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Morita equivalence classes of tame blocks of finite groups

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Morita equivalence classes of tame blocks of finite groups ☆

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ABSTRACT

We show that several Morita equivalence classes of tame algebras do not occur as blocks of finite groups. This refines classifications by Erdmann of classes of blocks with dihedral, semidihedral, and generalised quaternion defect groups. In particular we now have a complete classification of the Morita equivalence classes of blocks of finite groups with dihedral defect groups.

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1. Introduction

A finite-dimensional algebra over an algebraically closed field k has representation type finite, tame, or wild, depending on its indecomposable modules. It is called tame if it has infinitely many but, for each $n \in \mathbb{N}$, all but finitely many of the indecomposable modules of dimension n fall into finitely many one-parameter families. A block of a finite group has tame representation type if and only if its defect groups are dihedral, semidihedral, or generalised quaternion [3]. Two algebras are called Morita equivalent if their module categories are equivalent as k-linear categories (see [50]).

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We show that several classes of tame algebras do not occur as blocks of finite groups, improving existing classifications of tame blocks up to Morita equivalence by Erdmann [26]:

Theorem 1.1. *There are no blocks of finite groups with the following defect groups of order 2^n in the following Morita equivalence classes (according to the labellings in [26]):*

- *Dihedral: $D(3\mathcal{B})_1$ for $n \geq 4$;*
- *Generalised quaternion: $Q(3\mathcal{B})$ for $n \geq 5$;*
- *Semidihedral: $SD(2\mathcal{B})_3$ and $SD(3\mathcal{H})$ for any n; and $SD(2\mathcal{B})_1$, $SD(3\mathcal{B})_2$, and $SD(3\mathcal{C})_{2,1}$ for $n \geq 5$.*

In particular, the Morita equivalence classes that occur as blocks of finite groups with dihedral defect groups are known.

In a series of papers, the details of which are collected in [26], Erdmann gave a list of families of basic algebras – of so-called dihedral, semidihedral, and quaternion type – explicitly defined by quivers and relations such that any tame block of a finite group is Morita equivalent to one of these algebras. These algebras are labelled as in Theorem 1.1, for example $D(3\mathcal{B})_1$ denotes a particular algebra of dihedral type with three simple modules, and subscripts denote algebras with the same quiver that are defined by different relations; we instead use each label to denote the corresponding algebra's Morita equivalence class. Using necessary properties of blocks she deduced which of these algebras could possibly be Morita equivalent to blocks, however she was not able to say in each case whether such a block definitely exists. For each possible class she also determined the decomposition matrix that a block in that class must have, which is the information we use to identify the Morita equivalence classes of our blocks.

Holm later (almost completely) classified these families of algebras up to derived equivalence ([32] and [33]), finding many algebras that are not Morita equivalent but are derived equivalent. In doing so he was additionally able to show that all of the algebras in these families do indeed have tame representation type, something that was previously not known in all cases.

We improve Erdmann's classifications of tame blocks up to Morita equivalence by using the Classification of Finite Simple Groups to show that several classes do not occur as blocks of any quasi-simple groups, and hence, via a reduction to quasi-simple groups, do not occur as blocks of any finite groups.

Donovan's conjecture states that for any ℓ-group D there are, up to Morita equivalence, only finitely many blocks of finite groups with D as a defect group; this is known to be true for many families of groups. Erdmann's classifications already proved this for dihedral and semidihedral groups, however the Morita equivalence classes were not fully and precisely classified (other than for D_8), as we now have for all dihedral groups. This is perhaps the first 'interesting' family of non-abelian groups for which this has been done; interesting in that there are multiple classes and they include blocks of finite simple groups.

We give updated classifications of the remaining classes of blocks that may occur in Section 2, together with their decomposition matrices and those of the eliminated classes. Then, after recalling various background results in Section 3, we prove Theorem 1.1 by first proving reductions to quasi-simple groups in Section 4 (in fact we see it is sufficient to consider only odd central extensions of simple groups), then in Section 5 we work through the blocks of all these groups case by case, describing the blocks with dihedral or semidihedral defect groups and showing that the Morita equivalence classes in question do not appear and thus do not occur as blocks of any finite group. This is done primarily by deducing the decomposition matrices from the ordinary character degrees, using descriptions of blocks from various papers. The generalised quaternion class is eliminated as a corollary of the dihedral case. For the semidihedral class $SD(2\mathcal{B})_1$, however, quasi-simple groups are not quite sufficient, and this case is considered finally in Section 6.

1.1. Notation

Throughout, unless otherwise stated, we work over an algebraically closed field k of characteristic $\ell > 0$; usually we will have $\ell = 2$. Algebraic groups and finite groups of Lie type will be defined over an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$.

The number of irreducible ordinary and Brauer characters in a block B are denoted $k(B)$ and $l(B)$ respectively. For an integer $m \geq 1$ and a prime p, the largest power of p dividing m is denoted by $|m|_p$, and we write $|m|_{p'} = m/|m|_p$.

2. The classifications

We refine Erdmann's classifications of blocks with dihedral, semidihedral, and generalised quaternion defect groups, each of which can be found together in her book [26]; additionally see [16, Section 6.2] for a succinct description of the previously most up-to-date versions and example blocks. We obtain the following now complete classification of Morita equivalence classes of blocks with dihedral defect groups:

Theorem 2.1. *If B is a block of a finite group with dihedral defect groups of order 2^n for $n \geq 3$, then exactly one of the following holds:*

- $D(1)$: *B is Morita equivalent to kD_{2^n};*
- $D(2\mathcal{A})$: *B is Morita equivalent to the principal block of $\mathrm{PGL}_2(q)$, for $q \equiv 1 \bmod 4$ where $|q-1|_2 = 2^{n-1}$;*
- $D(2\mathcal{B})$: *B is Morita equivalent to the principal block of $\mathrm{PGL}_2(q)$, for $q \equiv -1 \bmod 4$ where $|q+1|_2 = 2^{n-1}$;*

$D(3\mathcal{A})_1$: *B is Morita equivalent to the principal block of* $\mathrm{PSL}_2(q)$*, for* $q \equiv 1 \bmod 4$ *where* $|q-1|_2 = 2^n$*;*

$D(3\mathcal{K})$: *B is Morita equivalent to the principal block of* $\mathrm{PSL}_2(q)$*, for* $q \equiv -1 \bmod 4$ *where* $|q+1|_2 = 2^n$*;*

$D(3\mathcal{B})_1$: $n = 3$ *and B is Morita equivalent to the principal block of* $\mathrm{Alt}(7)$*.*

The decomposition matrix of B is then accordingly one of the following, in order; in each case the last row is repeated $2^{n-2} - 1$ *times:*

$$\begin{pmatrix}1\\1\\1\\1\\2\end{pmatrix},\quad \begin{pmatrix}1&.\\1&.\\1&1\\1&1\\2&1\end{pmatrix},\quad \begin{pmatrix}1&.\\1&.\\1&1\\1&1\\.&1\end{pmatrix},\quad \begin{pmatrix}1&.&.\\1&1&.\\1&.&1\\1&1&1\\2&1&1\end{pmatrix},\quad \begin{pmatrix}1&.&.\\.&1&.\\.&.&1\\1&1&1\\.&1&1\end{pmatrix},\quad \begin{pmatrix}1&.&.\\1&1&.\\1&.&1\\1&1&1\\.&1&.\end{pmatrix}.$$

Note that '.' denote zero entries. The class we eliminate is $D(3\mathcal{B})_1$ for $n \geq 4$. In Erdmann's original classification there was also a parameter resulting in two possible classes for each decomposition matrix with two simple modules, but these additional classes were eliminated by Eisele in [20]. Now there is an example of a block occurring in each possible class and for each possible n, so we have a complete classification.

Throughout when we say dihedral we usually refer to groups of order at least 8, though blocks with Klein four defect groups are also tame. In that case only classes $D(1)$, $D(3\mathcal{A})_1$, and $D(3\mathcal{K})$ in Theorem 2.1 occur, as the principal blocks of V_4, $\mathrm{Alt}(5) \cong \mathrm{PSL}_2(5)$, and $\mathrm{Alt}(4) \cong \mathrm{PSL}_2(3)$ respectively [26]; note that since $n = 2$ the decomposition matrices have only four rows.

We rule out several possible Morita equivalence classes of blocks with semidihedral defect groups, though in this case we have a less complete result.

Theorem 2.2. *If B is a block of a finite group with semidihedral defect groups of order* 2^n *for* $n \geq 4$ *and* $l(B) = 3$*, then exactly one of the following holds:*

$SD(3\mathcal{D})$: *B is Morita equivalent to the principal block of* $\mathrm{PSL}_3(q)$*, for* $q \equiv -1 \bmod 4$ *where* $|q+1|_2 = 2^{n-2}$*;*

$SD(3\mathcal{B})_1$: *B has the same decomposition matrix as in* $SD(3\mathcal{D})$*, but with a different Ext-quiver; no example of such a block is known;*

$SD(3\mathcal{A})_1$: *B is Morita equivalent to the principal block of* $\mathrm{PSU}_3(q)$*, for* $q \equiv 1 \bmod 4$ *where* $|q-1|_2 = 2^{n-2}$*;*

$SD(3\mathcal{C})_{2,2}$: *no example of a block in the Morita equivalence class of B is known;*

$(*)$: $n = 4$ *and B is Morita equivalent to a certain non-principal block of the Monster group.*

The block $(*)$ *of the Monster is either in* $SD(3\mathcal{B})_2$ *or* $SD(3\mathcal{C})_{2,1}$*, and thus for* $n = 4$ *there are blocks in one of these classes but not both, and there are no blocks in either* *class for $n \geq 5$. The decomposition matrix of B is then one of the following, with the last row repeated $2^{n-2}-1$ times; from left to right: $SD(3\mathcal{D})$ or $SD(3\mathcal{B})_1$, $SD(3\mathcal{A})_1$, $SD(3\mathcal{C})_{2,2}$, $SD(3\mathcal{B})_2$, $SD(3\mathcal{C})_{2,1}$:*

$$\begin{pmatrix} 1 & . & . \\ 1 & 1 & . \\ 1 & . & 1 \\ 1 & 1 & 1 \\ . & . & 1 \\ . & 1 & . \end{pmatrix}, \quad \begin{pmatrix} 1 & . & . \\ 1 & 1 & . \\ 1 & . & 1 \\ 1 & 1 & 1 \\ . & . & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ 1 & . & 1 \\ . & 1 & 1 \\ . & . & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & . & . \\ 1 & 1 & . \\ 1 & . & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ . & 1 & . \end{pmatrix}, \quad \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ 1 & . & 1 \\ . & 1 & 1 \\ 1 & 1 & 1 \\ . & . & 1 \end{pmatrix}.$$

Note that [26] incorrectly lists the principal block of $\mathrm{PSL}_3(q)$ in $SD(3\mathcal{B})_1$ instead of $SD(3\mathcal{D})$. The classes we eliminate are $SD(3\mathcal{B})_2$ and $SD(3\mathcal{C})_{2,1}$ for $n \geq 5$, and $SD(3\mathcal{H})$ for all n which has the following decomposition matrix with the last row repeated $2^{n-2}-1$ times:

$$\begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \\ 1 & 1 & 1 \\ . & 1 & 1 \\ 1 & 1 & . \end{pmatrix}.$$

Here, unlike the dihedral case, the Morita equivalence class cannot always be identified based only on the degrees of the ordinary characters in the block. The same list of ordinary degrees could give rise to the decomposition matrices of both $SD(3\mathcal{A})_1$ and $SD(3\mathcal{C})_{2,2}$, and similarly for $SD(3\mathcal{B})_2$ and $SD(3\mathcal{C})_{2,1}$, while $SD(3\mathcal{D})$ and $SD(3\mathcal{B})_1$ even have identical decomposition matrices.

Theorem 2.3. *If B is a block of a finite group with semidihedral defect groups of order 2^n for $n \geq 4$ and $l(B) \leq 2$, then exactly one of the following holds:*

- $SD(1)$: *B is Morita equivalent to kSD_{2^n};*
- $SD(2\mathcal{A})_1$: *B is Morita equivalent to the principal block of $\mathrm{GU}_2(q)$, for $q \equiv 1 \bmod 4$ where $|q-1|_2 = 2^{n-2}$;*
- $SD(2\mathcal{B})_2$: *B is Morita equivalent to the principal block of $\mathrm{GL}_2(q)$, for $q \equiv -1 \bmod 4$ where $|q+1|_2 = 2^{n-2}$;*
- $SD(2\mathcal{A})_2$: *B is Morita equivalent to the principal block of $\mathrm{PSL}_2(q^2).2$, for q odd where $|q^2-1|_2 = 2^{n-1}$;*
- $SD(2\mathcal{B})_1$: *$n = 4$ and B is Morita equivalent to a non-principal block of $3 \cdot M_{10}$;*
- $(*)$: *B is Morita equivalent to an algebra with the same decomposition matrix and Ext-quiver as one of the four previous two-module classes, but with a different parameter in the relations for the path algebra; no example of such a block is known.*

The decomposition matrix of B is then accordingly one of the following, with the last row repeated $2^{n-2}-1$ times:

$$\begin{pmatrix}1\\1\\1\\1\\2\end{pmatrix}, \quad \begin{pmatrix}1&.\\1&.\\1&1\\1&1\\.&1\\2&1\end{pmatrix}, \quad \begin{pmatrix}1&.\\1&.\\1&1\\1&1\\2&1\\.&1\end{pmatrix}, \quad \begin{pmatrix}1&.\\1&.\\1&1\\1&1\\2&1\end{pmatrix}, \quad \begin{pmatrix}1&.\\1&.\\1&1\\1&1\\.&1\end{pmatrix}.$$

Note that [26] has the decomposition matrices of $SD(2\mathcal{B})_1$ and $SD(2\mathcal{B})_2$ the wrong way round. The group $\mathrm{PSL}_2(q^2).2$ above is the extension of $\mathrm{PSL}_2(q^2)$ by a field automorphism. For each possible quiver (hence decomposition matrix) with two simple modules we have an example of a block, but there is a parameter c that can possibly take the value 0 or 1, giving two possible Morita equivalence classes [26, Sect. VIII.4]. The classes we eliminate are (both families of) $SD(2\mathcal{B})_1$ for $n \geq 5$, and $SD(2\mathcal{B})_3$ for all n which has the following decomposition matrix with the last row repeated $2^{n-2}-1$ times:

$$\begin{pmatrix}1&.\\1&.\\.&1\\.&1\\1&1\end{pmatrix}.$$

Note that $SD(2\mathcal{B})_3$ is the same Morita equivalence class as that of $SD(2\mathcal{B})_4$ elsewhere in the literature (see [33, Prop. 4.2] for example).

