. In this case we will define $\mathscr{Q}^{(j)}$ satisfying (Q1)$_j$–(Q3)$_j$. For each $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$ and $b \in [a_j]$, we define

$$Q^{(j)}(\hat{\mathbf{x}}, b) := \begin{cases} P^{(j)}(\hat{\mathbf{x}}, b) \cap \mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}})) & \text{if } b \in [a_j - 1], \\ \mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}})) \setminus \bigcup_{b \in [a_j - 1]} P^{(j)}(\hat{\mathbf{x}}, b) & \text{otherwise.} \end{cases}$$

Let

$$\mathscr{Q}^{(j)} := \{Q^{(j)}(\hat{\mathbf{x}}, b) : \hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}), b \in [a_j]\}.$$

Then for any fixed $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$, it is obvious that $Q^{(j)}(\hat{\mathbf{x}}, 1), \dots, Q^{(j)}(\hat{\mathbf{x}}, a_j)$ forms a partition of $\mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}}))$. Thus (Q2)$_j$ holds.

For each $\hat{\mathbf{z}} \in \hat{A}(j+1, j, \mathbf{a})$, let

$$\hat{Q}^{(j)}(\hat{\mathbf{z}}) := \bigcup_{\hat{\mathbf{y}} \leq_{j,j-1} \hat{\mathbf{z}}} Q^{(j)}(\hat{\mathbf{y}}, \mathbf{z}^{(j)}_{\mathbf{y}^{(1)}_*}).$$

Then (Q3)$_j$ also holds.

Note that for any fixed $(j-1)$-set $J' \in \hat{P}^{(j-1)}(\hat{\mathbf{x}}) \triangle \hat{Q}^{(j-1)}(\hat{\mathbf{x}})$, there are at most $(1+\lambda)m$ distinct j-sets in $\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}})) \triangle \mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}}))$ containing J'. Thus for $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a})$ and $b \in [a_j]$, we obtain

$$\begin{aligned} |P^{(j)}(\hat{\mathbf{x}}, b) \triangle Q^{(j)}(\hat{\mathbf{x}}, b)| &\leq |\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}})) \triangle \mathcal{K}_j(\hat{Q}^{(j-1)}(\hat{\mathbf{x}}))| \\ &\leq (1+\lambda)m |\hat{P}^{(j-1)}(\hat{\mathbf{x}}) \triangle \hat{Q}^{(j-1)}(\hat{\mathbf{x}})| \\ &\overset{(2.12),(\mathrm{Q3})_{j-1}}{\leq} \sum_{\hat{\mathbf{y}} \leq_{j-1,j-2} \hat{\mathbf{x}}} 2m |P^{(j-1)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\mathbf{y}^{(1)}_*}) \triangle Q^{(j-1)}(\hat{\mathbf{y}}, \mathbf{x}^{(j)}_{\mathbf{y}^{(1)}_*})| \\ &\overset{(\mathrm{Q1})_{j-1}}{\leq} 2^j j! \lambda n^j. \end{aligned}$$

Thus (Q1)$_j$ holds and we obtain $\{\mathscr{Q}^{(i)}\}_{i=1}^{j}$ satisfying (Q1)$_j$–(Q3)$_j$. Inductively we obtain $\mathscr{Q} = \{\mathscr{Q}^{(i)}\}_{i=1}^{k-1}$ satisfying (Q1)$_{k-1}$–(Q3)$_{k-1}$.

Note that since $R = (\varepsilon/3, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance, we have the following inequality.

$$\varepsilon \leq \|\mathbf{a}\|_\infty^{-4^k} \varepsilon_{5.4}(\|\mathbf{a}\|_\infty^{-1}, \|\mathbf{a}\|_\infty^{-1}, k-1, k).$$

Thus Lemma 5.4 (together with Lemma 5.5(ii)) implies for any $j \in [k-1] \setminus \{1\}$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}) \times [a_j]$ that $|P^{(j)}(\hat{\mathbf{x}}, b)| \geq \varepsilon^{1/2} n^j$. This with (Q1)$_{k-1}$ shows that for any $j \in [k-1] \setminus \{1\}$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}) \times [a_j]$, the j-graph $Q^{(j)}(\hat{\mathbf{x}}, b)$ is nonempty. Together with properties (Q2)$_{k-1}$ and (Q3)$_{k-1}$ this in turn ensures that we can apply Lemma 4.2 to show that $\mathscr{Q}$ is a family of partitions.

By (Q1)$_{k-1}$ and the assumption that $R = (\varepsilon/3, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance, we can apply Lemma 5.11 with $\mathscr{P}, \mathscr{Q}, H^{(k)}, H^{(k)}, \lambda, \lambda^{9/10}$ playing the roles of $\mathscr{P}, \mathscr{Q}, H^{(k)}, G^{(k)}, \lambda, \nu$ to obtain that $\mathscr{Q} = \mathscr{Q}(k-1, \mathbf{a})$ is an $(\varepsilon + \lambda^{1/10}, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $H^{(k)}$. □

6. Regular approximations of partitions and hypergraphs

The main aim of this section is to prove a strengthening of the partition version (Lemma 4.1) of the regular approximation lemma. As described in Section 2.3, Lemma 4.1 outputs for a given equitable family of partitions $\mathscr{Q}$ another family of partitions $\mathscr{P}$ that refines $\mathscr{Q}$. In Lemma 6.1 $\mathscr{P}$ has the additional feature that it almost refines a further given (arbitrary) family of partitions $\mathscr{O}$. Observe that we cannot hope to refine $\mathscr{O}$ itself, as for example some sets in $\mathscr{O}^{(1)}$ may be very small. We also prove two further tools: Lemma 6.2 allows us to transfer the large scale structure of a hypergraph to another one on a different vertex set and Lemma 6.3 concerns suitable perturbations of a given partition. Lemmas 6.1–6.3 will all be used in the proof of Lemma 7.1.

Lemma 6.1. *For all $k, o \in \mathbb{N} \setminus \{1\}$, $s \in \mathbb{N}$, all $\eta, \nu > 0$, and every function $\varepsilon : \mathbb{N}^{k-1} \to (0, 1]$, there are $\mu = \mu_{6.1}(k, o, s, \eta, \nu, \varepsilon) > 0$ and $t = t_{6.1}(k, o, s, \eta, \nu, \varepsilon) \in \mathbb{N}$ and $n_0 = n_{6.1}(k, o, s, \eta, \nu, \varepsilon) \in \mathbb{N}$ such that the following hold. Suppose*

(O1)$_{6.1}$ *V is a set and $|V| = n \geq n_0$,*
(O2)$_{6.1}$ *$\mathscr{O}(k-1, \mathbf{a}^{\mathscr{O}}) = \{\mathscr{O}^{(j)}\}_{j=1}^{k-1}$ is an o-bounded family of partitions on V,*
(O3)$_{6.1}$ *$\mathscr{Q} = \mathscr{Q}(k, \mathbf{a}^{\mathscr{Q}})$ is a $(1/a_1^{\mathscr{Q}}, \mu, \mathbf{a}^{\mathscr{Q}})$-equitable o-bounded family of partitions on V, and*
(O4)$_{6.1}$ *$\mathscr{H}^{(k)} = \{H_1^{(k)}, \dots, H_s^{(k)}\}$ is a partition of $\binom{V}{k}$ so that $\mathscr{H}^{(k)} \prec \mathscr{Q}^{(k)}$.*

Then there exist a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ and a partition $\mathscr{G}^{(k)} = \{G_1^{(k)}, \dots, G_s^{(k)}\}$ of $\binom{V}{k}$ satisfying the following for each $j \in [k-1]$ and $i \in [s]$.

(P1)$_{6.1}$ *$\mathscr{P}$ is a t-bounded $(\eta, \varepsilon(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}})$-equitable family of partitions, and $a_j^{\mathscr{Q}}$ divides $a_j^{\mathscr{P}}$,*
(P2)$_{6.1}$ *$\mathscr{P}^{(j)} \prec \mathscr{Q}^{(j)}$ and $\mathscr{P}^{(j)} \prec_\nu \mathscr{O}^{(j)}$,*
(G1)$_{6.1}$ *$G_i^{(k)}$ is perfectly $\varepsilon(\mathbf{a}^{\mathscr{P}})$-regular with respect to $\mathscr{P}$,*
(G2)$_{6.1}$ *$\sum_{i=1}^{s} |G_i^{(k)} \triangle H_i^{(k)}| \leq \nu \binom{n}{k}$, and*
(G3)$_{6.1}$ *$\mathscr{G}^{(k)} \prec \mathscr{Q}^{(k)}$ and if $H_i^{(k)} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$ then $G_i^{(k)} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$.*

We believe that Lemma 6.1 will have additional applications. As we mentioned it is also used in the proof Corollary 9.3 in [17].

In Lemma 6.1 we may assume without loss of generality that $1/\mu, t, n_0$ are non-decreasing in k, o, s and non-increasing in η, ν.

To prove Lemma 6.1 we proceed by induction on k. In the induction step, we first construct an 'intermediate' family of partitions $\{\mathscr{L}^{(i)}\}_{i=1}^{k-1}$ which satisfies (P1)$_{6.1}$ and (P2)$_{6.1}$. The partitions $\mathscr{L}^{(1)}, \dots, \mathscr{L}^{(k-1)}$ are constructed via the inductive assumption of Lemma 6.1 (see Claim 2). We then construct a partition $\mathscr{L}^{(k)}$ via appropriate applications of the slicing lemma (see Claim 3). Finally, we apply Lemma 4.1 with $\mathscr{L}_* = \{\mathscr{L}^{(i)}\}_{i=1}^{k}$ playing the role of $\mathscr{Q}$ to obtain our desired family of partitions $\mathscr{P}$ and construct $G_i^{(k)}$ based on the k-graphs guaranteed by Lemma 4.1.

Proof of Lemma 6.1. First of all, by decreasing the value of η if necessary, we may assume that $\eta < 1/(10k!)$. We may also assume that $\nu \leq \eta$.

We use induction on k. For each $k \in \mathbb{N} \setminus \{1\}$, let $L_{6.1}(k)$ be the statement of the lemma. Let $L_{6.1}(1)$ be the following statement (Claim 1).

Claim 1 ($L_{6.1}(1)$)**.** *For all $o, s \in \mathbb{N}$, all $\eta, \nu > 0$, there are $t = t_{6.1}(1, o, s, \nu) := so\lceil 2\nu^{-2} \rceil$ and $n_0 = n_{6.1}(1, o, s, \nu) \in \mathbb{N}$ such that the following hold. Suppose*

(O1)$^1_{6.1}$ *V is a set and $|V| = n \geq n_0$,*
(O2)$^1_{6.1}$ *$\mathscr{Q}^{(1)}$ is an equipartition of V into $a_1^{\mathscr{Q}} \leq o$ parts,*
(O3)$^1_{6.1}$ *$\mathscr{H}^{(1)} = \{H_1^{(1)}, \dots, H_s^{(1)}\}$ is a partition of V so that $\mathscr{H}^{(1)} \prec \mathscr{Q}^{(1)}$.*

Then there exists a partition $\mathscr{P}^{(1)}$ of V satisfying the following.

(P1)$^1_{6.1}$ *$\mathscr{P}^{(1)}$ is an equipartition of V into $a_1^{\mathscr{P}} \leq t$ parts, and $a_1^{\mathscr{Q}}$ divides $a_1^{\mathscr{P}}$,* (P2)$^1_{6.1}$ $\mathscr{P}^{(1)} \prec \mathscr{Q}^{(1)}$ *and* $\mathscr{P}^{(1)} \prec_{\nu^2} \mathscr{H}^{(1)}$.

Proof. Write $\mathscr{Q}^{(1)} = \{Q_i^{(1)} : i \in [a_1^{\mathscr{Q}}]\}$. Let $a_1^{\mathscr{P}} := s a_1^{\mathscr{Q}} \lceil 2\nu^{-2} \rceil$, let $m := \min\{|Q_i^{(1)}| : i \in [a_1^{\mathscr{Q}}]\}$, and let $m' := \lfloor |V|/a_1^{\mathscr{P}} \rfloor$. Thus $|Q_i^{(1)}| \in \{m, m+1\}$ for each $i \in [a_1^{\mathscr{Q}}]$. The sets in $\mathscr{P}^{(1)}$ will have size m' or $m'+1$. Note that

$$m' \cdot \frac{a_1^{\mathscr{P}}}{a_1^{\mathscr{Q}}} \leq \left\lfloor \frac{|V|}{a_1^{\mathscr{P}}} \cdot \frac{a_1^{\mathscr{P}}}{a_1^{\mathscr{Q}}} \right\rfloor = m < (m'+1) \cdot \frac{a_1^{\mathscr{P}}}{a_1^{\mathscr{Q}}}.$$

In particular, as $\frac{a_1^{\mathscr{P}}}{a_1^{\mathscr{Q}}}$ is an integer, we have

$$0 \leq m \pmod{m'} \leq \frac{a_1^{\mathscr{P}}}{a_1^{\mathscr{Q}}}. \tag{6.1}$$

To obtain $\mathscr{P}^{(1)}$ we further (almost) refine $\mathscr{H}^{(1)}$. For each $i \in [s]$, we define $\ell_i := \lfloor |H_i^{(1)}|/m' \rfloor$. We arbitrarily partition $H_i^{(1)}$ into $\mathscr{L}(i,0), \dots, \mathscr{L}(i,\ell_i)$ such that $|\mathscr{L}(i,r)| = m'$ for all $r \in [\ell_i]$ and $|\mathscr{L}(i,0)| < m'$. For each $j \in [a_1^{\mathscr{Q}}]$, let $\mathscr{L}'(j,0) := \bigcup_{H_i^{(1)} \subseteq Q_j^{(1)}} \mathscr{L}(i,0)$. Let $\ell'_j := \lfloor |\mathscr{L}'(j,0)|/m' \rfloor$. We arbitrarily partition $\mathscr{L}'(j,0)$ into $\mathscr{L}''(j,0), \mathscr{L}'(j,1), \dots, \mathscr{L}'(j,\ell'_j)$ such that $|\mathscr{L}'(j,r)| = m'$ for all $r \in [\ell'_j]$ and $|\mathscr{L}''(j,0)| < m'$. Note that since $\mathscr{H}^{(1)} \prec \mathscr{Q}^{(1)}$, for all $j \in [a_1^{\mathscr{Q}}]$, we have

$$Q_j^{(1)} = \bigcup_{i:\, H_i^{(1)} \subseteq Q_j^{(1)}} H_i^{(1)} = \mathscr{L}''(j,0) \cup \bigcup_{r \in [\ell'_j]} \mathscr{L}'(j,r) \cup \bigcup_{i:\, H_i^{(1)} \subseteq Q_j^{(1)}, r \in [\ell_i]} \mathscr{L}(i,r).$$

As all sets $\mathscr{L}'(j,r)$ and $\mathscr{L}(i,r)$ with $r > 0$ have size exactly m', the fact that $|\mathscr{L}''(j,0)| < m'$ implies that we have $|\mathscr{L}''(j,0)| = m \pmod{m'}$. On the other hand, this together with (6.1) implies that $|\mathscr{L}''(j,0)| \leq a_1^{\mathscr{P}}/a_1^{\mathscr{Q}} = \ell'_j + \sum_{i:\, H_i^{(1)} \subseteq Q_j^{(1)}} \ell_i$.

Hence, by distributing at most one vertex from $\mathscr{L}''(j,0)$ into each of the sets in $\{\mathscr{L}'(j,r) : r \in [\ell'_j]\} \cup \{\mathscr{L}(i,r) : H_i^{(1)} \subseteq Q_j^{(1)}, r \in [\ell_i]\}$, we can obtain the following collection

$$\{L^{(1)}(j,1), \dots, L^{(1)}(j, a_1^{\mathscr{P}}/a_1^{\mathscr{Q}})\}$$

of sets of size m' or $m'+1$, which forms an equipartition of $Q_j^{(1)}$. Let

$$\mathscr{P}^{(1)} := \{L^{(1)}(j,r) : j \in [a_1^{\mathscr{Q}}], r \in [a_1^{\mathscr{P}}/a_1^{\mathscr{Q}}]\}.$$

Then (P1)$^1_{6.1}$ holds. By construction, for each $L^{(1)} \in \mathscr{P}^{(1)}$, either there exists $(i,r) \in [s] \times [\ell_i]$ such that $|L^{(1)} \setminus \mathscr{L}(i,r)| \leq 1$ or there exists $(j,r) \in [a_1^{\mathscr{Q}}] \times [\ell'_j]$ such that $|L^{(1)} \setminus \mathscr{L}'(j,r)| \leq 1$. In the former case, let $f(L^{(1)}) := H_i^{(1)}$ and the latter case, let $f(L^{(1)})$ be an arbitrary set in $\mathscr{H}^{(1)}$. Then

$$\begin{aligned}\sum_{L^{(1)} \in \mathscr{P}^{(1)}} |L^{(1)} \setminus f(L^{(1)})| &\leq |\mathscr{P}^{(1)}| + \sum_{(j,r) \in [a_1^{\mathscr{Q}}] \times [\ell'_j]} |\mathscr{L}'(j,r)| \leq a_1^{\mathscr{P}} + \sum_{j \in [a_1^{\mathscr{Q}}], H_i^{(1)} \subseteq Q_j^{(1)}} |\mathscr{L}(i,0)| \\ &\leq a_1^{\mathscr{P}} + s a_1^{\mathscr{Q}} m' \leq \nu^2 |V|.\end{aligned}$$

The final inequality follows since $n \geq n_0$. This and the construction of $\mathscr{P}^{(1)}$ shows that $\mathscr{P}^{(1)} \prec_{\nu^2} \mathscr{H}^{(1)}$ and $\mathscr{P}^{(1)} \prec \mathscr{Q}^{(1)}$. This shows that (P2)$^1_{6.1}$ holds and thus completes the proof of Claim 1. □

So assume that $k \geq 2$ and $L_{6.1}(k-1)$ holds. Let $\mu_{4.1}, t_{4.1}, n_{4.1}$ be the functions defined in Lemma 4.1. By decreasing the value of $\varepsilon(\mathbf{a})$ if necessary, we may assume that for all $\mathbf{a} \in \mathbb{N}^{k-1}$, we have

$$\varepsilon(\mathbf{a}) \ll 1/s, 1/k, 1/\|\mathbf{a}\|_\infty. \tag{6.2}$$

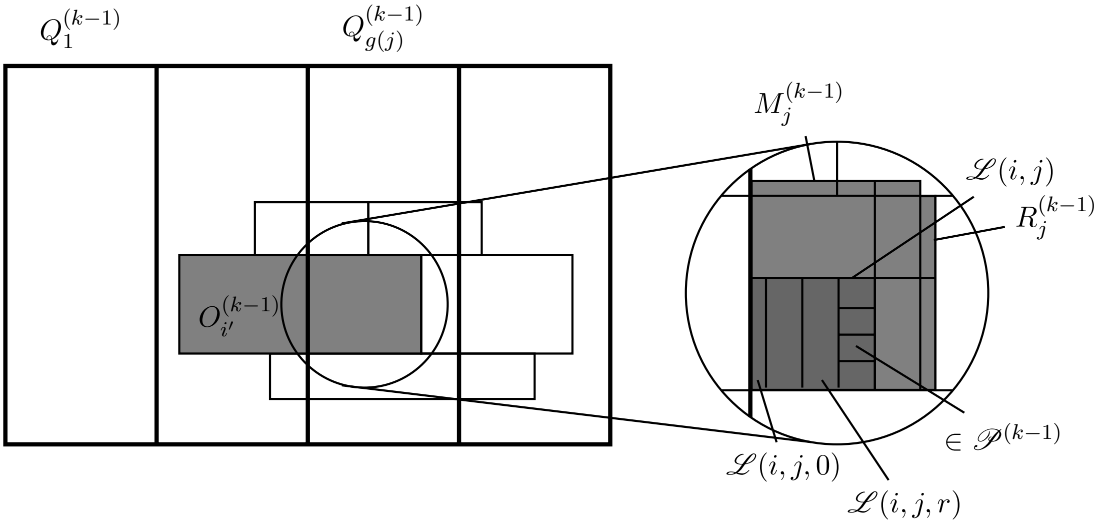

FIGURE 1. An illustration of the cascade of partitions in the proof of Lemma 6.1.

If $k = 2$, let $T := o^{4^k+1}\lceil 2\nu^{-2}\rceil$. If $k \geq 3$, for each $\mathbf{a} \in \mathbb{N}^{k-2}$, let $T = T(\mathbf{a}, o, \nu) = \max\{\|\mathbf{a}\|_\infty, o^{4^k+1}\lceil 2\nu^{-2}\rceil\}$. If $k \geq 3$, then we also let $\mu' : \mathbb{N}^{k-2} \to (0, 1]$ be a function such that for any $\mathbf{a} \in \mathbb{N}^{k-2}$, we have

$$\mu'(\mathbf{a}) \ll \nu, 1/k, 1/o, 1/\|\mathbf{a}\|_\infty \quad \text{and} \quad \mu'(\mathbf{a}) < (\mu_{4.1}(k, T, 2sT^{2^k}, \eta, \nu/3, \varepsilon^2))^2. \tag{6.3}$$

For all $k \geq 2$, let

$$t_{k-1} := \begin{cases} so^{4^k+1}\lceil 2\nu^{-2}\rceil & \text{if } k = 2, \\ \max\{t_{6.1}(k-1, o, o^{4^k}, \eta, \nu/3, \mu'), o^{4^k+1}\lceil 2\nu^{-2}\rceil\} & \text{if } k \geq 3, \end{cases} \tag{6.4}$$

which exists by the induction hypothesis. Choose an integer t such that

$$1/t \ll 1/t_{4.1}(k, t_{k-1}, 2st_{k-1}^{2^k}, \eta, \nu/3, \varepsilon^2), 1/t_{k-1}, \tag{6.5}$$

and choose $\mu > 0$ such that

$$\mu \ll \begin{cases} 1/t, \mu_{4.1}(k, T, 2sT^{2^k}, \eta, \nu/3, \varepsilon^2) & \text{if } k = 2, \\ 1/t, \mu'(\mathbf{a}), \varepsilon(\mathbf{a}'), \mu_{6.1}(k-1, o, o^{4^k}, \eta, \nu/3, \mu') & \text{if } k \geq 3. \\ \quad \text{for any } \mathbf{a} \in [t]^{k-2}, \mathbf{a}' \in [t]^{k-1} & \end{cases} \tag{6.6}$$

Finally, choose an integer n_0 such that

$$1/n_0 \ll \begin{cases} 1/n_{4.1}(k, t_{k-1}, 2st_{k-1}^{2^k}, \eta, \nu/3, \varepsilon^2), 1/n_{6.1}(k-1, o, o^{4^k}, \nu), 1/\mu. & \text{if } k = 2, \\ 1/n_{4.1}(k, t_{k-1}, 2st_{k-1}^{2^k}, \eta, \nu/3, \varepsilon^2), 1/n_{6.1}(k-1, o, o^{4^k}, \eta, \nu/3, \mu'), 1/\mu & \text{if } k \geq 3. \end{cases} \tag{6.7}$$

Suppose $\mathscr{O}(k-1, \mathbf{a}^{\mathscr{O}})$, $\mathscr{Q}(k, \mathbf{a}^{\mathscr{Q}})$ and $\mathscr{H}^{(k)}$ are given (families of) partitions satisfying (O1)$_{6.1}$–(O4)$_{6.1}$ with μ, t, n_0 as defined above. Write

$$\mathscr{O}^{(k-1)} = \{O_1^{(k-1)}, \dots, O_{s_O}^{(k-1)}\} \quad \text{and} \quad \mathscr{Q}^{(k-1)} = \{Q_1^{(k-1)}, \dots, Q_{s_Q}^{(k-1)}\}.$$

Let

$$O_{s_O+1}^{(k-1)} := \binom{V}{k-1} \setminus \mathcal{K}_{k-1}(\mathscr{O}^{(1)}), \quad Q_{s_Q+1}^{(k-1)} := \binom{V}{k-1} \setminus \mathcal{K}_{k-1}(\mathscr{Q}^{(1)}), \text{ and} \tag{6.8}$$

$$\mathscr{R}^{(k-1)} := \{O_i^{(k-1)} \cap Q_j^{(k-1)} : i \in [s_O + 1], j \in [s_Q + 1]\} \setminus \{\emptyset\}.$$

We also write $\mathscr{R}^{(k-1)} = \{R_1^{(k-1)}, \dots, R_{s''}^{(k-1)}\}$. See Figure 1 for an illustration of the relationship of the different partitions defined in the proof.

