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RESEARCH ARTICLE

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Hamilton transversals in random Latin squares

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Abstract

Gyárfás and Sárközy conjectured that every $n \times n$ Latin square has a “cycle-free” partial transversal of size $n-2$. We confirm this conjecture in a strong sense for almost all Latin squares, by showing that as $n \rightarrow \infty$, all but a vanishing proportion of $n \times n$ Latin squares have a *Hamilton* transversal, that is, a full transversal for which any proper subset is cycle-free. In fact, we prove a counting result that in almost all Latin squares, the number of Hamilton transversals is essentially that of Taranenko’s upper bound on the number of full transversals. This result strengthens a result of Kwan (which in turn implies that almost all Latin squares also satisfy the famous Ryser–Brualdi–Stein conjecture).

KEYWORDS

arc-colouring, distributive absorption, Hamilton transversals, Latin squares, rainbow cycle, transversals

1 | INTRODUCTION

1.1 | Transversals in Latin squares

An $n \times n$ Latin square is an arrangement of n symbols into n rows and n columns, such that each row and each column contains precisely one instance of each symbol. A (*full*) *transversal* in an $n \times n$ Latin square is a collection of n positions of the Latin square that use each row, column, and symbol exactly once, and a *partial transversal* is a collection of at most n positions not using any row, column, or symbol more than once. The most famous open problem on the topic of transversals in Latin squares is the following.

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Conjecture 1.1 (Ryser, Brualdi, and Stein [8, 32, 34]). *All $n \times n$ Latin squares have a partial transversal of size $n - 1$.*

Conjecture 1.1 would be best-possible, because for even n the addition table of the integers modulo n is a Latin square which has no transversal. If n is odd, it is actually conjectured that all $n \times n$ Latin squares have a full transversal. For nearly forty years the best result toward Conjecture 1.1 was the theorem of Hatami and Shor [19, 33] (improving [7, 37]) that all $n \times n$ Latin squares have a partial transversal of size $n - O(\log^2 n)$. Recently however, Keevash, Pokrovskiy, Sudakov, and Yepremyan [22] improved the error term to $O(\log n / \log \log n)$.

Conjecture 1.1 is related to the following conjecture of Andersen [3]. An edge-colored graph is *rainbow* if all of its edges have different colors, and an edge-coloring is *proper* if no two edges of the same color share a vertex.

Conjecture 1.2 (Andersen [3]). *All proper edge-colorings of K_n , the complete graph on n vertices, admit a rainbow path of length $n - 2$.*

In light of the result of Maamoun and Meyniel [27] that for infinitely many n there are proper edge-colorings of K_n without a rainbow Hamilton path, Conjecture 1.2 would be best-possible. Similarly to Conjecture 1.1, progress toward Conjecture 1.2 has largely focused on increasing the length of the longest rainbow path known to exist for any proper edge-coloring of K_n (see, e.g., [10, 13, 16, 17]). Alon, Pokrovskiy, and Sudakov [1] were the first to asymptotically prove Conjecture 1.2 by exhibiting the existence of a rainbow path of length $n - O(n^{3/4})$, with the best known error bound now being $O(n^{1/2} \log n)$, provided by Balogh and Molla [4].

Let \vec{K}_n be the digraph obtained from the complete n -vertex graph K_n by replacing each edge with two arcs (one in each direction) and adding a directed loop at each vertex. For every $n \times n$ Latin square, we can uniquely associate an arc-coloring of \vec{K}_n as follows: for every position (i, j) of the Latin square, assign the symbol of (i, j) as a color to the arc in \vec{K}_n with tail i and head j . Importantly, a partial transversal corresponds to a rainbow subgraph of \vec{K}_n with maximum in-degree and out-degree one. A set of positions is a *cycle* if the corresponding subgraph of \vec{K}_n is a directed cycle, and a partial transversal is *cycle-free* if it contains no cycle. Thus, cycle-free partial transversals correspond to linear directed forests in \vec{K}_n . Gyarfas and Sarkozy [18] proposed the following conjecture, which combines aspects of Conjectures 1.1 and 1.2.

Conjecture 1.3 (Gyarfas and Sarkozy [18]). *All $n \times n$ Latin squares have a cycle-free partial transversal of size $n - 2$.*

A *proper k -arc-coloring* of a digraph is a coloring of its arcs with k colors such that no two arcs of the same color have a common head, or a common tail. The set of $n \times n$ Latin squares is in fact in bijection with the set of proper n -arc-colorings of \vec{K}_n , with the correspondence described above. Thus, Conjecture 1.3 is equivalent to the following: all proper n -arc-colorings of \vec{K}_n contain a rainbow directed linear forest with at least $n - 2$ arcs. No undirected analog of this conjecture is known—Balogh and Molla [4] proved that for every proper edge-coloring of K_n , there is a rainbow linear forest with at least $n - O(\log^2 n)$ edges, and this bound is the best known.

Less is known in the directed setting. Gyarfas and Sarkozy [18] proved that every $n \times n$ Latin square has a cycle-free partial transversal of size $n - O(n \log \log n / \log n)$, and Benzing, Pokrovskiy, and Sudakov [5] improved the error bound to $O(n^{2/3})$. Benzing, Pokrovskiy, and Sudakov [5] also

proved that every proper arc-coloring of \vec{K}_n contains a rainbow directed cycle of size $n - O(n^{4/5})$ and asked by how much this bound can be improved. We believe it is also interesting to consider by how much this bound can be improved if one considers both rainbow directed cycles and paths, that is, rainbow *connected* subgraphs of maximum in-degree and out-degree at most one. We conjecture the following.

Conjecture 1.4. *All proper arc-colorings of \vec{K}_n admit a rainbow directed cycle or path of length at least $n - 1$.*

We define a set of positions in a Latin square to be *connected* if the corresponding subgraph of \vec{K}_n is (weakly) connected, and we say a transversal is *Hamilton* if it is both full and connected. For the case of n -arc-colorings, Conjecture 1.4 is equivalent to the following: all $n \times n$ Latin squares have a connected transversal of size $n - 1$. If true, Conjecture 1.3 implies that every $n \times n$ Latin square has a partial transversal of size one less than what is predicted by Conjecture 1.1 and also that every proper n -edge-coloring of K_n contains either a rainbow path of length $n - 2$, as predicted by Conjecture 1.2, or a spanning rainbow forest with two components. Conjecture 1.4, if true, implies all of Conjectures 1.1–1.3.

1.2 | Random Latin squares

In this article, we study the above conjectures in the probabilistic setting. Recently, Kwan [23] proved that at most a vanishing proportion of Latin squares fail to satisfy the statement of Conjecture 1.1, even finding (many) full transversals in most Latin squares, as follows.

Theorem 1.5 (Kwan [23]). *Almost all $n \times n$ Latin squares have at least*

$$\left((1 - o(1)) \frac{n}{e^2} \right)^n$$

transversals.

Equivalently, a uniformly random $n \times n$ Latin square has at least $((1 - o(1))n/e^2)^n$ transversals with high probability. We note that it was proven by Taranenko [35] (with a simpler proof later found by Glebov and Luria [14]) that $n \times n$ Latin squares can have at most $((1 + o(1))n/e^2)^n$ transversals, so that the counting term given in Theorem 1.5 is best possible, up to the exponential error term. Analogously, the authors, together with Kühn and Osthus [15], proved that almost all optimal edge-colorings (proper edge-colorings using the minimum possible number of colors) of K_n admit a rainbow Hamilton path, which proves a stronger statement than Conjecture 1.2 for all but a vanishing proportion of such colorings.

The main result of this article is the following strengthening of Theorem 1.5.

Theorem 1.6. *Almost all proper n -arc-colorings of \vec{K}_n contain at least*

$$\left((1 - o(1)) \frac{n}{e^2} \right)^n$$

rainbow directed Hamilton cycles. Equivalently, almost all $n \times n$ Latin squares have at least $((1 - o(1))n/e^2)^n$ Hamilton transversals.

Theorem 1.6 implies that a uniformly random proper n -arc-coloring satisfies Conjecture 1.4 with high probability, which in turn implies that a uniformly random $n \times n$ Latin square satisfies Conjectures 1.1 and 1.3 with high probability as well. We note that the number of optimal edge-colorings of K_n is a vanishing fraction of the number of $n \times n$ Latin squares, so Theorem 1.6 does not imply the result of [15].

Random Latin squares can be difficult to analyze, in part due to their “rigidity” and lack of independence. To prove Theorem 1.5, Kwan [23]—using Keevash’s [20, 21] breakthrough results on the existence of combinatorial designs—developed a method for approximating a uniformly random Latin square by an outcome of the “triangle-removal process,” which is in comparison much easier to analyze. Prior to Kwan’s [23] work, a limited number of results (e.g., [9, 26, 29, 36]) were proved using so-called “switching” methods. Our proof, notably, does not rely on Keevash’s [20, 21] results and instead introduces new techniques for analyzing “switchings” to study Latin squares, thus providing a more elementary proof of Theorem 1.5.

1.3 | Organization of the article

In Section 2, we clarify some notation and definitions that we will use throughout the article. We overview the proof of Theorem 1.6 in Section 3 and give some preliminary probabilistic results and useful theorems of other authors in Section 4. Sections 5–7 are devoted to the proof of Theorem 1.6.

2 | NOTATION

For a natural number n , we define $[n] := \{1, 2, \dots, n\}$ and $[n]_0 := [n] \cup \{0\}$. We say that a partition $\mathcal{P} = \{D_i\}_{i=1}^m$ of a finite set D into m parts is *equitable* if $|D_i| \in \{\lfloor |D|/m \rfloor, \lceil |D|/m \rceil\}$ for all $i \in [m]$, and when $|D|$ is large we assume that each part D_i has the same size $|D|/m$, where this does not affect the argument.