Theorem 2.1 implies a similar improvement on Erdmann's classification of blocks with generalised quaternion defect groups, completely classifying those with three simple modules by eliminating the corresponding class $Q(3\mathcal{B})$ for $n \geq 5$ (this class was known not to occur for $n = 3$):

Corollary 2.4. *If B is a block of a finite group with generalised quaternion defect groups of order 2^n for $n \geq 3$ and $l(B) = 3$, then exactly one of the following holds:*

$Q(3\mathcal{A})_2$: *B is Morita equivalent to the principal block of $\mathrm{SL}_2(q)$, for $q \equiv 1 \bmod 4$ where $|q-1|_2 = 2^{n-1}$;*

$Q(3\mathcal{K})$: *B is Morita equivalent to the principal block of $\mathrm{SL}_2(q)$, for $q \equiv -1 \bmod 4$ where $|q+1|_2 = 2^{n-1}$;*

$Q(3\mathcal{B})$: *$n = 4$ and B is Morita equivalent to the principal block of $2 \cdot \mathrm{Alt}(7)$.*

The decomposition matrix of B is then accordingly one of the following; in each case the last row is repeated $2^{n-2}-1$ times:

$$\begin{pmatrix} 1 & . & . \\ 1 & 1 & . \\ 1 & . & 1 \\ 1 & 1 & 1 \\ . & 1 & . \\ . & . & 1 \\ 2 & 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \\ 1 & 1 & 1 \\ 1 & 1 & . \\ 1 & . & 1 \\ . & 1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & . & . \\ 1 & 1 & . \\ 1 & . & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \\ . & . & 1 \\ . & 1 & . \end{pmatrix}.$$

This follows from a result of Kessar and Linckelmann [39]: a block B of a finite group G with generalised quaternion defect group D and $l(B) = 3$ must be Morita equivalent to its Brauer correspondent B' in $C_G(Z(D))$. This implies that $C_G(Z(D))/Z(D)$ has a block b contained in B' with defect group $D/Z(D)$, which is dihedral. Thus if B were in $Q(3\mathcal{B})$ with defect $n \geq 5$, then b would have to be in $D(3\mathcal{B})_1$ with defect $n-1 \geq 4$, which is impossible by Theorem 2.1.

For completeness, the possible decomposition matrices of blocks with generalised quaternion defect groups and $l(B) \leq 2$ are as follows, each with the last row repeated $2^{n-2}-1$ times:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & . \\ 1 & . \\ 1 & 1 \\ 1 & 1 \\ . & 1 \\ 2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & . \\ 1 & . \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \\ . & 1 \end{pmatrix}.$$

The classes for these are labelled $Q(1)$, $Q(2\mathcal{A})$, and $Q(2\mathcal{B})_1$, and occur as kQ_{2^n} and, only for $n \geq 4$, as the principal blocks of $\mathrm{SL}_2(q).2$ for $q \equiv 1$ or $-1 \bmod 4$ respectively; note that these were originally listed as $\mathrm{SL}_2(q^2).2$ in [16] but were later corrected. There is another possible class, $Q(2\mathcal{B})_2$, listed in [26] with decomposition matrix

$$\begin{pmatrix} 1 & . \\ 1 & . \\ . & 1 \\ . & 1 \\ 1 & 1 \end{pmatrix},$$

with the last row repeated $2^{n-2}-1$ times, but this cannot occur as a block, since all blocks with generalised quaternion defect groups have $k(B) - l(B) = 2^{n-2} + 2$ by [52]; this was already known, noted in [32, (4.1)] for example. While we have an example of a block with each possible decomposition matrix, for each matrix with $l(B) = 2$ there is a parameter c which can take infinitely many possible values, giving infinitely many possible Morita equivalence classes, so we do not have a complete classification and even Donovan's conjecture is unknown.

Remark 2.5. We gather together here some discrepancies in [26] and its related papers, some of which were noted above. It has the decomposition matrices (and a given value of some integer k) of $SD(2\mathcal{B})_1$ and $SD(2\mathcal{B})_2$ the wrong way round; this can be seen from [24, Table 1]. It also lists the principal block of $\mathrm{PSL}_3(q)$ in $SD(3\mathcal{B})_1$, but it is in $SD(3\mathcal{D})$ as was originally calculated (see [22, Table II] and [25, Table 1]). This mistake was carried into [16], though it was noted that the principal block of M_{11} can be checked to be Morita equivalent to that of $\mathrm{PSL}_3(q)$; this block of M_{11} was in fact correctly given as $SD(3\mathcal{D})$ in [26], but originally as $SD(3\mathcal{B})_1$ in [25, (11.5)].

Additionally [23, Prop. 7.5.1] claimed that $D(3\mathcal{B})$ can only occur with defect groups of order 8, though the proof of this was later noted to be false in [26, (X.4)]; the same therefore applies to the proof of [24, Lemma 8.14] claiming that $SD(2\mathcal{B})_1$ can only occur with defect groups of order 16. Finally [24, Lemma 8.16] claimed that $SD(2\mathcal{B})_3$ cannot occur as a block, however the proof of this is incorrect; there are only two non-zero d_i and the necessary relations are in fact satisfied. The correct decomposition matrix is calculated in [26] and the family is reinstated as a possible block, but [32, (3.1)] uses the claim. Nonetheless we have now shown that each of these three claims is in fact true.

3. Preliminaries

3.1. Lusztig series

We assume basic knowledge of algebraic groups and finite groups of Lie type; see [49] for details. Throughout this section let $\mathbf{G}$ be a connected reductive algebraic group in characteristic $p > 0$, where $p \neq \ell$, with a Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$ so that $\mathbf{G}^F$ is a finite group of Lie type.

For $\mathbf{L}$ an F-stable Levi subgroup of $\mathbf{G}$, **Deligne-Lusztig induction** is a functor from virtual characters of $\mathbf{L}^F$ to those of $\mathbf{G}^F$ generalising Harish-Chandra induction (the details are not so important here but can be found in [18] for example). For ease of notation we will denote this functor by $R_{\mathbf{L}}^{\mathbf{G}}$, instead of $R_{\mathbf{L}^F}^{\mathbf{G}^F}$. Note that this functor is defined via a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{L}$, but is independent of the choice of $\mathbf{P}$ whenever the Mackey formula holds, which is always the case unless possibly $\mathbf{G}^F$ contains a component ${}^2E_6(2)$ or $E_8(2)$ (see [5] and [57]); we will not be concerned with $q = 2$, so will omit any mention of $\mathbf{P}$. If $(\mathbf{L}, \lambda)$ is a cuspidal pair (as in Theorem 3.10 below) then the set of constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ will be called a Harish-Chandra series.

Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$, and let $\mathbf{G}^*$ be a group dual to $\mathbf{G}$ around $\mathbf{T}$ and $\mathbf{T}^*$, with Steinberg endomorphism also denoted by F. The duality defines a correspondence between classes of F-stable Levi subgroups $\mathbf{L}$ of $\mathbf{G}$ and those $\mathbf{L}^*$ of $\mathbf{G}^*$, as well as a canonical isomorphism between $\mathrm{Irr}(\mathbf{T}^F)$ and $(\mathbf{T}^*)^F$ (see [13, Section 8.2]).

Definition 3.1. Let $\mathbf{T}, \mathbf{S}$ be F-stable maximal tori of $\mathbf{G}$, and let $\theta \in \mathrm{Irr}(\mathbf{T}^F)$ and $\psi \in \mathrm{Irr}(\mathbf{S}^F)$ correspond to $t \in (\mathbf{T}^*)^F$ and $s \in (\mathbf{S}^*)^F$ as above. Then the pairs $(\mathbf{T}, \theta)$ and $(\mathbf{S}, \psi)$ are said to be **rationally conjugate** whenever t and s are $(\mathbf{G}^*)^F$-conjugate.

Thus rational conjugacy classes of pairs $(\mathbf{T}, \theta)$ correspond to conjugacy classes of semisimple elements in $(\mathbf{G}^*)^F$.

Definition 3.2. Let $s \in (\mathbf{G}^*)^F$ be semisimple. The **rational Lusztig series** $\mathcal{E}(\mathbf{G}^F, s)$ associated to (the $(\mathbf{G}^*)^F$-conjugacy class of) s is the set of irreducible characters of $\mathbf{G}^F$ that occur as a constituent in some $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, where $(\mathbf{T}, \theta)$ is in the rational conjugacy class associated to s.

If s is a semisimple ℓ'-element of $(\mathbf{G}^*)^F$ define

$$\mathcal{E}_\ell(\mathbf{G}^F, s) = \bigcup_{t \in (C_{\mathbf{G}^*}(s)^F)_\ell} \mathcal{E}(\mathbf{G}^F, st),$$

so t runs over all ℓ-elements of $(\mathbf{G}^*)^F$ that commute with s.

These series describe the blocks of $\mathbf{G}^F$ in the following way, where here a block B is thought of as the set $\mathrm{Irr}(B)$:

Theorem 3.3 *([13, Thms 8.24 & 9.12]). The sets $\mathcal{E}(\mathbf{G}^F, s)$ for semisimple $s \in (\mathbf{G}^*)^F$ form a partition of $\mathrm{Irr}(\mathbf{G}^F)$. If $s \in (\mathbf{G}^*)^F$ is a semisimple ℓ'-element then:*

(i) $\mathcal{E}_\ell(\mathbf{G}^F, s)$ is a union of ℓ-blocks of $\mathbf{G}^F$;
(ii) each ℓ-block contained in $\mathcal{E}_\ell(\mathbf{G}^F, s)$ contains an element of $\mathcal{E}(\mathbf{G}^F, s)$.

Hence to parameterise the ℓ-blocks of $\mathbf{G}^F$ it suffices to decompose $\mathcal{E}(\mathbf{G}^F, s)$ into ℓ-blocks for each semisimple ℓ'-element $s \in (\mathbf{G}^*)^F$.

Definition 3.4. An irreducible character of $\mathbf{G}^F$ is called **unipotent** if it is a constituent of $R_{\mathbf{T}}^{\mathbf{G}}(1)$ for some F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$. Additionally if $\mathbf{G}$ is disconnected the unipotent characters of $\mathbf{G}$ are the constituents of $\left(R_{\mathbf{T}}^{\mathbf{G}^\circ}(1)\right)\uparrow^{\mathbf{G}^F}$ for F-stable maximal tori $\mathbf{T}$ of $\mathbf{G}^\circ$. A block is called unipotent if it contains a unipotent character.

Hence $\mathcal{E}(\mathbf{G}^F, 1)$ is the set of unipotent characters of $\mathbf{G}^F$. Enguehard [21] classified all unipotent blocks of finite simple groups of Lie type for bad primes, as 2 is unless $\mathbf{G}$ is of type A. The following so-called **Jordan correspondence** relates characters to unipotent characters of a usually smaller group:

Theorem 3.5 *([18, Thm 11.5.1, Prop. 11.5.6]). If $s \in (\mathbf{G}^*)^F$ is semisimple then there is a bijection between $\mathcal{E}(\mathbf{G}^F, s)$ and $\mathcal{E}\left(C_{\mathbf{G}^*}(s)^F, 1\right)$ such that if $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ corresponds to $\chi_u \in \mathrm{Irr}\left(C_{\mathbf{G}^*}(s)^F\right)$ then*

$$\chi(1) = \chi_u(1) \cdot \left|(\mathbf{G}^*)^F : C_{\mathbf{G}^*}(s)^F\right|_{p'}.$$

This was originally proved by Lusztig [48, (4.23)] when $\mathbf{G}$ has connected centre; in this case $C_{\mathbf{G}^*}(s)$ is always connected [15, Thm 4.5.9].

Remark 3.6. Since any $s \in (\mathbf{G}^*)^F_{\ell'}$ and $t \in C_{\mathbf{G}^*}(s)^F_\ell$ have coprime orders and commute, $C_{\mathbf{G}^*}(st)^F = C_{C_{\mathbf{G}^*}(s)}(t)^F \leq C_{\mathbf{G}^*}(s)^F$, so each character in $\mathcal{E}_\ell(\mathbf{G}^F, s)$ has a factor of $|(\mathbf{G}^*)^F : C_{\mathbf{G}^*}(s)^F|_{p'}$ in its degree. In particular (provided that $C_{\mathbf{G}^*}(s)^*$ is connected) there are bijections as above between $\mathcal{E}(\mathbf{G}^F, st)$ and $\mathcal{E}\left(C_{\mathbf{G}^*}(st)^F, 1\right) = \mathcal{E}\left(C_{C_{\mathbf{G}^*}(s)}(t)^F, 1\right)$ and $\mathcal{E}\left((C_{\mathbf{G}^*}(s)^*)^F, t\right)$; it follows that there is also a bijection between $\mathcal{E}_\ell(\mathbf{G}^F, s)$ and $\mathcal{E}_\ell\left((C_{\mathbf{G}^*}(s)^*)^F, 1\right)$.

Bonnafé and Rouquier gave an important reduction showing that if the centraliser is contained in a proper F-stable Levi subgroup then there is a Morita equivalence between the related blocks:

Theorem 3.7 *([6]). Let $s \in (\mathbf{G}^*)^F$ be a semisimple ℓ'-element and suppose that $C_{\mathbf{G}^*}(s)$ is contained in an F-stable Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$. The ℓ-blocks of $\mathbf{G}^F$ in $\mathcal{E}(\mathbf{G}^F, s)$ are in bijection via Jordan correspondence with the ℓ-blocks of $\mathbf{L}^F$ in $\mathcal{E}(\mathbf{L}^F, s)$, and the corresponding blocks are Morita equivalent.*

This Morita equivalence preserves decomposition matrices and, as shown further with Dat in [7], also defect groups. The result motivates the following definition:

Definition 3.8. A semisimple element $s \in \mathbf{G}$ is **quasi-isolated** if its centraliser $C_{\mathbf{G}}(s)$ is not contained in any proper Levi subgroup of $\mathbf{G}$; further, it is **isolated** if the identity component $C_{\mathbf{G}}(s)^\circ$ is not contained in a proper Levi subgroup. A block is then (quasi)-isolated if its semisimple label is (quasi)-isolated.

Unipotent blocks are isolated, since $C_{\mathbf{G}^*}(1) = \mathbf{G}^*$, and if $\mathbf{G}$ has connected centre then $C_{\mathbf{G}^*}(s)$ is always connected so the terms isolated and quasi-isolated coincide. Note that if $s \in \mathbf{G}^F$ and $C_{\mathbf{G}}(s)$ is contained in a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$, then $C_{\mathbf{G}}(s)$ is F-stable and contained in each element in the F-orbit of $\mathbf{L}$, so is also contained in the intersection of this orbit, which is an *F-stable* Levi subgroup of $\mathbf{G}$.