Since $\mathscr{O}$ and $\mathscr{Q}$ are both o-bounded, Proposition 2.7(viii) implies that

$$s'' \leq \left(o^{2^k}\right)^2 \leq o^{4^k}. \tag{6.9}$$

Now our aim is to construct a family of partitions $\mathscr{L}$ as follows.

Claim 2. *There exist $\{\mathscr{L}^{(i)}\}_{i=1}^{k-1}$ and $\mathbf{a}^{\mathscr{L}}_{\text{long}} = (a^{\mathscr{L}}_1, \dots, a^{\mathscr{L}}_{k-1}) \in [t_{k-1}]^{k-1}$ satisfying the following for all $j \in [k-1]$, where $\mathbf{a}^{\mathscr{L}} := (a^{\mathscr{L}}_1, \dots, a^{\mathscr{L}}_{k-2})$.*

- (L*1) *$\{\mathscr{L}^{(i)}\}_{i=1}^{k-1}$ forms an $(\eta, \mu'(\mathbf{a}^{\mathscr{L}})^{1/2}, \mathbf{a}^{\mathscr{L}}_{\text{long}})$-equitable t_{k-1}-bounded family of partitions, and $a^{\mathscr{Q}}_j$ divides $a^{\mathscr{L}}_j$,*
- (L*2) *$\mathscr{L}^{(j)} \prec \mathscr{Q}^{(j)}$.*
- (L*3) *$\mathscr{L}^{(j)} \prec_{\nu/2} \mathscr{O}^{(j)}$.*

Note that if $k = 2$, then the function μ' is not defined, but in this case, $\mu'(\mathbf{a}^{\mathscr{L}})^{1/2}$ plays no role in the definition of an equitable family of partitions (Definition 2.3) since Definition 2.3(iii) is vacuously true.

Proof. First we prove the claim for $k = 2$. We apply $L_{6.1}(1)$ with $\mathscr{Q}^{(1)}, \mathscr{R}^{(1)}, s'', \nu/2$ playing the roles of $\mathscr{Q}^{(1)}, \mathscr{H}^{(1)}, s, \nu$. (This is possible by (6.7).) Then we obtain a partition $\mathscr{L}^{(1)}$ of V which satisfies $(\text{P1})^1_{6.1}$ and $(\text{P2})^1_{6.1}$. Moreover, (6.4) implies that $\mathscr{L}^{(1)}$ is t_1-bounded, i.e. $\mathbf{a}^{\mathscr{L}}_{\text{long}} \in [t_{k-1}]$. Since $\mathscr{R}^{(1)} \prec \mathscr{O}^{(1)}$, this in turn implies (L*1)–(L*3).

Now we assume $k \geq 3$. First, we apply $L_{6.1}(k-1)$ with the following objects and parameters. (This is possible by (6.4) and (6.6)–(6.9).)

object/parameter	V	$\{\mathscr{O}^{(j)}\}_{j=1}^{k-2}$	$\{\mathscr{Q}^{(i)}\}_{i=1}^{k-1}$	$\mathscr{R}^{(k-1)}$	o	s''	η	$\nu/3$	μ'	t_{k-1}
playing the role of	V	$\mathscr{O}$	$\mathscr{Q}$	$\mathscr{H}^{(k)}$	o	s	η	ν	ε	t

Then we obtain a family of partitions $\mathscr{L} = \mathscr{L}(k-2, \mathbf{a}^{\mathscr{L}})$ and a partition $\mathscr{M}^{(k-1)}$ of $\binom{V}{k-1}$ with $\mathscr{M}^{(k-1)} = \{M^{(k-1)}_1, \dots, M^{(k-1)}_{s''}\}$ which satisfy the following for each $i \in [s'']$ and $j \in [k-2]$.

- (L′1) $\mathscr{L}$ is $(\eta, \mu'(\mathbf{a}^{\mathscr{L}}), \mathbf{a}^{\mathscr{L}})$-equitable and t_{k-1}-bounded, and $a^{\mathscr{Q}}_j$ divides $a^{\mathscr{L}}_j$,
- (L′2) $\mathscr{L}^{(j)} \prec \mathscr{Q}^{(j)}$ and $\mathscr{L}^{(j)} \prec_{\nu/3} \mathscr{O}^{(j)}$,
- (M′1) $M^{(k-1)}_i$ is perfectly $\mu'(\mathbf{a}^{\mathscr{L}})$-regular with respect to $\mathscr{L}$,
- (M′2) $\sum_{i=1}^{s''} |M^{(k-1)}_i \triangle R^{(k-1)}_i| \leq (\nu/3)\binom{n}{k-1}$, and
- (M′3) $\mathscr{M}^{(k-1)} \prec \mathscr{Q}^{(k-1)}$ and if $R^{(k-1)}_i \subseteq \mathcal{K}_{k-1}(\mathscr{Q}^{(1)})$, then $M^{(k-1)}_i \subseteq \mathcal{K}_{k-1}(\mathscr{Q}^{(1)})$.

Thus $\{\mathscr{L}^{(i)}\}_{i=1}^{k-2}$ satisfies (L*1)–(L*3) for $j \in [k-2]$ and it only remains to construct $\mathscr{L}^{(k-1)}$. Let

$$t' := \max\{\|\mathbf{a}^{\mathscr{L}}\|_\infty, a^{\mathscr{Q}}_{k-1} o^{4^k} \lceil 2\nu^{-2} \rceil\}. \tag{6.10}$$

Thus $\mathscr{L}$ is t'-bounded and $t' \leq \min\{t_{k-1}, T(\mathbf{a}^{\mathscr{L}}, o, \nu)\}$ by (6.4). For convenience, we write $\hat{\mathscr{L}}^{(k-2)} = \{\hat{L}^{(k-2)}_1, \dots, \hat{L}^{(k-2)}_{s'}\}$. Since $\mathscr{M}^{(k-1)} \prec \mathscr{Q}^{(k-1)}$ by (M′3), for each $j \in [s'']$ there exists a unique $g(j) \in [s_Q + 1]$ such that $M^{(k-1)}_j \subseteq Q^{(k-1)}_{g(j)}$. For each $i \in [s']$, $j \in [s'']$, we define

$$\mathscr{L}(i,j) := \mathcal{K}_{k-1}(\hat{L}^{(k-2)}_i) \cap M^{(k-1)}_j. \tag{6.11}$$

For each $i \in [s']$, we define $J(i) := \{j \in [s''] : \mathscr{L}(i,j) \neq \emptyset\}$. Note that $\mathscr{L}(i,j) \subseteq Q^{(k-1)}_{g(j)}$ for all $i \in [s']$.

Subclaim 1. *For each $i \in [s']$ and for each $j \in J(i)$, the $(k-1)$-graph $Q^{(k-1)}_{g(j)}$ is $(\mu'(\mathbf{a}^{\mathscr{L}}), d'_{g(j)})$-regular with respect to $\hat{L}^{(k-2)}_i$, where $d'_{g(j)} \in \{1/a^{\mathscr{Q}}_{k-1}, 1\}$.*

Proof. First, note that since $\mathscr{L} \prec \{\mathscr{Q}^{(j)}\}_{j=1}^{k-2}$, one of the following holds.

(LL1) There exists $\hat{Q}^{(k-2)} \in \hat{\mathscr{Q}}^{(k-2)}$ such that $\hat{L}_i^{(k-2)} \subseteq \hat{Q}^{(k-2)}$.

(LL2) $\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \subseteq \binom{V}{k-1} \setminus \mathcal{K}_{k-1}(\mathscr{Q}^{(1)})$.

If (LL1) holds, then by (L′1) and the fact that $\mu \ll \mu'(\mathbf{a}^{\mathscr{L}}) \ll \|\mathbf{a}^{\mathscr{L}}\|_\infty^{-k}$ we can apply Lemma 5.4 twice to conclude that

$$|\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)})| \geq t_{k-1}^{-2^k}|\mathcal{K}_{k-1}(\hat{Q}^{(k-2)})| \overset{(6.5),(6.6)}{\geq} \mu^{1/3}|\mathcal{K}_{k-1}(\hat{Q}^{(k-2)})|.$$

Note that, for each $j \in J(i)$, $Q_{g(j)}^{(k-1)}$ is $(\mu, 1/a_{k-1}^{\mathscr{Q}})$-regular with respect to $\hat{Q}^{(k-2)}$. Together with Lemma 5.1(ii) this implies that $Q_{g(j)}^{(k-1)}$ is $(\mu^{2/3}, 1/a_{k-1}^{\mathscr{Q}})$-regular with respect to $\hat{L}_i^{(k-2)}$ for each $j \in J(i)$.

If (LL2) holds, $j \in J(i)$ implies that $M_j^{(k-1)} \nsubseteq \mathcal{K}_{k-1}(\mathscr{Q}^{(1)})$. Thus (6.8) means that $M_j^{(k-1)} \cap Q_{s_Q+1}^{(k-1)} \neq \emptyset$, which implies $g(j) = s_Q+1$. Also (LL2) implies that $\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \subseteq Q_{s_Q+1}^{(k-1)}$. Thus $Q_{s_Q+1}^{(k-1)}$ is $(0,1)$-regular with respect to $\hat{L}_i^{(k-2)}$. This completes the proof of Subclaim 1. $\square$

Moreover, (M′1) implies that for each $i \in [s']$ and $j \in [s'']$, the $(k-1)$-graph $\mathscr{L}(i,j)$ is $\mu'(\mathbf{a}^{\mathscr{L}})$-regular with respect to $\hat{L}_i^{(k-2)}$, and thus it is $(2\mu'(\mathbf{a}^{\mathscr{L}}), d(\mathscr{L}(i,j) \mid \hat{L}_i^{(k-2)}))$-regular with respect to $\hat{L}_i^{(k-2)}$. Let

$$a_{k-1}^{\mathscr{L}} := a_{k-1}^{\mathscr{Q}} o^{4^k} \lceil 2\nu^{-2} \rceil, \tag{6.12}$$

and $\mathbf{a}_{\text{long}}^{\mathscr{L}} := (a_1^{\mathscr{L}}, \ldots, a_{k-1}^{\mathscr{L}})$. By (6.10), we conclude

$$\|\mathbf{a}_{\text{long}}^{\mathscr{L}}\|_\infty = t' \leq \min\{t_{k-1}, T(\mathbf{a}^{\mathscr{L}}, o, \nu)\}. \tag{6.13}$$

Let $\ell_{i,j} := \lfloor d(\mathscr{L}(i,j) \mid \hat{L}_i^{(k-2)}) a_{k-1}^{\mathscr{L}} \rfloor$, so $\ell_{i,j} = 0$ if $j \notin J(i)$. We now apply the slicing lemma (Lemma 5.3) to $\mathscr{L}(i,j)$ for each $i \in [s']$ and $j \in [s'']$. We obtain edge-disjoint $(k-1)$-graphs $\mathscr{L}(i,j,0), \ldots, \mathscr{L}(i,j,\ell_{i,j})$ such that

($\mathscr{L}$1) $\mathscr{L}(i,j) = \mathscr{L}(i,j,0) \cup \bigcup_{r=1}^{\ell_{i,j}} \mathscr{L}(i,j,r)$,

($\mathscr{L}$2) $\mathscr{L}(i,j,r)$ is $(6\mu'(\mathbf{a}^{\mathscr{L}}), 1/a_{k-1}^{\mathscr{L}})$-regular with respect to $\hat{L}_i^{(k-2)}$ for each $r \in [\ell_{i,j}]$, and

($\mathscr{L}$3) $\mathscr{L}(i,j,0)$ is $(6\mu'(\mathbf{a}^{\mathscr{L}}), d'_{i,j})$-regular with respect to $\hat{L}_i^{(k-2)}$, where $d'_{i,j} \leq 1/a_{k-1}^{\mathscr{L}}$.

Observe that $\mathscr{L}(i,j,0)$ may not have density $1/a_{k-1}^{\mathscr{L}}$. Since we would like to achieve this density for all classes, we now take the union of all these $(k-1)$-graphs and split this union into suitable pieces. For all $i \in [s']$ and $p \in [s_Q+1]$, let

$$\mathscr{L}'(i,p) := \bigcup_{j\colon g(j)=p} \mathscr{L}(i,j,0) = \left(\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \cap Q_p^{(k-1)}\right) \setminus \left(\bigcup_{j,r\colon g(j)=p, r\in[\ell_{i,j}]} \mathscr{L}(i,j,r)\right).$$

Note that if $p \notin g(J(i))$, then $\mathscr{L}'(i,p) = \emptyset$. So suppose that $p \in g(J(i))$. Then

$$\textit{$\mathscr{L}'(i,p)$ is $(\mu'(\mathbf{a}^{\mathscr{L}})^{2/3}, \ell'_{i,p}/a_{k-1}^{\mathscr{L}})$-regular with respect to $\hat{L}_i^{(k-2)}$ for some $\ell'_{i,p} \in \mathbb{N}$.} \tag{6.14}$$

Indeed the union lemma (Lemma 5.2) (applied with $\sum_{j\colon g(j)=p} \ell_{i,j} \leq s'' a_{k-1}^{\mathscr{L}} \leq o^{4^k} a_{k-1}^{\mathscr{L}}$ playing the role of s) implies that $\bigcup_{j,r\colon g(j)=p, r\in[\ell_{i,j}]} \mathscr{L}(i,j,r)$ is $(\mu'(\mathbf{a}^{\mathscr{L}})^{3/4}, \sum_{j\colon g(j)=p} \ell_{i,j}/a_{k-1}^{\mathscr{L}})$-regular with respect to $\hat{L}_i^{(k-2)}$. In addition, by Subclaim 1, $\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \cap Q_p^{(k-1)}$ is $(\mu'(\mathbf{a}^{\mathscr{L}}), d'_p)$-regular with respect to $\hat{L}_i^{(k-2)}$ for some $d'_p \in \{1/a_{k-1}^{\mathscr{Q}}, 1\}$. Note that (6.11) implies

$$\bigcup_{j,r\colon g(j)=p, r\in[\ell_{i,j}]} \mathscr{L}(i,j,r) \subseteq \mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \cap Q_p^{(k-1)}.$$

So Lemma 5.1(iii) implies that (6.14) holds where $\ell'_{i,p} := a_{k-1}^{\mathscr{L}} d'_p - \sum_{j:g(j)=p} \ell_{i,j}$. (Note $\ell'_{i,p} \in \mathbb{N}$ since $d'_p \in \{1/a_{k-1}^{\mathscr{Q}}, 1\}$ and $a_{k-1}^{\mathscr{Q}} \mid a_{k-1}^{\mathscr{L}}$.)

In addition, for all $i \in [s']$ and $p \in g(J(i))$, we have

$$|\mathscr{L}'(i,p)| \overset{(\mathscr{L}3)}{\leq} |g^{-1}(p)| \cdot \frac{5}{4a^{\mathscr{L}}_{k-1}} \cdot |\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)})| \overset{(6.9),(6.12)}{\leq} \nu^2 |\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \cap Q_p^{(k-1)}|. \quad (6.15)$$

Again, we apply the slicing lemma (Lemma 5.3), this time to $\mathscr{L}'(i,p)$. By (6.14), we obtain edge-disjoint $(k-1)$-graphs $\mathscr{L}'(i,p,1),\dots,\mathscr{L}'(i,p,\ell'_{i,p})$ such that

($\mathscr{L}'1$) $\mathscr{L}'(i,p) = \bigcup_{\ell=1}^{\ell'_{i,p}} \mathscr{L}'(i,p,\ell)$,
($\mathscr{L}'2$) $\mathscr{L}'(i,p,\ell)$ is $(\mu'(\mathbf{a}^{\mathscr{L}})^{1/2}, 1/a^{\mathscr{L}}_{k-1})$-regular with respect to $\hat{L}_i^{(k-2)}$ for each $\ell \in [\ell'_{i,p}]$.

Thus for each $i \in [s']$, (6.11), ($\mathscr{L}1$) and ($\mathscr{L}'1$) imply that

$$\bigcup_{p\in g(J(i))} \left(\{\mathscr{L}(i,j,r) : j,r \text{ with } g(j)=p, r\in[\ell_{i,j}]\} \cup \{\mathscr{L}'(i,p,1),\dots,\mathscr{L}'(i,p,\ell'_{i,p})\}\right)$$

forms a partition of $\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)})$ into $a^{\mathscr{L}}_{k-1}$ edge-disjoint $(k-1)$-graphs, each of which is $(\mu'(\mathbf{a}^{\mathscr{L}})^{1/2}, 1/a^{\mathscr{L}}_{k-1})$-regular with respect to $\hat{L}_i^{(k-2)}$, where the latter follows from ($\mathscr{L}2$) and ($\mathscr{L}'2$). We define

$$\mathscr{L}^{(k-1)} := \bigcup_{i\in[s'],j\in J(i)} \{\mathscr{L}(i,j,r) : r\in[\ell_{i,j}]\} \cup \bigcup_{i\in[s'],p\in g(J(i))} \{\mathscr{L}'(i,p,\ell) : \ell\in[\ell'_{i,p}]\}.$$

Then (L*1) follows from (L$'$1) and the construction of $\mathscr{L}^{(k-1)}$ (t_{k-1}-boundedness follows by (6.13)). Note that for all $i\in[s']$, $j\in[s'']$, $r\in[\ell_{i,j}]$, $p\in[s_Q+1]$, $\ell\in[\ell'_{i,p}]$, we have $\mathscr{L}(i,j,r)\subseteq Q^{(k-1)}_{g(j)}$ and $\mathscr{L}'(i,p,\ell)\subseteq Q^{(k-1)}_p$, and so (L*2) holds.

Subclaim 2. $\mathscr{L}^{(k-1)} \prec_{\nu/2} \mathscr{O}^{(k-1)}$.

Proof. To prove the subclaim, we define a suitable function $f_{k-1} : \mathscr{L}^{(k-1)} \to \mathscr{O}^{(k-1)}$. For each $j\in[s'']$, let $h(j)\in[s_O+1]$ be the index such that $R_j^{(k-1)} = O_{h(j)}^{(k-1)} \cap Q_p^{(k-1)}$ for some $p\in[s_Q+1]$. For each $i\in[s']$, $j\in J(i)$, $r\in[\ell_{i,j}]$, $\ell\in[\ell'_{i,g(j)}]$, let

$$f_{k-1}(\mathscr{L}(i,j,r)) := O_{h(j)}^{(k-1)} \text{ and } f_{k-1}(\mathscr{L}'(i,g(j),\ell)) := O_{h(j)}^{(k-1)}.$$

For fixed $j\in[s'']$, (6.11) and ($\mathscr{L}1$) imply that

$$\bigcup_{i\in[s'],r\in[\ell_{i,j}]} \mathscr{L}(i,j,r) \subseteq M_j^{(k-1)}. \quad (6.16)$$

Hence

$$\begin{aligned}
\sum_{L^{(k-1)}\in\mathscr{L}^{(k-1)}} |L^{(k-1)} \setminus f_{k-1}(L^{(k-1)})| &\leq \sum_{i,j,r} |\mathscr{L}(i,j,r)\setminus f_{k-1}(\mathscr{L}(i,j,r))| + \sum_{i,p,\ell} |\mathscr{L}'(i,p,\ell)| \\
&\overset{(6.16),(\mathscr{L}'1)}{\leq} \sum_{j\in[s'']} |M_j^{(k-1)} \setminus O_{h(j)}^{(k-1)}| + \sum_{i,p} |\mathscr{L}'(i,p)| \\
&\overset{(6.15)}{\leq} \sum_{j\in[s'']} |M_j^{(k-1)} \setminus R_j^{(k-1)}| + \nu^2 \sum_{i,p} |\mathcal{K}_{k-1}(\hat{L}_i^{(k-2)}) \cap Q_p^{(k-1)}| \\
&\overset{(\mathrm{M}'2)}{\leq} \frac{\nu}{3}\binom{n}{k-1} + \nu^2\binom{n}{k-1} \leq \frac{2\nu}{5}\binom{n}{k-1}.
\end{aligned}$$

The fact that $a_1^{\mathscr{L}} \geq \eta^{-1}$ and (2.5) together imply that $|\mathcal{K}_{k-1}(\mathscr{L}^{(1)})| \geq \frac{4}{5}\binom{n}{k-1}$, so the subclaim follows. □

This shows that (L*3) holds and completes the proof of Claim 2. □

Note that $\{\mathscr{L}^{(i)}\}_{i=1}^{k-1}$ obtained in Claim 2 naturally defines $\hat{\mathscr{L}}^{(k-1)}$. Write $\hat{\mathscr{L}}^{(k-1)} = \{\hat{L}_1^{(k-1)}, \dots, \hat{L}_{\hat{s}_{\mathscr{L}}}^{(k-1)}\}$. We now construct $\mathscr{L}^{(k)}$ by refining $\mathcal{K}_k(\hat{L}_i^{(k-1)})$ for all $i \in [\hat{s}_{\mathscr{L}}]$.

Claim 3. *For each $i \in [\hat{s}_{\mathscr{L}}]$, there is a partition $\{L^{(k)}(i,1), \dots, L^{(k)}(i, a_k^{\mathscr{Q}})\}$ of $\mathcal{K}_k(\hat{L}_i^{(k-1)})$ such that $L^{(k)}(i,r)$ is $(\mu^{1/2}, 1/a_k^{\mathscr{Q}})$-regular with respect to $\hat{L}_i^{(k-1)}$ for each $r \in [a_k^{\mathscr{Q}}]$. Moreover, we can ensure that $\{L(i,r) : i \in [\hat{s}_{\mathscr{L}}], r \in [a_k^{\mathscr{Q}}]\} \prec \mathscr{Q}^{(k)}$.*

Proof. Since $\mathscr{L}^{(1)} \prec \mathscr{Q}^{(1)}$, for each $\hat{L}_i^{(k-1)} \in \hat{\mathscr{L}}^{(k-1)}$, either $\mathcal{K}_k(\hat{L}_i^{(k-1)}) \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$ or $\mathcal{K}_k(\hat{L}_i^{(k-1)}) \subseteq \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})$.

Suppose first that $\mathcal{K}_k(\hat{L}_i^{(k-1)}) \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$. As $\mathscr{L}^{(k-1)} \prec \mathscr{Q}^{(k-1)}$, there exists a (unique) $\hat{Q}_j^{(k-1)} \in \hat{\mathscr{Q}}^{(k-1)}$ such that $\hat{L}_i^{(k-1)} \subseteq \hat{Q}_j^{(k-1)}$. In addition, there are exactly $a_k^{\mathscr{Q}}$ many k-graphs $Q^{(k)}(j,1), \dots, Q^{(k)}(j, a_k^{\mathscr{Q}})$ in $\mathscr{Q}^{(k)}$ that partition $\mathcal{K}_k(\hat{Q}_j^{(k-1)})$. For each $r \in [a_k^{\mathscr{Q}}]$, let

$$L^{(k)}(i,r) := Q^{(k)}(j,r) \cap \mathcal{K}_k(\hat{L}_i^{(k-1)}).$$

Hence $\{L^{(k)}(i,1), \dots, L^{(k)}(i, a_k^{\mathscr{Q}})\}$ forms a partition of $\mathcal{K}_k(\hat{L}_i^{(k-1)})$. We can now apply Lemma 5.4 twice and use (L*1) as well as $(\mathrm{O3})_{6.1}$ to obtain that

$$|\mathcal{K}_k(\hat{L}_i^{(k-1)})| \geq t_{k-1}^{-2^k} |\mathcal{K}_k(\hat{Q}_j^{(k-1)})|.$$

Thus Lemma 5.1(ii), $(\mathrm{O3})_{6.1}$ and (6.6) imply that $L^{(k)}(i,r)$ is $(\mu^{1/2}, 1/a_k^{\mathscr{Q}})$-regular with respect to $\hat{L}_i^{(k-1)}$ for each $r \in [a_k^{\mathscr{Q}}]$.

Suppose next that we have $\mathcal{K}_k(\hat{L}_i^{(k-1)}) \subseteq \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})$. We apply the slicing lemma (Lemma 5.3) with $\mathcal{K}_k(\hat{L}_i^{(k-1)}), \hat{L}_i^{(k-1)}, 1, 1/a_k^{\mathscr{Q}}$ playing the roles of $H^{(k)}, H^{(k-1)}, d, p_i$ respectively. We obtain a partition $\{L^{(k)}(i,1), \dots, L^{(k)}(i, a_k^{\mathscr{Q}})\}$ of $\mathcal{K}_k(\hat{L}_i^{(k-1)})$ such that $L^{(k)}(i,r)$ is $(\mu^{1/2}, 1/a_k^{\mathscr{Q}})$-regular with respect to $\hat{L}_i^{(k-1)}$ for each $r \in [a_k^{\mathscr{Q}}]$.