For a digraph G , we write the arc set of G as $E(G)$, and we denote an arc from a vertex u to a vertex v as uv , and we say that u is the *tail* of the arc $e = uv$, denoted $u = \text{tail}(e)$, and that v is the *head* of e , denoted $v = \text{head}(e)$. We say that any vertex v such that $uv \in E(G)$ is an *out-neighbor* of u in G , and that any v such that $vu \in E(G)$ is an *in-neighbor* of u in G . We define $N_G^+(v)$ to be the set of out-neighbors of v in G , sometimes dropping the subscript G when G is clear from context, and we call $N_G^+(v)$ the *out-neighborhood* of v in G . We define the *in-neighborhood* of v in G , denoted $N_G^-(v)$, analogously, and we define the *neighborhood* of v in G to be $N_G(v) := N_G^+(v) \cup N_G^-(v)$. We define $d_G^+(v) := |N_G^+(v)|$ and $d_G^-(v) := |N_G^-(v)|$. For (not necessarily distinct) vertex sets $A, B \subseteq V(G)$ we define $E_G(A, B) := \{ab \in E(G) : a \in A, b \in B\}$, and $e_G(A, B) := |E_G(A, B)|$. Suppose now that G is equipped with an arc-coloring ϕ_G in color set C . Then for a color $c \in C$ and an arc $e \in E(G)$ we write $\phi_G(e) = c$ to mean that e has color c in the coloring ϕ_G of G . We frequently drop the notation G when G is clear from context. Further, if $\phi(e) = c$ then we say that e is a c -arc, and in the case that e is a loop we say that e is a c -loop. We write $E_c(G)$ for the set of c -arcs in G (including c -loops), and we refer to $E_c(G)$ as the *color class* of c . Fix $u \in V(G)$. If $d \in C$ is such that there is a d -arc uv in G , then the (unique) vertex v is called the *d -out-neighbor* of u , which we denote by $N_d^+(u)$. We define the *d -in-neighbor* $N_d^-(u)$ of u analogously. For $D \subseteq C$ we define $N_D^+(u) := \{N_d^+(u) : d \in D\}$ and $N_D^-(u) := \{N_d^-(u) : d \in D\}$, and for $A, B \subseteq V$ we define $E_{G,D}(A, B) := \{e \in E_G(A, B) : \phi(e) \in D\}$ and $e_{G,D}(A, B) := |E_{G,D}(A, B)|$. For a subdigraph $H \subseteq G$, we define $\phi_G(H) := \{\phi_G(e) : e \in E(H)\}$.

With a slight abuse of notation, we often refer to a pair (H, ϕ) where H is a digraph and ϕ is a proper arc-coloring of H as a “colored digraph” H implicitly equipped with a proper arc-coloring ϕ_H

(or simply ϕ if it is clear from the context). Using this convention, we let $\Phi(\vec{K}_n)$ denote the set of all properly n -arc-colored digraphs $G \cong \vec{K}_n$ with vertex set and color set $[n]$. (That is, the set of pairs (G, ϕ) where $G \cong \vec{K}_n$ and ϕ is a proper n -arc-coloring of G .) For a colored digraph $G \in \Phi(\vec{K}_n)$ and a set of colors $D \subseteq [n]$, we define $G|_D$ to be the colored digraph obtained by deleting all arcs of G having colors not in D , and we set $\mathcal{G}_D^n := \{G|_D : G \in \Phi(\vec{K}_n)\}$, though we always drop the n in the superscript as n will be clear from context. By symmetry of the roles of rows, columns, and symbols in Latin squares, the correspondence between Latin squares and elements $G \in \Phi(\vec{K}_n)$, and the well-known result that any Latin rectangle has a completion to a Latin square, it is clear that \mathcal{G}_D could be equivalently defined as the set of all pairs (H, ϕ_H) , where H is a $|D|$ -regular digraph on vertices $[n]$, and ϕ_H is a proper arc-coloring of H in colors D . Throughout the article, we will use the letter G for an element of $\Phi(\vec{K}_n)$, and the letter H for an element of \mathcal{G}_D (for any D). We often write random variables and objects in bold notation. For an event \mathcal{E} in any probability space, we use the notation $\bar{\mathcal{E}}$ to denote the complement of \mathcal{E} .

3 | OVERVIEW OF THE PROOF

The proof of Theorem 1.6 proceeds in two key steps. We first analyze uniformly random $\mathbf{G} \in \Phi(\vec{K}_n)$ and show that with high probability, \mathbf{G} satisfies three key properties. It then suffices to suppose that a fixed $G \in \Phi(\vec{K}_n)$ satisfies these three properties, and use that hypothesis to build many rainbow directed Hamilton cycles in G . Before describing these properties, we discuss our strategy for building rainbow directed Hamilton cycles. To that end, we introduce the following definition.

Definition 3.1. A subgraph $H \subseteq G \in \Phi(\vec{K}_n)$ is *robustly rainbow-Hamiltonian* (with respect to flexible sets $V_{\text{flex}} \subseteq V(H)$ and $C_{\text{flex}} \subseteq \phi(H)$ of vertices and colors, and initial vertex $u \in V(H)$ and terminal vertex $v \in V(H)$), if for any pair of equal-sized subsets $X \subseteq V_{\text{flex}}$ and $Y \subseteq C_{\text{flex}}$ of size at most $\min\{|V_{\text{flex}}|/2, |C_{\text{flex}}|/2\}$, the graph $H - X$ contains a rainbow directed Hamilton path with tail u and head v , not containing a color in Y .

We show that for almost all $G \in \Phi(\vec{K}_n)$ and arbitrary sets $V_{\text{flex}}, C_{\text{flex}} \subseteq [n]$ of sizes $|V_{\text{flex}}| = |C_{\text{flex}}| = \Omega(n/\log^3 n)$, G contains a robustly rainbow-Hamiltonian subgraph H with flexible sets V_{flex} and C_{flex} , such that H has $O(n/\log^3 n)$ vertices and arcs in total. We construct rainbow directed Hamilton cycles by using the popular “absorption” method, and H will form the key absorbing structure. More precisely, we find a rainbow directed path P having the terminal vertex v of H as its tail, the initial vertex u of H as its head, such that $V(G) \setminus V(H) \subseteq V(P)$, $V(P) \cap V(H)$ is a subset of V_{flex} of size at most $|V_{\text{flex}}|/2$, and likewise for the colors. Letting $X := V(P) \cap V(H)$ and $Y := \phi(P) \cap \phi(H)$, the robust rainbow-Hamiltonicity of H guarantees there is a rainbow directed Hamilton path P' in $H - X$ with tail u and head v , not containing a color in Y , and $P \cup P'$ is a rainbow directed Hamilton cycle.

We find H by piecing together smaller building blocks we call “absorbers” in a delicate way, where each absorber has the ability to “absorb” a vertex v and a color c not used by P . We delay a definition of a (v, c) -absorber to Definition 6.1, but we give a figure now (see Figure 1). Notice that a (v, c) -absorber has a rainbow directed Hamilton path with tail x_1 and head x_6 , and a rainbow directed path with the same head and tail using all vertices except v and all colors except c . This is the key property of a (v, c) -absorber, and by piecing these together in a precise way we ensure that the resulting union of absorbers (with the sets of specified vertices “ v ” and colors “ c ” forming V_{flex} and C_{flex} , respectively) has the desired robustly rainbow-Hamiltonian property. For technical reasons, we find (v, c) -absorbers by piecing together two smaller structures we call (v, c) -absorbing gadgets and (y, z) -bridging gadgets

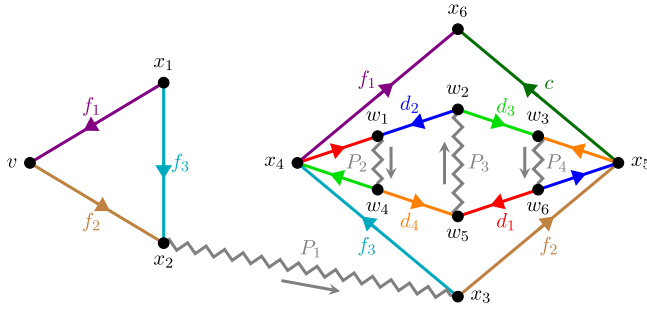


FIGURE 1 A (v, c) -absorber. Here $\phi(x_4w_1) = d_1$, $\phi(w_6x_5) = d_2$, $\phi(w_4x_4) = d_3$, and $\phi(x_5w_3) = d_4$. P_1, \dots, P_4 are rainbow directed paths with directions as indicated, sharing no colors with each other or with the rest of the (v, c) -absorber.

(see Definitions 5.1 and 5.2, respectively), together with the short rainbow directed paths P_1, P_2, P_3, P_4 as in Figure 1.

Thus, the first key property that we need almost all $G \in \Phi(\vec{K}_n)$ to satisfy, is that G contains many absorbing gadgets and bridging gadgets, in a “well-spread” way that enables us to construct an appropriate robustly rainbow-Hamiltonian subgraph. We prove this in Section 5 using “switchings” in Latin rectangles, then using permanent estimates (see [6, 11, 12], encapsulated by Proposition 4.4 in the current article) to compare a uniformly random $k \times n$ Latin rectangle to the first k rows of a uniformly random $n \times n$ Latin square. Lemma 5.6 ensures the existence of the absorbing gadgets we need, and Lemma 5.10 accomplishes the same for the bridging gadgets. This approach of using permanent estimates to translate statements between these probability spaces was pioneered by McKay and Wanless [29], who investigated the typical prevalence of 2×2 Latin subsquares (also called “intercalates”) in a uniformly random Latin square. For further insight into the usage of this method to study intercalates in random Latin squares, see, for example, [24–26]. As the substructures we seek are more complex than intercalates, and we moreover require that they are “well-spread,” our proof introduces new techniques for switching arguments in Latin rectangles. We note that in [15], the authors, with Kühn and Osthus, used switching arguments to analyze a uniformly random 1-factorization of K_n and show that with high probability there is a large collection of subgraphs of a form analogous to that of our (v, c) -absorbing gadgets in the undirected setting. Fortunately, this argument also works in the directed setting with only minor changes that we do not cover here, instead providing the proof of Lemma 5.6 in the appendix of the arXiv version of the paper. Thus, Section 5 is primarily devoted to the proof of Lemma 5.9.

The second property of almost all $G \in \Phi(\vec{K}_n)$ that we will need concerns the colors of the loops. Clearly, if we seek to find any rainbow directed Hamilton cycle of $G \in \Phi(\vec{K}_n)$, we need to know that there is no color appearing only on loops in G , and this is given for almost all $G \in \Phi(\vec{K}_n)$ (in the context of Latin squares and in considerably stronger form) by Lemma 4.6.