Additionally, the above equivalence is preserved taking quotients by central ℓ-subgroups:

Theorem 3.9 *([19, Prop. 4.1]). If B and C are ℓ-blocks of $\mathbf{G}^F$ and $\mathbf{L}^F$ in correspondence by Theorem 3.7, and Z is a central ℓ-subgroup of $\mathbf{G}^F$, then the ℓ-blocks $\overline{B}$ and $\overline{C}$ of $\mathbf{G}^F/Z$ and $\mathbf{L}^F/Z$ contained in B and C respectively are Morita equivalent.*

In light of the above, if a block B of $\mathbf{G}^F$ is (quasi)-isolated, then we will also say that the block $\overline{B}$ of $\mathbf{G}^F/Z$ contained in B is (quasi)-isolated. Quasi-isolated blocks of finite groups of Lie type have been well studied; in particular, Kessar and Malle describe the quasi-isolated ℓ-blocks of exceptional groups of Lie type when ℓ is a bad prime (see [49, Table 14.1]) for $\mathbf{G}$, obtaining the following description:

Theorem 3.10 *([40, Thm 1.2]). Let $\mathbf{G}$ be a simply connected simple exceptional algebraic group, $\ell \neq p$ be a bad prime for $\mathbf{G}$, and $1 \neq s \in (\mathbf{G}^*)^F$ be a quasi-isolated ℓ'-element.*

(i) There is a bijection between ℓ-blocks of $\mathbf{G}^F$ in $\mathcal{E}_\ell(\mathbf{G}^F, s)$ and pairs $(\mathbf{L}, \lambda)$ where $\mathbf{L}$ is an e-split Levi subgroup of $\mathbf{G}$ and $\lambda \in \mathcal{E}(\mathbf{L}^F, s)$ is e-cuspidal of quasi-central ℓ-defect.
(ii) If a block corresponds to $(\mathbf{L}, \lambda)$ then it has a defect group D such that $Z(\mathbf{L})_\ell^F = Z \trianglelefteq P \trianglelefteq D$, where D/P is isomorphic to a Sylow ℓ-subgroup of the Weyl group $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ and P/Z is isomorphic to a Sylow ℓ-subgroup of $\mathbf{L}^F/Z[\mathbf{L}, \mathbf{L}]^F$.

See [40] for definitions of the terminology used in (i); we will not use them further. For each exceptional group they list all $(\mathbf{L}, \lambda)$ that occur, and give further information.

Ennola duality, formally swapping q with $-q$ (see [12, (3A)]), is used in [40] and will be used later to translate arguments to different values of q mod 4 (which correspond to different values of e in the above theorem); all groups in question are exchanged with their Ennola duals, so for example $E_6(q)$ becomes ${}^2E_6(q)$ and $q-1$ becomes $q+1$, while $G_2(q)$ is its own Ennola dual.

3.2. Normal subgroups

Let N be a normal subgroup of a finite group G. We collect some facts relating the characters and blocks of N to those of G that will be used throughout.

Of particular note, if $|G : N| = 2$ then, from Clifford's Theorem [37, Thm 20.8], for any $\varphi \in \mathrm{Irr}(N)$: either $\varphi\uparrow^G = \chi_1 + \chi_2$ and $\chi_1\downarrow_N = \chi_2\downarrow_N = \varphi$ for some distinct $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ of equal degree; or $\varphi\uparrow^G = \psi\uparrow^G = \chi$ and $\chi\downarrow_N = \varphi + \psi$ for some $\chi \in \mathrm{Irr}(G)$ and $\varphi \neq \psi \in \mathrm{Irr}(N)$ with φ, ψ of equal degree. We will say these characters either split or fuse on induction or restriction accordingly.

Lemma 3.11 *([45, Thm 2.4.7]). Let $\phi \in \mathrm{Irr}(N)$. Then $\phi\uparrow^G \in \mathrm{Irr}(G)$ if and only if $\phi^g \neq \phi$ for all $g \in G \setminus N$.*

Lemma 3.12 *([51, Thm 9.4]). Let B be a block of G covering a block b of N and $\varphi \in \mathrm{Irr}(b) \cup \mathrm{IBr}(b)$. Then φ is a constituent of $\chi\downarrow_N$ for some $\chi \in \mathrm{Irr}(B) \cup \mathrm{IBr}(B)$.*

Lemma 3.13 *([51, Thm 8.11, Cor. 9.6, Thm 9.17]). Let $|G : N|$ be a power of ℓ.*

(i) If $\varphi \in \mathrm{IBr}(N)$ then there is a unique $\chi \in \mathrm{IBr}(G)$ covering φ and $\chi\downarrow_N$ is the sum of the distinct G-conjugates of φ.
(ii) If b is an ℓ-block of N then there is a unique ℓ-block B of G covering b.
(iii) Furthermore, if b is G-invariant with defect group D, then the defect groups of B have order $|D| \cdot |G : N|$.

Now let $\overline{G} = G/N$. The characters of $\overline{G}$ inflate to characters of G, so $\mathrm{Irr}(\overline{G})$ and $\mathrm{IBr}(\overline{G})$ are viewed as subsets of $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$, and similarly for their blocks; each block of $\overline{G}$ is contained in a unique block of G.

Lemma 3.14 *([51, Thm 9.9]).*

(i) If N is an ℓ-group, then each ℓ-block B of G contains an ℓ-block of $\overline{G}$ whose defect groups are D/N for defect groups D of B.
(ii) If N is an ℓ'-group and $\overline{B}$ is an ℓ-block of $\overline{G}$ contained in an ℓ-block B of G, then $\mathrm{Irr}(\overline{B}) = \mathrm{Irr}(B)$ and $\mathrm{IBr}(\overline{B}) = \mathrm{IBr}(B)$.

Lemma 3.15 *([41]). Let B be a block of G covering a block b of N. If P is a defect group of b then there is a defect group D of B such that $P = D \cap N$. Conversely, if D is a defect group of B then $D \cap N$ is a defect group of a block b^g for some $g \in G$.*

Lemma 3.16 *(Fong's first reduction [16, Thm 7.4.2]). Let b be a block of N and define the inertia subgroup of b in G as $T = \{g \in G \mid b^g = b\}$. There is a bijection between blocks of G covering b and blocks of T covering b, with corresponding blocks being Morita equivalent and having isomorphic defect groups.*

Note that $N \leq T$ and, since covered blocks are conjugate, b is the unique block of N covered by these blocks of T.

Lemma 3.17 *(Fong's second reduction [16, Thm 7.4.4]). Let B be a block of G with a defect group D, such that $D \cap N = 1$. Then there is a block $\hat{B}$ of some central extension of G/N such that B and $\hat{B}$ are Morita equivalent.*

3.3. Tame blocks

As described in [52], for any block B with dihedral, semidihedral, or generalised quaternion defect groups of order 2^n the numbers of irreducible ordinary and Brauer characters are $2^{n-2} + 3 \leq k(B) \leq 2^{n-2} + 5$ and $1 \leq l(B) \leq 3$ respectively. There are four ordinary characters of height zero (the first four rows in each of the decomposition matrices in Section 2) and $2^{n-2} - 1$ height one characters (the repeated row), and any additional characters have height $n-2$; blocks with dihedral defect groups have none of these 'large height' characters, semidihedral blocks have at most one, and generalised quaternion blocks have at most two. Semidihedral defect groups are of order at least 2^4, so the 'large height' is in particular greater than one.

If B has dihedral defect groups then its decomposition matrix, hence Morita equivalence class, can be determined – out of those in Theorem 2.1 and including $D(3\mathcal{B})_1$ as a possibility for any n – using only its ordinary character degrees, such as by checking the following properties: if the degree of the height one character is the largest, then

B is in one of $D(1)$, $D(2\mathcal{A})$, or $D(3\mathcal{A})_1$, depending on whether it has one, two, or at least three distinct height zero degrees respectively; otherwise, if the largest degree is repeated then B is in $D(2\mathcal{B})$; if the largest degree is the sum of the other three height zero degrees then B is in $D(3\mathcal{K})$; otherwise B is in $D(3\mathcal{B})_1$. Therefore given a block of a finite group G with dihedral defect groups, provided that $|G|$ is sufficiently small, its ordinary character degrees can be obtained using GAP [29] or Magma [8] and its Morita equivalence class can always be deduced. This is sometimes, but not always, possible for a block with semidihedral defect groups.

4. Reduction to quasi-simple groups

A quasi-simple group is a perfect central extension of a simple group, i.e., a group G such that $G = [G, G]$ and $G/Z(G)$ is simple. For every non-abelian finite simple group S, the largest quasi-simple group G such that $G/Z(G) \cong S$ is called the Schur cover of S and is unique up to isomorphism. If G is a finite group of Lie type then in all but finitely many cases its Schur cover is the corresponding finite group of simply connected type (see Section 5.5).

In order to reduce our problem from all groups to quasi-simple groups, we first show a Morita equivalence for odd-index normal subgroups. Note that this is not true for Klein four defect groups, as we require at least one height one character.

Lemma 4.1. *Let $N \trianglelefteq G$ be finite groups with $|G : N|$ odd, and let b be a block of N with dihedral or semidihedral defect groups (of order at least 8). Then each block of G covering b is Morita equivalent to b.*

Proof. Let B be a block of G covering b; note that these blocks have the same defect groups. By Lemma 3.16 we may assume without loss of generality that b is G-invariant. We can also assume by the Feit-Thompson theorem [28] that $|G : N| = p$, an odd prime.

For each $\chi \in \mathrm{Irr}(B)$, by Clifford's Theorem either χ restricts irreducibly to N or $\chi\downarrow_N = \phi_1 + \cdots + \phi_p$ for some $\phi_1, \ldots, \phi_p \in \mathrm{Irr}(b)$ that are conjugate in G; note that each constituent of $\chi\downarrow_N$ has the same height as χ. Since the height one characters of b have the same degrees, as do those of B, if any one of them is G-invariant then they all are. By Lemma 3.11 there is a unique $\chi \in \mathrm{Irr}(G)$ covering $\phi \in \mathrm{Irr}(b)$ if and only if ϕ is not G-invariant; in this case χ covers several characters of b. But b and B have the same number of height one characters, so all those of b must be G-invariant and in bijection with those of B. Since each height one character of b has more than one character of G above it, there must be multiple blocks of G covering b. Then by Lemma 3.12 each height zero character of b also has multiple characters of G above it, so must also be G-invariant.

Therefore restriction induces a bijection between $\mathrm{Irr}(B)$ and $\mathrm{Irr}(b)$, so B and b are Morita equivalent by [54, Cor.]. $\square$

Now we give a reduction to quasi-simple groups for any of the classes with three simple modules, based on Brauer's [9] and Olsson's [52] analyses of blocks with dihedral and semidihedral defect groups respectively.

Proposition 4.2. *If B is a block of a finite group G with dihedral or semidihedral defect groups (of order at least 8) and $l(B) = 3$, then B is Morita equivalent to a block of a finite quasi-simple group.*

Proof. Let $N \trianglelefteq G$ be a maximal normal subgroup and b be a block of N covered by B. If P is a defect group of b then by Lemma 3.15 there is a defect group D of B such that $P = D \cap N$. Since N is a normal subgroup, P is strongly closed in D with respect to G (that is, if $x \in P$ and $g \in G$ such that $x^g \in D$, then $x^g \in P$).

First, if B has dihedral defect groups, let X_1, X_2 be representatives of the two conjugacy classes of Klein four-groups in D. Since $l(B) = 3$, as in [9, Section 4] case (aa), there are elements of order 3 in G that permute the non-trivial elements of X_1 and X_2 respectively. This means that the involution in $X_1 \cap X_2 = Z(D)$ is G-conjugate to elements in both non-central conjugacy classes of involutions in D; hence all involutions in D – of which there are more than $|D|/2$ – are G-conjugate. Therefore, since P is strongly closed in D with respect to G, either $P = 1$ or $P = D$.

If B has semidihedral defect groups, then there are three conjugacy classes of subgroups of order 4 in D: one of Klein four-groups, one of cyclic groups, and another single cyclic group C contained in the index-2 cyclic subgroup of D. Since $l(B) = 3$, as in case (aa) of [52], again there is an element of order 3 in G permuting the non-trivial elements of a Klein four-group. Hence if $|P| > 1$ then P contains all the involutions in D, and so contains the index-2 dihedral subgroup of D; in particular $C \leq P$. Also, by [52, Lemma 2.4] we see that C is G-conjugate to all of the cyclic groups of order 4 in D, which are therefore also in P; this is then enough to generate D.

So for both dihedral and semidihedral defect groups we have that either $P = 1$ or $P = D$. If $P = 1$ then by Lemma 3.17 there is a block of a central extension of G/N Morita equivalent to B.

If $P = D$, first we can assume by Lemma 3.16 that b is G-invariant. By Lemma 3.13, since the defects are the same, $|G : N| \neq 2$, and if $|G : N|$ is odd then B and b are Morita equivalent by Lemma 4.1.

If G/N is non-abelian simple, then define $G[b]$ as in [42] as the set of $g \in G$ such that the algebra automorphism $b \to b,\ x \mapsto g^{-1}xg$ is an inner automorphism of b; this is a normal subgroup of G containing N. As in [46, Ex. 1.2] automorphisms of b can be considered as b-b-bimodules, so (their images in $\mathrm{Out}(b)$) are elements of the Picard group of b, and those induced by group automorphisms are elements of $\mathcal{T}(b)$, the subgroup of the Picard group consisting of bimodules with trivial source. By [2, Thm 1.1] if $\mathrm{Aut}(P)$ is solvable – as is the case when P is dihedral or semidihedral – then $\mathcal{T}(b)$ is also solvable, and hence so is its subgroup $G/G[b]$. Then since G/N is non-abelian simple, $G[b] = G$. Therefore B and b are Morita equivalent by [42, Thm 7].

Repeatedly taking maximal (non-central when possible) normal subgroups in this way gives a group whose only proper normal subgroups are central – which is non-abelian and therefore quasi-simple – with a block Morita equivalent to B. □

We now give two less general reductions, each for a specific semidihedral class with two simple modules. Note that each proof essentially builds on the previous ones.

Proposition 4.3. *If B is a block of a finite group G with semidihedral defect groups in either of the $SD(2\mathcal{B})_3$ Morita equivalence classes, then B is Morita equivalent to a block of a finite quasi-simple group.*

Proof. Again let $N \trianglelefteq G$ be a maximal normal subgroup, let b be a block of N covered by B, with defect group $P = D \cap N$ for a defect group D of B, and again assume by Lemma 3.16 that b is G-invariant. Since $k(B) = 2^{n-2}+3$ and $l(B) = 2$ we are in case (ab) of [52], and there is again an element of order 3 in G permuting the non-trivial elements of a Klein four-group. So if $|P| > 1$ then P contains the index-2 dihedral subgroup of D. The cases $P = 1$ and $P = D$ are the same as in the previous proof.

Suppose that $1 < P < D$, so P is dihedral of index 2 in D. Since the blocks have different defects, $|G : N|$ cannot be odd. If $|G : N| = 2$, then as B has an odd number of height one characters, one must split on restriction to N. As it is the sum of two distinct Brauer characters this would force both of them to split; but $l(b) \leq 3$, a contradiction.