The moreover part of Claim 3 is immediate from the construction in both cases. □

Let

$$\mathscr{L}^{(k)} := \{L^{(k)}(i,r) : i \in [\hat{s}_{\mathscr{L}}], r \in [a_k^{\mathscr{Q}}]\}, \quad \mathbf{a}^{\mathscr{L}_*} := (a_1^{\mathscr{L}}, \dots, a_{k-1}^{\mathscr{L}}, a_k^{\mathscr{Q}}) \quad \text{and} \quad \mathscr{L}_* := \{\mathscr{L}^{(i)}\}_{i=1}^{k},$$

$$\mathscr{J}^{(k)} := \Big(\{H_i^{(k)} \cap L_*^{(k)} : i \in [s], L_*^{(k)} \in \mathscr{L}^{(k)}\} \cup \{H_i^{(k)} \setminus \mathcal{K}_k(\mathscr{L}^{(1)}) : i \in [s]\}\Big) \setminus \{\emptyset\}.$$

$$\{J_1^{(k)}, \dots, J_{s_J}^{(k)}\} := \mathscr{J}^{(k)}, \text{ and } J'(i) := \{j' \in [s_J] : J_{j'}^{(k)} \subseteq H_i^{(k)}\} \text{ for each } i \in [s]. \tag{6.17}$$

Then $\mathscr{L}^{(k)} \prec \mathscr{Q}^{(k)}$. Let $\mu_* := \mu^{1/2}$ if $k = 2$ and $\mu_* := \mu'(\mathbf{a}^{\mathscr{L}})^{1/2} > \mu^{1/2}$ if $k \geq 3$. Then by (L*1), Claim 3, and (6.13), we have

$$\mathscr{L}_* \textit{ is a } (1/a_1^{\mathscr{L}}, \mu_*, \mathbf{a}^{\mathscr{L}_*})\textit{-equitable } t'\textit{-bounded family of partitions.} \tag{6.18}$$

Moreover, $s_J \leq 2st'^{2^k}$ by (6.13) and Proposition 2.7(viii). Also we have $\mathscr{J}^{(k)} \prec \mathscr{H}^{(k)}$ and $\{J'(1), \dots, J'(s)\}$ forms a partition of $[s_J]$.

Our next aim is to apply Lemma 4.1 with the following objects and parameters.

object/parameter	$\mathscr{L}_*$	$\mathscr{J}^{(k)}$	t	t'	s_J	η	$\nu/3$	ε^2	μ^*
playing the role of	$\mathscr{Q}$	$\mathscr{H}^{(k)}$	t	o	s	η	ν	ε	μ

Indeed, we can apply Lemma 4.1: (6.7) ensures that $(\mathrm{O1})_{4.1}$ holds; by (6.3), (6.6), (6.13) and (6.18), $\mathscr{L}_*$ satisfies $(\mathrm{O2})_{4.1}$. By construction, $\mathscr{J}^{(k)} \prec \mathscr{L}_*^{(k)}$, thus $(\mathrm{O3})_{4.1}$ also holds. We obtain $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ and $\mathscr{G}'^{(k)} = \{G_1'^{(k)}, \dots, G_{s_J}'^{(k)}\}$ satisfying the following.

(P′1) $\mathscr{P}$ is $(\eta, \varepsilon(\mathbf{a}^{\mathscr{P}})^2, \mathbf{a}^{\mathscr{P}})$-equitable and t-bounded, and $a_j^{\mathscr{L}_*}$ divides $a_j^{\mathscr{P}}$ for all $j \in [k-1]$,

(P′2) for each $j \in [k-1]$, $\mathscr{P}^{(j)} \prec \mathscr{L}^{(j)}$,

(P′3) $G_i'^{(k)}$ is perfectly $\varepsilon(\mathbf{a}^{\mathscr{P}})^2$-regular with respect to $\mathscr{P}$ for all $i \in [s_J]$, (P′4) $\sum_{i\in[s_J]} |G_i'^{(k)} \triangle J_i^{(k)}| \leq (\nu/3)\binom{n}{k}$, and

(P′5) $\mathscr{G}'^{(k)} \prec \mathscr{L}^{(k)}$ and if $J_i^{(k)} \subseteq \mathcal{K}_k(\mathscr{L}^{(1)})$, then $G_i'^{(k)} \subseteq \mathcal{K}_k(\mathscr{L}^{(1)})$.

Here we obtain (P′1) from (6.5). In addition, we also have the following.

$$Q_{i'}^{(k)} \textit{ is perfectly } \varepsilon(\mathbf{a}^{\mathscr{P}})^2\textit{-regular with respect to } \mathscr{P} \textit{ for all } i' \in [s_Q+1]. \tag{6.19}$$

Indeed, $\mathscr{P}^{(k-1)} \prec \mathscr{L}^{(k-1)} \prec \mathscr{Q}^{(k-1)}$. Thus for each $\hat{P}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$, either there exists unique $\hat{Q}^{(k-1)} \in \hat{\mathscr{Q}}^{(k-1)}$ such that $\mathcal{K}_k(\hat{P}^{(k-1)}) \subseteq \mathcal{K}_k(\hat{Q}^{(k-1)})$, or $\mathcal{K}_k(\hat{P}^{(k-1)}) \subseteq \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})$.

In the former case, by two applications of Lemma 5.4 and (P′1), it is easy to see that

$$|\mathcal{K}_k(\hat{P}^{(k-1)})| \geq t^{-2^k}|\mathcal{K}_k(\hat{Q}^{(k-1)})|.$$

Thus (O3)$_{6.1}$ with Lemma 5.1(ii) and (6.6) implies that $Q_{i'}^{(k)}$ is $\varepsilon(\mathbf{a}^{\mathscr{P}})^2$-regular with respect to $\hat{P}^{(k-1)}$.

Now suppose that $\mathcal{K}_k(\hat{P}^{(k-1)}) \subseteq \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})$. If $i' \in [s_Q]$, then we have $Q_{i'}^{(k)} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$. Thus $Q_{i'}^{(k)} \cap \mathcal{K}_k(\hat{P}^{(k-1)}) = \emptyset$, and $Q_{i'}^{(k)}$ is $(\varepsilon(\mathbf{a}^{\mathscr{P}})^2, 0)$-regular with respect to $\hat{P}^{(k-1)}$. If $i' = s_Q + 1$, then $Q_{i'}^{(k)} = \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}^{(1)})$, thus $Q_{i'}^{(k)}$ is $(\varepsilon(\mathbf{a}^{\mathscr{P}})^2, 1)$-regular with respect to $\hat{P}^{(k-1)}$. Thus we have (6.19).

It is easy to see that (P′1) and (L*1) imply (P1)$_{6.1}$. The statements (P′2), (2.1) together with (L*2) and (L*3) imply (P2)$_{6.1}$.

As $\mathscr{L}^{(k)} \prec \mathscr{Q}^{(k)}$ and (P′5) holds, we obtain $\mathscr{G}'^{(k)} \prec \mathscr{Q}^{(k)}$. For each $i' \in [s_Q+1]$, let

$$G'(i') := \{j' \in [s_J] : G_{j'}'^{(k)} \subseteq Q_{i'}^{(k)}\}, \text{ and } \quad H'(i') := \{i \in [s] : H_i^{(k)} \subseteq Q_{i'}^{(k)}\}.$$

Note that $\{G'(1), \dots G'(s_Q+1)\}$ forms a partition of $[s_J]$. Also by (O4)$_{6.1}$, $\{H'(1), \dots H'(s_Q+1)\}$ forms a partition of $[s]$. Moreover, both $G'(i')$ and $H'(i')$ are non-empty sets. For each $i' \in [s_Q+1]$, we arbitrarily choose a representative $h'_{i'} \in H'(i')$.

Recall that $J'(i)$ was defined in (6.17). For each $i' \in [s_Q+1]$ and $i \in H'(i') \setminus \{h'_{i'}\}$, we define

$$G_i^{(k)} := \bigcup_{j' \in J'(i) \cap G'(i')} G_{j'}'^{(k)} \quad \text{and} \quad G_{h'_{i'}}^{(k)} := Q_{i'}^{(k)} \setminus \bigcup_{\ell \in H'(i') \setminus h'_{i'}} G_\ell^{(k)}.$$

Let

$$\mathscr{G}^{(k)} := \{G_i^{(k)} : i \in [s]\}.$$

By the construction, $\mathscr{G}^{(k)}$ forms a partition of $\binom{V}{k}$. Moreover, we have the following:

$$\textit{Suppose that } i' \in [s_Q+1],\ i \in H'(i') \textit{ and } j' \in J'(i) \cap G'(i').\ \textit{Then } G_{j'}'^{(k)} \subseteq G_i^{(k)}. \tag{6.20}$$

Note that the construction of $\mathscr{G}^{(k)}$, (P′3), the union lemma (Lemma 5.2), (6.2) and (6.13) together imply that for each $i' \in [s_Q+1]$ and $i \in H'(i') \setminus \{h'_{i'}\}$, $G_i^{(k)}$ is perfectly $\varepsilon(\mathbf{a}^{\mathscr{P}})^{3/2}$-regular with respect to $\mathscr{P}$. In particular, together with (6.19), the union lemma (Lemma 5.2) and Lemma 5.1(iii), this implies that for each $i' \in [s_Q+1]$, $G_{h'_{i'}}^{(k)}$ is also perfectly $\varepsilon(\mathbf{a}^{\mathscr{P}})$-regular with respect to $\mathscr{P}$. Thus we obtain (G1)$_{6.1}$.

By the definition of $G_i^{(k)}$, we conclude that for every $i \in [s]$, there exists $i' \in [s_Q+1]$ such that $G_i^{(k)} \subseteq Q_{i'}^{(k)}$. Thus $\mathscr{G}^{(k)} \prec \mathscr{Q}^{(k)}$. Moreover, if $H_i^{(k)} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$, then $i \in H'(i')$ for some $i' \neq s_Q+1$. Hence in this case $G_i^{(k)} \subseteq Q_{i'}^{(k)}$ with $i' \neq s_Q+1$, and so $G_i^{(k)} \subseteq \mathcal{K}_k(\mathscr{Q}^{(1)})$. Thus (G3)$_{6.1}$ holds.

We now verify (G2)$_{6.1}$. Consider any edge $e \in G_i^{(k)} \setminus H_i^{(k)}$ for some $i \in [s]$. We claim that

$$e \in \bigcup_{j' \in [s_J]} J_{j'}^{(k)} \setminus G_{j'}'^{(k)}. \tag{6.21}$$

To prove (6.21) note that since $\mathscr{J}^{(k)}$ is a partition of $\binom{V}{k}$, there exists $j' \in [s_J]$ such that $e \in J^{(k)}_{j'}$. So (6.21) holds if $e \notin G'^{(k)}_{j'}$. Thus assume for a contradiction that $e \in G'^{(k)}_{j'}$. Let $i_* \in [s]$ be the index such that $j' \in J'(i_*)$. Then $J^{(k)}_{j'} \subseteq H^{(k)}_{i_*}$.

Since $\{H'(1), \dots, H'(s_Q+1)\}$ forms a partition of $[s]$, there exists $i' \in [s_Q+1]$ such that $i_* \in H'(i')$. Thus $e \in J^{(k)}_{j'} \subseteq H^{(k)}_{i_*} \subseteq Q^{(k)}_{i'}$. Hence $Q^{(k)}_{i'} \cap G'^{(k)}_{j'} \neq \emptyset$. Since $\mathscr{G}'^{(k)} \prec \mathscr{Q}^{(k)}$, this implies that $G'^{(k)}_{j'} \subseteq Q^{(k)}_{i'}$ and so $j' \in G'(i')$. Consequently, we have $i_* \in H'(i')$ and $j' \in J'(i_*) \cap G'(i')$. This together with (6.20) implies that $e \in G'^{(k)}_{j'} \subseteq G^{(k)}_{i_*}$. Since $\mathscr{G}^{(k)}$ is a partition of $\binom{V}{k}$, this implies that $i = i_*$. But then $e \in H^{(k)}_{i_*} = H^{(k)}_i$, a contradiction. This proves (6.21).

Then

$$\sum_{i\in[s]} |G^{(k)}_i \setminus H^{(k)}_i| \overset{(6.21)}{\leq} \sum_{j'\in[s_J]} |J'^{(k)}_{j'} \setminus G'^{(k)}_{j'}|. \tag{6.22}$$

Since all of $\mathscr{H}^{(k)}, \mathscr{G}^{(k)}$, $\mathscr{J}^{(k)}$ and $\mathscr{G}'^{(k)}$ are partitions of $\binom{V}{k}$, we obtain

$$\sum_{i\in[s]} |G^{(k)}_i \triangle H^{(k)}_i| = 2\sum_{i\in[s]} |G^{(k)}_i \setminus H^{(k)}_i|, \text{ and } \sum_{i\in[s_J]} |G'^{(k)}_i \triangle J^{(k)}_i| = 2\sum_{i\in[s_J]} |J^{(k)}_i \setminus G'^{(k)}_i|.$$

Thus we conclude

$$\sum_{i\in[s]} |G^{(k)}_i \triangle H^{(k)}_i| = 2\sum_{i\in[s]} |G^{(k)}_i \setminus H^{(k)}_i| \overset{(6.22)}{\leq} 2\sum_{j'\in[s_J]} |J'^{(k)}_{j'} \setminus G'^{(k)}_{j'}| \overset{(\mathrm{P}'4)}{\leq} \nu\binom{n}{k}.$$

Thus (G2)$_{6.1}$ holds. □

Suppose we are given a $(k-1)$-graph H on a vertex set V. In the next lemma we apply Lemma 4.1 to show that, given a different vertex set V', there exists another $(k-1)$-graph F on V' whose large scale structure is very close to that of H.

Lemma 6.2. *Suppose* $0 < 1/m, 1/n \ll \varepsilon \ll \nu, 1/o, 1/k \leq 1$ *and* $k, o \in \mathbb{N} \setminus \{1\}$. *Suppose* $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ *and* $\mathscr{Q} = \mathscr{Q}(k-1, \mathbf{a})$ *are both o-bounded* $(1/a_1, \varepsilon, \mathbf{a})$*-equitable families of partitions of* V *and* V' *respectively with* $|V| = n$ *and* $|V'| = m$. *Suppose that* $H^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{P}^{(1)})$. *Then there exists a* $(k-1)$*-graph* $G^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{P}^{(1)})$ *on* V *and a* $(k-1)$*-graph* $F^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{Q}^{(1)})$ *on* V' *such that*

(F1)$_{6.2}$ $|H^{(k-1)} \triangle G^{(k-1)}| \leq \nu\binom{n}{k-1}$,

(F2)$_{6.2}$ $d(\mathcal{K}_k(G^{(k-1)}) \mid \hat{P}^{(k-1)}(\hat{\mathbf{z}})) = d(\mathcal{K}_k(F^{(k-1)}) \mid \hat{Q}^{(k-1)}(\hat{\mathbf{z}})) \pm \nu$ *for each* $\hat{\mathbf{z}} \in \hat{A}(k, k-1, \mathbf{a})$.

To prove Lemma 6.2, we first apply Lemma 4.1 to obtain a family of partitions $\mathscr{R} = \mathscr{R}(k-2, \mathbf{a}^{\mathscr{R}})$ and a k-graph $G^{(k-1)}$ as in (F1)$_{6.2}$. We then 'project' $\mathscr{R}$ onto V' (so that it refines $\mathscr{Q}$). This results in a partition $\mathscr{L}$. We then apply the slicing lemma to construct $F^{(k-1)}$ which respects $\mathscr{L}$ (and in particular has the appropriate densities).

Proof of Lemma 6.2. First suppose $k = 2$. Then $H^{(1)} \subseteq V$. Let $G^{(1)} := H^{(1)}$. Thus (F1)$_{6.2}$ holds. Recall that for each $b \in [a_1]$, the vertex sets $P^{(1)}(b,b)$ and $Q^{(1)}(b,b)$ denote the b-th parts in $\mathscr{P}^{(1)}$ and $\mathscr{Q}^{(1)}$, respectively. For each $b \in [a_1]$, let $F^{(1)}(b,b)$ be a subset of $Q^{(1)}(b,b)$ with

$$|F^{(1)}(b,b)| = \left\lfloor \frac{m}{n} |H^{(1)} \cap P^{(1)}(b,b)| \right\rfloor$$

and let $F^{(1)} := \bigcup_{b\in[a_1]} F^{(1)}(b,b)$. For each $\mathbf{z} = (\alpha_1,\alpha_2) \in \hat{A}(2,1,\mathbf{a})$, we have

$$\begin{aligned} d(\mathcal{K}_2(F^{(1)}) \mid \hat{Q}^{(1)}(\hat{\mathbf{z}})) &= \frac{|F^{(1)}(\alpha_1,\alpha_1)||F^{(1)}(\alpha_2,\alpha_2)|}{(m/a_1 \pm 1)^2} \\ &= \frac{(|H^{(1)}\cap P^{(1)}(\alpha_1,\alpha_1)| \pm n/m)(|H^{(1)}\cap P^{(1)}(\alpha_2,\alpha_2)| \pm n/m)}{(n/a_1 \pm n/m)^2} \\ &= d(\mathcal{K}_2(G^{(1)}) \mid \hat{P}^{(1)}(\hat{\mathbf{z}})) \pm \nu. \end{aligned}$$

Thus (F2)$_{6.2}$ holds.

Now we show the lemma for $k \geq 3$. Let η' be a constant such that $\varepsilon \ll \eta' \ll \nu, 1/o, 1/k$. Let $\varepsilon' : \mathbb{N}^{k-2} \to (0,1]$ be a function such that

$$\varepsilon'(\mathbf{b}) \ll \nu, 1/o, 1/k, 1/\|\mathbf{b}\|_\infty \text{ for all } \mathbf{b} \in \mathbb{N}^{k-2}. \tag{6.23}$$

Let $t := t_{4.1}(k-1, o, o^{4^k}, \eta', \nu, \varepsilon')$. Since $\varepsilon \ll \nu, 1/o, 1/k, \eta'$, we may assume that

$$0 < \varepsilon \ll \mu_{4.1}(k-1, o, o^{4^k}, \eta', \nu, \varepsilon'), 1/t, \min\{\varepsilon'(\mathbf{b}) : \mathbf{b} \in [t]^{k-2}\}, \tag{6.24}$$

and we may assume that $n, m > n_0 := n_{4.1}(k-1, o, o^{4^k}, \eta', \nu, \varepsilon')$. Let

$$\begin{aligned} \{H_1^{(k-1)}, \dots, H_s^{(k-1)}\} &:= \{P^{(k-1)} \cap H^{(k-1)} : P^{(k-1)} \in \mathscr{P}^{(k-1)}\} \setminus \{\emptyset\}, \\ \{H_{s+1}^{(k-1)}, \dots, H_{s+s'}^{(k-1)}\} &:= \Big(\{P^{(k-1)} \setminus H^{(k-1)} : P^{(k-1)} \in \mathscr{P}^{(k-1)}\} \\ &\qquad \cup \Big\{\binom{V}{k-1} \setminus \mathcal{K}_{k-1}(\mathscr{P}^{(1)})\Big\}\Big) \setminus \{\emptyset\}, \\ \mathscr{H}^{(k-1)} &:= \{H_1^{(k-1)}, \dots, H_{s+s'}^{(k-1)}\}. \end{aligned}$$

Hence $\mathscr{H}^{(k-1)}$ is a partition of $\binom{V}{k-1}$ such that $\mathscr{H}^{(k-1)} \prec \mathscr{P}^{(k-1)}$ and $s + s' \leq 2o^{2^k} + 1 \leq o^{4^k}$ by Proposition 2.7(viii). We first construct $G^{(k-1)}$. By (6.24), we may apply Lemma 4.1 with the following objects and parameters.

object/parameter	$\mathscr{P}$	$\mathscr{H}^{(k-1)}$	o	$s+s'$	η'	ν	ε'	$k-1$	t
playing the role of	$\mathscr{Q}$	$\mathscr{H}^{(k)}$	o	s	η	ν	ε	k	t

We obtain $\mathscr{R} = \mathscr{R}(k-2, \mathbf{a}^{\mathscr{R}})$ and $\mathscr{G}^{(k-1)} = \{G_1^{(k-1)}, \dots, G_{s+s'}^{(k-1)}\}$ satisfying the following.

(R1)$_{6.2}$ $\mathscr{R}$ is $(\eta', \varepsilon'(\mathbf{a}^{\mathscr{R}}), \mathbf{a}^{\mathscr{R}})$-equitable and t-bounded and for each $j \in [k-2]$, a_j divides $a_j^{\mathscr{R}}$,
(R2)$_{6.2}$ $\{\mathscr{R}^{(j)}\}_{j=1}^{k-2} = \mathscr{R} \prec \{\mathscr{P}^{(j)}\}_{j=1}^{k-2}$,
(R3)$_{6.2}$ for each $i \in [s+s']$, $G_i^{(k-1)}$ is perfectly $\varepsilon'(\mathbf{a}^{\mathscr{R}})$-regular with respect to $\mathscr{R}$,
(R4)$_{6.2}$ $\sum_{i=1}^{s+s'} |G_i^{(k-1)} \triangle H_i^{(k-1)}| \leq \nu\binom{n}{k-1}$, and
(R5)$_{6.2}$ $\mathscr{G}^{(k-1)} \prec \mathscr{P}^{(k-1)}$, and if $H_i^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{P}^{(1)})$, then $G_i^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{P}^{(1)})$.

Observe that $a_1^{\mathscr{R}} > \eta'^{-1}$ by (R1)$_{6.2}$. Thus

$$1/a_1^{\mathscr{R}} \ll \nu, 1/o, 1/k. \tag{6.25}$$

Let $G^{(k-1)} := \bigcup_{i=1}^{s} G_i^{(k-1)}$. Then (F1)$_{6.2}$ holds and $G^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{P}^{(1)})$.

Next we show how to construct $F^{(k-1)}$. To this end we define a family of partitions $\mathscr{L}$ on V' which has the same number of parts as $\mathscr{R}$. We apply Lemma 5.10 with $\{\mathscr{Q}^{(j)}\}_{j=1}^{k-2}, \varepsilon, \mathbf{a}^{\mathscr{R}}$ playing the roles of $\mathscr{P}, \varepsilon, \mathbf{b}$ to obtain $\mathscr{L}$ so that

$\mathscr{L} = \mathscr{L}(k-2, \mathbf{a}^{\mathscr{R}})$ is an $(\eta', \varepsilon^{1/3}, \mathbf{a}^{\mathscr{R}})$-equitable family of partitions such that $\mathscr{L} \prec \{\mathscr{Q}\}_{j=1}^{k-2}$.