The third and final property of almost all $G \in \Phi(\vec{K}_n)$ that we will need is an appropriate notion of “lower-quasirandomness,” which roughly states that for any two subsets U_1, U_2 of vertices of G and any set D of colors, the number of arcs in G with tail in U_1 , head in U_2 , and color in D , is close to what we would expect if the colors of the arcs of G were assigned independently and uniformly at random. We delay the precise definition of lower-quasirandomness of $G \in \Phi(\vec{K}_n)$ to Definition 7.1. The desired property that almost all $G \in \Phi(\vec{K}_n)$ are lower-quasirandom will follow immediately from [26, Theorem 2] (see Theorem 4.7 of the current article), originally stated in the context of “discrepancy” of random Latin squares.

Armed with the three properties of typical $G \in \Phi(\vec{K}_n)$ described above, it then suffices to fix such a G and build many rainbow directed Hamilton cycles. In Section 6, we show that the existence of many well-spread absorbing and bridging gadgets enables us to greedily build a small robustly rainbow-Hamiltonian subgraph $H \subseteq G$ with arbitrary flexible sets V_{flex} and C_{flex} of size $\Theta(n/\log^3 n)$, and in Section 7, we use this to prove Theorem 1.6. The rough idea is to first choose the flexible sets V_{flex} and C_{flex} randomly. Next, we use the lower-quasirandomness property of G to build a rainbow directed spanning path forest Q of $G - H$, one arc at a time, until Q has very few components. Then, we use the random choice of V_{flex} and C_{flex} , together with Lemma 4.6, to find short rainbow directed paths linking the components of Q and the designated start and end of H , which use all remaining colors of $G - H$, and at most half of V_{flex} and C_{flex} . Finally, we use the key robustly rainbow-Hamiltonian property of H to absorb the remaining vertices and colors in V_{flex} and C_{flex} as described above, completing the rainbow directed Hamilton cycle of G . To obtain the counting result on the number of rainbow directed Hamilton cycles in G , it suffices to count the number of choices we can make while building the rainbow directed spanning path forest Q of $G - H$.

We remark that this particular absorption strategy, wherein we create an absorbing structure with “flexible” sets, is an instance of the “distributive absorption” method, which was introduced by Montgomery [30] in 2018 and has been found to have several applications since. In particular, this method is also used in [15, 23] to find transversals in random Latin squares and rainbow Hamilton paths in random 1-factorizations, respectively. Our approach differs from that of [15, 23] in a few key ways. First, the “asymmetry” of proper n -arc-colorings of \vec{K}_n (in comparison to proper edge-colorings of K_n with at most n colors, which correspond to proper n -arc-colorings of \vec{K}_n with monochromatic digons) and “connectedness” of rainbow Hamilton cycles/Hamilton transversals (in comparison to general transversals in Latin squares) necessitate a more complex absorbing structure than the one of either [23] or [15], which is more challenging to create and construct. Nevertheless, as mentioned, we show that switching arguments are sufficient for finding our absorbing structure, yielding a more elementary proof than that of [23], and moreover, by choosing our flexible sets randomly, we avoid complications involving vertices with few out- or in-neighbors in V_{flex} on arcs with color in C_{flex} , providing a further simplification of the approach in [23]. In [15], results [2, 31] on nearly perfect matchings in nearly regular hypergraphs are applied to auxiliary hypergraphs to construct both the absorbing structure and a nearly spanning rainbow path in a random 1-factorization of K_n , but since the absorbers we use here (minus the internal vertices of the linking paths P_1, \dots, P_4) are not regular, the analogous approach fails in the directed setting (as the corresponding auxiliary hypergraphs are not regular). However, as we show, the “lower-quasirandomness” of typical $G \in \Phi(\vec{K}_n)$ is enough for us to find Q , the nearly spanning rainbow path forest, without these hypergraph matching results, and our absorbing structure is robust enough to augment it to a rainbow directed path.

4 | PRELIMINARIES

In this brief section, we state some results that we will use in the proof of Theorem 1.6. We begin with a well-known concentration inequality for independent random variables.

Let X_1, \dots, X_m be independent random variables taking values in \mathcal{X} , and let $f : \mathcal{X}^m \rightarrow \mathbb{R}$. If for all $i \in [m]$ and $x'_i, x_1, \dots, x_m \in \mathcal{X}$, we have

$$|f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_m) - f(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_m)| \leq c_i,$$

then we say X_i affects f by at most c_i .

Theorem 4.1. (McDiarmid's inequality [28]). *If X_1, \dots, X_m are independent random variables taking values in \mathcal{X} and $f : \mathcal{X}^m \rightarrow \mathbb{R}$ is such that X_i affects f by at most c_i for all $i \in [m]$, then for all $t > 0$,*

$$\mathbb{P} \left[|f(X_1, \dots, X_m) - \mathbb{E} [f(X_1, \dots, X_m)]| \geq t \right] \leq \exp \left(-\frac{2t^2}{\sum_{i=1}^m c_i^2} \right).$$

Next, we need the notion of “robustly matchable” bipartite graphs, which will form a key part of our absorption argument.

Definition 4.2. Let T be a bipartite graph with bipartition (A, B) such that $|A| = |B|$.

- We say T is *robustly matchable* with respect to *flexible sets* $A' \subseteq A$ and $B' \subseteq B$, if for every pair of equal-sized subsets $X \subseteq A'$ and $Y \subseteq B'$ of size at most $\min\{|A'|/2, |B'|/2\}$, there is a perfect matching in $T - (X \cup Y)$.
- For $m \in \mathbb{N}$, we say T is a $2RMBG(7m, 2m)$ if $|A| = |B| = 7m$ and T is robustly matchable with respect to flexible sets $A' \subseteq A$ and $B' \subseteq B$ where $|A'| = |B'| = 2m$.

The concept of using robustly matchable bipartite graphs in absorption arguments was first introduced by Montgomery [30]. We need the following observation of the authors, Kühn, and Osthus [15, Lemma 4.5], which is based on the work of Montgomery.

Lemma 4.3 (Gould, Kelly, Kühn, and Osthus [15]). *For all sufficiently large m , there is a $2RMBG(7m, 2m)$ that is 256-regular.*

For a colored digraph $H \in \mathcal{G}_D$, we define $\text{comp}(H)$ to be the number of distinct ways to complete H to an element $G \in \Phi(\vec{K}_n)$, or more precisely the number of $H' \in \mathcal{G}_{[n] \setminus D}$ having $E(H) \cap E(H') = \emptyset$ (and therefore $E(H) \cup E(H') = E(\vec{K}_n)$). We will use the following proposition to compare the probabilities of events in the probability spaces corresponding to uniformly random $\mathbf{H} \in \mathcal{G}_D$ (for some small $D \subseteq [n]$) and uniformly random $\mathbf{G} \in \Phi(\vec{K}_n)$ (see, e.g., the proof of Lemma 5.10).

Proposition 4.4. *For any $D \subseteq [n]$ and $H, H' \in \mathcal{G}_D$ we have*

$$\frac{\text{comp}(H)}{\text{comp}(H')} \leq \exp(O(n \log^2 n)).$$

Proposition 4.4 follows immediately from, for example, [26, Proposition 5] as \mathcal{G}_D can easily be seen to be equivalent to the set of $|D| \times n$ Latin rectangles.

Next, we show (in the context of Latin squares) that a uniformly random $\mathbf{G} \in \Phi(\vec{K}_n)$ does not have too many loops of a fixed color. We first need the following well-known result on the number of fixed points of a random permutation.

Lemma 4.5. *Let σ be a uniformly random permutation of $[n]$, and let \mathbf{X} denote the number of fixed points of σ . Then, for $k \in [n]_0$, we have $\mathbb{P}[\mathbf{X} = k] = \frac{1}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!}$.*

Lemma 4.6. *Let \mathbf{L} be a uniformly random $n \times n$ Latin square with entries in $[n]$, and suppose $t \geq 3 \log n / \log \log n$. Let \mathbf{X} be the random variable which returns the maximum (over the symbol set $[n]$) number of times that any symbol appears on the leading diagonal, in \mathbf{L} . Then $\mathbb{P}[\mathbf{X} \geq t] \leq \exp(-\Omega(t \log t))$.*

Proof. Let \mathcal{L}_n be the set of $n \times n$ Latin squares with symbols $[n]$, and for $L, L' \in \mathcal{L}_n$, write $L \sim L'$ if L' can be obtained from L via a permutation of the rows. Clearly, \sim is an equivalence relation on \mathcal{L}_n . Note that \mathbf{L} can be obtained by first choosing an equivalence class $\mathbf{S} \in \mathcal{L}_n / \sim$ uniformly at random and then choosing $\mathbf{L} \in \mathbf{S}$ uniformly at random. We actually prove the stronger statement that for every equivalence class $S \in \mathcal{L}_n / \sim$, if $\mathbf{L} \in S$ is chosen uniformly at random, then $\mathbb{P}[\mathbf{X} \geq t] \leq \exp(-\Omega(t \log t))$.

Each equivalence class $S \in \mathcal{L}_n / \sim$ has size $n!$ and contains a unique representative $L_{S,i}$ with every symbol on the leading diagonal being i , for each $i \in [n]$. Applying a uniformly random row permutation σ to $L_{S,i}$ yields a uniformly random element \mathbf{L} of S , and the number of appearances \mathbf{X}_i of i on the leading diagonal of \mathbf{L} is equal to the number of fixed points of σ . Then, if $t \geq 3 \log n / \log \log n$ and n is sufficiently large, we have by Lemma 4.5 and Stirling’s formula that

$$\mathbb{P}[\mathbf{X}_i \geq t] = \sum_{k=t}^n \frac{1}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \leq \sum_{k=t}^n \frac{1}{k!} \leq \frac{n}{t!} \leq \exp\left(-\frac{1}{2}t \log t\right),$$

where we have used the simple observation that $\sum_{j=0}^{n-k} \frac{(-1)^j}{j!} \leq 1$ for all $k \in [n]_0$. A union bound over symbols $i \in [n]$ now completes the proof. ■

Finally, we need the following theorem of Kwan and Sudakov [26, Theorem 2], originally stated in the context of “discrepancy” of random Latin squares. Theorem 4.7 ensures in particular that almost all $G \in \Phi(\vec{K}_n)$ are “lower-quasirandom” (see Definition 7.1), which we will use when building and counting the almost-spanning rainbow directed path forests (see Lemma 7.2) that we later absorb into rainbow directed Hamilton cycles.