Suppose that G/N is non-abelian simple. Any odd-order element $g \in G \setminus N$ does not permute the ordinary characters of b, by Lemma 4.1 considering $N \trianglelefteq \langle g\rangle N$. Then since any non-abelian simple group G/N can be generated by odd-order elements, each $\phi \in \mathrm{Irr}(b)$ is G-invariant. The unique block B_{DN} of DN covering b has, by [38, Ch. 10 Thm 5.10] (see also [51, Ex. 9.4]), semidihedral defect group D. But $|DN : N| = 2$ and each character of b is DN-invariant, so must split on induction to DN giving eight height zero characters of B_{DN}, which is impossible. Hence P cannot have index 2 in D. □

The following case is more complicated:

Proposition 4.4. *If B is a block of a finite group G with semidihedral defect groups of order 2^n in either of the $SD(2\mathcal{B})_1$ Morita equivalence classes, then there is a quasi-simple group S with a block b_S such that either B is Morita equivalent to b_S, or b_S has dihedral defect groups in $D(3\mathcal{K})$ and is covered by a block of a group $S.2$ with semidihedral defect groups of order 2^n also in one of the $SD(2\mathcal{B})_1$ classes.*

Proof. Again let $N \trianglelefteq G$ be a maximal normal subgroup, let b be a block of N covered by B, with defect group $P = D \cap N$ for a defect group D of B, and assume by Lemma 3.16 that b is G-invariant. As in the previous proof, P is either 1, D, or dihedral of index 2 in D, and the first two cases are as in the proof of Proposition 4.2.

Suppose then that P is dihedral of index 2 in D. By the same arguments as in the previous proof, G/N cannot be non-abelian simple or of odd order, so $|G : N| = 2$ and

$G = DN$. Then one of the height one characters of B, and hence also the second Brauer character, must split on restriction to N, giving that b is in $D(3\mathcal{K})$.

Now let $H \trianglelefteq N$, and first suppose in addition that $H \trianglelefteq G$. Let b_H be a block of H covered by b with defect group $P \cap H$; as in the proof of Proposition 4.2 this is either 1 or P. If $P \cap H = 1$ then $D \cap H = 1$ or 2. But $D \cap H$ must also be 1, D, or index 2 in D, so $D \cap H = 1$, and thus B is Morita equivalent to a block of a central extension of G/H by Lemma 3.17. If $P \cap H = P$ then b_H is Morita equivalent to b, as in the proof of Proposition 4.2. Then the unique block B_{DH} of DH covering b_H has defect group D and, as with b, the second and third Brauer characters of b_H must fuse on induction to DH, so B_{DH} is also in one of the $SD(2\mathcal{B})_1$ classes.

Using the above we now assume that the only normal subgroups $H \trianglelefteq N$ that are also normal in G are N and those contained in $Z(G)$; then $N/Z(G)$ must be a direct product of simple groups. Let $H \trianglelefteq N$ be such that $H/Z(G)$ is simple, and suppose that $H \neq N$. Then N is a central product of the G-conjugates of H, which are H and H^x, where $x \in D \setminus N$.

Let b_H again be the block of H covered by b. The block $(b_H)^x$ of H^x conjugate to b_H is covered by B, and b is the unique block of N covered by B, so $(b_H)^x$ is the unique block of H^x covered by b, and b is a central product of b_H and $(b_H)^x$. Since the centre of G must have odd order, the defect groups of b are a direct product of the defect groups of b_H and $(b_H)^x$; they are dihedral, so must be a product of a dihedral and trivial group. But b_H and $(b_H)^x$ are conjugate in G, so must have isomorphic defect groups, a contradiction. Therefore no such $H \neq N$ exists and N is quasi-simple. $\square$

In Section 6 we show that there are no blocks in the $SD(2\mathcal{B})_1$ classes with defect at least 5 for groups with quasi-simple subgroups of index 2, as in the latter half of Proposition 4.4. Hence if $|D| \geq 32$ then P cannot have index 2 in D.

Other than for $SD(2\mathcal{B})_1$, it is sufficient to show that there are no blocks of quasi-simple groups in the previously mentioned Morita equivalence classes to show that there are no blocks of any finite groups in those classes. In fact, since for any of these classes the non-trivial central element of D is G-conjugate to non-central elements, the centre of G must have odd order; hence we need only consider odd covers of the finite simple groups.

We highlight several facts from the previous proofs for future reference:

Remark 4.5. Let B be a block of a finite group G with dihedral or semidihedral defect groups and three simple modules, or semidihedral defect groups in one of the $SD(2\mathcal{B})_3$ or $SD(2\mathcal{B})_1$ classes. Then:

(i) $|Z(G)|$ is odd;
(ii) G is unsolvable.

Let b be a block of a normal subgroup $N \trianglelefteq G$ covered by B, and let P and D be defect groups of b and B respectively such that $P = D \cap N$.

(iii) Unless B is in $SD(2\mathcal{B})_1$ with $|D| = 16$, either $P = 1$ or $P = D$;
(iv) If $P = D$ then b and B are Morita equivalent.

4.1. Reduction to quasi-isolated blocks

Let $\mathbf{G}$ be a simple algebraic group, with Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$. We describe in part the form the defect groups of blocks of Levi subgroups of $\mathbf{G}^F$ take:

Lemma 4.6. *Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$ with $L = \mathbf{L}^F/Z$ for some $Z \leq Z(\mathbf{L}^F)$, and let B be a block of L with defect group D. Then D has a subgroup P such that:*

- *P is strongly closed in D with respect to L (that is, if $x \in P$ and $g \in L$ such that $x^g \in D$, then $x^g \in P$);*
- *D/P is abelian;*
- *P is a central product of defect groups of blocks of the quasi-simple normal subgroups of L (or some of the finitely many solvable exceptions in [49, Thm 24.17]) covered by B.*

Proof. Take $P = D \cap [L, L]$ to be a defect group of a block B' of $[L, L]$ covered by B. Then P is strongly closed since $[L, L] \trianglelefteq L$, and D/P is abelian since $L/[L, L]$ is. Also $[L, L]$ is a central product of quasi-simple groups of Lie type (or solvable exceptions), so B' is a central product of blocks with defect groups central products of the defect groups of these blocks. $\square$

The above conditions are particularly strong for the blocks we consider. Let $\mathbf{G}$ have characteristic $p \neq 2$, and let $G = \mathbf{G}^F/Z$, for some $Z \leq Z(\mathbf{G}^F)$; we can assume that $|Z|$ is a power of 2.

Corollary 4.7. *Let B be a block of G with dihedral or semidihedral defect group D (of order at least 8) such that: $l(B) = 3$, B is in $SD(2\mathcal{B})_3$, or B is in $SD(2\mathcal{B})_1$ and $|D| \geq 32$. Then B is Morita equivalent to a quasi-isolated block of a quasi-simple group.*

Proof. If B is itself quasi-isolated then we are done. Otherwise by Theorem 3.9 it is Morita equivalent to a block B_L of some $L = \mathbf{L}^F/Z$, where $\mathbf{L}$ is a proper F-stable Levi subgroup of $\mathbf{G}$, also with defect group D. Consider a block B'_L of $[L, L]$ covered by B_L and let $P = D \cap [L, L]$ be a defect group of B'_L as in Lemma 4.6. Then P is strongly closed in D with respect to L.

Remark 4.5(iii) implies that $P = 1$ or $P = D$ (note for $SD(2\mathcal{B})_1$ that while Section 6 relies on Section 5 and Lemma 4.6, it does not rely on this result). But D/P is abelian, so $P = D$. Since P cannot be expressed as a non-trivial central product, the block B_S of some quasi-simple normal subgroup S of $[L, L]$ covered by B'_L has defect group P, so is therefore Morita equivalent to B_L, and hence to B, by Remark 4.5(iv); note that S cannot be solvable by Remark 4.5(ii).

The rank of S as a group of Lie type is at least 1 but strictly less than the rank of G. Therefore, repeating the above now with S and B_S in place of G and B, this process must terminate at some quasi-isolated block of a quasi-simple group that is Morita equivalent to B. $\square$

Hence to identify which of the Morita equivalence classes in question occur among finite groups of Lie type in cross characteristic it is sufficient to consider only the quasi-isolated blocks of quasi-simple groups.

5. Blocks of quasi-simple groups

We go through the finite simple groups and their odd covers case by case, and study the blocks with dihedral or semidihedral defect groups.

5.1. Defining characteristic

First consider the defining characteristic case, that is where G is a finite group of Lie type of characteristic $p = \ell = 2$. Humphreys [34] proved that every p-block of G has defect groups either trivial or the Sylow p-subgroups of G. The groups with dihedral Sylow 2-subgroups were classified by Gorenstein and Walter [30]; in particular any finite quasi-simple group with dihedral Sylow 2-subgroups is isomorphic to an odd cover of either $\mathrm{Alt}(7)$ or $\mathrm{PSL}_2(q)$ for q odd. Similarly, by a result of Alperin, Brauer, and Gorenstein [1] any quasi-simple group with semidihedral Sylow 2-subgroups is isomorphic to M_{11} or an odd cover of $\mathrm{PSL}_3(q)$ for $q \equiv -1 \bmod 4$ or $\mathrm{PSU}_3(q)$ for $q \equiv 1 \bmod 4$. Hence we need not further consider blocks in defining characteristic; note that while $\mathrm{PSL}_3(2)$ has dihedral Sylow 2-subgroups it is isomorphic to $\mathrm{PSL}_2(7)$, which will be considered later.

5.2. Types B, C, D

Now let $\mathbf{G}$ be a simple simply connected group of Lie type of characteristic $p \neq \ell = 2$, with F a Steinberg endomorphism so that $\mathbf{G}^F$ is a finite group of Lie type, and let $G = \mathbf{G}^F/Z$, where Z is a central subgroup of $\mathbf{G}^F$. First let $\mathbf{G}$ be of type B_n, C_n, or D_n for $n > 1$. Then $\mathbf{G}$ has no non-trivial quasi-isolated $2'$-elements [4, Table II], and the only unipotent 2-blocks of $\mathbf{G}^F$ are the principal blocks [21, Prop. 6]. Hence the only quasi-isolated block of G is the principal block, which has defect groups the Sylow 2-subgroups of G, which are neither dihedral nor semidihedral for any such G. Note that this includes the twisted groups ${}^2D_n(q)$ and ${}^3D_4(q)$, while ${}^2B_2(q)$ only exists in characteristic 2.

5.3. Exceptional groups

5.3.1. Unipotent blocks

Enguehard [21] classified all unipotent blocks of finite groups of Lie type for bad primes, as 2 is other than for type A. There are none with semidihedral defect groups, and only $E_7(q)$ has unipotent blocks with dihedral defect groups.

Theorem 5.1. *The simple group $E_7(q)$ for $q \equiv 1 \bmod 4$ (resp. $-1 \bmod 4$) has two unipotent blocks with dihedral defect groups of order $|q-1|_2$ (resp. $|q+1|_2$) in $D(3\mathcal{A})_1$ (resp. $D(3\mathcal{K})$).*

Proof. Let $G = E_7(q)_{ad}$, so $[G, G] = E_7(q)$ and $\big|G : [G, G]\big| = 2$. By [21, Section 3.2] there are two non-principal unipotent blocks of G with dihedral defect groups of order $2|q \pm 1|_2$, corresponding to unipotent characters labelled $E_6[\theta]$ and $E_6[\theta^2]$ of a Levi subgroup $E_6(q).(q-1)$ of G. Let B be one of these blocks, which contains the unipotent characters denoted in [15, Section 13.9] by $E_6[\theta^i], 1$ and $E_6[\theta^i], \varepsilon$, for $i = 1$ or 2.

For an algebraic group $\mathbf{G}$, restriction via the natural isogeny $\mathbf{G}_{sc} \to \mathbf{G}_{ad}$ induces a bijection between unipotent characters $\mathcal{E}(\mathbf{G}_{ad}^F, 1) \to \mathcal{E}(\mathbf{G}_{sc}^F, 1)$ by [13, Prop. 15.9], meaning that unipotent characters of G must restrict irreducibly to $[G, G]$. Hence B covers a unique block b of $[G, G]$ containing these two labelled unipotent characters, which each split on induction up to G into pairs of irreducible characters χ_1, χ_1' and $\chi_\varepsilon, \chi_\varepsilon'$ respectively in B.

From [15, Section 13.9] the degrees of these characters are related by $\chi_\varepsilon(1) = q^9\chi_1(1)$. Since χ_1, χ_ε have the same height but different degrees, and B has dihedral defect groups, they both have height zero, as do χ_1' and χ_ε'. Then, since among these four height zero characters there are exactly two distinct degrees, B must be in $D(2\mathcal{A})$ or $D(2\mathcal{B})$. Since B has defect at least 3, it contains at least one height one character, whose degree is $\chi_\varepsilon(1)+\chi_1(1) = (q^9+1)\chi_1(1)$ for $D(2\mathcal{A})$, which must therefore be the case if $q \equiv 1 \bmod 4$, and $\chi_\varepsilon(1) - \chi_1(1) = (q^9-1)\chi_1(1)$ for $D(2\mathcal{B})$ if $q \equiv -1 \bmod 4$.

Note that the defect groups of b are also dihedral (or Klein four), of order $|q \pm 1|_2$. Indeed by Lemma 3.13 they are index-2 subgroups of defect groups of B, so supposing they are not dihedral they must be cyclic, so have only height zero characters by [40, Thm 1.1]. So then all of the height one characters of B must split on restriction to $[G, G]$, which would lead to either none of the rows or two distinct rows of the decomposition matrix of b being repeated. But since the defect groups are of order at least 4, there must be exactly one repeated row as in [16, Thm 5.1.2].

There are an odd number of them, so one of the height one characters of B must split on restriction to $[G, G]$, and thus so must the second Brauer character. Then all the other ordinary characters must fuse, and thus from the resulting decomposition matrix b must be in $D(3\mathcal{A})_1$ if $q \equiv 1 \bmod 4$, and $D(3\mathcal{K})$ if $q \equiv -1 \bmod 4$. □

Note from the proof that $E_7(q)_{ad}$ has unipotent blocks with dihedral defect groups in $D(2\mathcal{A})$ or $D(2\mathcal{B})$ for $q \equiv 1$ or $-1 \bmod 4$ respectively.