Let $\mathbf{a}' := (a_1, \dots, a_{k-2})$, where $\mathbf{a} := (a_1, \dots, a_{k-1})$. By taking an appropriate $\mathbf{a}^{\mathscr{R}}$-labelling for $\mathscr{L}$, we may also assume that for each $\hat{\mathbf{x}} \in \hat{A}(k-1, k-2, \mathbf{a}')$,

$$\begin{aligned} A(\hat{\mathbf{x}}) &:= \{\hat{\mathbf{y}} \in \hat{A}(k-1, k-2, \mathbf{a}^{\mathscr{R}}) : \hat{R}^{(k-2)}(\hat{\mathbf{y}}) \subseteq \hat{P}^{(k-2)}(\hat{\mathbf{x}})\} \\ &= \{\hat{\mathbf{y}} \in \hat{A}(k-1, k-2, \mathbf{a}^{\mathscr{R}}) : \hat{L}^{(k-2)}(\hat{\mathbf{y}}) \subseteq \hat{Q}^{(k-2)}(\hat{\mathbf{x}})\}. \end{aligned} \tag{6.26}$$

For each $\hat{\mathbf{x}} \in \hat{A}(k-1, k-2, \mathbf{a}')$ and $\hat{\mathbf{y}} \in A(\hat{\mathbf{x}})$, Lemma 5.4 implies that

$$\begin{aligned} |\mathcal{K}_{k-1}(\hat{L}^{(k-2)}(\hat{\mathbf{y}}))| &\geq (1-\nu)\prod_{j=1}^{k-1}(a_j^{\mathscr{R}})^{-\binom{k-1}{j}} m^{k-1} \overset{(6.24)}{\geq} \varepsilon^{1/3}(1+\nu)\prod_{j=1}^{k-1}(a_j^{\mathscr{Q}})^{-\binom{k-1}{j}} m^{k-1} \\ &\geq \varepsilon^{1/3}|\mathcal{K}_{k-1}(\hat{Q}^{(k-2)}(\hat{\mathbf{x}}))|. \end{aligned} \tag{6.27}$$

We would like the relative densities of $F^{(k-1)}$ (with respect to the polyads of $\mathscr{L}$) to reflect the relative densities of $G^{(k-1)}$ (with respect to the polyads of $\mathscr{R}$). For this, we first determine the relative densities of $G^{(k-1)}$ (see (6.32)). For each $\hat{\mathbf{y}} \in \hat{A}(k-1, k-2, \mathbf{a}^{\mathscr{R}})$, $b \in [a_{k-1}]$, and the unique vector $\hat{\mathbf{x}}$ with $\hat{\mathbf{y}} \in A(\hat{\mathbf{x}})$, we define

$$\begin{aligned} Q_*^{(k-1)}(\hat{\mathbf{y}}, b) &:= Q^{(k-1)}(\hat{\mathbf{x}}, b) \cap \mathcal{K}_{k-1}(\hat{L}^{(k-2)}(\hat{\mathbf{y}})), \\ P_*^{(k-1)}(\hat{\mathbf{y}}, b) &:= P^{(k-1)}(\hat{\mathbf{x}}, b) \cap \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})). \end{aligned} \tag{6.28}$$

Since each $Q^{(k-1)}(\hat{\mathbf{x}}, b) \in \mathscr{Q}^{(k-1)}$ is $(\varepsilon, 1/a_{k-1})$-regular with respect to $\hat{Q}^{(k-2)}(\hat{\mathbf{x}})$ for each $b \in [a_{k-1}]$, Lemma 5.1(ii) and (6.27) with the definition of $A(\hat{\mathbf{x}})$ imply that

$$Q_*^{(k-1)}(\hat{\mathbf{y}}, b) \textit{ is } (\varepsilon^{2/3}, 1/a_{k-1})\textit{-regular with respect to } \hat{L}^{(k-2)}(\hat{\mathbf{y}}). \tag{6.29}$$

Similarly,

$$P_*^{(k-1)}(\hat{\mathbf{y}}, b) \textit{ is } (\varepsilon^{2/3}, 1/a_{k-1})\textit{-regular with respect to } \hat{R}^{(k-2)}(\hat{\mathbf{y}}). \tag{6.30}$$

For each $\hat{\mathbf{y}} \in \hat{A}(k-1, k-2, \mathbf{a}^{\mathscr{R}})$ and $b \in [a_{k-1}]$, let

$$G(\hat{\mathbf{y}}, b) := G^{(k-1)} \cap P_*^{(k-1)}(\hat{\mathbf{y}}, b). \tag{6.31}$$

Thus $G(\hat{\mathbf{y}}, b) \subseteq \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}}))$ by (6.28). Since $\mathscr{G}^{(k-1)} \prec \mathscr{P}^{(k-1)}$ by (R5)$_{6.2}$, we know that $G(\hat{\mathbf{y}}, b)$ is the union of some $(k-1)$-graphs in $\{G_i^{(k-1)} \cap \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})) : i \in [s]\}$. Thus (R3)$_{6.2}$ and the union lemma (Lemma 5.2) with the fact that $\varepsilon'(\mathbf{a}^{\mathscr{R}}) \ll 1/s$ imply that $G(\hat{\mathbf{y}}, b)$ is $\varepsilon'(\mathbf{a}^{\mathscr{R}})^{2/3}$-regular with respect to $\hat{R}^{(k-2)}(\hat{\mathbf{y}})$. As $G(\hat{\mathbf{y}}, b) \subseteq P_*^{(k-1)}(\hat{\mathbf{y}}, b)$ and (6.30) holds, there exists a number $d(\hat{\mathbf{y}}, b) \in [0, 1/a_{k-1}]$ such that

$$G(\hat{\mathbf{y}}, b) \textit{ is } (\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}, d(\hat{\mathbf{y}}, b))\textit{-regular with respect to } \hat{R}^{(k-2)}(\hat{\mathbf{y}}). \tag{6.32}$$

Now we use the values $d(\hat{\mathbf{y}}, b)$ to construct $F^{(k-1)}$. We apply the slicing lemma (Lemma 5.3) with the following objects and parameters.

object/parameter	$Q_*^{(k-1)}(\hat{\mathbf{y}}, b)$	$\hat{L}^{(k-2)}(\hat{\mathbf{y}})$	$1/a_{k-1}$	$\max\{d(\hat{\mathbf{y}}, b)a_{k-1}, 1 - d(\hat{\mathbf{y}}, b)a_{k-1}\}$	1
playing the role of	$H^{(k)}$	$H^{(k-1)}$	d	p_1	s

By (6.29) we obtain a partition of $Q_*^{(k-1)}(\hat{\mathbf{y}}, b)$ into two $(k-1)$-graphs such that for one of these, say $F(\hat{\mathbf{y}}, b)$, we have that

$$F(\hat{\mathbf{y}}, b) \textit{ is } (\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}, d(\hat{\mathbf{y}}, b))\textit{-regular with respect to } \hat{L}^{(k-2)}(\hat{\mathbf{y}}) \textit{ and } F(\hat{\mathbf{y}}, b) \subseteq Q_*^{(k-1)}(\hat{\mathbf{y}}, b). \tag{6.33}$$

Let

$$F^{(k-1)} := \bigcup_{\hat{\mathbf{y}} \in \hat{A}(k-1, k-2, \mathbf{a}^{\mathscr{R}}), b \in [a_{k-1}]} F(\hat{\mathbf{y}}, b).$$

Thus $F^{(k-1)} \subseteq \mathcal{K}_{k-1}(\mathscr{Q}^{(1)})$.

Now we have defined $F^{(k-1)}$ and $G^{(k-1)}$. It only remains to show that these two $(k-1)$-graphs satisfy (F2)$_{6.2}$. Fix any vector $\hat{\mathbf{z}} \in \hat{A}(k, k-1, \mathbf{a})$. Consider $\hat{\mathbf{y}}$ and $\hat{\mathbf{x}}$ such that $\hat{\mathbf{y}} \in A(\hat{\mathbf{x}})$ and $\hat{\mathbf{x}} \leq_{k-1,k-2} \hat{\mathbf{z}}$. By (2.12) we have

$$\hat{P}^{(k-1)}(\hat{\mathbf{z}}) = \bigcup_{\hat{\mathbf{w}} \leq_{k-1,k-2} \hat{\mathbf{z}}} P^{(k-1)}(\hat{\mathbf{w}}, \mathbf{z}^{(k-1)}_{\mathbf{w}^{(1)}_*}). \tag{6.34}$$

By (6.26), $\hat{\mathbf{y}} \in A(\hat{\mathbf{x}})$ implies that

$$\mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_{k-1}(\hat{P}^{(k-2)}(\hat{\mathbf{x}})) \quad \text{and}$$
$$\mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})) \cap \mathcal{K}_{k-1}(\hat{P}^{(k-2)}(\hat{\mathbf{w}})) = \emptyset \text{ for } \hat{\mathbf{w}} \neq \hat{\mathbf{x}}.$$

Also $P^{(k-1)}(\hat{\mathbf{w}}, \mathbf{z}^{(k-1)}_{\mathbf{w}^{(1)}_*}) \subseteq \mathcal{K}_{k-1}(\hat{P}^{(k-2)}(\hat{\mathbf{w}}))$ whenever $\hat{\mathbf{w}} \leq_{k-1,k-2} \hat{\mathbf{z}}$. Together this implies

$$\hat{P}^{(k-1)}(\hat{\mathbf{z}}) \cap \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})) \overset{(6.34)}{=} P^{(k-1)}(\hat{\mathbf{x}}, \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}) \cap \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})) \overset{(6.28)}{=} P^{(k-1)}_*(\hat{\mathbf{y}}, \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}).$$

Thus

$$G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}}) \cap \mathcal{K}_{k-1}(\hat{R}^{(k-2)}(\hat{\mathbf{y}})) = G^{(k-1)} \cap P^{(k-1)}_*(\hat{\mathbf{y}}, \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}) \overset{(6.31)}{=} G(\hat{\mathbf{y}}, \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}). \tag{6.35}$$

Together with (R2)$_{6.2}$ this implies that

$$G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}}) = \bigcup_{\hat{\mathbf{x}} \leq_{k-1,k-2} \hat{\mathbf{z}}} \bigcup_{\hat{\mathbf{y}} \in A(\hat{\mathbf{x}})} G(\hat{\mathbf{y}}, \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}).$$

Similarly

$$F^{(k-1)} \cap \hat{Q}^{(k-1)}(\hat{\mathbf{z}}) = \bigcup_{\hat{\mathbf{x}} \leq_{k-1,k-2} \hat{\mathbf{z}}} \bigcup_{\hat{\mathbf{y}} \in A(\hat{\mathbf{x}})} F(\hat{\mathbf{y}}, \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}).$$

For each $\hat{\mathbf{y}} \in \hat{A}(k-1, k-2, \mathbf{a}^{\mathscr{R}})$ let

$$d_{\mathbf{a}^{\mathscr{R}}, \hat{\mathbf{z}}, k-1}(\hat{\mathbf{y}}) := \begin{cases} d(\hat{\mathbf{y}}, b) & \text{if } \hat{\mathbf{y}} \in A(\hat{\mathbf{x}}) \text{ for some } \hat{\mathbf{x}} \leq_{k-1,k-2} \hat{\mathbf{z}} \text{ and } b = \mathbf{z}^{(k-1)}_{\mathbf{x}^{(1)}_*}, \\ 0 & \text{otherwise.} \end{cases}$$

The properties (6.32) and (6.35) together imply that for each $\hat{R}^{(k-2)}(\hat{\mathbf{y}}) \in \hat{\mathscr{R}}^{(k-2)}$,

> $G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}})$ *is* $(\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}, d_{\mathbf{a}^{\mathscr{R}}, \hat{\mathbf{z}}, k-1}(\hat{\mathbf{y}}))$*-regular with respect to* $\hat{R}^{(k-2)}(\hat{\mathbf{y}})$.

Analogously using (6.33), we obtain that for each $\hat{L}^{(k-2)}(\hat{\mathbf{y}}) \in \hat{\mathscr{L}}^{(k-2)}$,

> $F^{(k-1)} \cap \hat{Q}^{(k-1)}(\hat{\mathbf{z}})$ *is* $(\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}, d_{\mathbf{a}^{\mathscr{R}}, \hat{\mathbf{z}}, k-1}(\hat{\mathbf{y}}))$*-regular with respect to* $\hat{L}^{(k-2)}(\hat{\mathbf{y}})$.

In other words, $\mathscr{R}$ is an $(\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}, d_{\mathbf{a}^{\mathscr{R}}, \hat{\mathbf{z}}, k-1})$-partition of $G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}})$. From (6.23) and (6.25), we know

$$\varepsilon'(\mathbf{a}^{\mathscr{R}}) \ll 1/\|\mathbf{a}^{\mathscr{R}}\|_\infty \leq 1/a_1^{\mathscr{R}} \ll \nu, 1/o, 1/k.$$

In particular, this means that $G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}})$ satisfies the following regularity instance.

$$R := (\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}, \mathbf{a}^{\mathscr{R}}, d_{\mathbf{a}^{\mathscr{R}}, \hat{\mathbf{z}}, k-1}).$$

Similarly, $F^{(k-1)} \cap \hat{Q}^{(k-1)}(\hat{\mathbf{z}})$ also satisfies the regularity instance R. Thus we can apply Lemma 5.8 twice with the following objects and parameters, once with $G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}})$ playing the role of H and once more with $F^{(k-1)} \cap \hat{Q}^{(k-1)}(\hat{\mathbf{z}})$ playing the role of H.

object/parameter	$\varepsilon'(\mathbf{a}^{\mathscr{R}})^{1/2}$	$\|a^{\mathscr{R}}\|_\infty$	$a_1^{\mathscr{R}}$	$\nu^2 o^{-4^k}$	$\mathbf{a}^{\mathscr{R}}$	$k-1$	$K_k^{(k-1)}$
playing the role of	ε	t	a_1	γ	$\mathbf{a}$	k	F

Thus we obtain

$$\frac{|\mathcal{K}_k(G^{(k-1)} \cap \hat{P}^{(k-1)}(\hat{\mathbf{z}}))|}{\binom{n}{k}} = IC(K_k^{(k-1)}, d_{\mathbf{a}^{\mathscr{R}},\hat{\mathbf{z}},k-1}) \pm \nu^2 o^{-4^k} \quad \text{and} \tag{6.36}$$

$$\frac{|\mathcal{K}_k(F^{(k-1)} \cap \hat{Q}^{(k-1)}(\hat{\mathbf{z}}))|}{\binom{m}{k}} = IC(K_k^{(k-1)}, d_{\mathbf{a}^{\mathscr{R}},\hat{\mathbf{z}},k-1}) \pm \nu^2 o^{-4^k}. \tag{6.37}$$

On the other hand, we can apply Lemma 5.4 to show that for every $\hat{\mathbf{z}} \in \hat{A}(k, k-1, \mathbf{a})$

$$|\mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{z}}))| = (1 \pm \nu^2) \prod_{j=1}^{k-1} a_j^{-\binom{k}{j}} n^k \geq o^{-4^k} \binom{n}{k} \quad \text{and}$$

$$|\mathcal{K}_k(\hat{Q}^{(k-1)}(\hat{\mathbf{z}}))| = (1 \pm \nu^2) \prod_{j=1}^{k-1} a_j^{-\binom{k}{j}} m^k \geq o^{-4^k} \binom{m}{k}.$$

This together with (6.36) and (6.37) implies that

$$d(\mathcal{K}_k(G^{(k-1)}) \mid \hat{P}^{(k-1)}(\hat{\mathbf{z}})) = d(\mathcal{K}_k(F^{(k-1)}) \mid \hat{Q}^{(k-1)}(\hat{\mathbf{z}})) \pm \nu.$$

□

Suppose we are given two families of partitions $\mathscr{P}, \mathscr{O}$ such that $\mathscr{P}$ almost refines $\mathscr{O}$ and such that $\mathscr{O}$ is an equitable partition of some k-graph H. Roughly speaking, the next lemma shows that there is a family of partitions $\mathscr{O}'$ such that $\mathscr{P} \prec \mathscr{O}'$ and such that $\mathscr{O}'$ is still an equitable partition of H (with a somewhat larger regularity constant).

Lemma 6.3. *Suppose $0 < 1/m, 1/n \ll \varepsilon \ll \nu \ll \varepsilon_0 \leq 1$ and $k \in \mathbb{N} \setminus \{1\}$. Suppose V is a vertex set of size n. Suppose $R = (\varepsilon_0/2, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$ is a regularity instance and $\mathscr{O} = \mathscr{O}(k-1, \mathbf{a}^{\mathscr{O}})$ is an $(\varepsilon_0, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$-equitable partition of a k-graph $H^{(k)}$ on V. Suppose there exists an $(\eta, \varepsilon, \mathbf{a}^{\mathscr{P}})$-equitable family of partitions $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a}^{\mathscr{P}})$ on V such that $\mathscr{P} \prec_\nu \mathscr{O}$. Then there exists a family of partitions $\mathscr{O}'$ on V such that*

(O′1)$_{6.3}$ *$\mathscr{P} \prec \mathscr{O}'$,*

(O′2)$_{6.3}$ *$\mathscr{O}'$ is a $(1/a_1^{\mathscr{O}}, \varepsilon_0 + \nu^{1/20}, \mathbf{a}^{\mathscr{O}}, \nu^{1/20})$-equitable family of partitions and it is an $(\varepsilon_0 + \nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k})$-partition of $H^{(k)}$,*

(O′3)$_{6.3}$ *for each $j \in [k-1]$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$, we have $|O'^{(j)}(\hat{\mathbf{x}}, b) \triangle O^{(j)}(\hat{\mathbf{x}}, b)| \leq \nu^{1/2}\binom{n}{j}$.*

We construct $\mathscr{O}'$ by induction on $j \in [k-1]$. When constructing $\mathscr{O}'^{(j-1)}$ a natural approach is as follows. For a given class $O^{(j)}(\hat{\mathbf{x}}, b)$ of $\mathscr{O}^{(j)}$ we can let $O'^{(j)}(\hat{\mathbf{x}}, b)$ consist e.g. of all classes of $\mathscr{P}^{(j)}$ which lie (mostly) in $O^{(j)}(\hat{\mathbf{x}}, b)$. This is formalized by the function f_j in (6.38). However, this construction may not fit with the existing polyads of $\hat{P}^{(j-1)}$ (i.e. it may violate Definition 2.1(ii)). This issue requires some adjustments, whose overall effect can be shown to be negligible.

Proof of Lemma 6.3. For any function $f : \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}}) \times [a_j^{\mathscr{P}}] \to \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$, let

$$d(f) := \sum_{(\hat{\mathbf{x}},b) \in \hat{A}(j,j-1,\mathbf{a}^{\mathscr{P}}) \times [a_j^{\mathscr{P}}]} |P^{(j)}(\hat{\mathbf{x}}, b) \setminus O^{(j)}(f(\hat{\mathbf{x}}, b))|.$$

Note that $\mathscr{P} \prec_\nu \mathscr{O}$ implies that for each $j \in [k-1]$, there exists a function $f_j : \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}}) \times [a_j^{\mathscr{P}}] \to \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$ such that

$$d(f_j) \leq \nu \binom{n}{j}. \tag{6.38}$$

Moreover, note that since R is a regularity instance (see Definition 2.9), we have $\varepsilon_0 \leq \|\mathbf{a}^{\mathscr{O}}\|_\infty^{-4^k}\varepsilon_{5.4}(\|\mathbf{a}^{\mathscr{O}}\|_\infty^{-1}, \|\mathbf{a}^{\mathscr{O}}\|_\infty^{-1}, k-1, k)$. Thus Lemma 5.4 and the definition of an equitable family of partitions (see Definition 2.3) imply that for any $j \in [k-1] \setminus \{1\}$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$, we have

$$|O^{(j)}(\hat{\mathbf{x}}, b)| \geq \frac{1}{2\|\mathbf{a}^{\mathscr{O}}\|_\infty^{2^k}} n^j \geq \varepsilon_0^{1/2} n^j. \tag{6.39}$$

For each $i \in [a_1^{\mathscr{O}}]$, let

$$O'^{(1)}(i,i) := \bigcup_{s \in [a_1^{\mathscr{P}}], f_1(s)=i} P^{(1)}(s,s) \quad \text{and let} \quad \mathscr{O}'^{(1)} := \{O'^{(1)}(i,i) : i \in [a_1^{\mathscr{O}}]\}.$$

Note that (6.38) implies that $|O'^{(1)}(i,i)| = (1 \pm a_1^{\mathscr{O}}\nu)n/a_1^{\mathscr{O}} = (1 \pm \nu^{1/2})n/a_1^{\mathscr{O}}$. For all distinct $i, i' \in [a_1^{\mathscr{O}}]$, let $O'^{(1)}(i,i') := \emptyset$. Hence $\mathscr{O}'^{(1)}$ satisfies properties $(\mathrm{O}'1)_1$–$(\mathrm{O}'4)_1$ below. (Here, $(\mathrm{O}'2)_1$ and $(\mathrm{O}'4)_1$ are vacuous.) Assume for some $j \in [k-1] \setminus \{1\}$ we have defined $\mathscr{O}'^{(1)}, \dots, \mathscr{O}'^{(j-1)}$ satisfying the following for each $i \in [j-1]$:

$(\mathrm{O}'1)_i$ $\mathscr{O}'^{(i)}$ forms a partition of $\mathcal{K}_i(\mathscr{O}'^{(1)})$ and $\mathscr{P}^{(i)} \prec \mathscr{O}'^{(i)}$,

$(\mathrm{O}'2)_i$ if $i > 1$, then $\mathscr{O}'^{(i)} = \{O'^{(i)}(\hat{\mathbf{x}}, b) : (\hat{\mathbf{x}}, b) \in \hat{A}(i, i-1, \mathbf{a}^{\mathscr{O}}) \times [a_i^{\mathscr{O}}]\}$,

$(\mathrm{O}'3)_i$

$$\sum_{(\hat{\mathbf{x}}, b) \in \hat{A}(i,i-1,\mathbf{a}^{\mathscr{O}}) \times [a_i^{\mathscr{O}}]} |O'^{(i)}(\hat{\mathbf{x}}, b) \setminus O^{(i)}(\hat{\mathbf{x}}, b)| \leq i\nu n^i,$$

$(\mathrm{O}'4)_i$ if $i > 1$, then for each $\hat{\mathbf{x}} \in \hat{A}(i, i-1, \mathbf{a}^{\mathscr{O}})$, the collection $\{O'^{(i)}(\hat{\mathbf{x}}, 1), \dots, O'^{(i)}(\hat{\mathbf{x}}, a_i^{\mathscr{O}})\}$ forms a partition of $\mathcal{K}_i(\hat{O'}^{(i-1)}(\hat{\mathbf{x}}))$, where

$$\hat{O'}^{(i-1)}(\hat{\mathbf{x}}) := \bigcup_{\hat{\mathbf{y}} \leq_{i-1,i-2} \hat{\mathbf{x}}} O'^{(i-1)}(\hat{\mathbf{y}}, \mathbf{x}^{(i-1)}_{\mathbf{y}_*^{(1)}}).$$

We will now construct $\mathscr{O}'^{(j)}$ satisfying $(\mathrm{O}'1)_j$–$(\mathrm{O}'4)_j$. So assume that $k \geq 3$. Note that $(\mathrm{O}'3)_1$–$(\mathrm{O}'3)_{j-1}$ with (6.39) shows that for any $i \in [j-1]$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(i, i-1, \mathbf{a}^{\mathscr{O}}) \times [a_i^{\mathscr{O}}]$, the i-graph $O'^{(i)}(\hat{\mathbf{x}}, b)$ is nonempty. Together with $(\mathrm{O}'1)_1$–$(\mathrm{O}'1)_{j-1}$, $(\mathrm{O}'2)_1$–$(\mathrm{O}'2)_{j-1}$, $(\mathrm{O}'3)_1$–$(\mathrm{O}'3)_{j-1}$, $(\mathrm{O}'4)_1$–$(\mathrm{O}'4)_{j-1}$, (6.39), and Lemma 4.2 this implies that $\{\mathscr{O}'^{(i)}\}_{i=1}^{j-1}$ forms a family of partitions. Let $\hat{\mathscr{O}'}^{(j-1)}$ be the collection of all the $\hat{O'}^{(j-1)}(\hat{\mathbf{x}})$ with $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$. Note that Proposition 2.7(iv) and (vi) implies that

$$\{\mathcal{K}_j(\hat{O}'^{(j-1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})\} \textit{ forms a partition of } \mathcal{K}_j(\mathscr{O}'^{(1)}). \tag{6.40}$$

By Proposition 2.7(xi)

$$\{\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}})\} \prec \{\mathcal{K}_j(\hat{O}'^{(j-1)}(\hat{\mathbf{x}})) : \hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})\}. \tag{6.41}$$

Let

$$A := \{\hat{\mathbf{y}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}}) : \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_j(\mathscr{O}'^{(1)})\}.$$

Then (6.41) implies that

$$\bigcup_{\hat{\mathbf{y}} \in A} \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) = \mathcal{K}_j(\mathscr{O}'^{(1)}). \tag{6.42}$$

By (6.40) and (6.41), for each $\hat{\mathbf{y}} \in A$, there exists $g(\hat{\mathbf{y}}) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$ such that $\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}}))$ is a subset of $\mathcal{K}_j(\hat{O'}^{(j-1)}(g(\hat{\mathbf{y}})))$.