Theorem 4.7 (Kwan and Sudakov [26]). *Let $\mathbf{G} \in \Phi(\vec{K}_n)$ be chosen uniformly at random. Then with high probability, for all (not necessarily distinct) sets $U_1, U_2, D \subseteq [n]$, we have that*

$$\left| e_{G,D}(U_1, U_2) - \frac{|U_1||U_2||D|}{n} \right| = O\left(\sqrt{|U_1||U_2||D|} \log n + n \log^2 n\right).$$

5 | ABSORBERS VIA SWITCHINGS

The aim of this section is to prove that almost all $G \in \Phi(\vec{K}_n)$ have many well-distributed absorbing gadgets and bridging gadgets, which we define now (see also Figure 2).

Definition 5.1. For a vertex v and a color c , a (v, c) -*absorbing gadget* is a digraph A having vertex set $V(A) = \{v, x_1, x_2, \dots, x_6\}$ and arcs $E(A) = \{x_1v, vx_2, x_1x_2, x_3x_4, x_3x_5, x_4x_6, x_5x_6\}$, equipped with a proper arc-coloring ϕ_A , such that the following holds:

- $\phi_A(x_1v) = \phi_A(x_4x_6) =: f_1$.
- $\phi_A(vx_2) = \phi_A(x_3x_5) =: f_2$.
- $\phi_A(x_1x_2) = \phi_A(x_3x_4) =: f_3$.
- $\phi_A(x_5x_6) = c$.
- The colors f_1, f_2, f_3, c are distinct.

In this case, we say (x_4, x_5) is the pair of *abutment vertices* of A .

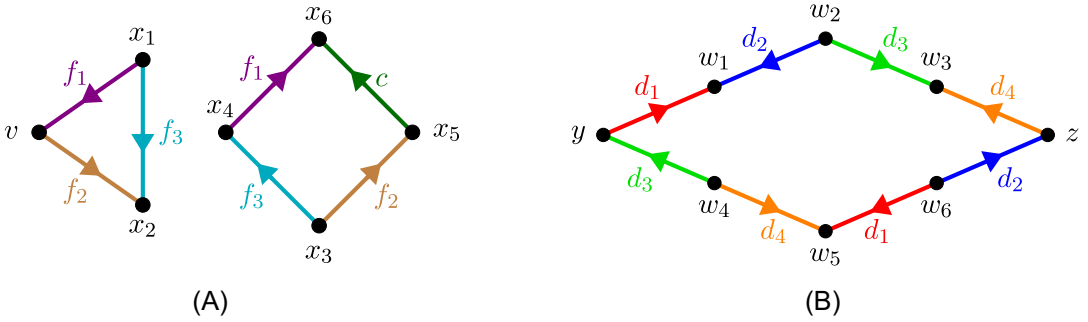


FIGURE 2 The key building blocks for the absorbing structure we build in Section 6. (A) A (v, c) -absorbing gadget. (B) A (y, z) -bridging gadget

Definition 5.2. For distinct vertices y and z , a (y, z) -bridging gadget is a digraph B such that $V(B) = \{y, z, w_1, w_2, \dots, w_6\}$ and $E(B) = \{yw_1, w_2w_1, w_2w_3, zw_3, w_4y, w_4w_5, w_6w_5, w_6z\}$, equipped with a proper arc-coloring ϕ_B , such that the following holds:

- $\phi_B(yw_1) = \phi_B(w_6w_5) =: d_1$.
- $\phi_B(w_2w_1) = \phi_B(w_6z) =: d_2$.
- $\phi_B(w_4y) = \phi_B(w_2w_3) =: d_3$.
- $\phi_B(w_4w_5) = \phi_B(zw_3) =: d_4$.
- The colors d_1, \dots, d_4 are distinct.

As discussed in Section 3, the union of a (v, c) -absorbing gadget and an (x_4, x_5) -bridging gadget (together with some short rainbow directed paths) forms a structure we will call a (v, c) -absorber (see Figure 1 and Definition 6.1), which is the key building block of our absorption structure. To show that almost all $G \in \Phi(\vec{K}_n)$ contain the gadgets we need, we analyze switchings in the probability space corresponding to uniformly random $\mathbf{H} \in \mathcal{G}_D$ (recall that \mathcal{G}_D is the set of digraphs obtained from the digraphs in $\Phi(\vec{K}_n)$ by deleting all arcs with color not in D) for small $D \subseteq [n]$, before applying Proposition 4.4 to compare this probability space with that of uniformly random $\mathbf{G} \in \Phi(\vec{K}_n)$ (see the proof of Lemma 5.10).

First, we need the following lemma, which asserts that for small $D \subseteq [n]$, a uniformly random $\mathbf{H} \in \mathcal{G}_D$ does not have too many more arcs than we would expect between any pair of vertex sets, each of size $|D|$.

Definition 5.3. For $D \subseteq [n]$, we say that $H \in \mathcal{G}_D$ is ℓ -upper-quasirandom if $e_H(A, B) \leq (1 + \ell)|D|^3/n$ for all (not necessarily distinct) vertex sets $A, B \subseteq V(H)$ of sizes $|A| = |B| = |D|$. We define $\mathcal{Q}_D^\ell := \{H \in \mathcal{G}_D : H \text{ is } \ell\text{-upper-quasirandom}\}$.

For a color $c \in D$ and uniformly random $\mathbf{H} \in \mathcal{G}_D$, we write $\mathbf{F}_c = F_c(\mathbf{H})$ for the random color class of c in \mathbf{H} (F here standing for “factor”), so that \mathbf{H} is determined by the random variables $\{\mathbf{F}_c\}_{c \in D}$.

Lemma 5.4. Suppose $D \subseteq [n]$ has size $|D| = n/10^6$. Fix $c \in D$, let $\mathbf{H} \in \mathcal{G}_D$ be chosen uniformly at random, and let $\mathbf{F}_c = F_c(\mathbf{H})$. Then for any outcome F of \mathbf{F}_c we have

$$\mathbb{P}[\mathbf{H} \in \mathcal{Q}_D^1 | \mathbf{F}_c = F] \geq 1 - \exp(-\Omega(n^2)).$$

The authors of [15] proved a lemma [15, Lemma 6.3] analogous to Lemma 5.4 in the undirected setting. The proof of Lemma 5.4 is similar so we omit it here. In the appendix of the arXiv version

of the paper, we describe how the proof of [15, Lemma 6.3] can be modified to obtain a proof of Lemma 5.4.

We condition on versions of upper-quasirandomness when we are using switching arguments to show that almost all $H \in \mathcal{G}_D$ admit many absorbing gadgets and bridging gadgets. Further, we will need that H does not have many c -loops in order to find many (v, c) -absorbing gadgets, for any $v \in V(H)$. Lemma 5.4 enables us to “uncondition” from these two events, so as to study simply the probability that a uniformly random \mathbf{H} has many absorbing gadgets.

Since, as discussed in Section 3, we eventually piece together gadgets in a greedy fashion to build an absorbing structure in a typical $G \in \Phi(\vec{K}_n)$, it will be important to know that we can find collections \mathcal{A} of gadgets which are “well-spread,” in that no vertex or color of G is contained in too many $A \in \mathcal{A}$. We formalize this notion in the following definition.

Definition 5.5. Suppose that G is an n -vertex directed, arc-colored digraph with vertices V and colors C . Fix $v \in V$, $c \in C$, and fix $y, z \in V$ distinct. We say that a collection \mathcal{A} of (v, c) -absorbing gadgets in G is *well-spread* if for all $u \in V \setminus \{v\}$ and $d \in C \setminus \{c\}$, there are at most n distinct $A \in \mathcal{A}$ which contain u , and at most n distinct $A \in \mathcal{A}$ which contain d . We say that a collection \mathcal{B} of (y, z) -bridging gadgets in G is *well-spread* if for all $u \in V \setminus \{y, z\}$ and $d \in C$, there are at most n distinct $B \in \mathcal{B}$ which contain u , and at most n distinct $B \in \mathcal{B}$ which contain d .

The next lemma ensures that almost all $G \in \Phi(\vec{K}_n)$ contain the collections of well-spread absorbing gadgets that we need.

Lemma 5.6. Let $\mathbf{G} \in \Phi(\vec{K}_n)$ be chosen uniformly at random, and let \mathcal{E} be the event that for all $v, c \in [n]$, \mathbf{G} contains a well-spread collection of at least $n^2/2^{100}$ (v, c) -absorbing gadgets. Let \mathcal{C} be the event that no color class of \mathbf{G} has more than $n/10^9$ loops. Then $\mathbb{P}[\mathcal{E}|\mathcal{C}] \geq 1 - \exp(-\Omega(n^2))$, and in particular, $\mathbb{P}[\mathcal{E}] \geq 1 - \exp(-\Omega(n \log n))$ by Lemma 4.6.

As with Lemma 5.4, the authors of [15] proved an analogous lemma [15, Lemma 3.8] in the undirected setting with a similar proof, so we omit it here but provide details in the appendix of the arXiv version of the paper of how the proof of [15, Lemma 3.8] may be modified to prove Lemma 5.6.

The rest of this section is dedicated to showing that almost all $G \in \Phi(\vec{K}_n)$ have large well-spread collections of bridging gadgets (recall Figure 2B). For technical reasons that make the switching argument a little easier to analyze, we instead actually look for a slightly more special structure. In particular, we add some extra arcs so that all vertices we find are in the neighborhood of y or of z , we partition the colors to limit the number of “roles” certain arcs can play when we apply the switching operation, and we introduce the notion of *distinguishability*, which will be useful when arguing that the gadgets we find are well-spread.