5.3.2. Quasi-isolated blocks

Kessar and Malle [40] described the non-unipotent quasi-isolated blocks of exceptional groups of Lie type for bad primes $\ell \neq p$. Let $\mathbf{G}$ be a simply connected simple exceptional algebraic group, with F a Steinberg endomorphism so that $\mathbf{G}^F$ is a finite group of Lie type, and $\mathbf{G}^*$ is a group dual to $\mathbf{G}$. They list all quasi-isolated ℓ'-elements $s \in \mathbf{G}^*$ and describe the quasi-isolated ℓ-blocks in each Lusztig series $\mathcal{E}(\mathbf{G}^F, s)$, in particular proving Theorem 3.10. For $\ell = 2$ in most cases the corresponding $Z(\mathbf{L})_2^F$, so also the defect groups of the block, contains a 4×4 subgroup, so they are neither dihedral nor semidihedral; nor are the defect groups for $E_7(q)$ after quotienting by the simply connected group's centre of order 2 (see [49, Table 24.2] for the possible centres of $\mathbf{G}^F$). The exceptions are the blocks of $E_6(q)$ and ${}^2E_6(q)$ in line 8 of [40, Table 3], the block of $E_8(q)$ in line 6 of [40, Table 5], and the block of $G_2(q)$ in line 2 of [40, Table 9]. We show that the defect groups of these blocks cannot be dihedral, and deduce the possible classes of the block if they are semidihedral. Note that for each of these exceptions the blocks are isolated and the centraliser $C_{\mathbf{G}^*}(s)^F$ has a unipotent block with semidihedral defect groups.

Hiss and Shamash [31] described the 2-blocks of $G_2(q)$ for q odd, including character degrees, defect groups, and decomposition matrices. The case in question only occurs when $q \equiv \pm 5 \bmod 12$. By [31, (2.2.3)], if $q \equiv 5 \bmod 12$ then $G_2(q)$ has a block with semidihedral defect groups of order $4|q-1|_2$ and decomposition matrix that of $SD(3\mathcal{A})_1$. This is the block corresponding to line 2 in [40, Table 9], since we know that block contains the character corresponding to the trivial character of $C_{\mathbf{G}^*}(s)^F = {}^2A_2(q)$, whose degree is $\left|G_2(q) : {}^2A_2(q)_{ad}\right|_{p'} = q^3 - 1$. Similarly by [31, (2.3.2)], replacing ${}^2A_2(q)$ with $A_2(q)$, if $q \equiv -5 \bmod 12$ then $G_2(q)$ has an isolated block with semidihedral defect groups of order $4|q+1|_2$ and decomposition matrix that of $SD(3\mathcal{D})$ or $SD(3\mathcal{B})_1$.

Theorem 5.2. *If B is a quasi-isolated block of $E_6(q)_{sc}$ with semidihedral defect groups, then $q \equiv -1 \bmod 4$, the defect groups are of order $4|q+1|_2$, and B is in $SD(3\mathcal{D})$ or $SD(3\mathcal{B})_1$. If B is a quasi-isolated block of ${}^2E_6(q)_{sc}$ with semidihedral defect groups, then $q \equiv 1 \bmod 4$, the defect groups are of order $4|q-1|_2$, and B is in $SD(3\mathcal{A})_1$ or $SD(3\mathcal{C})_{2,2}$.*

Proof. First let $\mathbf{G}^F = E_6(q)_{sc}$. Block 8 in [40, Table 3] for $q \equiv -1 \bmod 4$ has $C_{\mathbf{G}^*}(s)^F = A_2(q^3).3$ and defect groups of the form $|q+1|_2.2.2$. Since $C_{\mathbf{G}^*}(s)$ is disconnected its unipotent characters are those of $(C_{\mathbf{G}^*}(s)^\circ)^F = A_2(q^3)$ induced up to $C_{\mathbf{G}^*}(s)^F$. Since $\left|C_{\mathbf{G}^*}(s) : C_{\mathbf{G}^*}(s)^\circ\right| = 3$, it follows that $\mathcal{E}(\mathbf{G}^F, s)$ consists of three blocks, each containing Jordan correspondents of the unipotent characters of $A_2(q^3)$, denoted ϕ_3, ϕ_{21}, and ϕ_{1^3} according to the partitions of 3.

Consider one of these blocks B; the degrees of the three known characters are $\phi_3(1) = 1$, $\phi_{21}(1) = q^3(q^3+1)$, and $\phi_{1^3}(1) = q^9$ each multiplied by a factor of $m = \left|(\mathbf{G}^*)^F : C_{\mathbf{G}^*}(s)^F\right|_{p'}$. Each character of B has this factor in its degree by Remark 3.6, so the ϕ_3 character has height zero. Then the ϕ_{21} character has height at least 2, so the defect groups of B are not dihedral. Suppose they are semidihedral.

The two known height zero character degrees are distinct and neither their sum nor difference is equal to the large height character degree, which, observing the possible decomposition matrices, implies that $l(B) = 3$. We can then rule out the decomposition matrices of $SD(3\mathcal{B})_2$ and $SD(3\mathcal{C})_{2,1}$ since the large height character does not have the largest degree. For the decomposition matrices of $SD(3\mathcal{A})_1$ and $SD(3\mathcal{C})_{2,2}$ we exhaustively consider each pair of height zero characters and suppose their degrees are m and mq^9; each case implies a contradiction that the large height degree $q^3(q^3+1)$ is q^9-1 or q^9+1, or that the repeated degree has height greater than one. We do the same for the decomposition matrix of $SD(3\mathcal{H})$, and the only pair not giving either of the previous contradictions has ϕ_3 as the third row, giving the second row degree $m\left(q^3(q^3+1)-1\right)$, which is notably not a cyclotomic polynomial. Using GAP and [47], we see that there are no ordinary character degrees of $E_6(q)_{sc}$ which coincide with this degree for any q. (This was done by listing all character degrees as polynomials in q, subtracting the polynomial degree $m(q^6+q^3-1)$, and checking for any positive integer roots up to a sufficiently high q; a polynomial's roots are bounded by twice the absolute value of the largest coefficient divided by its leading coefficient.) Therefore B is in $SD(3\mathcal{D})$ or $SD(3\mathcal{B})_1$; these classes have the same decomposition matrix.

Similarly, by Ennola duality and using the same arguments, $\mathbf{G}^F = {}^2E_6(q)_{sc}$ with $q \equiv 1 \bmod 4$ has three blocks with $C_{\mathbf{G}^*}(s)^F = {}^2A_2(q^3).3$ that may have semidihedral defect groups, and if so are in $SD(3\mathcal{A})_1$ or $SD(3\mathcal{C})_{2,2}$, whose decomposition matrices are indistinguishable based only on ordinary character degrees. $\square$

Theorem 5.3. *If B is an isolated block of $E_8(q)$ with semidihedral defect groups, then either $q \equiv 5 \bmod 12$, the defect groups are of order $4|q-1|_2$, and B is in $SD(3\mathcal{A})_1$ or $SD(3\mathcal{C})_{2,2}$; or $q \equiv 7 \bmod 12$, the defect groups are of order $4|q+1|_2$, and B is in $SD(3\mathcal{D})$ or $SD(3\mathcal{B})_1$.*

Proof. Let $\mathbf{G}^F = E_8(q)$; note that $\mathbf{G}^* \cong \mathbf{G}$. When $q \equiv 1 \bmod 4$, block 6 of [40, Table 5] occurs when also $q \equiv -1 \bmod 3$, and has $C_{\mathbf{G}}(s)^F = {}^2E_6(q).{}^2A_2(q)$; here $\mathcal{E}_2(\mathbf{G}^F, s)$ is a union of three blocks. Two of these blocks have defect groups of the form P with subgroups $A \leq D \leq P$ such that $D \cong 2|q-1|_2$ is the unique cyclic subgroup of index 2 in P, and any $\sigma \in P \setminus D$ inverts $A \cong |q-1|_2$; this means that P is dihedral, semidihedral, or generalised quaternion, of order $4|q-1|_2$. These two blocks, corresponding to $i = 1$ and 2, each contain two Harish-Chandra series: one series has $\mathbf{L}^F = \Phi_1.E_7(q)$, $\lambda = {}^2E_6[\theta^i]$, and $W_{\mathbf{G}^F}(\mathbf{L}, \lambda) = A_1 = 2$; and the other has $\mathbf{L}^F = E_8(q)$, $\lambda = {}^2E_6[\theta^i] \otimes \phi_{21}$, and $W_{\mathbf{G}^F}(\mathbf{L}, \lambda) = 1$ (here λ is labelled by the Jordan corresponding unipotent character in $C_{\mathbf{G}}(s)^F$). The latter series consists only of the character ${}^2E_6[\theta^i] \otimes \phi_{21}$, since $\mathbf{L}^F = \mathbf{G}^F$, and the former contains ${}^2E_6[\theta^i]$ and one other character, since $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ has order 2 and hence $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ has norm 2. Assume $i = 1$ (all degrees are identical for $i = 2$) and let B be the corresponding block of $\mathbf{G}^F$.

By Remark 3.6 each character of $\mathrm{Irr}(B)$ corresponds to an ordinary character of ${}^2E_6(q).{}^2A_2(q)$ and has $m = \left|\mathbf{G}^F : C_{\mathbf{G}}(s)^F\right|_{p'} = \left|E_8(q) : {}^2E_6(q).{}^2A_2(q)\right|_{p'}$ as a common factor in its degree; in particular, the 2-part of this is $2^3\left|(q-1)^3\right|_2$. The 2-part of $\left|E_8(q)\right|$ is $2^{14}\left|(q-1)^8\right|_2$ and the defect groups of B have order $2^2|q-1|_2$. Then since $\left|{}^2E_6[\theta](1)\right|_2 = 2^9\left|(q-1)^4\right|_2$, the correspondent of ${}^2E_6[\theta]$ has height zero and that of ${}^2E_6[\theta]\otimes\phi_{21}$ has height at least 2, since $\phi_{21}(1) = q(q-1)$; therefore the defect groups are not dihedral.

Of all the other unipotent characters of $C_{\mathbf{G}}(s)^F$, found in [15, Section 13.9], only ${}^2E_6[\theta]\otimes\phi_{1^3}$ has a large enough 2-part given the defect of B – except also perhaps ${}^2E_6[1]\otimes\phi_{21}$, but [40, Table 5] lists this character in a Harish-Chandra series of the other of the three blocks contained in $\mathcal{E}_2(\mathbf{G}^F, s)$ – and it has height zero since $\phi_{1^3}(1) = q^3$.

Suppose B has semidihedral defect groups. In the same way as in the previous proof, the three known character degrees imply that $l(B) = 3$, and that B cannot have decomposition matrix that of $SD(3\mathcal{C})_{2,1}$ or $SD(3\mathcal{B})_2$ since the large height character does not have largest degree, or that of $SD(3\mathcal{D})$ or $SD(3\mathcal{B})_1$ checking the possible combinations of height zero characters. Similarly for $SD(3\mathcal{H})$, the third row is forced to have degree $m\cdot{}^2E_6[\theta](1)\cdot(q^2-q-1)$, again not a cyclotomic polynomial, which does not coincide with any ordinary character degree of ${}^2E_6(q).{}^2A_2(q)$ for any q (checking all products of degrees of ${}^2E_6(q)$ with those of ${}^2A_2(q)$ up to sufficiently large q). Therefore B is in $SD(3\mathcal{A})_1$ or $SD(3\mathcal{C})_{2,2}$.

If $q \equiv -1 \bmod 4$ then by Ennola duality and using the same arguments, with $C_{\mathbf{G}}(s)^F = E_6(q).A_2(q)$, we get that B is in $SD(3\mathcal{D})$ or $SD(3\mathcal{B})_1$. □

Note that ${}^2F_4(q)$ only exists in characteristic 2, and ${}^2G_2(q)$ has elementary abelian Sylow 2-subgroups.

5.4. Type A

Now consider the groups of type A, that is the linear and unitary groups. The general unitary group will be denoted $\mathrm{GL}_n(-q)$ or $\mathrm{GU}_n(q)$ (as opposed to $\mathrm{GU}_n(q^2)$), and similarly for the special and projective unitary groups. Let b be a 2-block of an odd cover of $\mathrm{PSL}_n(\varepsilon q)$, where $\varepsilon = \pm 1$, contained in a block b_{SL} of $\mathrm{SL}_n(\varepsilon q)$, with semisimple label $s \in \mathrm{PGL}_n(\varepsilon q)$ according to Lusztig series, and let B be a block of $G = \mathrm{GL}_n(\varepsilon q)$ covering b_{SL}. The defect groups of the blocks of the general linear and unitary groups were described by Broué in [11, (3.7)], and those of B are the Sylow 2-subgroups of $C_G(\tilde{s}) \cong \prod_{i\in I}\mathrm{GL}_{n_i}\left((\varepsilon q)^{a_i}\right)$ for some suitable n_i and a_i, where $\tilde{s}$ is a representative of s in G.

Suppose that b is quasi-isolated and has dihedral or semidihedral defect groups. By [4, Table II] quasi-isolated semisimple elements $s \in \mathrm{PGL}_n(\overline{\mathbb{F}_q})$ have connected centraliser of the form $(\mathrm{PGL}_{n/d}(\overline{\mathbb{F}_q}))^d$, where d is the order of s which is therefore odd. Therefore $C_G(\tilde{s}) \cong \prod_{i\in I}\mathrm{GL}_{n/d}\left((\varepsilon q)^{a_i}\right)$.

We must have $n/d > 1$, otherwise the defect groups of B would be abelian. Additionally if $|I| > 1$ then $C_G(\tilde{s})$ would at least contain $\mathrm{GL}_2\left((\varepsilon q)^{a_1}\right)\times\mathrm{GL}_2\left((\varepsilon q)^{a_2}\right)$ as a subgroup, so the defect groups of b_{SL} would contain the Sylow 2-subgroups of $\mathrm{SL}_2\left((\varepsilon q)^{a_1}\right)\times\mathrm{SL}_2\left((\varepsilon q)^{a_2}\right)$ – a product of two generalised quaternion groups – which after quotienting by a cyclic subgroup cannot be dihedral or semidihedral.

Therefore $C_G(\tilde{s}) \cong \mathrm{GL}_{n/d}\left((\varepsilon q)^d\right) = \mathrm{GL}_{n/d}(\varepsilon q^d)$; let $\sigma : \mathrm{GL}_{n/d}(\varepsilon q^d) \to \mathrm{GL}_n(\varepsilon q)$ denote an embedding. For any x we have $\det_n(x\sigma) = \left(\det_{n/d}(x)\right)^m$, where $m = (q^d - \varepsilon)/(q - \varepsilon)$ which is odd since d is, so if x is a 2-element then $\det_n(x\sigma) = 1$ precisely when $\det_{n/d}(x) = 1$. Hence, with the following subscript-2 notation denoting a Sylow 2-subgroup, $\left(\mathrm{GL}_{n/d}(\varepsilon q^d)_2\right)\sigma \cap \mathrm{SL}_n(\varepsilon q) = \left(\mathrm{SL}_{n/d}(\varepsilon q^d)_2\right)\sigma$, so the defect groups of b_{SL} are isomorphic to $\mathrm{SL}_{n/d}(\varepsilon q^d)_2$, and those of b are isomorphic to $\mathrm{PSL}_{n/d}(\varepsilon q^d)_2$. In particular the defect groups of b are dihedral (or Klein four) if and only if $n/d = 2$, and semidihedral if and only if $n/d = 3$ and $q \equiv -\varepsilon \bmod 4$.