Claim 1.

$$\sum_{\hat{\mathbf{y}} \in A} |\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) \setminus \mathcal{K}_j(\hat{O}^{(j-1)}(g(\hat{\mathbf{y}})))| \leq (j-1)\nu n^j.$$

Proof. Observe that by (6.41) for each $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$, we have

$$\bigcup_{\hat{\mathbf{y}}\colon g(\hat{\mathbf{y}})=\hat{\mathbf{x}}} \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) = \mathcal{K}_j(\hat{O'}^{(j-1)}(\hat{\mathbf{x}})).$$

This implies that

$$\sum_{\hat{\mathbf{y}}\in A} |\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) \setminus \mathcal{K}_j(\hat{O}^{(j-1)}(g(\hat{\mathbf{y}})))| = \sum_{\hat{\mathbf{x}}\in\hat{A}(j,j-1,\mathbf{a}^{\mathscr{O}})} |\mathcal{K}_j(\hat{O'}^{(j-1)}(\hat{\mathbf{x}})) \setminus \mathcal{K}_j(\hat{O}^{(j-1)}(\hat{\mathbf{x}}))|.$$

Any j-set counted on the right hand side lies in $\mathcal{K}_j(\mathscr{O}'^{(1)})$ and contains a $(j-1)$-set $J \in O'^{(j-1)}(\hat{\mathbf{z}}, b) \setminus O^{(j-1)}(\hat{\mathbf{z}}, b)$ for some $\hat{\mathbf{z}} \leq_{j-1,j-2} \hat{\mathbf{x}}$ and $b = \mathbf{x}^{(j-1)}_{\mathbf{z}_*^{(1)}}$. Note that $(\mathrm{O}'3)_{j-1}$ implies that there are at most $(j-1)\nu n^{j-1}$ such sets J. For such a fixed $(j-1)$-set J, there are at most n j-sets in $\mathcal{K}_j(\mathscr{O}'^{(1)})$ containing J. Thus at most $(j-1)\nu n^{j-1}\cdot n = (j-1)\nu n^j$ j-sets are counted in the above expression. This proves the claim. □

Ideally, for every $\mathbf{x} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$ and $b \in [a_j^{\mathscr{O}}]$, we would like to define $O'^{(j)}(\mathbf{x}, b)$ as the union of all $P^{(j)}(\hat{\mathbf{y}}, b')$ for which $f_j(\hat{\mathbf{y}}, b') = (\hat{\mathbf{x}}, b)$ holds. However, we may have $f_j(\hat{\mathbf{y}}, b') \neq (g(\hat{\mathbf{y}}), b)$ for all $b \in [a_j^{\mathscr{O}}]$. This leads to difficulties when attempting to prove $(\mathrm{O}'4)_j$. We resolve this problem by defining a function f'_j, which is a slight modification of f_j. To this end, let

$$W := \{(\hat{\mathbf{y}}, b') \in A \times [a_j^{\mathscr{P}}] : f_j(\hat{\mathbf{y}}, b') \neq (g(\hat{\mathbf{y}}), b) \text{ for all } b \in [a_j^{\mathscr{O}}]\}.$$

Thus if $(\hat{\mathbf{y}}, b') \in W$, then $O^{(j)}(f_j(\hat{\mathbf{y}}, b')) \cap \mathcal{K}_j(\hat{O}^{(j-1)}(g(\hat{\mathbf{y}}))) = \emptyset$. This and the fact that $P^{(j)}(\hat{\mathbf{y}}, b') \subseteq \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}}))$ imply that for $(\hat{\mathbf{y}}, b') \in W$

$$P^{(j)}(\hat{\mathbf{y}}, b') \cap O^{(j)}(f_j(\hat{\mathbf{y}}, b')) \subseteq \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) \setminus \mathcal{K}_j(\hat{O}^{(j-1)}(g(\hat{\mathbf{y}}))). \tag{6.43}$$

We define a function $f'_j : A \times [a_j^{\mathscr{P}}] \to \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$ by

$$f'_j(\hat{\mathbf{y}}, b') := \begin{cases} (g(\hat{\mathbf{y}}), b) \text{ for an arbitrary } b \in [a_j^{\mathscr{O}}] & \text{if } (\hat{\mathbf{y}}, b') \in W, \\ f_j(\hat{\mathbf{y}}, b') & \text{otherwise.} \end{cases}$$

For each $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$ and $b \in [a_j^{\mathscr{O}}]$, let

$$O'^{(j)}(\hat{\mathbf{x}}, b) := \bigcup_{\substack{(\hat{\mathbf{y}},b')\in A\times[a_j^{\mathscr{P}}]:\\ f'_j(\hat{\mathbf{y}},b')=(\hat{\mathbf{x}},b)}} P^{(j)}(\hat{\mathbf{y}}, b'). \tag{6.44}$$

Let $\mathscr{O}'^{(j)}$ be as described in $(\mathrm{O}'2)_j$. By (6.40), (6.41), and the fact that f'_j is defined for all $A \times [a_j^{\mathscr{P}}]$, we obtain $(\mathrm{O}'1)_j$.

We now verify $(\mathrm{O}'3)_j$. For this, we estimate $d(f'_j)$, namely

$$\begin{aligned} d(f'_j) &= \sum_{(\hat{\mathbf{y}},b')\in\hat{A}(j,j-1,\mathbf{a}^{\mathscr{P}})\times[a_j^{\mathscr{P}}]} |P^{(j)}(\hat{\mathbf{y}}, b') \setminus O^{(j)}(f'_j(\hat{\mathbf{y}}, b'))| \\ &\leq \sum_{(\hat{\mathbf{y}},b')\in\hat{A}(j,j-1,\mathbf{a}^{\mathscr{P}})\times[a_j^{\mathscr{P}}]} |P^{(j)}(\hat{\mathbf{y}}, b') \setminus O^{(j)}(f_j(\hat{\mathbf{y}}, b'))| \\ &\qquad + \sum_{(\hat{\mathbf{y}},b')\in W} |P^{(j)}(\hat{\mathbf{y}}, b') \cap O^{(j)}(f_j(\hat{\mathbf{y}}, b'))| \\ &\overset{(6.43)}{\leq} d(f_j) + \sum_{\hat{\mathbf{y}}\in A} |\mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) \setminus \mathcal{K}_j(\hat{O}^{(j-1)}(g(\hat{\mathbf{y}})))| \\ &\overset{\text{Claim 1}}{\leq} d(f_j) + (j-1)\nu n^j \overset{(6.38)}{\leq} j\nu n^j. \end{aligned} \tag{6.45}$$

This in turn implies that

$$\sum_{(\hat{\mathbf{x}},b)\in \hat{A}(j,j-1,\mathbf{a}^{\mathscr{O}})\times[a_j^{\mathscr{O}}]} |O'^{(j)}(\hat{\mathbf{x}},b)\setminus O^{(j)}(\hat{\mathbf{x}},b)|$$

$$\overset{(6.44)}{=} \sum_{(\hat{\mathbf{x}},b)\in \hat{A}(j,j-1,\mathbf{a}^{\mathscr{O}})\times[a_j^{\mathscr{O}}]} \sum_{\substack{(\hat{\mathbf{y}},b')\in A\times[a_j^{\mathscr{P}}]:\\ f_j'(\hat{\mathbf{y}},b')=(\hat{\mathbf{x}},b)}} |P^{(j)}(\hat{\mathbf{y}},b')\setminus O^{(j)}(\hat{\mathbf{x}},b)|$$

$$= \sum_{(\hat{\mathbf{y}},b')\in A\times[a_j^{\mathscr{P}}]} |P^{(j)}(\hat{\mathbf{y}},b')\setminus O^{(j)}(f_j'(\hat{\mathbf{y}},b'))| \le d(f_j') \overset{(6.45)}{\le} j\nu n^j.$$

Thus $(\mathrm{O}'3)_j$ holds.

Suppose $\hat{\mathbf{x}} \in \hat{A}(j,j-1,\mathbf{a}^{\mathscr{O}})$ and $b\in[a_j^{\mathscr{O}}]$. Note that $P^{(j)}(\hat{\mathbf{y}},b') \subseteq \mathcal{K}_j(\hat{P}^{(j-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_j(\hat{O'}^{(j-1)}(g(\hat{\mathbf{y}})))$ for each $\hat{\mathbf{y}}\in A$ and $b'\in[a_j^{\mathscr{P}}]$. Together with (6.44) and the definition of f_j', we obtain that $O'^{(j)}(\hat{\mathbf{x}},b)\subseteq \mathcal{K}_j(\hat{O'}^{(j-1)}(\hat{\mathbf{x}}))$. By this and (6.40)–(6.42), the collection $\{O'^{(j)}(\hat{\mathbf{x}},1),\dots,O'^{(j)}(\hat{\mathbf{x}},a_i^{\mathscr{O}})\}$ forms a partition of $\mathcal{K}_j(\hat{O'}^{(j-1)}(\hat{\mathbf{x}}))$. Thus $(\mathrm{O}'4)_j$ holds.

By repeating this procedure, we obtain $\mathscr{O}'^{(1)},\dots,\mathscr{O}'^{(k-1)}$. Let $\mathscr{O}' := \{\mathscr{O}'^{(j)}\}_{j=1}^{k-1}$. As observed before (6.40), $\mathscr{O}'$ is a family of partitions. Properties $(\mathrm{O}'1)_1$–$(\mathrm{O}'1)_{k-1}$ and imply $(\mathrm{O}'1)_{6.3}$.

Note that $(\mathrm{O}'3)_1$ implies that for each $j\in[k-1]$ we have $|\mathcal{K}_j(\mathscr{O}^{(1)})\triangle\mathcal{K}_j(\mathscr{O}'^{(1)})|\le 2\nu n^j$. Thus for each $j\in[k-1]$ and $(\hat{\mathbf{x}},b)\in\hat{A}(j,j-1,\mathbf{a}^{\mathscr{O}})\times[a_j^{\mathscr{O}}]$, this implies that

$$|O'^{(j)}(\hat{\mathbf{x}},b)\triangle O^{(j)}(\hat{\mathbf{x}},b)| \le |\mathcal{K}_j(\mathscr{O}^{(1)})\triangle\mathcal{K}_j(\mathscr{O}'^{(1)})| + \sum_{\hat{\mathbf{x}}\in\hat{A}(j,j-1,\mathbf{a}),b\in[a_j^{\mathscr{O}}]} |O'^{(j)}(\hat{\mathbf{x}},b)\setminus O^{(j)}(\hat{\mathbf{x}},b)|$$

$$\overset{(\mathrm{O}'3)_j}{\le} (j+2)\nu n^j \le \nu^{1/2}\binom{n}{j}.$$

Thus we have $(\mathrm{O}'3)_{6.3}$. Finally, since R is a regularity instance, $(\mathrm{O}'3)_{6.3}$ enables us to apply Lemma 5.11 with the following objects and parameters.

object/parameter	$\mathscr{O}$	$\mathscr{O}'$	$\nu^{1/2}$	0	ε_0	$d_{\mathbf{a}^{\mathscr{O}},k}$	$H^{(k)}$	$H^{(k)}$
playing the role of	$\mathscr{P}$	$\mathscr{Q}$	ν	λ	ε	$d_{\mathbf{a},k}$	$H^{(k)}$	$G^{(k)}$

This implies $(\mathrm{O}'2)_{6.3}$. □

7. Sampling a regular partition

In this section we prove Theorem 3.1. In Section 7.1 we provide the main tool (Lemma 7.1) for this result and in Section 7.2 we deduce Theorem 3.1.

7.1. **Building a family of partitions from three others.** In this subsection we prove our key tool (Lemma 7.1) for the proof of Theorem 3.1. Roughly speaking Lemma 7.1 states the following. Suppose there are two k-graphs H_1, H_2 with vertex set V_1, V_2, respectively, and there are two ε-equitable families of partitions of these k-graphs which have the same parameters. Suppose further that there is another ε_0-equitable family of partitions $\mathscr{O}_1$ for H_1. Then there is an equitable family of partitions $\mathscr{O}_2$ of H_2 which has the (roughly) same parameters as $\mathscr{O}_1$ provided $\varepsilon \ll \varepsilon_0$. Even more loosely, the result says that if two hypergraphs share a single 'high-quality' regularity partition, then they share any 'low-quality' regularity partition.

Lemma 7.1. *Suppose $0 < 1/n, 1/m \ll \varepsilon \ll 1/T, 1/a_1^{\mathscr{Q}} \ll \delta \ll \varepsilon_0 \le 1$ and $k\in\mathbb{N}\setminus\{1\}$. Suppose $\mathbf{a}^{\mathscr{Q}}\in[T]^{k-1}$. Suppose that $R = (\varepsilon_0/2, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$ is a regularity instance. Suppose V_1, V_2 are sets of size n, m, and $H_1^{(k)}, H_2^{(k)}$ are k-graphs on V_1, V_2, respectively. Suppose*

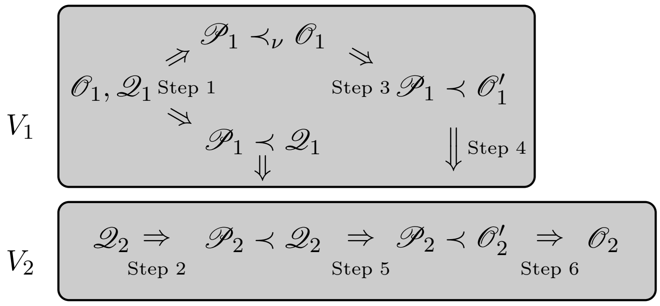

FIGURE 2. The proof strategy for Lemma 7.1.

(P1)$_{7.1}$ *$\mathscr{Q}_1 = \mathscr{Q}_1(k-1, \mathbf{a}^{\mathscr{Q}})$ is an $(\varepsilon, \mathbf{a}^{\mathscr{Q}}, d_{\mathbf{a}^{\mathscr{Q}},k})$-equitable partition of $H_1^{(k)}$,*
(P2)$_{7.1}$ *$\mathscr{Q}_2 = \mathscr{Q}_2(k-1, \mathbf{a}^{\mathscr{Q}})$ is an $(\varepsilon, \mathbf{a}^{\mathscr{Q}}, d_{\mathbf{a}^{\mathscr{Q}},k})$-equitable partition of $H_2^{(k)}$, and*
(P3)$_{7.1}$ *$\mathscr{O}_1 = \mathscr{O}_1(k-1, \mathbf{a}^{\mathscr{O}})$ is an $(\varepsilon_0, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$-equitable partition of $H_1^{(k)}$.*

Then there exists an $(\varepsilon_0 + \delta, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$-equitable partition $\mathscr{O}_2$ of $H_2^{(k)}$.

A crucial point here is that the construction of $\mathscr{O}_2$ incurs only an additive increase (by δ) of the regularity parameter of $\mathscr{O}_1$.

For an illustration of the proof strategy of Lemma 7.1 see Figure 2. Our strategy is first to apply Lemma 6.1 to $\mathscr{Q}_1, \mathscr{O}_1$ to obtain a family of partitions $\mathscr{P}_1$ that refines $\mathscr{Q}_1$ and almost refines $\mathscr{O}_1$ (see Step 1). Moreover, we refine $\mathscr{Q}_2$ and obtain $\mathscr{P}_2$ in such a way that $\mathscr{P}_2$ has the same number of partition classes as $\mathscr{P}_1$ (see Step 2). We then apply Lemma 6.3 to construct a family $\mathscr{O}_1'$ of partitions that is very similar to $\mathscr{O}_1$ and satisfies $\mathscr{P}_1 \prec \mathscr{O}_1'$ (see Step 3). Then we analyse how $\mathscr{P}_1$ refines $\mathscr{O}_1'$ (see Step 4). We then use Lemma 6.2 to mimic this structure in order to build $\mathscr{O}_2'$ from $\mathscr{P}_2$ (see Step 5). Finally in Step 6 we apply Lemma 5.12 to show that $\mathscr{O}_2'$ can be slightly modified to obtain the desired $\mathscr{O}_2$.

Proof of Lemma 7.1. We start with several definitions. Choose a new constant ν such that $1/T, 1/a_1^{\mathscr{Q}} \ll \nu \ll \delta$. Let $\overline{\varepsilon} : \mathbb{N}^{k-1} \to (0,1]$ be a function such that for any $\mathbf{a} = (a_1, \dots, a_{k-1}) \in \mathbb{N}^{k-1}$, we have $0 < \overline{\varepsilon}(\mathbf{a}) \ll \nu, \|\mathbf{a}\|_\infty^{-k}$. Now given $\overline{\varepsilon}$, we define

$$t' := t_{6.1}(k, T, T^{4^k}, 1/a_1^{\mathscr{Q}}, \nu, \overline{\varepsilon}).$$

Observe that $0 < \varepsilon \ll 1/k, 1/T, 1/a_1^{\mathscr{Q}}, \nu, 1/t'$. Thus we may assume that for any $\mathbf{a} \in [t']^{k-1}$, we have

$$0 < \varepsilon \ll \overline{\varepsilon}(\mathbf{a}), \mu_{6.1}(k, T, T^{4^k}, 1/a_1^{\mathscr{Q}}, \nu, \overline{\varepsilon}).$$

Step 1. *Constructing $\mathscr{P}_1$ as a refinement of $\mathscr{Q}_1$.*

Let

$$\mathscr{Q}_1^{(k)} := \{\mathcal{K}_k(\hat{Q}_1^{(k-1)}) : \hat{Q}_1^{(k-1)} \in \hat{\mathscr{Q}}_1^{(k-1)}\} \quad \text{and}$$
$$\mathscr{Q}_1'^{(k)} := \left(\mathscr{Q}_1^{(k)} \cup \left\{\binom{V_1}{k} \setminus \mathcal{K}_k(\mathscr{Q}_1^{(1)})\right\}\right) \setminus \{\emptyset\}.$$

Since $\mathscr{Q}_1$ is T-bounded, $|\mathscr{Q}_1^{(k)}| \leq T^{2^k}$ by Proposition 2.7(viii). Thus $|\mathscr{Q}_1'^{(k)}| \leq T^{4^k}$. Moreover, the fact that R is a regularity instance (and the definition of $\varepsilon_{2.9}$) implies that $\mathbf{a}^{\mathscr{O}} \in [T]^{k-1}$. We can apply Lemma 6.1 with the following objects and parameters.

object/parameter	V_1	$\mathscr{O}_1$	$\mathscr{Q}_1'^{(k)}$	$\{\mathscr{Q}_1^{(i)}\}_{i=1}^k$	T	T^{4^k}	$1/a_1^{\mathscr{Q}}$	ν	$\overline{\varepsilon}$
playing the role of	V	$\mathscr{O}$	$\mathscr{H}^{(k)}$	$\mathscr{Q}$	o	s	η	ν	ε

Observe that $\mathscr{O}_1$, $\{\mathscr{Q}_1^{(i)}\}_{i=1}^k$ and $\mathscr{Q}_1'^{(k)}$ playing the roles of $\mathscr{O}$, $\mathscr{Q}$ and $\mathscr{H}^{(k)}$, respectively, satisfy (O1)$_{6.1}$–(O4)$_{6.1}$ in Lemma 6.1. We obtain a family of partitions $\mathscr{P}_1 = \mathscr{P}_1(k-1, \mathbf{a}^{\mathscr{P}})$ such that the following hold.

(P11) $\mathscr{P}_1$ is $(1/a_1^{\mathscr{Q}}, \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}})$-equitable and t'-bounded, and $a_j^{\mathscr{Q}}$ divides $a_j^{\mathscr{P}}$ for each $j \in [k-1]$.

(P12) $\mathscr{P}_1^{(j)} \prec \mathscr{Q}_1^{(j)}$ and $\mathscr{P}_1^{(j)} \prec_\nu \mathscr{O}_1^{(j)}$ for each $j \in [k-1]$.

Let

$$\varepsilon' := \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), \quad \text{and} \quad a_k^{\mathscr{P}} = a_k^{\mathscr{Q}} = a_k^{\mathscr{O}} := 1.$$

Hence $\varepsilon' \ll \nu, \|\mathbf{a}^{\mathscr{P}}\|_\infty^{-k}$ by the definition of $\overline{\varepsilon}$.

By (P12), for each $j \in [k-1]$, $\hat{\mathbf{y}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}})$, and $b' \in [a_j^{\mathscr{P}}]$, either there exists $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{Q}})$ and $b \in [a_j^{\mathscr{Q}}]$ such that $P_1^{(j)}(\hat{\mathbf{y}}, b') \subseteq Q_1^{(j)}(\hat{\mathbf{x}}, b)$ or $P_1^{(j)}(\hat{\mathbf{y}}, b') \cap Q_1^{(j)}(\hat{\mathbf{x}}, b) = \emptyset$ for all $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{Q}})$ and $b \in [a_j^{\mathscr{Q}}]$. This allows us to describe $\mathscr{P}_1$ in terms of $\mathscr{Q}_1$ in the following way. For each $j \in [k-1]$, $\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{Q}})$, and $b \in [a_j^{\mathscr{Q}}]$, we define

$$\begin{aligned} A_j(\hat{\mathbf{x}}, b) &:= \{(\hat{\mathbf{y}}, b') \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}}) \times [a_j^{\mathscr{P}}] : P_1^{(j)}(\hat{\mathbf{y}}, b') \subseteq Q_1^{(j)}(\hat{\mathbf{x}}, b)\} \text{ and} \\ A_j &:= \bigcup_{\hat{\mathbf{x}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{Q}}), b \in [a_j^{\mathscr{Q}}]} A_j(\hat{\mathbf{x}}, b). \end{aligned} \tag{7.1}$$

For each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}})$, let

$$\begin{aligned} \hat{A}_k(\hat{\mathbf{x}}) &:= \{\hat{\mathbf{y}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{P}}) : \hat{P}_1^{(k-1)}(\hat{\mathbf{y}}) \subseteq \hat{Q}_1^{(k-1)}(\hat{\mathbf{x}})\}, \text{ and} \\ \hat{A}_k &:= \bigcup_{\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}})} \hat{A}_k(\hat{\mathbf{x}}). \end{aligned}$$

The density function $d_{\mathbf{a}^{\mathscr{Q}},k}$ for $\mathscr{Q}_1$ naturally gives rise to a density function for $\mathscr{P}_1$. Indeed, for each $\hat{\mathbf{y}} \in A(k, k-1, \mathbf{a}^{\mathscr{P}})$, we define

$$d_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}}) := \begin{cases} d_{\mathbf{a}^{\mathscr{Q}},k}(\hat{\mathbf{x}}) & \text{if } \hat{\mathbf{y}} \in \hat{A}_k(\hat{\mathbf{x}}) \text{ for some } \hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}}) \text{ and} \\ 0 & \text{if } \hat{\mathbf{y}} \notin \hat{A}_k. \end{cases}$$

Recall that $\mathscr{P}_1$ is a $(1/a_1^{\mathscr{Q}}, \varepsilon', \mathbf{a}^{\mathscr{P}})$-equitable family of partitions, $\mathscr{Q}_1$ is a $(1/a_1^{\mathscr{Q}}, \varepsilon, \mathbf{a}^{\mathscr{Q}})$-equitable family of partitions, and $\varepsilon \ll \varepsilon' \ll \nu, \|\mathbf{a}^{\mathscr{P}}\|_\infty^{-k}$. Thus Lemma 5.4 implies that for each $j \in [k-1]$, $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})$, and $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{Q}})$, we have

$$|\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))| = (1 \pm \nu) \prod_{i=1}^{j} (a_i^{\mathscr{P}})^{-\binom{j+1}{i}} n^{j+1} \text{ and } |\mathcal{K}_{j+1}(\hat{Q}_1^{(j)}(\hat{\mathbf{x}}))| = (1 \pm \nu) \prod_{i=1}^{j} (a_i^{\mathscr{Q}})^{-\binom{j+1}{i}} n^{j+1}. \tag{7.2}$$

By (P11), for each $j \in [k-2]$, $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})$ and $b \in [a_{j+1}^{\mathscr{P}}]$, we have

$$|P_1^{(j+1)}(\hat{\mathbf{y}}, b)| = (1/a_{j+1}^{\mathscr{P}} \pm \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}))|\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))| = (1 \pm 2\nu) \prod_{i=1}^{j+1} (a_i^{\mathscr{P}})^{-\binom{j+1}{i}} n^{j+1}. \tag{7.3}$$

It will be convenient to restrict our attention to the k-graph $G_1^{(k)}$ which consists of the crossing k-sets of $H_1^{(k)}$ with respect to $\mathscr{Q}_1^{(1)}$ (rather than $H_1^{(k)}$ itself).

Claim 1. *Let* $G_1^{(k)} := H_1^{(k)} \cap \bigcup_{\hat{\mathbf{y}} \in \hat{A}_k} \mathcal{K}_k(\hat{P}_1^{(k-1)}(\hat{\mathbf{y}}))$. *Then*

(G11) $\mathscr{P}_1$ *is an* $(\varepsilon', \mathbf{a}^{\mathscr{P}}, d_{\mathbf{a}^{\mathscr{P}},k})$*-equitable partition of* $G_1^{(k)}$.

(G12) $|G_1^{(k)} \triangle H_1^{(k)}| \leq \nu\binom{n}{k}$.