Definition 5.7. Let $D \subseteq [n]$, let $H \in \mathcal{G}_D$, and let $\mathcal{P} = (D_i)_{i=1}^6$ be an equitable (ordered) partition of D into six parts. Let $y, z \in [n]$ be distinct vertices.

- We say that a subgraph $B \subseteq H$ is a (y, z, \mathcal{P}) -bridge (see Figure 3) if B is the union of a (y, z) -bridging gadget B' (with vertex- and color-labeling as in Definition 5.2) and the extra arcs yw_2, zw_5 , such that $d_i \in D_i$ for all $i \in [4]$, $\phi_H(yw_2) \in D_5$, and $\phi_H(zw_5) \in D_6$.
- We say that a (y, z, \mathcal{P}) -bridge B is *distinguishable* in H if B is the only (y, z, \mathcal{P}) -bridge in H containing any of the arcs $w_2w_1, w_2w_3, w_4w_5, w_6w_5$.

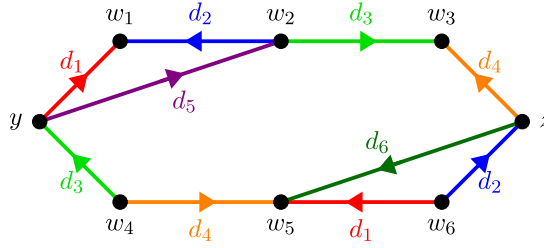


FIGURE 3 A (y, z, \mathcal{P}) -bridge, with $\mathcal{P} = (D_i)_{i=1}^6$ and $d_i \in D_i$ for each $i \in [6]$

- We write $r_{(y,z,\mathcal{P})}(H)$ for the number of distinguishable (y, z, \mathcal{P}) -bridges in H .
- For $s \in [n|D|]_0$, we write $M_s^{(y,z,\mathcal{P})}$ for the set of $H \in \mathcal{G}_D$ such that $r_{(y,z,\mathcal{P})}(H) = s$ and $e_H(A, B) \leq 2|D|^3/n + 12s$ for all $A, B \subseteq [n]$ of size $|A| = |B| = |D|$, and we define $\widehat{Q}_D^{(y,z,\mathcal{P})} := \bigcup_{s=0}^{n|D|} M_s^{(y,z,\mathcal{P})}$.

We frequently drop the (y, z, \mathcal{P}) -notation in the terminology introduced above when the tuple (y, z, \mathcal{P}) is clear from context. For every distinct $y, z \in [n]$ and equitable partition $\mathcal{P} = (D_i)_{i=1}^6$,

$$Q_D^1 \subseteq \widehat{Q}_D, \text{ and if } s \leq |D|^4/(10^{24}n^2), \text{ then } M_s \subseteq Q_D^2. \quad (5.1)$$

In Lemma 5.9, we use switchings on some $H \in \mathcal{G}_D$ to produce some $H' \in \mathcal{G}_D$ having $r(H') = r(H) + 1$. As mentioned earlier, we condition on upper-quasirandomness in this lemma; more specifically, we will condition that $\mathbf{H} \in \widehat{Q}_D$. The notion of distinguishability of (y, z, \mathcal{P}) -bridges is useful because, as we show in Lemma 5.10, Claim 7, a collection of distinguishable (y, z, \mathcal{P}) -bridges in $G|_D$ is necessarily well-spread (recall Definition 5.5).

We now discuss the switching operation that forms the backbone of the proof of Lemma 5.9.

Definition 5.8. Let $D \subseteq [n]$, let $H \in \mathcal{G}_D$, let $\mathcal{P} = (D_i)_{i=1}^6$ be a partition of D and suppose $y, z \in [n]$ are distinct. Let $u_1, \dots, u_6, u'_1, \dots, u'_8, u''_1, \dots, u''_8 \in [n] \setminus \{y, z\}$, where $u_1, \dots, u_6, u'_1, \dots, u'_8$ are distinct and $\{u'_1, \dots, u'_8\} \cap \{u_1, \dots, u_6, u'_1, \dots, u'_8\} = \emptyset$. Let $U^{\text{int}} := \{u_1, u_2, \dots, u_6\}$, $U^{\text{mid}} := \{u'_1, u'_2, \dots, u'_8\}$, $U^{\text{ext}} := \{u''_1, u''_2, \dots, u''_8\}$. Then we say that a subgraph $T \subseteq H[\{y, z\} \cup U^{\text{int}} \cup U^{\text{mid}} \cup U^{\text{ext}}]$ is a *twist system* (see Figure 4) of H if:

- $E(T) = \{yu_1, yu_2, u_4y, zu_3, zu_5, u_6z, u'_1u_1, u_2u'_2, u_2u'_3, u'_4u_3, u_4u'_5, u'_6u_5, u'_7u_5, u_6u'_8, u'_2u'_1, u'_3u'_4, u'_5u'_6, u'_8u'_7\}$.
- $\phi_H(yu_1) = \phi_H(u'_7u_5) = \phi_H(u'_8u'_7) = \phi_H(u_6u'_8) \in D_1$.
- $\phi_H(u_6z) = \phi_H(u_2u'_2) = \phi_H(u'_2u'_1) = \phi_H(u'_1u_1) \in D_2$.
- $\phi_H(u_4y) = \phi_H(u_2u'_3) = \phi_H(u'_3u'_4) = \phi_H(u'_4u_3) \in D_3$.
- $\phi_H(zu_3) = \phi_H(u_4u'_5) = \phi_H(u'_5u'_6) = \phi_H(u'_6u_5) \in D_4$.
- $\phi_H(yu_2) \in D_5$ and $\phi_H(zu_5) \in D_6$.
- $u_2u_1, u'_1u'_1, u'_2u'_2, u_2u_3, u'_3u'_3, u'_4u'_4, u_4u_5, u'_5u'_5, u'_6u'_6, u_6u_5, u'_7u'_7, u'_8u'_8 \notin E(H)$.

For a twist system $T \subseteq H$, we define $\text{twist}_T(H)$ to be the colored digraph obtained from H by deleting the arcs $u'_1u_1, u'_2u'_1, u_2u'_2, u_2u'_3, u'_3u'_4, u'_4u_3, u_4u'_5, u'_5u'_6, u'_6u_5, u_6u'_8, u'_8u'_7, u'_7u_5$, and adding the arcs $u_6u_5, u'_7u'_7, u'_8u'_8$ each in color $\phi_H(yu_1)$, the arcs $u_4u_5, u'_6u'_6, u'_5u'_5$ each in color $\phi_H(zu_3)$, the arcs $u_2u_3, u'_4u'_4, u'_3u'_3$ each in color $\phi_H(u_4y)$, and the arcs $u_2u_1, u'_1u'_1, u'_2u'_2$ each in color $\phi_H(u_6z)$. The (y, z, \mathcal{P}) -bridge in $\text{twist}_T(H)$ with arc set $\{yu_1, yu_2, u_2u_1, u_2u_3, zu_3, u_6z, zu_5, u_6u_5, u_4u_5, u_4y\}$ is called the *canonical* (y, z, \mathcal{P}) -bridge of the twist.

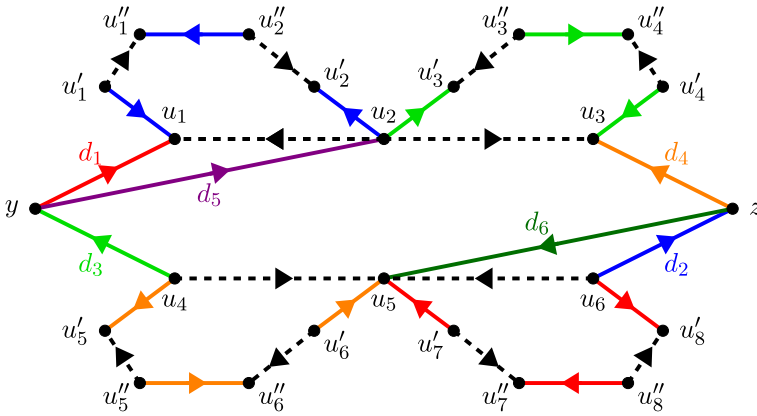


FIGURE 4 A twist system. Here, $d_i \in D_i$ for each $i \in [6]$, and dashed arcs indicate an arc which is absent in H .

Notice that if $H \in \mathcal{G}_D$ and T is a twist system of H , then $\text{twist}_T(H) \in \mathcal{G}_D$, even if u'_1, \dots, u'_8 are not distinct. We now use the twist switching operation to argue that almost all $H \in \widehat{\mathcal{Q}}_D$ have many distinguishable (y, z, \mathcal{P}) -bridges, for fixed y, z, \mathcal{P} , and appropriately sized D .

Lemma 5.9. *Suppose $D \subseteq [n]$ has size $|D| = n/10^6$. Let $y, z \in [n]$ be distinct, and let $\mathcal{P} = (D_i)_{i=1}^6$ be an equitable partition of D . Let $\mathbf{H} \in \mathcal{G}_D$ be chosen uniformly at random. Then*

$$\mathbb{P} \left[r(\mathbf{H}) \leq \frac{n^2}{10^{50}} \mid \mathbf{H} \in \widehat{\mathcal{Q}}_D \right] \leq \exp(-\Omega(n^2)).$$

Proof. Let $k := |D|$. Recall that M_1, \dots, M_{nk} is a partition of $\widehat{\mathcal{Q}}_D$ (see Definition 5.7). For each $s \in [nk - 1]_0$, we define an auxiliary bipartite digraph B_s with vertex bipartition (M_s, M_{s+1}) by putting an arc HH' whenever $H \in M_s$ contains a twist system T for which the canonical (y, z, \mathcal{P}) -bridge of the twist is distinguishable in $\text{twist}_T(H) =: H'$ and $H' \in M_{s+1}$. Define $\delta_s^+ := \min_{H \in M_s} d_{B_s}^+(H)$ and $\Delta_{s+1}^- := \max_{H' \in M_{s+1}} d_{B_s}^-(H')$, and note that $|M_s|/|M_{s+1}| \leq \Delta_{s+1}^-/\delta_s^+$. We will show that $|M_s|/|M_{s+1}| \leq 1/10$, if M_s is non-empty. To that end, we first obtain an upper bound for Δ_{s+1}^- . Fix $H' \in M_{s+1}$. There are $s + 1$ choices of a distinguishable (y, z, \mathcal{P}) -bridge B in H' which could have been the canonical (y, z, \mathcal{P}) -bridge of a twist of a graph $H \in M_s$ producing H' . There are then at most n^8 choices for the eight additional arcs added by a twist whose canonical bridge is B since the colors of these arcs are determined by B and there are n arcs of each color in H' . For any such sequence of choices, there is a unique $H \in M_s$ and twist system $T \subseteq H$ such that $\text{twist}_T(H) = H'$, so we determine that $\Delta_{s+1}^- \leq (s + 1)n^8$, for all $s \in [nk - 1]_0$.