Remark 5.4. Note for reference in Section 6 that if $P \leq D$ are defect groups of b_{SL} and B respectively then $|D : P|_2 = \left|G : \mathrm{SL}_n(\varepsilon q)\right|_2$.

We first look at the semidihedral case, as it is more straightforward. By non-exceptional cover we mean that the group is a quotient of $\mathrm{SL}_n(\varepsilon q)$; see Section 5.5 for the exceptional covers.

Theorem 5.5. *Let b be a quasi-isolated block of (a non-exceptional odd cover of) $\mathrm{PSL}_n(\varepsilon q)$ with semidihedral defect groups. Then $q \equiv -\varepsilon \bmod 4$, the defect groups of b are of order $4|q+\varepsilon|_2$, and b is in $SD(3\mathcal{D})$ or $SD(3\mathcal{B})_1$ if $\varepsilon = 1$ and $SD(3\mathcal{A})_1$ if $\varepsilon = -1$.*

Proof. Using the previous notation of this section, we have that $C_G(\tilde{s}) = \mathrm{GL}_3(\varepsilon q^{n/3})$, with $n/3$ odd and $q^{n/3} \equiv q \equiv -\varepsilon \bmod 4$. By [26] (and as in Theorem 2.2) the principal blocks of $\mathrm{PSL}_3(q^{n/3})$ for $q \equiv -1 \bmod 4$ and $\mathrm{PSU}_3(q^{n/3})$ for $q \equiv 1 \bmod 4$ have decomposition matrices

$$\begin{pmatrix} 1 & . & . \\ 1 & 1 & . \\ 1 & . & 1 \\ 1 & 1 & 1 \\ . & . & 1 \\ . & 1 & . \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & . & . \\ 1 & 1 & . \\ 1 & . & 1 \\ 1 & 1 & 1 \\ . & . & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

respectively, with the last rows repeated $|q+\varepsilon|_2 - 1$ times. Since $\left|Z(\mathrm{SL}_3(\varepsilon q^{n/3}))\right|$ is odd, the principal block of $\mathrm{SL}_3(\varepsilon q^{n/3})$ is isomorphic to that of $\mathrm{PSL}_3(\varepsilon q^{n/3})$. Define H' to be the subgroup of $\mathrm{GL}_3(\varepsilon q^{n/3})$ consisting of elements whose determinant has odd order in $\mathbb{F}_{q^{n/3}}^\times$. Then since $\left|q^{n/3} - \varepsilon\right|_2 = 2$ we have that $\left|H' : \mathrm{SL}_3(\varepsilon q^{n/3})\right|$ is odd, so by Lemma 4.1 the principal block of H' is Morita equivalent to that of $\mathrm{SL}_3(\varepsilon q^{n/3})$, and $\mathrm{GL}_3(\varepsilon q^{n/3}) = H' \times 2$, so the principal block of $\mathrm{GL}_3(\varepsilon q^{n/3})$ has the corresponding decomposition matrix above with each row occurring twice.

Similarly, since n is odd and $|q-\varepsilon|_2 = 2$, if H is the subgroup of $G = \mathrm{GL}_n(\varepsilon q)$ of elements with determinant of odd order then $G = H \times 2$ and b is Morita equivalent to the block B_H of H covered by the block B of G.

Since $C_{\mathbf{G}}(\tilde{s})$ must be a Levi subgroup of $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}_q})$, Theorem 3.7 implies that B is Morita equivalent to the principal block of $C_G(\tilde{s}) = \mathrm{GL}_3(\varepsilon q^{n/3})$ with decomposition matrix as described above. Then since $G = H \times 2$, the ordinary characters of B restrict irreducibly in pairs to those of B_H, so B_H, and hence also b, has decomposition matrix that of the principal block of $\mathrm{PSL}_3(\varepsilon q^{n/3})$. Therefore b is in $SD(3\mathcal{D})$ or $SD(3\mathcal{B})_1$ if $\varepsilon = 1$, and is in $SD(3\mathcal{A})_1$ if $\varepsilon = -1$. □

For the dihedral case we first calculate a decomposition matrix we will use in the proof. Throughout the rest of this section set $a = |q+1|_2$ and $c = |q-1|_2$.

Proposition 5.6. *The decomposition matrix of the principal block B of $\mathrm{GU}_2(q)$ is*

$$\begin{array}{c} 1 \\ q \\ q-1 \\ q+1 \end{array} \begin{pmatrix} 1 & . \\ 1 & 1 \\ . & 1 \\ 2 & 1 \end{pmatrix} \begin{array}{c} a \textit{ times} \\ a \textit{ times} \\ \frac{1}{2}a\,(a-1) \textit{ times} \\ \frac{1}{2}a\,(c-1) \textit{ times} \end{array};$$

the character degrees are shown on the left, and each row is repeated the number of times shown on the right.

Proof. Note that $\mathrm{SL}_2(q) \cong \mathrm{SU}_2(q)$. First let $q \equiv -1 \bmod 4$. By [26] (and as in Corollary 2.4) the decomposition matrix of the principal block b of $\mathrm{SU}_2(q)$ is

$$\begin{pmatrix} 1 & . & . \\ . & 1 & . \\ . & . & 1 \\ 1 & 1 & 1 \\ 1 & 1 & . \\ 1 & . & 1 \\ . & 1 & 1 \end{pmatrix}$$

with the final row repeated $a-1$ times. The central product $A = \mathrm{SU}_2(q) * (q+1)$ is an index-2 subgroup of $\mathrm{GU}_2(q)$, and the decomposition matrix of the principal block B_A of A is that of b with each row occurring $a/2$ times.

By [27, (4.3)] $l(B) = 2$ and $k(B) = a^2/2 + 2a$. Therefore, on induction to $\mathrm{GU}_2(q)$, the second and third Brauer characters of B_A must fuse, which forces the ordinary characters relating to the second and third rows and also the fifth and sixth rows above to fuse in pairs, while all the other ordinary characters must split to give the correct value for $k(B)$. Hence the decomposition matrix of B is as claimed.

If $q \equiv 1 \bmod 4$ then the argument is identical with a slightly different matrix for $\mathrm{SU}_2(q)$; note also that then B is as in Theorem 2.3. □

Theorem 5.7. *Let b be a quasi-isolated block of (a non-exceptional odd cover of)* $\mathrm{PSL}_n(\varepsilon q)$*, where $\varepsilon = pm1$, with dihedral defect groups (of order at least 8). Then $n/2$ is odd and either: $q \equiv 1 \bmod 4$, the defect groups are of order $|q-1|_2$, and b is in $D(3\mathcal{A})_1$; or $q \equiv -1 \bmod 4$, the defect groups are of order $|q+1|_2$, and b is in $D(3\mathcal{K})$.*

Proof. Using the previous notation of this section, by those considerations, $C_G(\tilde{s}) = \mathrm{GL}_2(\varepsilon q^{n/2})$ with $n/2$ odd. Without loss of generality, consider b as a block of $\mathrm{SL}_n(\varepsilon q)/2$, an odd cover of $\mathrm{PSL}_n(\varepsilon q)$ since $\left|(q,n)\right|_2 = 2$. Set H as the subgroup of $G = \mathrm{GL}_n(\varepsilon q)$ consisting of elements with odd-order determinant, let $B_{H/2}$ be a block of $H/2$ covering b, which by Lemma 4.1 is Morita equivalent to b, and let B_H be the block of H containing $B_{H/2}$ (choose $B_{H/2}$ so that the block B of G covers B_H). The central product $A = H * (q-\varepsilon)$ is an index-2 subgroup of G, and $|A : H| = |q-\varepsilon|_2/2$. Let B_A be the unique block of A covering B_H, so B is the unique block of G covering B_A.

Since $C_{\mathbf{G}}(\tilde{s})$ must be a Levi subgroup of $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}_q})$, Theorem 3.7 implies that B is Morita equivalent to the principal block of $C_G(\tilde{s}) = \mathrm{GL}_2(\varepsilon q^{n/2})$ with the same decomposition matrix. The decomposition matrix of the principal block of $\mathrm{GU}_2(q^{n/2})$ is as in Proposition 5.6, replacing q with $t = q^{n/2}$, and that of $\mathrm{GL}_2(q^{n/2})$, calculated using [35], is

$$\begin{array}{c} 1 \\ t \\ t-1 \\ t+1 \end{array} \begin{pmatrix} 1 & . \\ 1 & 1 \\ . & 1 \\ 2 & 1 \end{pmatrix} \begin{array}{l} c \text{ times} \\ c \text{ times} \\ \frac{1}{2}c\,(a-1) \text{ times} \\ \frac{1}{2}c\,(c-1) \text{ times} \end{array} ;$$

note that since $n/2$ is odd, $|t \pm 1|_2 = |q \pm 1|_2$.

The decomposition matrix of B_A is that of B_H with each row repeated $c/2$ or $a/2$ times, depending on ε; the characters of $B_{H/2}$ are then a subset of those of B_H, and $B_{H/2}$ has dihedral defect groups. Since the defects of B and B_A are different, all the height zero characters of B must fuse on restriction to B_A, giving two possible height zero characters of $B_{H/2}$. Then one of the height one characters, and hence the second Brauer character, of B must split on restriction to A. It follows that the decomposition matrix of $B_{H/2}$, hence that of b, must be that of $D(3\mathcal{A})_1$ if $q \equiv 1 \bmod 4$ or $D(3\mathcal{K})$ if $q \equiv -1 \bmod 4$. □

5.5. Exceptional Schur covers

The following are the maximal odd covers of finite simple groups of Lie type that are not quotients of the corresponding simply connected group: $3 \cdot \mathrm{PSL}_2(9)$, $3^2 \cdot \mathrm{PSU}_4(3)$, $3 \cdot B_3(3)$, and $3 \cdot G_2(3)$ (see [49, Table 24.3]). The decomposition matrices of these groups are all in the Modular Atlas [10]. The new 2-blocks occurring – that is those that are not blocks of the simple group – for $3^2 \cdot \mathrm{PSU}_4(3)$ and $3 \cdot G_2(3)$ all have either maximal or zero defect, while $3 \cdot B_3(3) = 3 \cdot O_7(3)$ has two blocks with dihedral defect groups of order 8 in $D(2\mathcal{A})$. Finally, $3 \cdot \mathrm{PSL}_2(9) \cong 3 \cdot \mathrm{Alt}(6)$ is considered in the following section.

5.6. Alternating groups

The alternating groups are investigated via the representation theory of the symmetric groups, which is well understood (see [36] for example). The ordinary characters of $\mathrm{Sym}(n)$ correspond to the partitions of n, and the result of repeatedly removing ℓ-hooks – partitions of the form $(\ell - m, 1^m)$ for some m – from a partition is called the ℓ-core, which is uniquely defined. Two characters are in the same ℓ-block if and only if the ℓ-cores of their partitions are the same [36, (6.1.21)], so the blocks are labelled by ℓ-cores. The weight of a character, and thus the weight of its block, is the number of ℓ-hooks removed to get to its ℓ-core. The defect groups of a block of weight w are then conjugate to Sylow p-subgroups of $\mathrm{Sym}(\ell w)$ by [36, (6.2.45)].

Theorem 5.8. *If b is a block of an alternating group $\mathrm{Alt}(n)$ with dihedral defect groups (of order at least 8), then they must be of order 8 and either:*

(i) $n = 6$ and b is in $D(3\mathcal{A})_1$, or;
(ii) $n = t + 6$ where $t \geq 1$ is a triangular number and b is in $D(3\mathcal{B})_1$.

Proof. There is a unique block B of $\mathrm{Sym}(n)$ covering b. If B has weight w then its defect groups are conjugate to the Sylow 2-subgroups of $\mathrm{Sym}(2w)$, so the defect groups of b are isomorphic to the Sylow 2-subgroups of $\mathrm{Alt}(2w)$. Therefore we must have $w = 3$ and the defect groups are of order 8.

The 2-blocks of $\mathrm{Sym}(n)$ are labelled by 2-cores, which are precisely the triangular partitions. In [55, Lem. 2.4, Thm 4.2] Scopes gives a method of showing that certain blocks of $\mathrm{Sym}(n)$ and $\mathrm{Sym}(n+m)$ of equal weight w with $m \geq w$ are Morita equivalent with the same decomposition matrices. From this we get that all 2-blocks of symmetric groups of weight 3, other than perhaps the blocks labelled $\emptyset$ of $\mathrm{Sym}(6)$ and (1) of $\mathrm{Sym}(7)$, are Morita equivalent to the block $(2, 1)$ of $\mathrm{Sym}(9)$.

The (principal) blocks of $\mathrm{Alt}(6)$ and $\mathrm{Alt}(7)$ covered by these blocks $\emptyset$ and (1) can be checked individually and found to be as stated; note also that $\mathrm{Alt}(6) \cong \mathrm{PSL}_2(9)$. Suppose then that B is Morita equivalent to the block $(2, 1)$ of $\mathrm{Sym}(9)$, so as in [10] has the following decomposition matrix:

$$\begin{pmatrix} 1 & . & . \\ 1 & . & . \\ 1 & 1 & . \\ 1 & 1 & . \\ 1 & . & 1 \\ 1 & . & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ . & 1 & . \\ . & 1 & . \end{pmatrix}.$$

Since b has dihedral defect groups and defect 3 it has exactly four height zero and one height one character, so the ten ordinary characters of B must fuse in the obvious way, giving that b has decomposition matrix that of $D(3\mathcal{B})_1$. $\square$

Additionally, the only odd covers of alternating groups are the exceptional covers $3 \cdot \mathrm{Alt}(6)$ and $3 \cdot \mathrm{Alt}(7)$, whose Sylow 2-subgroups are also dihedral of order 8. Checking these in GAP or the Modular Atlas shows that $3 \cdot \mathrm{Alt}(6)$ has two additional blocks of defect 3 in $D(3\mathcal{K})$, while $3 \cdot \mathrm{Alt}(7)$ has two in $D(2\mathcal{B})$.

The Sylow 2-subgroups of $\mathrm{Alt}(2w)$ are not semidihedral for any w, so there are no blocks of alternating groups with semidihedral defect groups.

5.7. Sporadic groups

None of the sporadic groups has dihedral Sylow 2-subgroups, so only non-principal blocks may have dihedral defect groups, and only the Mathieu group M_{11} has semidihedral Sylow 2-subgroups. Landrock [43] described the non-principal 2-blocks of all sporadic groups, and their defect groups. There are several blocks with dihedral defect groups – they are all of order only 8, but we give their decomposition matrices anyway – and there are two blocks with semidihedral defect groups, of order 16. The ordinary character degrees can be found using GAP and its Atlas database of ordinary character tables, and, in most cases this is sufficient to deduce the decomposition matrix and Morita equivalence class. We also check any additional 2-blocks of odd covers of the sporadic groups with defect at least 3 but not maximal; these only occur for $3 \cdot Fi_{24}'$, and it can be seen from the number of characters and heights that the defect groups of these blocks are indeed dihedral.