Proof. Consider $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{Q}})$ and $\hat{\mathbf{y}} \in \hat{A}_k(\hat{\mathbf{x}})$. Note that (7.2) implies that

$$|\mathcal{K}_k(\hat{P}_1^{(k-1)}(\hat{\mathbf{y}}))| \geq \varepsilon' |\mathcal{K}_k(\hat{Q}_1^{(k-1)}(\hat{\mathbf{x}}))|.$$

Also by (P1)$_{7.1}$, $H_1^{(k)}$ is $(\varepsilon, d_{\mathbf{a}^{\mathscr{Q}},k}(\hat{\mathbf{x}}))$-regular with respect to $\hat{Q}_1^{(k-1)}(\hat{\mathbf{x}})$. Thus by Lemma 5.1(ii) $H_1^{(k)}$ is $(\varepsilon', d_{\mathbf{a}^{\mathscr{Q}},k}(\hat{\mathbf{x}}))$-regular with respect to $\hat{P}_1^{(k-1)}(\hat{\mathbf{y}})$. Together with the definition of

$d_{\mathbf{a}^{\mathscr{Q}},k}(\hat{\mathbf{y}})$ this in turn shows that for all $\hat{\mathbf{y}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{P}})$ we have that $G_1^{(k)}$ is $(\varepsilon', d_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}}))$-regular with respect to $\hat{P}_1^{(k-1)}(\hat{\mathbf{y}})$. Thus (G11) holds.

Note that (P12) and the definition of $\hat{A}_k$ imply that

$$G_1^{(k)} \triangle H_1^{(k)} \subseteq \binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}_1^{(1)}).$$

Since $\mathscr{Q}_1$ is $(1/a_1^{\mathscr{Q}}, \varepsilon, \mathbf{a}^{\mathscr{Q}})$-equitable and $1/a_1^{\mathscr{Q}} \ll \nu, 1/k$, we obtain

$$\left|\binom{V}{k} \setminus \mathcal{K}_k(\mathscr{Q}_1^{(1)})\right| \overset{(2.5)}{\leq} \frac{k^2}{a_1^{\mathscr{Q}}}\binom{n}{k} \leq \nu\binom{n}{k}.$$

This proves (G12). □

Step 2. *Refining $\mathscr{Q}_2$ into $\mathscr{P}_2$ which mirrors $\mathscr{P}_1$.*

We have now set up the required definitions for the objects on V_1 and will now proceed with the objects on V_2. By using Lemma 5.10 with $\mathscr{Q}_2$, $\mathbf{a}^{\mathscr{P}}$, t' playing the roles of $\mathscr{P}, \mathbf{b}, t$, respectively, we can obtain a $(1/a_1^{\mathscr{Q}}, \varepsilon^{1/3}, \mathbf{a}^{\mathscr{P}})$-equitable family of partitions $\mathscr{P}_2 = \mathscr{P}_2(k-1, \mathbf{a}^{\mathscr{P}})$ such that $\mathscr{P}_2 \prec \mathscr{Q}_2$. By considering an appropriate $\mathbf{a}^{\mathscr{P}}$-labelling, we may assume that for each $j \in [k-1]$, $(\hat{\mathbf{y}}, b') \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}}) \times [a_j^{\mathscr{P}}]$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{Q}}) \times [a_j^{\mathscr{Q}}]$, we have

$$P_2^{(j)}(\hat{\mathbf{y}}, b') \subseteq Q_2^{(j)}(\hat{\mathbf{x}}, b) \text{ if and only if } (\hat{\mathbf{y}}, b') \in A_j(\hat{\mathbf{x}}, b).$$

Again Lemma 5.4 and the fact that $\varepsilon' \ll \nu, \|\mathbf{a}^{\mathscr{P}}\|_\infty^{-k}$ imply that for each $j \in [k-1]$, $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})$ and $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{Q}})$, we have

$$\begin{aligned} |\mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{y}}))| &= (1 \pm \nu)\prod_{i=1}^{j}(a_i^{\mathscr{P}})^{-\binom{j+1}{i}}m^{j+1} \quad \text{and} \\ |\mathcal{K}_{j+1}(\hat{Q}_2^{(j)}(\hat{\mathbf{x}}))| &= (1 \pm \nu)\prod_{i=1}^{j}(a_i^{\mathscr{Q}})^{-\binom{j+1}{i}}m^{j+1}. \end{aligned} \tag{7.4}$$

Let

$$G_2^{(k)} := H_2^{(k)} \cap \bigcup_{\hat{\mathbf{y}} \in \hat{A}_k} \mathcal{K}_k(\hat{P}_2^{(k-1)}(\hat{\mathbf{y}})).$$

Similarly as in Claim 1, we conclude the following.

(G21) $\mathscr{P}_2$ is an $(\varepsilon', \mathbf{a}^{\mathscr{P}}, d_{\mathbf{a}^{\mathscr{P}},k})$-equitable partition of $G_2^{(k)}$.

(G22) $|G_2^{(k)} \triangle H_2^{(k)}| \leq \nu\binom{n}{k}$.

Step 3. *Modifying $\mathscr{O}_1$ into $\mathscr{O}_1'$ with $\mathscr{P} \prec \mathscr{O}_1'$.*

Recall that $\mathscr{P}_1 \prec_\nu \mathscr{O}_1$ by (P12). We next replace $\mathscr{O}_1$ by a very similar family of partitions $\mathscr{O}_1'$ such that $\mathscr{P}_1 \prec \mathscr{O}_1'$. To this end we apply Lemma 6.3 with $\mathscr{O}_1, \mathscr{P}_1$ playing the roles of $\mathscr{O}, \mathscr{P}$, respectively, and obtain $\mathscr{O}_1' = \mathscr{O}_1'(k-1, \mathbf{a}^{\mathscr{O}})$ such that

(O′11) $\mathscr{P}_1 \prec \mathscr{O}_1'$.

(O′12) $\mathscr{O}_1'$ is a $(1/a_1^{\mathscr{O}}, \varepsilon_0 + \nu^{1/20}, \mathbf{a}^{\mathscr{O}}, \nu^{1/20})$-equitable family of partitions which is an $(\varepsilon_0 + \nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k})$-partition of $H_1^{(k)}$.

(O′13) for each $j \in [k-1]$ and $(\hat{\mathbf{x}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$, we have $|O'^{(j)}(\hat{\mathbf{x}}, b) \triangle O^{(j)}(\hat{\mathbf{x}}, b)| \leq \nu^{1/2}\binom{n}{j}$.

Note that since $(\varepsilon_0/2, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$ is a regularity instance and $\nu \ll \varepsilon_0$, we have

$$\varepsilon_0 + \nu^{1/20} \leq \|\mathbf{a}^{\mathscr{O}}\|_\infty^{-4^k} \cdot \varepsilon_{5.4}(\|\mathbf{a}^{\mathscr{O}}\|_\infty^{-1}, \|\mathbf{a}^{\mathscr{O}}\|_\infty^{-1}, k-1, k).$$

Thus Lemma 5.4 implies for any $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$, we have

$$|\mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))| \geq \varepsilon_0^{1/2} n^{j+1}. \tag{7.5}$$

Also, (O′12) implies that for all $j \in [k-2]$, $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$ and $b \in [a^{\mathscr{O}}_{j+1}]$, we have

$$|O'^{(j+1)}_1(\hat{\mathbf{w}}, b)| \geq (1/a^{\mathscr{O}}_{j+1} - 2\varepsilon_0)|\mathcal{K}_{j+1}(\hat{O}'^{(j)}_1(\hat{\mathbf{w}}))| \geq \varepsilon_0^{2/3} n^{j+1}. \tag{7.6}$$

Moreover, by (O′12), (O′13) and (G12), we can apply Lemma 5.11 with $\mathscr{O}'_1, \mathscr{O}'_1, H^{(k)}_1$ and $G^{(k)}_1$ playing the roles of $\mathscr{P}, \mathscr{Q}, H^{(k)}$ and $G^{(k)}$ to obtain that

$$\mathscr{O}'_1 \text{ is an } (\varepsilon_0 + 2\nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k})\text{-partition of } G^{(k)}_1. \tag{7.7}$$

Step 4. *Describing $\mathscr{O}'_1$ in terms of its refinement $\mathscr{P}_1$.*

We now describe how the partition classes and polyads of $\mathscr{O}'_1$ can be expressed in terms of $\mathscr{P}_1$. This description will be used to construct $\mathscr{O}'_2$ from $\mathscr{P}_2$ in Step 5.

For each $j \in [k-2]$, our next aim is to define $B_{j+1}(\hat{\mathbf{w}}, b)$ for $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$ and $b \in [a^{\mathscr{O}}_{j+1}]$ in a similar way as we defined $A_{j+1}(\mathbf{x}, b)$ for $\hat{\mathbf{x}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{Q}})$ and $b \in [a^{\mathscr{Q}}_{j+1}]$ in (7.1). To this end, for each $b \in [a^{\mathscr{O}}_1]$, let

$$B_1(b, b) := \{(b', b') \in \hat{A}(1, 0, \mathbf{a}^{\mathscr{P}}) \times [a^{\mathscr{P}}_1] : P^{(1)}_1(b', b') \subseteq O'^{(1)}_1(b, b)\}.$$

For each $j \in [k-1]$, let

$$\hat{B}_{j+1} := \left\{\hat{\mathbf{u}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}}) : \left|\mathbf{u}^{(1)}_* \cap \{b' : (b', b') \in B_1(b, b)\}\right| \leq 1 \text{ for each } b \in [a^{\mathscr{O}}_1]\right\}.$$

Note that this easily implies that

$$\hat{\mathbf{u}} \in \hat{B}_{j+1} \text{ if and only if } \mathcal{K}_{j+1}(\hat{P}^{(j)}_1(\hat{\mathbf{u}})) \subseteq \mathcal{K}_{j+1}(\mathscr{O}'^{(1)}_1). \tag{7.8}$$

Consider any $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$. Let

$$\hat{B}_{j+1}(\hat{\mathbf{w}}) := \left\{\hat{\mathbf{u}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}}) : \mathcal{K}_{j+1}(\hat{P}^{(j)}_1(\hat{\mathbf{u}})) \subseteq \mathcal{K}_{j+1}(\hat{O}'^{(j)}_1(\hat{\mathbf{w}}))\right\}. \tag{7.9}$$

Together with (O′11) and Proposition 2.7(xi) this implies that

$$\mathcal{K}_{j+1}(\hat{O'}^{(j)}_1(\hat{\mathbf{w}})) = \bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \mathcal{K}_{j+1}(\hat{P_1}^{(j)}(\hat{\mathbf{u}})). \tag{7.10}$$

Moreover, if $j \in [k-2]$ and $b \in [a^{\mathscr{O}}_{j+1}]$, let

$$B_{j+1}(\hat{\mathbf{w}}, b) := \left\{(\hat{\mathbf{u}}, b') \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}}) \times [a^{\mathscr{P}}_{j+1}] : P^{(j+1)}_1(\hat{\mathbf{u}}, b') \subseteq O'^{(j+1)}_1(\hat{\mathbf{w}}, b)\right\}. \tag{7.11}$$

Thus (O′11), (7.2) and (7.5) imply that for all $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$

$$|\hat{B}_{j+1}(\hat{\mathbf{w}})| \geq \frac{|\mathcal{K}_{j+1}(\hat{O}'^{(j)}_1(\hat{\mathbf{w}}))|}{(1+\nu)\prod_{i=1}^{j}(a^{\mathscr{P}}_i)^{-\binom{j+1}{i}} n^{j+1}} \geq \frac{1}{2}\varepsilon_0^{1/2} \prod_{i=1}^{j} (a^{\mathscr{P}}_i)^{\binom{j+1}{i}}. \tag{7.12}$$

Similarly, (O′11), (7.3) and (7.6) imply that for all $j \in [k-1] \setminus \{1\}$, $\hat{\mathbf{w}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$ and $b \in [a^{\mathscr{O}}_j]$,

$$|B_j(\hat{\mathbf{w}}, b)| \geq \frac{|O'^{(j-1)}_1(\hat{\mathbf{w}}, b)|}{(1+2\nu)\prod_{i=1}^{j}(a^{\mathscr{P}}_i)^{-\binom{j}{i}} n^j} \geq \frac{1}{2}\varepsilon_0^{2/3} \prod_{i=1}^{j} (a^{\mathscr{P}}_i)^{\binom{j}{i}} > 0. \tag{7.13}$$

Note that by Proposition 2.7(xi) and (O′11), for each $j \in [k-1]$, we have

$$\{\mathcal{K}_{j+1}(\hat{P}^{(j)}_1(\hat{\mathbf{u}})) : \hat{\mathbf{u}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})\} \prec \{\mathcal{K}_{j+1}(\hat{O'}^{(j)}_1(\hat{\mathbf{w}})) : \hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})\}. \tag{7.14}$$

Together with (7.8) and Proposition 2.7(vi) applied to $\mathscr{O}'_1$, this implies that $\hat{\mathbf{u}} \in \hat{B}_{j+1}$ if and only if $\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})$ for some $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$. Thus for each $j \in [k-1]$,

$$\{\hat{B}_{j+1}(\hat{\mathbf{w}}) : \hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})\} \text{ forms a partition of } \hat{B}_{j+1}. \tag{7.15}$$

The following observations relate polyads and partition classes of $\mathscr{O}'_1$ and $\mathscr{P}_1$. They will be used in the proof of Claim 3 to relate $\mathscr{O}'_2$ (which is constructed in Step 5) and $\mathscr{P}_2$.

Claim 2. (i) *For all* $j \in [k-1] \setminus \{1\}$ *and* $\hat{\mathbf{w}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$, *we have*

$$\bigcup_{b \in [a_j^{\mathscr{O}}]} B_j(\hat{\mathbf{w}}, b) = \hat{B}_j(\hat{\mathbf{w}}) \times [a_j^{\mathscr{P}}].$$

(ii) *For all* $j \in [k-1]$ *and* $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$, *we have*

$$\left\{(\hat{\mathbf{v}}, \mathbf{u}^{(j)}_{\mathbf{v}_*^{(1)}}) : \hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}}), \hat{\mathbf{v}} \leq_{j,j-1} \hat{\mathbf{u}}\right\} \subseteq \bigcup_{\hat{\mathbf{z}} \leq_{j,j-1} \hat{\mathbf{w}}} B_j(\hat{\mathbf{z}}, \mathbf{w}^{(j)}_{\mathbf{z}_*^{(1)}}).$$

Proof. We first prove (i). Note that for all $j \in [k-1] \setminus \{1\}$, $(\hat{\mathbf{u}}, b') \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{P}}) \times [a_j^{\mathscr{P}}]$ and $(\hat{\mathbf{w}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$ with $(\hat{\mathbf{u}}, b') \in B_j(\hat{\mathbf{w}}, b)$, we have

$$P_1^{(j)}(\hat{\mathbf{u}}, b') \subseteq O_1'^{(j)}(\hat{\mathbf{w}}, b) \subseteq \mathcal{K}_j(\hat{O}_1'^{(j-1)}(\hat{\mathbf{w}})).$$

Since $P_1^{(j)}(\hat{\mathbf{u}}, b') \subseteq \mathcal{K}_j(\hat{P}_1^{(j-1)}(\hat{\mathbf{u}}))$, this means $\mathcal{K}_j(\hat{P}_1^{(j-1)}(\hat{\mathbf{u}})) \cap \mathcal{K}_j(\hat{O}'^{(j-1)}(\hat{\mathbf{w}})) \neq \emptyset$. By (7.14) this in turn implies that $\mathcal{K}_j(\hat{P}_1^{(j-1)}(\hat{\mathbf{u}})) \subseteq \mathcal{K}_j(\hat{O}_1'^{(j-1)}(\hat{\mathbf{w}}))$, and thus $\hat{\mathbf{u}} \in \hat{B}_j(\hat{\mathbf{w}})$. On the other hand, if $\hat{\mathbf{u}} \in \hat{B}_j(\hat{\mathbf{w}})$, then (O′11) implies that for each $b' \in [a_j^{\mathscr{P}}]$ there exists $b \in [a_j^{\mathscr{O}}]$ such that $P_1^{(j)}(\hat{\mathbf{u}}, b') \subseteq O_1'^{(j)}(\hat{\mathbf{w}}, b)$, and thus $(\hat{\mathbf{u}}, b') \in B_j(\hat{\mathbf{w}}, b)$.

We now prove (ii). Recall that for each $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$, $\hat{O}_1'^{(j)}(\hat{\mathbf{w}})$ satisfies (2.12). Together with (O′11) this implies that

$$\hat{O}_1'^{(j)}(\hat{\mathbf{w}}) = \bigcup_{\hat{\mathbf{z}} \leq_{j,j-1} \hat{\mathbf{w}}} O_1'^{(j)}(\hat{\mathbf{z}}, \mathbf{w}^{(j)}_{\mathbf{z}_*^{(1)}}) \overset{(7.11)}{=} \bigcup_{\hat{\mathbf{z}} \leq_{j,j-1} \hat{\mathbf{w}}} \bigcup_{(\hat{\mathbf{v}}, b') \in B_j(\hat{\mathbf{z}}, \mathbf{w}^{(j)}_{\mathbf{z}_*^{(1)}})} P_1^{(j)}(\hat{\mathbf{v}}, b'). \tag{7.16}$$

Then

$$\begin{aligned}
& (\hat{\mathbf{v}}, b') \in \{(\hat{\mathbf{v}}, \mathbf{u}^{(j)}_{\mathbf{v}_*^{(1)}}) : \hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}}), \hat{\mathbf{v}} \leq_{j,j-1} \hat{\mathbf{u}}\} \\
\overset{(2.12),(7.9)}{\Longrightarrow}\ & \exists \hat{\mathbf{u}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}}) : \mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{u}})) \subseteq \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}})), P_1^{(j)}(\hat{\mathbf{v}}, b') \subseteq \hat{P}_1^{(j)}(\hat{\mathbf{u}}) \\
\overset{(2.2)}{\Longrightarrow}\ & \exists (I, J) \in P_1^{(j)}(\hat{\mathbf{v}}, b') \times \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}})) : I \subseteq J \\
\overset{(2.2),(O'11)}{\Longrightarrow}\ & P_1^{(j)}(\hat{\mathbf{v}}, b') \subseteq \hat{O}_1'^{(j)}(\hat{\mathbf{w}}) \\
\overset{(7.16)}{\Longrightarrow}\ & (\hat{\mathbf{v}}, b') \in \bigcup_{\hat{\mathbf{z}} \leq_{j,j-1} \hat{\mathbf{w}}} B_j(\hat{\mathbf{z}}, \mathbf{w}^{(j)}_{\mathbf{z}_*^{(1)}}).
\end{aligned}$$

This proves the claim. □

Step 5. *Constructing* $\mathscr{O}_2'$ *from* $\mathscr{P}_2$.

Together $B_j(\mathbf{w}, b)$ and $\hat{B}_j(\mathbf{w})$ encode how $\mathscr{O}_1'$ can be refined into $\mathscr{P}_1$. We now use this information to construct $\mathscr{O}_2'$ from $\mathscr{P}_2$. Claim 3 then shows that this construction indeed yields a family of partitions (whose polyads can be expressed in terms of those of $\mathscr{P}_2$). Finally, Claim 4 shows that the partition classes are appropriately regular.

For each $b \in [a_1^{\mathscr{O}}]$, we let

$$O_2'^{(1)}(b, b) := \bigcup_{(b', b') \in B_1(b, b)} P_2^{(1)}(b', b'). \tag{7.17}$$

We also let $\mathscr{O}_2'^{(1)} := \{O_2'^{(1)}(b, b) : b \in [a_1^{\mathscr{O}}]\}$. Again, as in (7.8), this easily implies that for each $j \in [k-1]$

$$\hat{\mathbf{u}} \in \hat{B}_{j+1} \text{ if and only if } \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}})) \subseteq \mathcal{K}_{j+1}(\mathscr{O}_2'^{(1)}). \tag{7.18}$$

Note that for each $b \in [a_1^{\mathscr{O}}]$, by (O′12), we have that

$$|O_2'^{(1)}(b,b)| \quad = \sum_{(b',b')\in B_1(b,b)} |P_2^{(1)}(b',b')| = (1 \pm 2\nu^{1/20})m/a_1^{\mathscr{O}}. \tag{7.19}$$

In analogy to (7.11), for each $j \in [k-1] \setminus \{1\}$, $\hat{\mathbf{w}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$, and $b \in [a_j^{\mathscr{O}}]$, we define

$$O_2'^{(j)}(\hat{\mathbf{w}}, b) := \bigcup_{(\hat{\mathbf{u}},b')\in B_j(\hat{\mathbf{w}},b)} P_2^{(j)}(\hat{\mathbf{u}}, b'), \tag{7.20}$$

and for each $j \in [k-1] \setminus \{1\}$, we let

$$\mathscr{O}_2'^{(j)} := \{O_2'^{(j)}(\hat{\mathbf{w}}, b) : \hat{\mathbf{w}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}), b \in [a_j^{\mathscr{O}}]\}.$$

Moreover, let $\mathscr{O}_2' := \{\mathscr{O}_2'^{(j)}\}_{j=1}^{k-1}$. Note that since $\mathscr{P}_2$ is a family of partitions, (7.13) and (7.20) imply that $O_2'^{(j)}(\hat{\mathbf{w}}, b)$ is nonempty for each $j \in [k-1] \setminus \{1\}$ and $(\hat{\mathbf{w}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$. The construction of $\mathscr{O}_2'$ also gives rise to a natural description of all polyads. Indeed, as in (2.12), we define for all $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$

$$\hat{O}_2'^{(j)}(\hat{\mathbf{w}}) \quad := \bigcup_{\hat{\mathbf{z}} \le_{j,j-1} \hat{\mathbf{w}}} O_2'^{(j)}(\hat{\mathbf{z}}, \mathbf{w}^{(j)}_{\mathbf{z}^{(1)}_*}) \tag{7.21}$$

$$\overset{(7.20)}{=} \bigcup_{\hat{\mathbf{z}} \le_{j,j-1} \hat{\mathbf{w}}} \; \bigcup_{(\hat{\mathbf{v}},b')\in B_j(\hat{\mathbf{z}}, \mathbf{w}^{(j)}_{\mathbf{z}^{(1)}_*})} P_2^{(j)}(\hat{\mathbf{v}}, b'). \tag{7.22}$$

Claim 3. *$\mathscr{O}_2'$ is a family of partitions on V_2. Moreover, for all $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$, we have*

$$\mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}})) = \bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}})). \tag{7.23}$$

Proof. We will prove Claim 3 by applying the criteria in Lemma 4.2. For each $j \in [k-1] \setminus \{1\}$, $\hat{\mathbf{w}} \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}})$ and $b \in [a_j^{\mathscr{O}}]$, let $\phi^{(j)}(O_2'^{(j)}(\hat{\mathbf{w}}, b)) := b$. Let $\ell \in [k-1]$ be the largest number satisfying the following.

(OP1)$_\ell$ $\{\mathscr{O}_2'^{(j)}\}_{j=1}^{\ell}$ is a family of partitions,

(OP2)$_\ell$ Let $O_*^{(j)}(\cdot,\cdot)$ and $\hat{O}_*^{(j)}(\cdot)$ be the maps defined as in (2.8)–(2.11) for $\{\mathscr{O}_2'^{(j)}\}_{j=1}^{k}$ and $\{\phi^{(j)}\}_{j=2}^{k}$. Then $\{\phi^{(j)}\}_{j=2}^{\ell}$ is an $(a_1^{\mathscr{O}}, \dots, a_\ell^{\mathscr{O}})$-labelling of $\{\mathscr{O}_2'^{(j)}\}_{j=1}^{\ell}$ such that for each $j \in [\ell] \setminus \{1\}$ and $(\hat{\mathbf{w}}, b) \in \hat{A}(j, j-1, \mathbf{a}^{\mathscr{O}}) \times [a_j^{\mathscr{O}}]$, we have

$$O_*^{(j)}(\hat{\mathbf{w}}, b) = O_2'^{(j)}(\hat{\mathbf{w}}, b),$$

and for each $j \in [\ell]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$ we have

$$\hat{O}_*^{(j)}(\hat{\mathbf{w}}) = \hat{O}_2'^{(j)}(\hat{\mathbf{w}}).$$

It is easy to check that (OP1)$_1$–(OP2)$_1$ hold and thus $\ell \ge 1$. Since $\{\mathscr{O}_2'^{(j)}\}_{j=1}^{\ell}$ is a family of partitions, $\hat{\mathscr{O}}_2'^{(j)}$ is well-defined for each $j \in [\ell]$. Claim 2(ii) now allows us to express (the cliques spanned by) these polyads in terms of those of $\mathscr{P}_2^{(j)}$.