We now find a lower bound for δ_s^+ , in the case where $s \leq k^4/(10^{24}n^2)$. Fix $H \in M_s$. We proceed by finding a large collection \mathcal{T} of distinct twist systems in H , such that for each $T \in \mathcal{T}$, the canonical (y, z, \mathcal{P}) -bridge of the twist is distinguishable in $\text{twist}_T(H) =: H'$ and $H' \in M_{s+1}$. We do this by ensuring that for any $T \in \mathcal{T}$, the arc deletions involved in twisting on T do not decrease $r(H)$, and that the only (y, z, \mathcal{P}) -bridge created by the arc additions involved in twisting on T is the canonical (y, z, \mathcal{P}) -bridge B of the twist (whence B is evidently distinguishable in H'). We first use the assumption on s to argue that H is not far from being upper-quasirandom (as per Definition 5.3). Indeed, since $H \in M_s$ and $s \leq k^4/(10^{24}n^2)$ we have by Definition 5.7 that for any sets $W_1, W_2 \subseteq [n]$ of sizes $|W_1| = |W_2| = k$,

$$e_H(W_1, W_2) \leq \frac{2k^3}{n} + 12s \leq \frac{3k^3}{n}. \tag{5.2}$$

We now use (5.2) to find a large set Λ of choices for a sequence of colors and arcs $\lambda = (d_1, d_2, \dots, d_6, e_1, \dots, e_4)$ with $d_i \in D_i$, and $e_i \in E_{d_i}(H)$, such that λ has a number of desirable properties. We simultaneously use such a sequence λ to choose vertices $u_1, \dots, u_6, u'_1, \dots, u'_8, u''_1, \dots, u''_8$ as in Definition 5.8 and a subgraph $T_\lambda \subseteq H[\{y, z\} \cup U_{\text{int}} \cup U^{\text{mid}} \cup U^{\text{ext}}]$, thus constructing a set \mathcal{T} of such T_λ by ranging over all $\lambda \in \Lambda$. We will then use the known properties of the sequences $\lambda \in \Lambda$ to verify that each $T_\lambda \in \mathcal{T}$ is a twist system for which the arc $\text{Htwist}_{T_\lambda}(H)$ is in B_s .

Claim 1. There is a set $D_{1,2}^{\text{good}}$ of pairs $(d_1, d_2) \in D_1 \times D_2$ such that $|D_{1,2}^{\text{good}}| \geq k^2/100$ and each $(d_1, d_2) \in D_{1,2}^{\text{good}}$ satisfies the following, where $u_1 := N_{d_1}^+(y)$, $u'_1 := N_{d_2}^-(u_1)$, $u_6 := N_{d_2}^-(z)$, $u'_8 := N_{d_1}^+(u_6)$.

- (D₁1) There are at most 10^8 loops with color d_1 in H .
- (D₁2) u_1 has at most $300k^2/n$ in-neighbors in the set $N_{D_5}^+(y)$.
- (D₁3) there are at most $k/100$ arcs e colored d_1 in H such that e is contained in a distinguishable (y, z, \mathcal{P}) -bridge in H .
- (D₂1) there are at most 10^8 loops with color d_2 in H .
- (D₂2) u_6 has at most $300k^2/n$ out-neighbors in the set $N_{D_6}^+(z)$.
- (D₂3) there are at most $k/100$ arcs e colored d_2 in H such that e is contained in a distinguishable (y, z, \mathcal{P}) -bridge in H .
- (V_{1,2}) the vertices y, z, u_1, u_6 are distinct, and $u'_1, u'_8 \notin \{y, z, u_1, u_6\}$.
- (R_{1,2}) there is no distinguishable (y, z, \mathcal{P}) -bridge in H containing the arc $u'_1 u_1$ or the arc $u_6 u'_8$.

Proof of Claim 1. For $i \in [3]$, let $D_{1,i}$ be the set of colors $d_1 \in D_1$ that fail to satisfy (D₁ i). Since H contains at most n loops, $|D_{1,1}| \leq n/10^8 = k/100$. Since $e_H(N_{D_5}^+(y), N_{D_1}^+(y)) \leq 3k^3/n$ by (5.2), $|D_{1,2}| \leq k/100$. Since any d_1 -arc of H whose tail is not y is contained in at most one distinguishable (y, z, \mathcal{P}) -bridge B and each B contains two such arcs, $r(H) \geq |D_{1,3}|(k/100 - 1)/2$. Thus, $|D_{1,3}| \leq k/1000$. Let $D'_1 := D_1 \setminus (D_{1,1} \cup D_{1,2} \cup D_{1,3})$, and notice that $|D'_1| \geq k/6 - k/50 - k/1000 \geq 7k/50$. Similarly there is a set $D'_2 \subseteq D_2$ of size at least $7k/50$ such that each $d_2 \in D'_2$ satisfies (D₂1)–(D₂3). At most two colors $d_1 \in D'_1$ yield $u_1 \in \{y, z\}$, and for any $d_1 \in D'_1$ there are at most three choices of $d_2 \in D'_2$ such that $u'_1 \in \{y, z, u_1\}$, and at most three choices of $d_2 \in D'_2$ such that $u_6 \in \{y, z, u_1\}$. For fixed $d_1 \in D'_1$, by (D₁1) there are at most 10^8 choices of $d_2 \in D'_2$ such that $u'_8 = u_6$, and at most three choices of d_2 such that $u'_8 \in \{y, z, u_1\}$. Finally, if u_1 and z are distinct, then we have $u'_1 \neq u_6$, since otherwise $u_1 = z$ has two distinct d_2 -out-neighbors. Since $|D'_1|, |D'_2| \leq k/6$, we deduce that we can remove at most $2|D'_2| + (10^8 + 9)|D'_1| \leq 10^8 k/3$ pairs from $D'_1 \times D'_2$ to ensure that all remaining pairs satisfy (V_{1,2}). To address (R_{1,2}), notice that for fixed $d_1 \in D'_1$, by (D₁3) there are at most $k/100$ choices of $d_2 \in D'_2$ such that $u_6 u'_8$ is contained in a distinguishable (y, z, \mathcal{P}) -bridge B in H . Handling $u'_1 u_1$ analogously we deduce that we may remove at most $2 \cdot \frac{k}{100} \cdot \frac{k}{6}$ pairs from $D'_1 \times D'_2$ to ensure all remaining pairs satisfy (R_{1,2}). In total the number of pairs in $D'_1 \times D'_2$ satisfying (V_{1,2}) and (R_{1,2}) is at least $(7k/50)^2 - 10^8 k/3 - k^2/300 \geq k^2/100$ as claimed. ■

Claim 2. For any $(d_1, d_2) \in D_{1,2}^{\text{good}}$ there is a set $D_{3,4}^{\text{good}} = D_{3,4}^{\text{good}}(d_1, d_2)$ of pairs $(d_3, d_4) \in D_3 \times D_4$ such that $|D_{3,4}^{\text{good}}| \geq k^2/100$ and each $(d_3, d_4) \in D_{3,4}^{\text{good}}$ satisfies the following, where $u_4 := N_{d_3}^-(y)$, $u'_5 := N_{d_4}^+(u_4)$, $u_3 := N_{d_4}^+(z)$, $u'_4 := N_{d_3}^-(u_3)$, and u_1, u'_1, u_6 , and u'_8 are defined as in Claim 1.

- (D₃1) There are at most 10^8 loops with color d_3 in H .
- (D₃2) u_4 has at most $300k^2/n$ out-neighbors in the set $N_{D_6}^+(z)$.
- (D₃3) there are at most $k/100$ arcs e colored d_3 in H such that e is contained in a distinguishable (y, z, \mathcal{P}) -bridge in H .

- (D₄1) there are at most 10^8 loops with color d_4 in H .
 (D₄2) u_3 has at most $300k^2/n$ in-neighbors in the set $N_{D_5}^+(y)$.
 (D₄3) there are at most $k/100$ arcs e colored d_4 in H such that e is contained in a distinguishable (y, z, \mathcal{P}) -bridge in H .
 (V_{3,4}) y, z, u_1, u_3, u_4, u_6 are distinct vertices, and $u'_1, u'_4, u'_5, u'_8 \notin \{y, z, u_1, u_3, u_4, u_6\}$.
 (R_{3,4}) there is no distinguishable (y, z, \mathcal{P}) -bridge in H containing the arc u'_4u_3 or the arc $u_4u'_5$.

The proof is similar to that of Claim 1, so we omit it.

Claim 3. For any $(d_1, d_2) \in D_{1,2}^{\text{good}}$ and $(d_3, d_4) \in D_{3,4}^{\text{good}}(d_1, d_2)$, there is a set $D_{5,6}^{\text{good}}$ (depending on (d_1, \dots, d_4)) of pairs $(d_5, d_6) \in D_5 \times D_6$ such that $|D_{5,6}^{\text{good}}| \geq k^2/100$ and each $(d_5, d_6) \in D_{5,6}^{\text{good}}$ satisfies the following, where $u_2 := N_{d_5}^+(y)$, $u'_2 := N_{d_2}^+(u_2)$, $u'_3 := N_{d_3}^+(u_2)$, $u_5 := N_{d_6}^+(z)$, $u'_6 := N_{d_4}^-(u_5)$, $u'_7 := N_{d_1}^-(u_5)$, and $u_1, u_3, u_4, u_6, u'_1, u'_4, u'_5, u'_8$ are defined as in Claims 1 and 2.