Theorem 5.9. *The blocks of sporadic groups with dihedral defect groups (of order at least* 8*), with their decomposition matrices and character degrees, are as follows.*

Fi_{23} D($2\mathcal{B}$)	97 976 320	166 559 744
97 976 320	1	.
166 559 744	.	1
166 559 744	.	1
264 536 064	1	1
264 536 064	1	1

B $D(2\mathcal{B})$	2 642 676 197 359 616	9 211 433 539 600 384
2 642 676 197 359 616	1	.
9 211 433 539 600 384	.	1
9 211 433 539 600 384	.	1
11 854 109 736 960 000	1	1
11 854 109 736 960 000	1	1

Fi'_{24} $D(3\mathcal{A})_1$	38 467 010 560	38 641 860 608	107 008 229 376
38 641 860 608	.	1	.
77 108 871 168	1	1	.
145 650 089 984	.	1	1
184 117 100 544	1	1	1
222 758 961 152	1	2	1

$O'N$ $D(3\mathcal{K})$	10 944	13 376	13 376
10 944	1	.	.
13 376	.	1	.
13 376	.	.	1
26 752	.	1	1
37 696	1	1	1

He $D(3\mathcal{B})_1$	1920	4352	4608
1920	1	.	.
4352	.	1	.
6272	1	1	.
6528	1	.	1
10 880	1	1	1

Suz $D(3\mathcal{B})_1$	66 560	79 872	102 400
66 560	1	.	.
79 872	.	1	.
146 432	1	1	.
168 960	1	.	1
248 832	1	1	1

Co_1 $D(3\mathcal{B})_1$	40 370 176	150 732 800	313 524 224
40 370 176	1	.	.
150 732 800	.	1	.
191 102 976	1	1	.
464 257 024	.	1	1
504 627 200	1	1	1

Additionally, $3 \cdot Fi'_{24}$ has two blocks with dihedral defect groups and the following decomposition matrix.

$3 \cdot Fi'_{24}$ $D(2\mathcal{A})$	55 349 084 160	80 256 172 032
80 256 172 032	.	1
80 256 172 032	.	1
135 605 256 192	1	1
135 605 256 192	1	1
215 861 428 224	1	2

Theorem 5.10. *The blocks of sporadic groups with semidihedral defect groups, with their decomposition matrices and character degrees, are as follows.*

M_{11} $SD(3\mathcal{D})$	1	10	44
1	1	.	.
10	.	1	.
10	.	1	.
10	.	1	.
11	1	1	.
44	.	.	1
45	1	.	1
55	1	1	1

HN $SD(3\mathcal{D})$ or $SD(3\mathcal{B})_1$	214 016	1 361 920	2 985 984
214 016	1	.	.
1 361 920	.	1	.
1 361 920	.	1	.
1 361 920	.	1	.
1 575 936	1	1	.
2 985 984	.	.	1
3 200 000	1	.	1
4 561 920	1	1	1

The Monster group has a block with one of the following decomposition matrices:

M $SD(3\mathcal{B})_2$ or $SD(3\mathcal{C})_{2,1}$	φ_1	φ_2	φ_3	φ_1	φ_2	φ_4
5 514 132 424 881 463 208 443 904	1	.	.	1	.	.
5 514 132 424 881 463 208 443 904	1	.	.	1	.	.
5 514 132 424 881 463 208 443 904	1	.	.	1	.	.
9 416 031 858 681 585 751 556 096	.	1	.	.	1	.
14 930 164 283 563 048 960 000 000	1	1	.	1	1	.
124 058 385 593 021 471 188 320 256	.	1	1	.	.	1
129 572 518 017 902 934 396 764 160	1	1	1	1	.	1
138 988 549 876 584 520 148 320 256	1	2	1	1	1	1

where $\varphi_1 = 5\,514\,132\,424\,881\,463\,208\,443\,904$, $\varphi_2 = 9\,416\,031\,858\,681\,585\,751\,556\,096$, $\varphi_3 = 114\,642\,353\,734\,339\,885\,436\,764\,160$, $\varphi_4 = 124\,058\,385\,593\,021\,471\,188\,320\,256$.

The block of HN is in one of the two Morita equivalence classes with its decomposition matrix, while the principal block of M_{11} is known to be Morita equivalent to that of $\mathrm{PSL}_3(3)$ (a calculation in Magma). As seen above, the decomposition matrix of the block of M cannot be identified from just the ordinary character degrees; neither can it be identified by tensoring any of the possible projective indecomposable characters of this block or those of the defect zero blocks of M with irreducible characters (this was checked in GAP; see [51, 25]).

6. Semidihedral class $SD(2\mathcal{B})_1$

As according to the latter half of Proposition 4.4, we consider the blocks of quasi-simple groups with dihedral defect groups in $D(3\mathcal{K})$, as have been described throughout Section 5, and show that those with defect at least 4 are not covered by blocks with semidihedral defect groups in $SD(2\mathcal{B})_1$. There are such blocks in $D(3\mathcal{K})$ with defect only 3 in $3\cdot\mathrm{Alt}(6)$ and $O'N$, and $SD(2\mathcal{B})_1$ for defect 4 does occur as a block of $3\cdot M_{10}$ covering this block of $3\cdot\mathrm{Alt}(6)$, while the covering block of $O'N.2$ has dihedral defect groups (a simple calculation in Magma). Other than these there are the quasi-isolated blocks of odd covers of $\mathrm{PSL}_n(\varepsilon q)$ and the unipotent blocks of $E_7(q)$, both for $q \equiv -1 \bmod 4$; we must also consider the non-quasi-isolated blocks of quasi-simple groups of Lie type that are Morita equivalent to these blocks via Theorem 3.7.

Proposition 6.1. *Let b be a block of a finite quasi-simple group of Lie type G with dihedral defect groups in $D(3\mathcal{K})$. If B is the block of a group $G.2$ covering b, then B does not have semidihedral defect groups in $SD(2\mathcal{B})_1$.*

Proof. Suppose that B is in $SD(2\mathcal{B})_1$. Then $G.2$ must stabilise b, otherwise B and b would be Morita equivalent by Theorem 3.16. Each of B and b has four height zero characters, so two of these characters of b are conjugate and fuse on induction to $G.2$. Therefore $G.2$ defines some $\sigma \in \mathrm{Aut}(G)$ that stabilises b such that the corresponding $\overline{\sigma} \in \mathrm{Out}(G)$ has order 2. We also consider σ as an automorphism of the simple group $G/Z(G)$, and recall that any such automorphism is a product of an inner, diagonal, field, and graph automorphism.

Let G be defined over $\mathbb{F}_q$, a power of a prime $p \neq 2$, and let $\mathbf{G}$ be the corresponding simply connected simple algebraic group, with Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$ so that $G = \mathbf{G}^F/Z$, where Z is a central subgroup; we may assume that $|Z|$ is a power of 2. Let $b_{\mathbf{G}^F}$ be the block of $\mathbf{G}^F$ containing b, with semisimple label $s \in (\mathbf{G}^*)^F$ according to Lusztig series. Theorem 3.7 implies that $b_{\mathbf{G}^F}$ is Morita equivalent to a quasi-isolated block $b_{\mathbf{L}^F}$ of $\mathbf{L}^F$ for some F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}^F$, so b is Morita equivalent to the quasi-isolated block b_L of $L = \mathbf{L}^F/Z$ contained in $b_{\mathbf{L}^F}$ by Theorem 3.9.

Considering $[L, L]$, as in Lemma 4.6, as a central product of quasi-simple groups of Lie type, the blocks of $[L, L]$ are a central product of quasi-isolated blocks; since $\left|Z(G)\right|$ must be odd, their defect groups are a direct product. Then one of these quasi-isolated blocks b_X of some quasi-simple normal subgroup X of L is Morita equivalent to b_L. Hence b_X must be a block appearing in Section 5, so X is either $X = \mathrm{SL}_m(\varepsilon q^a)/Z$ for some m, a, and $\varepsilon = \pm 1$ or $X = E_7(q)$. Since $\mathbf{G}$ is simply connected, so is $[\mathbf{L}, \mathbf{L}]$; hence Z must be non-trivial, so $\mathbf{G}$ is of type A, B, C, D, or E_7.

If $X = E_7(q)$ then, since $\mathbf{G} \neq E_8$, we have $\mathbf{G}^F = E_7(q)_{sc}$ and $X = G$. Then, b is unipotent as in Theorem 5.1 and $q \equiv -1 \bmod 4$, so q is an odd power of p. Hence σ is not a field automorphism and must be the diagonal automorphism, giving that $G.2 = E_7(q)_{ad}$ and B has dihedral defect groups as described in Section 5.3.1.

If $X = \mathrm{SL}_m(\varepsilon q^a)/Z$ then Theorem 5.7 implies that $m/2$ is odd and $q^a \equiv -1 \bmod 4$, so a is odd and q is an odd power of p. Hence we need not consider field automorphisms; note that if F is twisted then the field automorphism $(x) \mapsto (x^q) = (x)^{-T}$ on $\mathbf{G}^F$ can be considered as a graph automorphism. Note that by [49, Prop. 14.20], the exponent of $A(s) = C_{\mathbf{G}^*}(s)/C_{\mathbf{G}^*}(s)^\circ$ must divide the orders of both s and $Z(\mathbf{G})$. The order of s is odd, so if $\left|Z(\mathbf{G})\right|$ was even then $C_{\mathbf{G}^*}(s)$ must be connected; in this case b_X is an isolated block and it follows that it must be the principal block of $X = \mathrm{SL}_2(q^a)/Z$.

In case $[L, L]$ is not itself quasi-simple, consider the quasi-isolated 2-blocks of quasi-simple groups of Lie type with trivial defect groups, described in [19, Lemma 5.2]: there are such blocks of $G_2(q)$, $F_4(q)$, $E_6(q)$, and $E_8(q)$. But G_2, F_4, and E_8 cannot be a component of any proper Levi subgroup of a simple algebraic group, and if E_6 is a component of $\mathbf{L}$ then since $\mathbf{G} \neq E_8$ we would have $[L, L] = E_6(q)$, which is not possible. Additionally in [19, Lemma 5.2] there are quasi-isolated blocks of $\mathrm{SL}_n(\varepsilon q)$ for $\varepsilon = \pm 1$

and n odd with abelian defect groups that could contain blocks of $\mathrm{SL}_n(\varepsilon q)/Z$ with trivial defect groups; these blocks are not isolated, so cannot occur if $|Z(\mathbf{G})|$ is even.

Take T to be the torus part of $\mathbf{L}^F$, so $|\mathbf{L}^F| = |[\mathbf{L}^F, \mathbf{L}^F]| \cdot |T|$. The simple components of $[\mathbf{L}^F, \mathbf{L}^F]$ (in the algebraic group sense of the simple components of $[\mathbf{L}, \mathbf{L}]$) are all of type A, including $\mathrm{SL}_m(\varepsilon q^a)$ as above with $m/2$ and a odd. Since b and b_L have the same defect groups, $|Z| = |Z(\mathbf{L}^F)|$. It follows from Remark 5.4, since b_L and b_X also have the same defect groups, that $|L/[L, L]|$ must be odd, so

$$|Z| = \left|Z\Big([\mathbf{L}^F, \mathbf{L}^F]\Big)\right|_2 \cdot |T|_2 \geq 2|T|_2\,.$$

We consider the different possible $\mathbf{G}$:

(i) G of Type B or C: Carter [14] describes the maximal-rank subgroups of $\mathbf{G}^F$ for $\mathbf{G}$ classical. If G is of type B or C then $|T|_2 \geq 2$ by [14, Props 9 & 11]; but $|Z| = 2$, so this is not possible.

(ii) G of Type A: If $\mathbf{G}^F = \mathrm{SL}_n(\varepsilon q)$, then by [14, Props 7 & 8] we must have $[\mathbf{L}^F, \mathbf{L}^F] = \mathrm{SL}_m(\varepsilon q^{n/m})$ and $|T| = (q^{n/m} - \varepsilon)/(q - \varepsilon)$ with $n/2$ and $m/2$ odd.

Set $\mathbf{GL} = \mathrm{GL}_n(\overline{\mathbb{F}_q})$ and let B_{GL} be a block of $\mathbf{GL}^F$ covering $b_{\mathbf{G}^F}$. Since $b_{\mathbf{L}^F}$ is quasi-isolated, as described in Section 5.4 the semisimple part of $\big(C_{\mathbf{G}^*}(s)^\circ\big)^F$ is $\mathrm{PGL}_2(\varepsilon q^{n/2})$. Then $C_{\mathbf{GL}}(\tilde{s})^F = \mathrm{GL}_2(\varepsilon q^{n/2})$, where $\tilde{s}$ is a pre-image of s, and the defect groups of B_{GL} are the Sylow 2-subgroups of $\mathrm{GL}_2(\varepsilon q^{n/2})$. Consider the block $B_{GL/Z}$ of $\mathrm{GL}_n(\varepsilon q)/Z\big(\mathrm{GL}_n(\varepsilon q)\big)_2$ contained in B_{GL}. This group is an odd cover of $\mathrm{PGL}_n(\varepsilon q)$, and $|q^{n/2} - \varepsilon|_2 = |q - \varepsilon|_2$, so $B_{GL/Z}$ has defect groups the Sylow 2-subgroups of $\mathrm{PGL}_2(\varepsilon q^{n/2})$, which are dihedral. Supposing that σ is a diagonal automorphism, $G.2$ is (isomorphic to) an odd-index subgroup of $\mathrm{GL}_n(\varepsilon q)/Z\big(\mathrm{GL}_n(\varepsilon q)\big)_2$, and B_{GL} can be chosen so that $B_{GL/Z}$ covers B; hence B would also have dihedral defect groups by Lemma 4.1.

We now show that σ cannot be a graph automorphism. The graph automorphism τ of $\mathbf{GL}^F$, defined by $(a) \mapsto (a)^{-T}$, maps each conjugacy class to its inverse, so sends any $\chi \in \mathrm{Irr}(\mathbf{GL}^F)$ to its complex conjugate $\overline{\chi}$. By [56, Lemma 4.2], if $\chi \in \mathcal{E}(\mathbf{GL}^F, \tilde{s})$ then $\overline{\chi} \in \mathcal{E}(\mathbf{GL}^F, \tilde{s}^{-1})$; in particular if $\tilde{s}$ and $\tilde{s}^{-1}$ are not conjugate in $(\mathbf{GL}^*)^F$ then χ and $\overline{\chi}$ are in different blocks. Note that $\mathbf{GL} \cong \mathbf{GL}^*$ and $[\mathbf{GL}, \mathbf{GL}]$ is simply connected, so $\tilde{s} \in \mathbf{GL}^F$ is conjugate to $\tilde{s}^{-1}$ in $\mathbf{GL}^F$ if and only if they are conjugate in $\mathbf{GL}$.