Subclaim 1. *For each $j \in [\ell]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$, we have*

$$\mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}})) = \bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}})).$$

Proof. Consider $j \in [\ell]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$. Note that

$$\bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \hat{P}_2^{(j)}(\hat{\mathbf{u}}) \overset{(2.12)}{=} \bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \bigcup_{\hat{\mathbf{v}} \leq_{j,j-1} \hat{\mathbf{u}}} P_2^{(j)}(\hat{\mathbf{v}}, \mathbf{u}^{(j)}_{\mathbf{v}_*^{(1)}}).$$

This together with (7.22) and Claim 2(ii) implies that $\bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \hat{P}_2^{(j)}(\hat{\mathbf{u}}) \subseteq \hat{O}_2'^{(j)}(\hat{\mathbf{w}})$, thus we obtain

$$\bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}})) \subseteq \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}})). \tag{7.24}$$

On the other hand, we have

$$\bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}} \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}})) \overset{(7.18)}{=} \mathcal{K}_{j+1}(\mathscr{O}_2'^{(1)}) = \bigcup_{\hat{\mathbf{w}} \in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{O}})} \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}})). \tag{7.25}$$

(Here the final equality follows from $(\mathrm{OP1})_\ell$, $(\mathrm{OP2})_\ell$ and Proposition 2.7(vi) applied to $\{O_2'^{(j)}\}_{j=1}^{\ell}$.) Consider a $(j+1)$-set $J \in \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))$. Then by (7.25) there exists $\hat{\mathbf{u}}' \in \hat{B}_{j+1}$ such that $J \in \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}}'))$.

We claim that $\hat{\mathbf{u}}' \in \hat{B}_{j+1}(\hat{\mathbf{w}})$. Indeed if not, then by (7.15), there exists $\hat{\mathbf{w}}' \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}}) \setminus \{\hat{\mathbf{w}}\}$ such that $\hat{\mathbf{u}}' \in \hat{B}_{j+1}(\hat{\mathbf{w}}')$, thus we have

$$J \in \bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}}')} \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}})) \overset{(7.24)}{\subseteq} \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}')).$$

Hence $\mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}})) \cap \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}'))$ is nonempty (as it contains J). However, since $\{\mathscr{O}_2'^{(i)}\}_{i=1}^{j}$ is a family of partitions, this contradicts Proposition 2.7(vi) and (ix). Hence, we have $\hat{\mathbf{u}}' \in \hat{B}_{j+1}(\hat{\mathbf{w}})$, thus $J \in \bigcup_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}}))$. The fact that this holds for arbitrary $J \in \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))$ combined with (7.24) proves the subclaim. □

Now, if $\ell = k-1$, then $\mathscr{O}_2'$ is a family of partitions and Subclaim 1 implies the moreover part of Claim 3. Assume that $\ell < k-1$ for a contradiction. Now we show that $\{\mathscr{O}_2'^{(j)}\}_{j=1}^{\ell+1}$ and the maps $\{O_2'^{(j)}(\cdot,\cdot), \hat{O}_2'^{(j)}(\cdot)\}_{j=1}^{\ell+1}$ satisfy the conditions (FP1)–(FP3) in Lemma 4.2. Condition (FP1) follows from $(\mathrm{OP1})_\ell$, (7.13) and (7.20). Condition (FP3) also holds because of (7.21) and the assumption that $\ell < k-1$. Property $(\mathrm{OP1})_\ell$ implies that (FP2) holds when $j \in [\ell]$. To check that (FP2) also holds for $j = \ell+1$, consider $\hat{\mathbf{w}} \in \hat{A}(\ell+1, \ell, \mathbf{a}^{\mathscr{O}})$. By Claim 2(i) and (7.13), we have that $\{B_{\ell+1}(\hat{\mathbf{w}}, 1), \dots, B_{\ell+1}(\hat{\mathbf{w}}, a^{\mathscr{O}}_{\ell+1})\}$ forms a partition of $\hat{B}_{\ell+1}(\hat{\mathbf{w}}) \times [a^{\mathscr{P}}_{\ell+1}]$ into nonempty sets. Thus by (7.20) and Subclaim 1, $\{O_2'^{(\ell+1)}(\hat{\mathbf{w}}, 1), \dots, O_2'^{(\ell+1)}(\hat{\mathbf{w}}, a^{\mathscr{O}}_{\ell+1})\}$ forms a partition of $\mathcal{K}_{\ell+1}(\hat{O}_2'^{(\ell)}(\hat{\mathbf{w}}))$ into nonempty sets. Thus (FP2) holds for $j = \ell+1$.

Hence, by (7.17) we can apply Lemma 4.2 to see that $(\mathrm{OP1})_{\ell+1}$ and $(\mathrm{OP2})_{\ell+1}$ hold, a contradiction to the choice of ℓ. Thus $\ell = k-1$, which proves the claim. □

By Claim 3, $\mathscr{O}_2'$ is a family of partitions and (7.20) implies that $\mathscr{P}_2 \prec \mathscr{O}_2'$. Consider any $j \in [k-1]$ and $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$. Note that

$$|\mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))| \overset{(7.10)}{=} \sum_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} |\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{u}}))| \overset{(7.2)}{=} |\hat{B}_{j+1}(\hat{\mathbf{w}})|(1 \pm \nu) \prod_{i=1}^{j} (a_i^{\mathscr{P}})^{-\binom{j+1}{i}} n^{j+1},$$

$$|\mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))| \overset{(7.23)}{=} \sum_{\hat{\mathbf{u}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} |\mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{u}}))| \overset{(7.4)}{=} |\hat{B}_{j+1}(\hat{\mathbf{w}})|(1 \pm \nu) \prod_{i=1}^{j} (a_i^{\mathscr{P}})^{-\binom{j+1}{i}} m^{j+1}. \tag{7.26}$$

For notational convenience, for each $\hat{\mathbf{w}} \in \hat{A}(k, k-1, \mathbf{a}^{\mathscr{O}})$, let

$$O_1'^{(k)}(\hat{\mathbf{w}}, 1) := G_1^{(k)} \cap \mathcal{K}_k(\hat{O}_1'^{(k-1)}(\hat{\mathbf{w}})) \quad \text{and} \quad O_2'^{(k)}(\hat{\mathbf{w}}, 1) := G_2^{(k)} \cap \mathcal{K}_k(\hat{O}_2'^{(k-1)}(\hat{\mathbf{w}})).$$

Claim 4. *For all $j \in [k-1]$, $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$ and $b \in [a_{j+1}^{\mathscr{O}}]$, we have that $O_2'^{(j+1)}(\hat{\mathbf{w}}, b)$ is*

- *$(\varepsilon_0 + 3\nu^{1/20}, 1/a_{j+1}^{\mathscr{O}})$-regular with respect to $\hat{O}_2'^{(j)}(\hat{\mathbf{w}})$ if $j \leq k-2$, and*
- *$(\varepsilon_0 + 3\nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k}(\hat{\mathbf{w}}))$-regular with respect to $\hat{O}_2'^{(j)}(\hat{\mathbf{w}})$ if $j = k-1$.*

Proof. To prove Claim 4, we will apply Lemma 6.2 (see (J1) and (J2) and the preceding discussion), which allows us to transfer information about $\mathscr{O}_1'$ and $\mathscr{P}_1$ to $\mathscr{O}_2'$ and $\mathscr{P}_2$. Fix $j \in [k-1]$, $\hat{\mathbf{w}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{O}})$, and $b \in [a_{j+1}^{\mathscr{O}}]$. Let

$$d := \begin{cases} 1/a_{j+1}^{\mathscr{O}} & \text{if } j \leq k-2, \\ d_{\mathbf{a}^{\mathscr{O}},k}(\hat{\mathbf{w}}) & \text{if } j = k-1. \end{cases} \tag{7.27}$$

Consider an arbitrary j-graph $F^{(j)} \subseteq \hat{O}_2'^{(j)}(\hat{\mathbf{w}})$ with

$$|\mathcal{K}_{j+1}(F^{(j)})| \geq (\varepsilon_0 + 3\nu^{1/20})|\mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))|. \tag{7.28}$$

To prove the claim, it suffices to show that $d(O_2'^{(j+1)}(\hat{\mathbf{w}}, b) \mid F^{(j)}) = d \pm (\varepsilon_0 + 3\nu^{1/20})$. For each $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})$, let

$$d(\hat{\mathbf{y}}) := \begin{cases} \frac{1}{a_{j+1}^{\mathscr{P}}} |\{b' : (\hat{\mathbf{y}}, b') \in B_{j+1}(\hat{\mathbf{w}}, b)\}| & \text{if } j \leq k-2, \\ d_{\mathbf{a}^{\mathscr{P}},k}(\hat{\mathbf{y}}) & \text{if } \hat{\mathbf{y}} \in \hat{B}_{j+1}(\hat{\mathbf{w}}), j = k-1, \\ 0 & \text{if } \hat{\mathbf{y}} \notin \hat{B}_{j+1}(\hat{\mathbf{w}}), j = k-1. \end{cases} \tag{7.29}$$

Thus Claim 2(i) and the above definition implies that

$$\textit{if } \hat{\mathbf{y}} \notin \hat{B}_{j+1}(\hat{\mathbf{w}}), \textit{ then we have } d(\hat{\mathbf{y}}) = 0. \tag{7.30}$$

Subclaim 2. *For all $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})$ and each $i \in [2]$, we have that $O_i'^{(j+1)}(\hat{\mathbf{w}}, b)$ is $(\varepsilon'^{1/2}, d(\hat{\mathbf{y}}))$-regular with respect to $\hat{P}_i^{(j)}(\hat{\mathbf{y}})$.*

Proof. First, we note that if $j \leq k-2$, then by (7.11) and (7.20), for $i \in [2]$, we have

$$O_i'^{(j+1)}(\hat{\mathbf{w}}, b) \cap \mathcal{K}_{j+1}(\hat{P}_i^{(j)}(\hat{\mathbf{y}})) = \bigcup_{b' : (\hat{\mathbf{y}}, b') \in B_{j+1}(\hat{\mathbf{w}}, b)} P_i^{(j+1)}(\hat{\mathbf{y}}, b').$$

Together with (G11), (G21) and (7.29) this shows that $O_i'^{(j+1)}(\hat{\mathbf{w}}, b) \cap \mathcal{K}_{j+1}(\hat{P}_i^{(j)}(\hat{\mathbf{y}}))$ is the disjoint union of $a_{j+1}^{\mathscr{P}} d(\hat{\mathbf{y}}) \leq \|\mathbf{a}^{\mathscr{P}}\|_\infty$ hypergraphs, each of which is $(\varepsilon', 1/a_{j+1}^{\mathscr{P}})$-regular with respect to $\hat{P}_i^{(j)}(\hat{\mathbf{y}})$. Thus the union lemma (Lemma 5.2) together with the fact that $\varepsilon' \ll 1/\|\mathbf{a}^{\mathscr{P}}\|_\infty$ implies Subclaim 2 in this case.

If $j = k-1$, then we have $b = 1$. If $\hat{\mathbf{y}} \in \hat{B}_k(\hat{\mathbf{w}})$, then $\mathcal{K}_k(\hat{P}_i^{(k-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_k(\hat{O}_i'^{(k-1)}(\hat{\mathbf{w}}))$ by (7.9) and Claim 3. Thus

$$O_i'^{(k)}(\hat{\mathbf{w}}, 1) \cap \mathcal{K}_k(\hat{P}_i^{(k-1)}(\hat{\mathbf{y}})) = G_i^{(k)} \cap \mathcal{K}_k(\hat{P}_i^{(k-1)}(\hat{\mathbf{y}})).$$

Together with (G11), (G21) and (7.29) this implies Subclaim 2 in this case.

If $\hat{\mathbf{y}} \notin \hat{B}_k(\hat{\mathbf{w}})$, then by (7.9), (7.14) and Claim 3 we have

$$O_i'^{(k)}(\hat{\mathbf{w}}, 1) \cap \mathcal{K}_k(\hat{P}_i^{(k-1)}(\hat{\mathbf{y}})) \subseteq \mathcal{K}_k(\hat{O}_i'^{(k-1)}(\hat{\mathbf{w}})) \cap \mathcal{K}_k(\hat{P}_i^{(k-1)}(\hat{\mathbf{y}})) = \emptyset.$$

Since $d(\hat{\mathbf{y}}) = 0$ in this case, this proves Subclaim 2. □

In order to show that $O_2'^{(j+1)}(\hat{\mathbf{w}}, b)$ is $(\varepsilon_0 + 3\nu^{1/20})$-regular with respect to $\hat{O}_2'^{(j)}(\hat{\mathbf{w}})$, we will transfer all the calculations about hypergraph densities from the hypergraphs on V_2 to the hypergraphs on V_1, because there we have much better control over their structure. To this end we first use Lemma 6.2 to show the existence of two hypergraphs $J_1^{(j)}, J_2^{(j)}$ on V_1, V_2, respectively, that exhibit a very similar structure in terms of their densities and where $J_2^{(j)}$ is very close to $F^{(j)}$. Consequently, $J_1^{(j)}$ also resembles $F^{(j)}$.

More precisely, we now apply Lemma 6.2 with $F^{(j)}, \{\mathscr{P}_2^{(i)}\}_{i=1}^j, \{\mathscr{P}_1^{(i)}\}_{i=1}^j, j$ playing the roles of $H^{(k-1)}, \mathscr{P}, \mathscr{Q}, k-1$ respectively (we can do this by (P11), (G21) and Claim 3). We obtain j-graphs $J_2^{(j)} \subseteq \mathcal{K}_j(\mathscr{P}_2^{(1)})$ on V_2 and $J_1^{(j)} \subseteq \mathcal{K}_j(\mathscr{P}_1^{(1)})$ on V_1 such that

(J1) $|J_2^{(j)} \triangle F^{(j)}| \leq \nu\binom{m}{j}$, and

(J2) $d(\mathcal{K}_{j+1}(J_2^{(j)}) \mid \hat{P}_2^{(j)}(\hat{\mathbf{y}})) = d(\mathcal{K}_{j+1}(J_1^{(j)}) \mid \hat{P}_1^{(j)}(\hat{\mathbf{y}})) \pm \nu$ for each $\hat{\mathbf{y}} \in \hat{A}(j+1, j, \mathbf{a}^{\mathscr{P}})$.

Our next aim is to estimate $|\mathcal{K}_{j+1}(J_2^{(j)})|$ in terms of $|\mathcal{K}_{j+1}(J_1^{(j)})|$.

$$\begin{aligned}
\left|\mathcal{K}_{j+1}(J_2^{(j)})\right| &= \sum_{\hat{\mathbf{y}} \in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})} \left|\mathcal{K}_{j+1}(J_2^{(j)}) \cap \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{y}}))\right| \\
&\overset{(7.4)}{=} (1\pm\nu)\prod_{i=1}^{j}(a_i^{\mathscr{P}})^{-\binom{j+1}{i}} m^{j+1} \sum_{\hat{\mathbf{y}} \in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})} d(\mathcal{K}_{j+1}(J_2^{(j)}) \mid \hat{P}_2^{(j)}(\hat{\mathbf{y}})) \\
&\overset{(J2),(7.2)}{=} (1\pm 3\nu)\frac{m^{j+1}}{n^{j+1}}\left|\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right| \sum_{\hat{\mathbf{y}} \in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})} (d(\mathcal{K}_{j+1}(J_1^{(j)}) \mid \hat{P}_1^{(j)}(\hat{\mathbf{y}})) \pm \nu) \\
&= (1\pm 3\nu)\frac{m^{j+1}}{n^{j+1}}\left(\sum_{\hat{\mathbf{y}} \in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})} \left|\mathcal{K}_{j+1}(J_1^{(j)}) \cap \mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right| \pm \nu \sum_{\hat{\mathbf{y}} \in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})} \left|\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right|\right) \\
&= \frac{m^{j+1}}{n^{j+1}}(|\mathcal{K}_{j+1}(J_1^{(j)})| \pm 5\nu n^{j+1}). \qquad (7.31)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\left|O_2'^{(j+1)}(\hat{\mathbf{w}}, b) \cap \ \mathcal{K}_{j+1}(J_2^{(j)})\right| &= \sum_{\hat{\mathbf{y}} \in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})} \left|O_2'^{(j+1)}(\hat{\mathbf{w}}, b) \cap \mathcal{K}_{j+1}(J_2^{(j)}) \cap \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{y}}))\right| \\
&\overset{\text{Subcl. 2}}{=} \sum_{\hat{\mathbf{y}} \in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})} \left(d(\hat{\mathbf{y}})\left|\mathcal{K}_{j+1}(J_2^{(j)}) \cap \mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{y}}))\right| \pm \varepsilon'^{1/4}\left|\mathcal{K}_{j+1}(\hat{P}_2^{(j)}(\hat{\mathbf{y}}))\right|\right) \\
&\overset{(7.4),(7.30)}{=} \left((1\pm\nu)\prod_{i=1}^{j}(a_i^{\mathscr{P}})^{-\binom{j+1}{i}} m^{j+1} \sum_{\hat{\mathbf{y}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} d(\hat{\mathbf{y}}) d(\mathcal{K}_{j+1}(J_2^{(j)}) \mid \hat{P}_2^{(j)}(\hat{\mathbf{y}}))\right) \pm \varepsilon'^{1/4} m^{j+1} \\
&\overset{(J2),(7.2)}{=} \left((1\pm 3\nu)\frac{m^{j+1}}{n^{j+1}}\left|\mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right| \sum_{\hat{\mathbf{y}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} d(\hat{\mathbf{y}}) d(\mathcal{K}_{j+1}(J_1^{(j)}) \mid \hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right) \pm 4\nu m^{j+1} \\
&= (1\pm 3\nu)\frac{m^{j+1}}{n^{j+1}}\left(\sum_{\hat{\mathbf{y}} \in \hat{B}_{j+1}(\hat{\mathbf{w}})} d(\hat{\mathbf{y}})\left|\mathcal{K}_{j+1}(J_1^{(j)}) \cap \mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right|\right) \pm 4\nu m^{j+1} \\
&\overset{(7.30),\text{Subcl. 2}}{=} (1\pm 3\nu)\frac{m^{j+1}}{n^{j+1}}\left(\sum_{\hat{\mathbf{y}} \in \hat{A}(j+1,j,\mathbf{a}^{\mathscr{P}})} \left|O_1'^{(j+1)}(\hat{\mathbf{w}}, b) \cap \mathcal{K}_{j+1}(J_1^{(j)}) \cap \mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right|\right) \pm 5\nu m^{j+1} \\
&= \frac{m^{j+1}}{n^{j+1}}\left(\left|O_1'^{(j+1)}(\hat{\mathbf{w}}, b) \cap \mathcal{K}_{j+1}(J_1^{(j)})\right| \pm 10\nu n^{j+1}\right). \qquad (7.32)
\end{aligned}$$

Note that (J1) implies that

$$\left|\mathcal{K}_{j+1}(J_2^{(j)}))\triangle\mathcal{K}_{j+1}(F^{(j)})\right| \leq \nu\binom{m}{j} \cdot m \leq \nu m^{j+1} \qquad (7.33)$$

Since $F^{(j)} \subseteq \hat{O'}_2^{(j)}(\hat{\mathbf{w}})$ by assumption, (7.33) implies that

$$\left|\mathcal{K}_{j+1}(J_2^{(j)}) \setminus \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))\right| \le \nu m^{j+1}. \tag{7.34}$$

We can transfer (7.34) to the corresponding graphs on V_1 as follows:

$$\begin{aligned}
\left|\mathcal{K}_{j+1}(J_1^{(j)}) \setminus \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))\right| &\overset{(7.9)}{=} \sum_{\hat{\mathbf{y}} \notin \hat{B}_{j+1}(\hat{\mathbf{w}})} \left|\mathcal{K}_{j+1}(J_1^{(j)}) \cap \mathcal{K}_{j+1}(\hat{P}_1^{(j)}(\hat{\mathbf{y}}))\right| \\
&\overset{(7.2)}{\le} (1+\nu)\prod_{i=1}^{j}(a_i^{\mathscr{P}})^{-\binom{j+1}{i}} n^{j+1} \sum_{\hat{\mathbf{y}} \notin \hat{B}_{j+1}(\hat{\mathbf{w}})} d(\mathcal{K}_{j+1}(J_1^{(j)}) \mid \hat{P}_1^{(j)}(\hat{\mathbf{y}})) \\
&\overset{(J2)}{\le} \left((1+\nu)\prod_{i=1}^{j}(a_i^{\mathscr{P}})^{-\binom{j+1}{i}} n^{j+1} \sum_{\hat{\mathbf{y}} \notin \hat{B}_{j+1}(\hat{\mathbf{w}})} d(\mathcal{K}_{j+1}(J_2^{(j)}) \mid \hat{P}_2^{(j)}(\hat{\mathbf{y}}))\right) + 2\nu n^{j+1} \\
&\overset{(7.4),(7.23)}{\le} \frac{n^{j+1}}{m^{j+1}}\left|\mathcal{K}_{j+1}(J_2^{(j)}) \setminus \mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))\right| + 5\nu n^{j+1} \\
&\overset{(7.34)}{\le} 6\nu n^{j+1}.
\end{aligned} \tag{7.35}$$

Next we show that $|\mathcal{K}_{j+1}(J_1^{(j)}) \cap \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))|$ is not too small:

$$\begin{aligned}
\left|\mathcal{K}_{j+1}(J_1^{(j)}) \cap \mathcal{K}_{j+1}(\hat{O'}_1^{(j)}(\hat{\mathbf{w}}))\right| &\overset{(7.35)}{\ge} \left|\mathcal{K}_{j+1}(J_1^{(j)})\right| - 6\nu n^{j+1} \\
&\overset{(7.31)}{\ge} \frac{n^{j+1}}{m^{j+1}}\left|\mathcal{K}_{j+1}(J_2^{(j)})\right| - 11\nu n^{j+1} \\
&\overset{(7.33)}{\ge} \frac{n^{j+1}}{m^{j+1}}\left|\mathcal{K}_{j+1}(F^{(j)})\right| - 12\nu n^{j+1} \\
&\overset{(7.28)}{\ge} \frac{n^{j+1}}{m^{j+1}}(\varepsilon_0 + 3\nu^{1/20})\left|\mathcal{K}_{j+1}(\hat{O}_2'^{(j)}(\hat{\mathbf{w}}))\right| - 12\nu n^{j+1} \\
&\overset{(7.12),(7.26)}{\ge} (\varepsilon_0 + 2\nu^{1/20})\left|\mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))\right|.
\end{aligned} \tag{7.36}$$

Recall that d was defined in (7.27). We now can combine our estimates to conclude that

$$\begin{aligned}
\left|O_2'^{(j+1)}(\hat{\mathbf{w}}, b) \cap \mathcal{K}_{j+1}(F^{(j)})\right| &\overset{(7.32),(7.33)}{=} \frac{m^{j+1}}{n^{j+1}}\left(\left|O'_1{}^{(j+1)}(\hat{\mathbf{w}}, b) \cap \mathcal{K}_{j+1}(J_1^{(j)})\right| \pm 20\nu n^{j+1}\right) \\
&\overset{(7.7),(7.36),(O'12)}{=} \frac{m^{j+1}}{n^{j+1}}\left((d \pm (\varepsilon_0 + 2\nu^{1/20}))\left|\mathcal{K}_{j+1}(J_1^{(j)}) \cap \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))\right| \pm 20\nu n^{j+1}\right) \\
&= \frac{m^{j+1}}{n^{j+1}}(d \pm (\varepsilon_0 + 2\nu^{1/20}))\left(|\mathcal{K}_{j+1}(J_1^{(j)})| - |\mathcal{K}_{j+1}(J_1^{(j)}) \setminus \mathcal{K}_{j+1}(\hat{O}_1'^{(j)}(\hat{\mathbf{w}}))|\right) \pm 20\nu m^{j+1} \\
&\overset{(7.31),(7.35)}{=} (d \pm (\varepsilon_0 + 2\nu^{1/20}))\left|\mathcal{K}_{j+1}(J_2^{(j)})\right| \pm 40\nu m^{j+1} \\
&\overset{(7.33)}{=} (d \pm (\varepsilon_0 + 2\nu^{1/20}))\left|\mathcal{K}_{j+1}(F^{(j)})\right| \pm 50\nu m^{j+1} \\
&= (d \pm (\varepsilon_0 + 3\nu^{1/20}))\left|\mathcal{K}_{j+1}(F^{(j)})\right|.
\end{aligned} \tag{7.37}$$

Here, we obtain the final inequality since (7.12), (7.26) and (7.28) imply $|\mathcal{K}_{j+1}(F^{(j)})| \ge \varepsilon_0^2 m^{j+1}$ and $\nu \ll \varepsilon_0$. (7.37) holds for all $F^{(j)} \subseteq \hat{O}_2'^{(j)}(\hat{\mathbf{w}})$ satisfying (7.28), thus $O_2'^{(j+1)}(\hat{\mathbf{w}}, b)$ is $(\varepsilon_0 + 3\nu^{1/20}, d)$-regular with respect to $\hat{O}_2'^{(j)}(\hat{\mathbf{w}})$. This with the definition of d completes the proof of Claim 4. □

Claim 4 and (7.19) show that $\mathscr{O}_2'$ is a $(1/a_1^{\mathscr{O}}, \varepsilon_0 + 3\nu^{1/20}, \mathbf{a}^{\mathscr{O}}, 2\nu^{1/20})$-equitable family of partitions which is also an $(\varepsilon_0 + 3\nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k})$-partition of $G_2^{(k)}$ (as defined in Section 2.4.2).