- (D₅1) $u_2u_1, u_2u_3 \notin E(H)$.
 (D₆1) $u_4u_5, u_6u_5 \notin E(H)$.
 (R₅) There is no distinguishable (y, z, \mathcal{P}) -bridge in H containing the arc $u_2u'_2$ or the arc $u_2u'_3$.
 (R₆) There is no distinguishable (y, z, \mathcal{P}) -bridge in H containing the arc u'_6u_5 or the arc u'_7u_5 .
 (V_{5,6}) $y, z, u_1, u_2, \dots, u_6$ are distinct, and $u'_1, u'_2, \dots, u'_8 \notin \{y, z, u_1, u_2, \dots, u_6\}$.
 (A_{5,6}1) For each $(d'_1, d'_2) \in F_{1,2}(d_5)$, where $F_{1,2}(d_5)$ is the set of pairs $(d'_1, d'_2) \in D_1 \times D_2$ such that $N_{d'_1}^+(y) = N_{d'_2}^+(u_2)$, we have $N_{d'_1}^-(u_5) \neq N_{d'_2}^-(z)$.
 (A_{5,6}2) For each $(d'_3, d'_4) \in F_{3,4}(d_5)$, where $F_{3,4}(d_5)$ is the set of pairs $(d'_3, d'_4) \in D_3 \times D_4$ such that $N_{d'_4}^+(z) = N_{d'_3}^+(u_2)$, we have $N_{d'_4}^-(u_5) \neq N_{d'_3}^-(y)$.

Proof of Claim 3. Let \tilde{D}_5 be the set of colors $d_5 \in D_5$ which fail to satisfy (D₅1) and (R₅), let \tilde{D}_6 be the set of colors $d_6 \in D_6$ which fail to satisfy (D₆1) and (R₆), and define $D'_5 := D_5 \setminus \tilde{D}_5$ and $D'_6 := D_6 \setminus \tilde{D}_6$. By (D₁2) and (D₄2), there are at most $600k^2/n$ colors $d_5 \in D_5$ which fail to satisfy (D₅1). By (D₂3) and (D₃3), at most $k/50$ choices of $d_5 \in D_5$ give u_2 to be the tail of a d_2 -arc or d_3 -arc contained in a distinguishable (y, z, \mathcal{P}) -bridge in H , and all other choices of $d_5 \in D_5$ satisfy (R₅). Using (D₂2), (D₃2), (D₁3), and (D₄3) similarly, we deduce that $|D'_5|, |D'_6| \geq 7k/50$. By (D₂1) and (D₃1), for any $d_6 \in D'_6$ there are at most $2 \cdot 10^8 + 10$ choices of $d_5 \in D'_5$ such that $u_5 \in \{y, z, u_1, u_3, u_4, u_6, u'_1, u'_4, u'_5, u'_8\}$ or u_5 is incident to a loop of color d_2 or d_3 . Further, for any $d_6 \in D'_6$, there are at most 12 choices of d_5 such that $u_2 \in \{N_{d'}^-(w) : d \in \{d_2, d_3\}, w \in \{y, z, u_1, u_3, u_4, u_6\}\}$. Thus there are at most $(2 \cdot 10^8 + 22)|D_6|$ choices of pair $(d_5, d_6) \in D'_5 \times D'_6$ such that the choice of d_5 causes (V_{5,6}) to fail. Using (D₁1) and (D₄1) to address the choice of d_6 similarly, we conclude that we can remove a set of at most $(5 \cdot 10^8)k/6$ pairs $(d_5, d_6) \in D'_5 \times D'_6$ such that (V_{5,6}) holds for all remaining pairs. For (A_{5,6}1), notice that (5.2) implies $e_H(N_{D_5}^+(y), N_{D_1}^+(y)) \leq 3k^3/n$, so that there are at most $k/100$ colors $d_5 \in D'_5$ for which u_2 has at least $300k^2/n$ out-neighbors in the set $N_{D_1}^+(y)$. Deleting all pairs (d_5, d_6) using such a d_5 , we have in particular that all remaining pairs satisfy $|F_{1,2}(d_5)| \leq 300k^2/n$. For a remaining pair (d_5, d_6) and each $(d'_1, d'_2) \in F_{1,2}(d_5)$, we define the vertex $v_{d'_1, d'_2} := N_{d'_1}^+(N_{d'_2}^-(z))$. Now further deleting all (at most $\frac{k}{6} \cdot 300\frac{k^2}{n}$) pairs (d_5, d_6) such that $u_5 \in \{v_{d'_1, d'_2} : (d'_1, d'_2) \in F_{1,2}(d_5)\}$, all remaining pairs satisfy (A_{5,6}1). We address (A_{5,6}2) similarly. In total, the number of pairs in $D'_5 \times D'_6$ satisfying (V_{5,6}), (A_{5,6}1), and (A_{5,6}2) is at least $(7k/50)^2 - 5 \cdot 10^8k/6 - 2(k^2/600 + 50k^3/n) \geq k^2/100$, finishing the proof of the claim. ■

Claim 4. For any $(d_1, d_2) \in D_{1,2}^{\text{good}}$, $(d_3, d_4) \in D_{3,4}^{\text{good}}$, $(d_5, d_6) \in D_{5,6}^{\text{good}}$, there is a set $E^{\text{good}} = E^{\text{good}}(d_1, d_2, \dots, d_6) \subseteq \prod_{i \in [4]} E_{d_i}(H)$ such that $|E^{\text{good}}| \geq 9n^4/10$ and each $(e_1, \dots, e_4) \in E^{\text{good}}$ (with $e_i \in E_{d_i}(H)$ for each $i \in [4]$) satisfies the following, where $u_1'' := \text{head}(e_2)$, $u_2'' := \text{tail}(e_2)$, $u_3'' := \text{tail}(e_3)$, $u_4'' := \text{head}(e_3)$, $u_5'' := \text{tail}(e_4)$, $u_6'' := \text{head}(e_4)$, $u_7'' := \text{head}(e_1)$, $u_8'' := \text{tail}(e_1)$, and $u_1, \dots, u_6, u_1', \dots, u_8'$ are defined as in Claims 1–3.

(V_E) $u_1'', u_2'', \dots, u_8''$ are distinct vertices and $u_1'', \dots, u_8'' \notin \{y, z, u_1, \dots, u_6, u_1', \dots, u_8'\}$.

(E1) $u_1' u_1'', u_2' u_2'', u_3' u_3'', u_4' u_4'', u_5' u_5'', u_6' u_6'', u_7' u_7'', u_8' u_8'' \notin E(H)$.

(E2) $\{u_1'', u_2'', \dots, u_8''\} \cap (N(y) \cup N(z)) = \emptyset$.

(R_E) there is no distinguishable (y, z, \mathcal{P}) -bridge in H containing any of the arcs e_1, \dots, e_4 .

Proof of Claim 4. At most 32 d_2 -arcs of H have head or tail in $\{y, z, u_1, \dots, u_6, u_1', \dots, u_8'\}$, and by (D₂1), at most 10^8 arcs in $E_{d_2}(H)$ are loops. Choosing any other d_2 -arc to be e_2 , and proceeding similarly for e_3, e_4, e_1 (also avoiding the vertices of previously chosen such arcs), we deduce that we can delete at most $4 \cdot 10^9 n^3$ tuples from $\prod_{i=1}^4 E_{d_i}(H)$ so that the remaining tuples satisfy (V_E). Notice that at most $2k$ choices of $e_2 \in E_{d_2}(H)$ have head in $N^+(u_1')$ or tail in $N^-(u_2')$, and any other e_2 satisfies $u_1' u_1'', u_2' u_2'' \notin E(H)$. Dealing with e_3, e_4, e_1 similarly addresses (E1). Similarly, for (E2) it suffices to notice that at most $8k$ choices of e_2 , for example, have either head or tail in $N(y) \cup N(z)$. Finally, by (D_i3) for each $i \in [4]$, at most $k/100$ choices of each e_i fail to satisfy (R_E). In total, at least $n^4 - 4n^3(10^9 - 2k - 8k - k/100) \geq 9n^4/10$ tuples in $\prod_{i=1}^4 E_{d_i}(H)$ satisfy (V_E), (E1), (E2), and (R_E), as desired. ■

Let Λ be the set of tuples $\lambda = (d_1, \dots, d_6, e_1, \dots, e_4)$ satisfying all properties in Claims 1–4. For each $\lambda \in \Lambda$, we define a subgraph $T_\lambda \subseteq H$ with the vertices $V(T_\lambda) = \{y, z, u_1, \dots, u_6\} \cup \{u_1', \dots, u_8', u_1'', \dots, u_8''\}$ as defined in Claims 1–4 by the choice of λ , and whose arcs are as in Definition 5.8(i). (These arcs each exist in H by the way the vertices of T_λ were defined.) Then since $d_i \in D_i$ for each $i \in [6]$ and since (D₅1), (D₆1), and (E1) hold for each $\lambda \in \Lambda$, we have that each condition of Definition 5.8 is satisfied, so that T_λ is a twist system of H for each $\lambda \in \Lambda$. Further, clearly the T_λ are distinct. Define $\mathcal{T} := \{T_\lambda : \lambda \in \Lambda\}$, and notice that $|\mathcal{T}| \geq (k^2/100)^3 \cdot 9n^4/10 = 9k^6 n^4/10^7$.

Claim 5. For any $T_\lambda \in \mathcal{T}$, the only (y, z, \mathcal{P}) -bridge in $\text{twist}_{T_\lambda}(H)$ that is not in H is the canonical (y, z, \mathcal{P}) -bridge of the twist.