First suppose F is the standard Frobenius endomorphism, so $C_{\mathbf{GL}}(\tilde{s})^F = \mathrm{GL}_2(q^{n/2})$. Then $\tilde{s}$ is $\mathbf{GL}$-conjugate to a diagonal matrix of the form

$$x = \mathrm{diag}(x_1, x_1, x_2, x_2, \ldots, x_{n/2}, x_{n/2}) \in \mathbf{GL},$$

where $x_1^q = x_2, \ldots, x_{n/2}^q = x_1$, so $x_1^{q^{n/2}-1} = 1$. Suppose that x is conjugate to x^{-1}, which is equivalent to $\tilde{s}$ and $\tilde{s}^{-1}$ being conjugate in $\mathbf{GL}^F$. Conjugate elements have the same eigenvalues, so the entries of x^{-1} must be those of x permuted; hence $x_1 \cdot x_1^{q^d} = x_1^{q^d+1} = 1$ for some $d \leq n/2$. Let r be a prime dividing the order of x_1, hence also dividing both

q^d+1 and $q^{n/2}-1$; note that $r \neq 2$ since the order of $\tilde{s}$ is odd. Then $q^d \equiv -1 \bmod r$, so the order of $q \bmod r$ is even. But $q^{n/2} \equiv 1 \bmod r$ and $n/2$ is odd, a contradiction.

Now suppose F is the twisted Steinberg endomorphism sending (a_{ij}) to $(a_{ij}^q)^{-T}$, so $\mathbf{GL}^F = \mathrm{GU}_n(q)$ and $C_{\mathbf{GL}}(\tilde{s})^F = \mathrm{GU}_2(q^{n/2})$. Then similarly $\tilde{s}$ is conjugate to

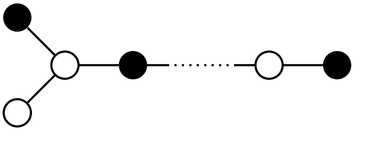
$$x = \mathrm{diag}(x_1, x_1, x_2, x_2, \ldots, x_{n/2}, x_{n/2}) \in \mathbf{GL},$$

and $x_1^{-q} = x_2, \ldots, x_{n/2}^{-q} = x_1$, so, since $n/2$ is odd, $x^{q^{n/2}+1} = 1$. Supposing x is conjugate to x^{-1} implies that $x_1^{(-q)^d+1} = 1$ for some $d \leq n/2$. Then for a prime $r \neq 2$ dividing the order of x_1 we have that $q^{n/2} \equiv -1 \bmod r$, so the order of $q \bmod r$ is divisible by 2 but not 4. But also $(-q)^d \equiv -1 \bmod r$, so if d is odd then the order of $q \bmod r$ is odd, and if d is even then the order is divisible by 4, a contradiction in either case.

Therefore $\tilde{s}$ is not conjugate to $\tilde{s}^{-1}$, so χ and $\overline{\chi}$ are in different blocks; hence τ does not stabilise B_{GL}, so restricted to $\mathbf{G}^F$ does not stabilise $b_{\mathbf{G}^F}$. Therefore the graph automorphisms of G do not stabilise b, so σ is not a graph automorphism. Additionally Remark 5.4 implies that the diagonal autmorphisms of order 2 stabilise b, so neither is σ the product of a diagonal and graph automorphism, and finally $\mathbf{G}$ cannot be of type A.

(iii) G of Type D: Since $|Z| \leq 4$ we have $|T|_2 \leq 2$, so by [14, Prop. 10] we must have $[\mathbf{L}^F, \mathbf{L}^F] = \mathrm{SL}_m(q^{n/m})$ and $|T| = q^{n/m}-1$ where n/m is odd. Note that $|Z(\mathbf{G})|$ is even, so b_X must be isolated, and the only component of L is $X = \mathrm{SL}_2(q^{n/2})/Z$. Additionally [14, Prop. 10] states that the total number of components of type ${}^2A_m$ with m even and ${}^2D_m$ for any m is even if and only if F is not twisted, so $\mathbf{G}^F = D_n(q)$.

The Dynkin diagram of $\mathbf{L}$ within that of $\mathbf{G}$ is given below. Note that it cannot contain the two leftmost nodes which would be considered in [14] as $D_2 \cong A_1 \times A_1$ or ${}^2D_2 \cong A_1(q^2)$.

D_n 

Considering σ as an automorphism of $\mathbf{G}^F$ that stabilises $b_{\mathbf{G}^F}$, by [53, Prop. 4.9] there is some automorphism $\sigma' \in \mathrm{Aut}(\mathbf{G}^F)$ in the coset $\overline{\sigma} \in \mathrm{Out}(\mathbf{G}^F)$ that also stabilises $\mathbf{L}^F$. As above, $\mathbf{L}$ contains one but not both of the leftmost nodes, so $[\mathbf{L}, \mathbf{L}]$ is contained in one but not both of the two Levi subgroups of $\mathbf{G}$ of the form A_{n-1}. The graph automorphism τ of $\mathbf{G}$ swaps the two leftmost nodes, and since n is even the two A_{n-1} Levi subgroups are not conjugate by [44, Prop. 2.6]; hence it does not stabilise $\mathbf{L}^F$, and therefore σ is not a graph automorphism.

If $\rho \in \mathrm{Aut}(\mathbf{G}^F)$ is a diagonal automorphism then there is some $g \in \mathrm{Inn}(\mathbf{G}^F)$ such that $g\rho$ stabilises $\mathbf{L}^F$, so, since the two A_{n-1} Levi subgroups are not conjugate and τ does not stabilise $\mathbf{L}^F$, neither does $g'\rho\tau$ for any $g' \in \mathrm{Inn}(\mathbf{G}^F)$; hence neither is σ the product of a diagonal automorphism and graph automorphism.

Therefore σ must be a diagonal automorphism, and there is a simple algebraic group $\mathbf{H}$ (one of the three intermediate groups of type D, a special orthogonal or half-spin group, whose fixed point groups are each of the form $2.G.2$) with a central subgroup Z' of order 2 such that $H = \mathbf{H}^F/Z' \cong G.2$. Consider B as a block of H with defect group D.

There are no non-trivial quasi-isolated $2'$-elements for type D and the principal blocks do not have dihedral or semidihedral defect groups for any isogeny type, so B is not quasi-isolated. Theorem 3.9 then implies that B is Morita equivalent to a block B_M of some $M = \mathbf{M}^F/Z'$, where $\mathbf{M}$ is a proper F-stable Levi subgroup of $\mathbf{H}$, also with defect group D.

Consider a block of $[M, M]$ covered by B_M with defect group $P = D \cap [M, M]$ as in Lemma 4.6. Then, as in the proof of Proposition 4.4, we have that P is 1, the index-2 dihedral subgroup of D, or D; but D/P is abelian so $P \neq 1$. In either case P cannot be expressed as a non-trivial central product, so there is a block B_Y of some quasi-simple $Y \trianglelefteq M$ with defect group P. If $P = D$ then B_Y is Morita equivalent to B by Remark 4.5; otherwise we see that B_Y is in $D(3\mathcal{K})$, as in Proposition 4.3.

Notice that the rank of Y as a group of Lie type is at least 1 but strictly less than G and H, and that Y must be of type A or D, with B_Y in $D(3\mathcal{K})$ or $SD(2\mathcal{B})_1$.

Consider Y of type A, so $Y = \mathrm{SL}_{m'}(\varepsilon' q^{a'})/Z'$ for some $\varepsilon' = \pm 1, m', a'$. Then if B_Y is in $SD(2\mathcal{B})_1$ it cannot be quasi-isolated, otherwise it would have appeared in Section 5. Hence we can do with Y as with H above, and get a block of a quasi-simple normal subgroup of some Levi subgroup of Y (which now must be of type A) that is in $D(3\mathcal{K})$ or $SD(2\mathcal{B})_1$. Therefore, by induction on the rank of H, we can assume that B_Y is in $D(3\mathcal{K})$ and is covered by a block of H in $SD(2\mathcal{B})_1$. Set $T = \{g \in M \mid {B_Y}^g = B_Y\}$; by Lemma 3.16 there is a block of T covering B_Y that is Morita equivalent to B_M with defect group $\tilde{D} \cong D$. Then, since $Y \trianglelefteq T$ and B_Y is T-invariant, we can consider the unique block $B_{\tilde{D}Y}$ of $\tilde{D}Y$ covering B_Y, which by [38, Ch. 10, Thm 5.10] also has semidihedral defect group $\tilde{D}$. Then, since $|\tilde{D}Y : Y| = 2$ and B_Y is in $D(3\mathcal{K})$, we see that $B_{\tilde{D}Y}$ must be in $SD(2\mathcal{B})_1$. But we have shown in *(ii)* that a block of Y, being of type A, in $D(3\mathcal{K})$ cannot be covered by a block of $Y.2$ in $SD(2\mathcal{B})_1$.

If Y is of type D, and if B_Y is in $SD(2\mathcal{B})_1$, then we can similarly use induction on the rank of H to assume that B_Y is in $D(3\mathcal{K})$ (with Y of type D since we have just shown that it cannot be of type A) and is covered by a block of $Y.2$ in $SD(2\mathcal{B})_1$. But then we can consider Y and B_Y in place of G and b throughout *(iii)*, and induction on the rank of G shows that this case is also not possible.

(iv) $\mathbf{G}$ *of Type* E_7: If $\mathbf{G}^F = E_7(q)_{sc}$ then σ must be a diagonal automorphism and $G.2 = E_7(q)_{ad} \cong (\mathbf{G}^*)^F$. If B is not quasi-isolated, then as above there is a block of a quasi-simple normal subgroup Y of some Levi subgroup of $(\mathbf{G}^*)^F$ in $D(3\mathcal{K})$ or $SD(2\mathcal{B})_1$, and we have shown that such Y cannot be of type A or D. Then Y is of type E_6, so the block cannot be in $D(3\mathcal{K})$ since the group's centre has odd order; using our earlier argument that Z must be non-trivial. But if it is in $SD(2\mathcal{B})_1$ then in the same way we can get a block of a quasi-simple normal subgroup of some Levi subgroup of Y in $D(3\mathcal{K})$

or $SD(2\mathcal{B})_1$; now this must be of type A or D, so we have shown that this case is not possible. Hence B must be quasi-isolated, but not unipotent by Section 5.3.1.

Again $\left|Z(\mathbf{G}^F)\right|$ is even, so b_X must be isolated, implying that $X = \mathrm{SL}_2(q^a)/Z$ and $[\mathbf{L}^F, \mathbf{L}^F] = \mathrm{SL}_2(q^a)$. Since $|Z| = 2$ we have that $|T|$ is odd, so from the possible connected centralisers of E_7 listed in [17, Table 1], we have that $C_{\mathbf{G}^*}(s)^F = \mathrm{PGL}_2(q^3).S$, where $|S| = \Phi_3^2$, Φ_6^2, or Φ_{12}. Then $b_{\mathbf{L}^F}$ is Morita equivalent to the principal block of $\mathrm{SL}_2(q^3)$ by Lemma 4.1, which is therefore Morita equivalent to $b_{\mathbf{G}^F}$; so the ordinary character degrees of $b_{\mathbf{G}^F}$ are those of the principal block of $\mathrm{SL}_2(q^3)$ multiplied by $\left|E_7(q)_{ad} : C_{\mathbf{G}^*}(s)^F\right|_{p'}$ according to Theorem 3.5. Hence the ordinary character degrees of b are among these, and those of B are also among these or twice these if they fuse on induction to $G.2$; let d be equal to or twice the degree of an ordinary character of the principal block of $\mathrm{SL}_2(q^3)$.

On the other hand, we can see from [4, Table III] that $C_{\mathbf{G}}(s')^F = \mathrm{SL}_6(\varepsilon_1 q).\mathrm{SL}_3(\varepsilon_2 q)$, where $\varepsilon_1, \varepsilon_2 = \pm 1$ and $s' \in E_7(q)_{sc}$ is the semisimple label for B; note that $C_{\mathbf{G}}(s')$ is connected since $\mathbf{G}^*$ has connected centre, and $C_{\mathbf{G}}(s') = [C_{\mathbf{G}}(s'), C_{\mathbf{G}}(s')]$ is simply connected since $(\mathbf{G}^*)^* \cong \mathbf{G}$ is. Then B must contain some character in $\mathcal{E}(E_7(q)_{ad}, s')$, corresponding to a unipotent character χ_u of $C_{\mathbf{G}}(s')^F$, with degree $\chi_u(1) \cdot \left|E_7(q)_{sc} : C_{\mathbf{G}}(s')^F\right|_{p'}$ by Theorem 3.5.

But the degree of each of the unipotent characters of $C_{\mathbf{G}}(s')^F$ is too small. Indeed, we must have

$$\chi_u(1) \cdot \left|E_7(q)_{sc} : C_{\mathbf{G}}(s')^F\right|_{p'} = d \cdot \left|E_7(q)_{ad} : C_{\mathbf{G}^*}(s)^F\right|_{p'},$$

that is,

$$\chi_u(1) \cdot \left|\mathrm{PGL}_2(q^3)\right|_{p'} \cdot |S| = d \cdot \left|\mathrm{SL}_6(\varepsilon_1 q).\mathrm{SL}_3(\varepsilon_2 q)\right|_{p'}. \tag{$*$}$$

Considering the 2-part of $(*)$ gives

$$\left|\chi_u(1) \cdot \Phi_1\Phi_2\right|_2 \geq \left|\mathrm{SL}_6(\varepsilon_1 q)\right|_2 \geq \left|\Phi_1^3\Phi_2^3\right|_2,$$

so we must have $\left|\chi_u(1)\right|_2 \geq \left|\Phi_1^2\Phi_2^2\right|_2$.

Note that χ_u is a product of unipotent characters χ_6 and χ_3 of $\mathrm{SL}_6(\varepsilon_1 q)$ and $\mathrm{SL}_3(\varepsilon_2 q)$ respectively. Calculating these degrees using [15, Section 13.8], each possible χ_3 has $\left|\chi_3(1)\right|_2 \leq |\Phi_2|_2$, and the only possible χ_6 whose degree has 2-part greater than 2 has degree $q^4\Phi_2^3\Phi_4\Phi_6$. But as in [47] for $q \equiv \bmod 2$ the power of q in any ordinary character degree of $\mathrm{SL}_2(q^3)$, and hence the power of q in d and the right-hand side of $(*)$, is at most q^3, so this is not possible.

Hence B is not quasi-isolated, $\mathbf{G}$ is not of type E_7, and finally B is not in $SD(2\mathcal{B})_1$. □

Therefore if B is in $SD(2\mathcal{B})_1$ with defect groups of order at least 32 then it does not cover a block with dihedral defect groups.

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