Note that $(\varepsilon_0 + 3\nu^{1/20})/3 \leq \varepsilon_0/2$, thus $((\varepsilon_0 + 3\nu^{1/20})/3, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$ is a regularity instance. Since $|G_2^{(k)} \triangle H_2^{(k)}| \leq \nu\binom{m}{k}$ by (G22), this means that we can apply Lemma 5.11 with the following objects and parameters.

object/parameter	$\mathscr{O}_2'$	$\mathscr{O}_2'$	ν	$\varepsilon_0 + 3\nu^{1/20}$	$d_{\mathbf{a}^{\mathscr{O}},k}$	$G_2^{(k)}$	$H_2^{(k)}$
playing the role of	$\mathscr{P}$	$\mathscr{Q}$	ν	ε	$d_{\mathbf{a},k}$	$H^{(k)}$	$G^{(k)}$

Hence $\mathscr{O}_2'$ is also an $(\varepsilon_0 + 4\nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k})$-partition of $H_2^{(k)}$.

Step 6. *Adjusting $\mathscr{O}_2'$ into an equipartition $\mathscr{O}_2$.*

Finally, we modify $\mathscr{O}_2'$ to turn it from an 'almost' equipartition into an equipartition $\mathscr{O}_2$. For this we apply Lemma 5.12 with $\mathscr{O}_2', H_2^{(k)}\varepsilon_0 + 4\nu^{1/20}, 2\nu^{1/20}, d_{\mathbf{a}^{\mathscr{O}},k}$ playing the roles of $\mathscr{P}, H^{(k)}, \varepsilon, \lambda, d_{\mathbf{a},k}$ respectively. This guarantees an $(\varepsilon_0 + 3\nu^{1/200}, \mathbf{a}^{\mathscr{O}}, d_{\mathbf{a}^{\mathscr{O}},k})$-equitable partition $\mathscr{O}_2$ of $H_2^{(k)}$, which completes the proof. □

7.2. **Random samples.** To prove our results about random samples of hypergraphs, we will need the following lemma due to Czygrinow and Nagle. It states that ε-regularity of a random complex is inherited by a random sample (but with significantly worse parameters).

Lemma 7.2 (Czygrinow and Nagle [9]). *Suppose $0 < 1/m_0, 1/s, \varepsilon \ll \varepsilon', d_0, 1/\ell, 1/k \leq 1$ and $k, \ell \in \mathbb{N}\setminus\{1\}$ with $\ell \geq k$. Suppose $\mathscr{H} = \{H^{(j)}\}_{j=1}^k$ is an $(\varepsilon, (d_2, \dots, d_k))$-regular (ℓ, k)-complex with $H^{(1)} = \{V_1, \dots, V_\ell\}$ such that $d_i \in [d_0, 1]$, and $|V_i| > m_0$ for all $i \in [\ell]$. Let $s_1, \dots, s_\ell \geq s$ be integers such that $|V_i| \geq s_i$. Then for subsets $S_i \in \binom{V_i}{s_i}$ chosen uniformly at random, $\{H^{(j)}[S_1 \cup S_2 \cup \dots \cup S_\ell]\}_{j=1}^k$ is an $(\varepsilon', (d_2, \dots, d_k))$-regular (ℓ, k)-complex with probability at least $1 - e^{-\varepsilon s}$.*

Note that in [9], the lemma is only stated for the case $\ell = k$, but the case $\ell \geq k$ follows via a union bound. The next lemma generalizes Lemma 7.2 and shows how an equitable partition of a k-graph transfers with high probability to a random sample.

Lemma 7.3. *Suppose $0 < 1/n < 1/q \ll \varepsilon \ll \varepsilon' \ll 1/t, 1/k$, and $k \in \mathbb{N} \setminus \{1\}$. Suppose that $\mathscr{P} = \mathscr{P}(k-1, \mathbf{a})$ is an $(\varepsilon, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of a k-graph H on vertex set V with $|V| = n$ and $\mathbf{a} \in [t]^{k-1}$. Then for a set $Q \in \binom{V}{q}$ chosen uniformly at random, with probability at least $1 - e^{-\varepsilon^3 q}$, there exists an $(\varepsilon', \mathbf{a}, d_{\mathbf{a},k})$-equitable family of partitions $\mathscr{Q}$ of $H[Q]$.*

The parameter ε' in Lemma 7.3 will be too large for our purposes. But we can combine Lemmas 7.1 and 7.3 to obtain the stronger assertion stated in $(\mathrm{Q1})_{3.1}$ of Theorem 3.1.

For the proof of Lemma 7.3 we also need the following lemma which is easy to show, for example using Azuma's inequality. We omit the proof.

Lemma 7.4. *Suppose $0 < 1/n \leq 1/q \ll 1/k \leq 1/2$ and $1/q \ll \nu$. Let H be an n-vertex k-graph on vertex set V. Let $Q \in \binom{V}{q}$ be a q-vertex subset of V chosen uniformly at random. Then*
$$\mathbb{P}\left[|H[Q]| = \frac{q^k}{n^k}|H| \pm \nu\binom{q}{k}\right] \geq 1 - 2e^{\frac{-\nu^2 q}{8k^2}}.$$

Proof of Lemma 7.3. We choose an additional constant ν such that
$$0 < \varepsilon \ll \nu \ll 1/t, 1/k, \varepsilon'.$$

Let Q be a set of q vertices selected uniformly at random in V. Write $\mathscr{P}^{(1)} = \{V_1, \dots, V_{a_1}\}$ and let $S_i := Q \cap V_i$. For $\mathbf{S} = (s_1, \dots, s_{a_1})$ with $\sum_{i=1}^{a_1} s_i = q$ and $s_i \in \mathbb{N} \cup \{0\}$, let $\mathcal{E}(\mathbf{S})$ be the event that $|S_i| = s_i$ for all $i \in [a_1]$, and let
$$I := \left\{\mathbf{S} : s_i = (1 \pm \varepsilon)\frac{q}{a_1} \text{ for each } i \in [a_1]\right\}.$$

By some standard concentration inequality, we conclude

$$\mathbb{P}\left[\bigvee_{\mathbf{S}\in I}\mathcal{E}(\mathbf{S})\right] \geq 1 - 2a_1 e^{-\varepsilon^2 q^2/(a_1^2 q)} \geq 1 - e^{-\varepsilon^{5/2} q}. \tag{7.38}$$

Recall that for $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$, $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ denotes the $(k, k-1)$-complex induced by $\hat{\mathbf{x}}$ in $\mathscr{P}$ as defined in (2.14) (as remarked at (2.14), for a family of partitions $\mathscr{P}$, $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ is indeed a $(k, k-1)$-complex). Let

$$A := \{\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}) : d_{\mathbf{a},k}(\hat{\mathbf{x}}) \geq \nu\} \quad \text{and} \quad G := \bigcup_{\hat{\mathbf{x}}\in A}\Big(H \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))\Big) \cup \Big(H \setminus \mathcal{K}_k(\mathscr{P}^{(1)})\Big).$$

It is easy to see that $G \subseteq H$ and $|G \triangle H| \leq 2\nu\binom{n}{k}$.

For each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$, let

$$\hat{\mathcal{P}}'(\hat{\mathbf{x}}) := \hat{\mathcal{P}}(\hat{\mathbf{x}}) \cup \left\{G \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}}))\right\}.$$

Note that $\hat{\mathcal{P}}'(\hat{\mathbf{x}})$ is an $(\varepsilon, (1/a_2, \ldots, 1/a_{k-1}, d_{\mathbf{a},k}(\hat{\mathbf{x}})))$-regular (k,k)-complex for each $\hat{\mathbf{x}} \in A$ and $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ is a $(\varepsilon, (1/a_2, \ldots, 1/a_{k-1}))$-regular $(k, k-1)$-complex for each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$.

For each $\hat{\mathbf{x}} \in A$, we define the following event:

($\hat{\mathcal{E}}(\hat{\mathbf{x}})$) $\hat{\mathcal{P}}'(\hat{\mathbf{x}})[Q]$ is an $(\varepsilon'/2, (1/a_2, \ldots, 1/a_{k-1}, d_{\mathbf{a},k}(\hat{\mathbf{x}})))$-regular (k,k)-complex.

For each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}) \setminus A$, we also define the following event:

($\hat{\mathcal{E}}(\hat{\mathbf{x}})$) *$\hat{\mathcal{P}}(\hat{\mathbf{x}})[Q]$ is an $(\varepsilon'/2, (1/a_2, \ldots, 1/a_{k-1}))$-regular $(k, k-1)$-complex.*

Note that for each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}) \setminus A$, the event $\hat{\mathcal{E}}(\hat{\mathbf{x}})$ implies that the complex $\hat{\mathcal{P}}'(\hat{\mathbf{x}})[Q]$ is an $(\varepsilon'/2, (1/a_2, \ldots, 1/a_{k-1}, d_{\mathbf{a},k}(\hat{\mathbf{x}}))$-regular complex as we have $d_{\mathbf{a},k}(\hat{\mathbf{x}}) \leq \nu \ll \varepsilon'$ and $(G \cap \mathcal{K}_k(\hat{P}^{(k-1)}(\hat{\mathbf{x}})))[Q] = \emptyset$. Thus we have that

$$\textstyle\bigwedge_{\hat{\mathbf{x}}\in\hat{A}(k,k-1,\mathbf{a})} \hat{\mathcal{E}}(\hat{\mathbf{x}}) \textit{ implies that } \mathscr{P}[Q] \textit{ is an } (\varepsilon'/2, d_{\mathbf{a},k})\textit{-partition of } G[Q]. \tag{7.39}$$

Consider any $\hat{\mathbf{x}} \in A$. Since q is sufficiently large, we may apply Lemma 7.2 with the following objects and parameters.

object/parameter	$\hat{\mathcal{P}}'(\hat{\mathbf{x}})$	S_i	$1/a_1, \ldots, 1/a_{k-1}$	$d_{\mathbf{a},k}(\hat{\mathbf{x}})$	$\varepsilon'/2$	$\nu/2$
playing the role of	$\mathcal{H}$	S_i	$d_1, \ldots, d_{k-1}$	d_k	ε'	d_0

We obtain for any fixed $\mathbf{S} \in I$, that

$$\mathbb{P}[\hat{\mathcal{E}}(\hat{\mathbf{x}}) \mid \mathcal{E}(\mathbf{S})] \geq 1 - e^{-\varepsilon^2 q}. \tag{7.40}$$

In a similar way, for each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a}) \setminus A$, we can apply Lemma 7.2 to $\hat{\mathcal{P}}(\hat{\mathbf{x}})$ to obtain that (7.40) holds, too. Thus for each $\hat{\mathbf{x}} \in \hat{A}(k, k-1, \mathbf{a})$, we obtain

$$\begin{aligned}\mathbb{P}[\hat{\mathcal{E}}(\hat{\mathbf{x}})] &= \sum_{\mathbf{S}\in I}\mathbb{P}[\hat{\mathcal{E}}(\hat{\mathbf{x}}) \mid \mathcal{E}(\mathbf{S})]\mathbb{P}[\mathcal{E}(\mathbf{S})] + \sum_{\mathbf{S}\notin I}\mathbb{P}[\hat{\mathcal{E}}(\hat{\mathbf{x}}) \mid \mathcal{E}(\mathbf{S})]\mathbb{P}[\mathcal{E}(\mathbf{S})] \\ &\overset{(7.40)}{\geq} (1 - e^{-\varepsilon^2 q})\sum_{\mathbf{S}\in I}\mathbb{P}[\mathcal{E}(\mathbf{S})] \overset{(7.38)}{\geq} (1 - e^{-\varepsilon^2 q})(1 - e^{-\varepsilon^{5/2} q}) \geq 1 - 2e^{-\varepsilon^{5/2} q}.\end{aligned}$$

Let $\mathcal{E}_0$ be the event that

$$|(H \setminus G)[Q]| \leq 3\nu\binom{q}{k}. \tag{7.41}$$

Since $|H \setminus G| \leq 2\nu\binom{n}{k}$, we may apply Lemma 7.4 with $n, H \setminus G, Q, \nu/2$ playing the roles of n, H, Q, ν to obtain

$$\mathbb{P}[\mathcal{E}_0] \geq 1 - e^{-\nu^3 q}.$$

As $|\hat{A}(k,k-1,\mathbf{a})| \le t^{2^k}$ by Proposition 2.7(viii), we conclude

$$\mathbb{P}\left[\mathcal{E}_0 \wedge \bigwedge_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} \hat{\mathcal{E}}(\hat{\mathbf{x}})\right] \ge 1 - e^{-\nu^3 q} - 2t^{2^k} e^{-\varepsilon^{5/2} q} \ge 1 - e^{-\varepsilon^{8/3} q}. \tag{7.42}$$

Now suppose that $\mathcal{E}(\mathbf{S})$ holds for some $\mathbf{S} \in I$ and that $\mathcal{E}_0 \wedge \bigwedge_{\hat{\mathbf{x}} \in \hat{A}(k,k-1,\mathbf{a})} \hat{\mathcal{E}}(\hat{\mathbf{x}})$ holds. Then $\mathscr{P}$ induces a family of partitions $\mathscr{P}[Q]$ on Q which is $(1/a_1, \varepsilon'/2, \mathbf{a}, \varepsilon)$-equitable. Note $\varepsilon' \ll 1/t, 1/k$, thus $(\varepsilon'/6, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance. Since $\nu \ll \varepsilon' \ll 1/t$, by using (7.39), we can apply Lemma 5.11 with the following objects and parameters.

object/parameter	$\mathscr{P}[Q]$	$\mathscr{P}[Q]$	3ν	$\varepsilon'/2$	$d_{\mathbf{a},k}$	$G[Q]$	$H[Q]$	ε
playing the role of	$\mathscr{P}$	$\mathscr{Q}$	ν	ε	$d_{\mathbf{a},k}$	$H^{(k)}$	$G^{(k)}$	λ

This implies that $\mathscr{P}[Q]$ is an $(1/a_1, \varepsilon'/2 + \nu^{1/7}, \mathbf{a}, \nu^{1/7})$-equitable family of partitions on Q which is also an $(\varepsilon'/2 + \nu^{1/7}, d_{\mathbf{a},k})$-partition of $H[Q]$.

Finally, since $\nu \ll \varepsilon'$, Lemma 5.12 implies that there exists a family of partitions $\mathscr{Q}$ which is an $(\varepsilon', \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $H[Q]$. By (7.38) and (7.42), this completes the proof. □

Next we proceed with the proof of Theorem 3.1. To prove (Q1)$_{3.1}$, we first apply the regular approximation lemma (Theorem 2.5) to obtain an ε-equitable partition $\mathscr{P}_1$ of a k-graph G that is very close to H. Lemma 7.3 implies that (with high probability) $G[Q]$ has a regularity partition $\mathscr{P}_2$ which has the same parameters as $\mathscr{P}_1$, except for a much worse regularity parameter ε'. However, we still have $\varepsilon' \ll \varepsilon_0$ and thus we can now apply Lemma 7.1 to G, $G[Q]$ and $\mathscr{P}_1, \mathscr{P}_2, \mathscr{O}_1$ to obtain an equitable partition $\mathscr{O}_2$ of $G[Q]$ which reflects $\mathscr{O}_1$. By Lemma 5.11, $\mathscr{O}_2$ is also an equitable partition of $H[Q]$. To prove (Q2)$_{3.1}$, we again apply Lemma 7.1 but with the roles of G and $G[Q]$ interchanged.

Proof of Theorem 3.1. Choose new constants η, ν so that $c \ll \eta \ll \nu \ll \delta$. Let $\overline{\varepsilon} : \mathbb{N}^{k-1} \to (0,1]$ be a function such that for all $\mathbf{b} \in \mathbb{N}^{k-1}$, we have

$$\overline{\varepsilon}(\mathbf{b}) \ll \|\mathbf{b}\|_\infty^{-k}.$$

Let $t_0 := t_{2.5}(\eta, \nu, \overline{\varepsilon})$.

By Theorem 2.5, there exists a t_0-bounded $(\eta, \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), \mathbf{a}^{\mathscr{P}})$-equitable family of partitions $\mathscr{P}_1 = \mathscr{P}_1(k-1, \mathbf{a}^{\mathscr{P}})$, a k-graph G and a density function $d_{\mathbf{a}^{\mathscr{P}},k}$ such that the following hold.

(G1)$_{3.1}$ $\mathscr{P}_1$ is an $(\overline{\varepsilon}(\mathbf{a}^{\mathscr{P}}), d_{\mathbf{a}^{\mathscr{P}},k})$-partition of G, and

(G2)$_{3.1}$ $|G \triangle H| \le \nu\binom{n}{k}$.

(Here (G1)$_{3.1}$ follows from Theorem 2.5(ii), (2.16), and Lemma 5.5.)

Let $\varepsilon := \overline{\varepsilon}(\mathbf{a}^{\mathscr{P}})$ and $T := \|\mathbf{a}^{\mathscr{P}}\|_\infty$. As t_0 only depends on $\eta, \nu, \overline{\varepsilon}$, we may assume that $c \ll \varepsilon$. Together with the choice of $\overline{\varepsilon}$ and the fact that $1/T \le 1/a_1^{\mathscr{P}} \le \eta$, this implies

$$0 < 1/n < 1/q \ll c \ll \varepsilon \ll 1/T, 1/a_1^{\mathscr{P}} \ll \nu \ll \delta \ll \varepsilon_0 \le 1.$$

Additionally, we choose ε' so that

$$0 < 1/n < 1/q \ll c \ll \varepsilon \ll \varepsilon' \ll 1/T, 1/a_1^{\mathscr{P}} \ll \nu \ll \delta \ll \varepsilon_0 \le 1. \tag{7.43}$$

Let $\mathcal{E}_0$ be the event that

$$|G[Q] \triangle H[Q]| \le 2\nu\binom{q}{k}.$$

Property (G2)$_{3.1}$ and Lemma 7.4 imply that

$$\mathbb{P}[\mathcal{E}_0] \ge 1 - e^{-\nu^3 q}. \tag{7.44}$$

Let $\mathcal{E}_1$ be the event that there exists a family of partitions $\mathscr{P}_2 = \mathscr{P}_2(k-1, \mathbf{a}^{\mathscr{P}})$ which is an $(\varepsilon', \mathbf{a}^{\mathscr{P}}, d_{\mathbf{a}^{\mathscr{P}},k})$-equitable partition of $G[Q]$. Since $\varepsilon \ll \varepsilon'$, Lemma 7.3 implies that

$$\mathbb{P}[\mathcal{E}_1] \ge 1 - e^{-\varepsilon^3 q}. \tag{7.45}$$

Thus (7.44) and (7.45) imply that

$$\mathbb{P}[\mathcal{E}_0 \wedge \mathcal{E}_1] \geq 1 - 2e^{-\varepsilon^3 q} \geq 1 - e^{-cq}. \tag{7.46}$$

Hence it suffices to show that the two statements $(\text{Q1})_{3.1}$ and $(\text{Q2})_{3.1}$ both hold if we condition on $\mathcal{E}_0 \wedge \mathcal{E}_1$.

First, assume $\mathcal{E}_0 \wedge \mathcal{E}_1$ holds and $\mathscr{O}_1$ exists as in $(\text{Q1})_{3.1}$. As $\nu \ll \delta \ll \varepsilon_0$, we can apply Lemma 5.11 with $\mathscr{O}_1$, $\mathscr{O}_1$, ν, ε_0, $d_{\mathbf{a},k}$, H and G playing the roles of $\mathscr{P}$, $\mathscr{Q}$, ν, ε, $d_{\mathbf{a},k}$, $H^{(k)}$ and $G^{(k)}$, respectively, to conclude that $\mathscr{O}_1$ is also an $(\varepsilon_0 + \delta/3, d_{\mathbf{a},k})$-partition of G.

Note that $(\varepsilon_0 + \delta/3)/2 \leq 2\varepsilon_0/3$, thus $((\varepsilon_0 + \delta/3)/2, \mathbf{a}, d_{\mathbf{a},k})$ is a regularity instance. By this and (7.43), we can apply Lemma 7.1 with the following objects and parameters.

object/parameter	n	q	$\mathscr{O}_1$	$\mathscr{P}_1$	$\mathscr{P}_2$	G	$G[Q]$	ε'	T	$\delta/3$	$\varepsilon_0 + \delta/3$	$d_{\mathbf{a}^{\mathscr{P}},k}$	$d_{\mathbf{a},k}$
playing the role of	n	m	$\mathscr{O}_1$	$\mathscr{Q}_1$	$\mathscr{Q}_2$	$H_1^{(k)}$	$H_2^{(k)}$	ε	T	δ	ε_0	$d_{\mathbf{a}^{\mathscr{Q}},k}$	$d_{\mathbf{a}^{\mathscr{O}},k}$

Hence there exists an $(\varepsilon_0 + 2\delta/3, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition $\mathscr{O}_2$ of $G[Q]$. Since $\mathcal{E}_0$ holds and $\nu \ll \delta \ll \varepsilon_0$, we can apply Lemma 5.11 with $\mathscr{O}_2$, $\mathscr{O}_2$, 2ν, $\varepsilon_0 + 2\delta/3$, $d_{\mathbf{a},k}$, $H[Q]$ and $G[Q]$ playing the roles of $\mathscr{P}$, $\mathscr{Q}$, ν, ε, $d_{\mathbf{a},k}$, $G^{(k)}$ and $H^{(k)}$, respectively. Then we conclude that $\mathscr{O}_2$ is an $(\varepsilon_0 + \delta, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $H[Q]$. Thus $\mathcal{E}_0 \wedge \mathcal{E}_1$ implies $(\text{Q1})_{3.1}$.

Now assume $\mathcal{E}_0 \wedge \mathcal{E}_1$ holds and $\mathscr{O}_2$ exists as in $(\text{Q2})_{3.1}$. As $\nu \ll \delta \ll \varepsilon_0$, we can apply Lemma 5.11 with $\mathscr{O}_2$, $\mathscr{O}_2$, 2ν, ε_0, $d_{\mathbf{a},k}$, $H[Q]$ and $G[Q]$ playing the roles of $\mathscr{P}$, $\mathscr{Q}$, ν, ε, $d_{\mathbf{a},k}$, $H^{(k)}$ and $G^{(k)}$, respectively. Thus $\mathscr{O}_2$ is an $(\varepsilon_0 + \delta/3, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of $G[Q]$. By (7.43) and the fact that R is a regularity instance, we can apply Lemma 7.1 with the following objects and parameters.

object/parameter	q	n	$\mathscr{O}_2$	$\mathscr{P}_2$	$\mathscr{P}_1$	$G[Q]$	G	ε'	T	$\delta/3$	$\varepsilon_0 + \delta/3$	$d_{\mathbf{a}^{\mathscr{P}},k}$	$d_{\mathbf{a},k}$
playing the role of	n	m	$\mathscr{O}_1$	$\mathscr{Q}_1$	$\mathscr{Q}_2$	$H_1^{(k)}$	$H_2^{(k)}$	ε	T	δ	ε_0	$d_{\mathbf{a}^{\mathscr{Q}},k}$	$d_{\mathbf{a}^{\mathscr{O}},k}$

Thus there exists a family of partitions $\mathscr{O}_1$ which is an $(\varepsilon_0+2\delta/3, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of G. By $(\text{G2})_{3.1}$ and the fact that $\nu \ll \delta \ll \varepsilon_0$, we can apply Lemma 5.11 with $\mathscr{O}_1$, $\mathscr{O}_1$, ν, $\varepsilon_0 + 2\delta/3$, $d_{\mathbf{a},k}$, H and G playing the roles of $\mathscr{P}$, $\mathscr{Q}$, ν, ε, $d_{\mathbf{a},k}$, $G^{(k)}$ and $H^{(k)}$, respectively. We conclude that $\mathscr{O}_1$ is an $(\varepsilon_0 + \delta, \mathbf{a}, d_{\mathbf{a},k})$-equitable partition of H. Thus $\mathcal{E}_0 \wedge \mathcal{E}_1$ implies $(\text{Q2})_{7.3}$. □

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