Proof of Claim 5. Fix $T_\lambda \in \mathcal{T}$ (fixing the notation of all the vertices and arcs as above), and let B be the canonical (y, z, \mathcal{P}) -bridge of the twist (which has vertices $V(B) = \{y, z, u_1, \dots, u_6\}$ and colors d_1, \dots, d_6). Suppose that B' is a (y, z, \mathcal{P}) -bridge in $\text{twist}_{T_\lambda}(H)$ that is not in H , and label the vertices of B' as $V(B') = \{y, z, v_1, \dots, v_6\}$ (where the role of v_i in B' corresponds to that of u_i in B), and label the colors of B' as $d_i' \in D_i$, for $i \in [6]$. By (V_{1,2}), (V_{3,4}), and (V_{5,6}), all arcs we add when producing $\text{twist}_{T_\lambda}(H)$ from H do not have y or z as an endpoint, and thus one (or more) of the arcs $v_2 v_1, v_2 v_3, v_4 v_5, v_6 v_5$ is added by the twist operation. Further, due to the color partition \mathcal{P} , $v_2 v_1$ must either be in H , or be one of the added arcs $u_2 u_1, u_1' u_1'', u_2'' u_2'$ with color $d_2' \in D_2$. But since we do not add any arcs incident to y , by (E2), u_1'' and u_2'' are not in the neighborhood of y in $\text{twist}_{T_\lambda}(H)$, whence u_1'' cannot be v_1 , and u_2'' cannot be v_2 . Thus $v_2 v_1$ must either be in H , or be $u_2 u_1$. Similarly $v_2 v_3, v_4 v_5, v_6 v_5$ must be $u_2 u_3, u_4 u_5, u_6 u_5$, respectively, or be in H , in some combination. We now split the analysis into cases, depending on how many arcs in $F := \{v_2 v_1, v_2 v_3, v_4 v_5, v_6 v_5\}$ are in H . In each case, we show either

that that case does not occur or that $B' = B$, which will complete the proof of the claim. Since $B' \not\subseteq H$, at most three arcs in F are in H .

Case 1: Precisely three arcs in F are in H .

Let e be the arc in F that is not in H . Suppose $e = v_2v_1$, which implies that $e = u_2u_1$, and $d'_2 = d_2$. Since $N_{d'_1}^+(y) = v_1 = u_1 = N_{d_1}^+(y)$, we have $d'_1 = d_1$. Then $v_6 = N_{d'_2}^-(z) = N_{d_2}^-(z) = u_6$, and v_5 is the d'_1 -out-neighbor of v_6 in $\text{twist}_{T_\lambda}(H)$, which is the d_1 -out-neighbor of u_6 in $\text{twist}_{T_\lambda}(H)$, namely, u_5 . This is a contradiction, since v_6v_5 is in H , but u_6u_5 is not. One similarly obtains a contradiction if e is v_2v_3 , v_4v_5 , or v_6v_5 , so we deduce that this case does not occur.

Case 2: Precisely two arcs in F are in H .

Suppose that $v_2v_1 = u_2u_1$ and $v_6v_5 = u_6u_5$. Then $d'_1 = d_1$, $d'_2 = d_2$, and v_2v_3, v_4v_5 are in H . Hence $v_3 \neq u_3$, and the arc u_2v_3 is in H with color d'_3 . Moreover the arc zv_3 is in H with color d'_4 . In particular, $(d'_3, d'_4) \in F_{3,4}(d_5)$. Similarly $v_4 \neq u_4$, we have $v_4y \in E(H)$ has color d'_3 , and $v_4u_5 \in E(H)$ has color d'_4 . That is, $N_{d'_4}^-(u_5) = v_4 = N_{d'_3}^-(y)$, which contradicts $(A_{5,6,2})$ of Claim 3. Similarly one can use $(A_{5,6,1})$ to show that assuming $v_2v_3 = u_2u_3$ and $v_4v_5 = u_4u_5$ yields a contradiction. Suppose instead that $v_2v_1 = u_2u_1$ and $v_2v_3 = u_2u_3$. Then $d'_2 = d_2$, and $v_1 = u_1$ so that $d'_1 = d_1$. Further, $d'_3 = d_3$, and $v_3 = u_3$ so that $d'_4 = d_4$. But this now also determines that $v_4v_5 = u_4u_5$ and $v_6v_5 = u_6u_5$, a contradiction since then no arcs in F are in H . All remaining possibilities yield a contradiction similarly, whence this case does not occur.

Case 3: Precisely one arc in F is in H .

In particular, either we have $v_2v_1 = u_2u_1$ and $v_2v_3 = u_2u_3$ or we have $v_4v_5 = u_4u_5$ and $v_6v_5 = u_6u_5$. But as in Case 2, either way yields that no arcs in F are in H . We deduce that this case does not occur.

Case 4: No arcs in F are in H .

It is easy to see in this case that $v_i = u_i$ for all $i \in [6]$ whence $B' = B$. ■

For any $T_\lambda \in \mathcal{T}$, we have by Claim 5 that the canonical (y, z, \mathcal{P}) -bridge of the twist is distinguishable in $\text{twist}_{T_\lambda}(H)$, since the four arcs $u_2u_1, u_2u_3, u_4u_5, u_6u_5$ are each added by the twist, and do not create any other (y, z, \mathcal{P}) -bridge. Further, since the twist operation only adds 12 arcs, we obtain from (5.2) that $e_{\text{twist}_{T_\lambda}(H)}(W_1, W_2) \leq 2k^3/n + 12(s + 1)$ for all $W_1, W_2 \subseteq [n]$ of sizes $|W_1| = |W_2| = k$. Thus by $(R_{1,2}), (R_{3,4}), (R_5), (R_6), (R_E)$, and Claim 5, we have that $\text{twist}_{T_\lambda}(H) \in M_{s+1}$. We now give a final claim which ensures that $|\{\text{twist}_{T_\lambda}(H) : T_\lambda \in \mathcal{T}\}| = |\mathcal{T}|$.

Claim 6. Fix $H' \in M_{s+1}$, let $\lambda, \lambda' \in \Lambda$, and suppose that $\text{twist}_{T_\lambda}(H) = \text{twist}_{T_{\lambda'}}(H) = H'$. Then $\lambda = \lambda'$.

Proof of Claim 6. Let $\lambda = (d_1, \dots, d_6, e_1, \dots, e_4)$, $\lambda' = (d'_1, \dots, d'_6, e'_1, \dots, e'_4)$, with corresponding vertices $V(T_\lambda) = \{y, z, u_1, \dots, u_6\} \cup \{u'_1, \dots, u'_6, u''_1, \dots, u''_6\}$, $V(T_{\lambda'}) = \{y, z, v_1, \dots, v_6\} \cup \{v'_1, \dots, v'_6, v''_1, \dots, v''_6\}$. Let B and B' be the canonical (y, z, \mathcal{P}) -bridges of the twists corresponding to λ and λ' , respectively. By Claim 5, B and B' are each the unique (y, z, \mathcal{P}) -bridge that is in H' but not in H , and thus $B = B'$. In particular, $d_i = d'_i$ and $u_i = v_i$ for all $i \in [6]$. By considering the partition \mathcal{P} of D , the four arcs in $E(H) \setminus E(H')$ with no endvertex in $\{u_1, \dots, u_6\}$ must be $e_1 = e'_1, e_2 = e'_2, e_3 = e'_3$, and $e_4 = e'_4$. We conclude that $\lambda = \lambda'$, as required. ■

We determine that if $s \leq k^4/(10^{24}n^2)$ and M_s is non-empty, then M_{s+1} is non-empty, and $\delta_s^+ \geq |\mathcal{T}| \geq 9k^6n^4/10^7$, whence $|M_s|/|M_{s+1}| \leq \Delta_{s+1}^-/\delta_s^+ \leq 1/10$. Recalling that $\mathbf{H} \in \mathcal{G}_D$ is uniformly random, it follows that if $s \leq k^4/(10^{25}n^2)$ then

$$\begin{aligned} \mathbb{P} \left[r(\mathbf{H}) = s \mid \mathbf{H} \in \widehat{\mathcal{Q}}_D \right] &\leq \frac{|M_s|}{|M_{k^4/(10^{24}n^2)}|} = \prod_{t=s}^{k^4/(10^{24}n^2)-1} \frac{|M_t|}{|M_{t+1}|} \leq \left(\frac{1}{10} \right)^{k^4/(10^{24}n^2)-s} \\ &\leq \exp \left(-\frac{9k^4 \log 10}{10^{25}n^2} \right). \end{aligned}$$

Moreover, we note that if M_s is empty, then clearly $\mathbb{P} \left[r(\mathbf{H}) = s \mid \mathbf{H} \in \widehat{\mathcal{Q}}_D \right] = 0$. Since $k = n/10^6$, we obtain that $\mathbb{P} \left[r(\mathbf{H}) \leq k^4/(10^{25}n^2) \mid \mathbf{H} \in \widehat{\mathcal{Q}}_D \right] \leq (k^4/(10^{25}n^2) + 1) \exp(-\Omega(n^2)) = \exp(-\Omega(n^2))$. Since $n^2/10^{50} \leq k^4/(10^{25}n^2)$, the result follows. \blacksquare

We are now ready to argue that almost all $G \in \Phi(\vec{K}_n)$ contain large well-spread collections of bridging gadgets, which will complete our study of the properties we need to be satisfied by uniformly random $\mathbf{G} \in \Phi(\vec{K}_n)$.

Lemma 5.10. *Let $\mathbf{G} \in \Phi(\vec{K}_n)$ be chosen uniformly at random, and let \mathcal{E} be the event that for all distinct $y, z \in [n]$, \mathbf{G} contains a well-spread collection of at least $n^2/10^{50}$ distinct (y, z) -bridging gadgets. Then $\mathbb{P}[\mathcal{E}] \geq 1 - \exp(-\Omega(n^2))$.*

Proof. For $D \subseteq [n]$, let $\mathcal{E}|_D$ denote the event (in $\Phi(\vec{K}_n)$) that $\mathbf{G}|_D$ contains a well-spread collection of $n^2/10^{50}$ distinct (y, z) -bridging gadgets, for each distinct $y, z \in [n]$. Let $\mathbf{H} \in \mathcal{G}_D$ be chosen uniformly at random, let \mathbb{P}_D denote the measure for this probability space, and let $\mathcal{E}_D^{(y,z)}$ denote the event (in \mathcal{G}_D) that \mathbf{H} contains a well-spread collection of $n^2/10^{50}$ distinct (y, z) -bridging gadgets, and define $\mathcal{E}_D := \bigcap_{y,z \in [n]} \text{distinct } \mathcal{E}_D^{(y,z)}$.

Claim 7. Suppose $D \subseteq [n]$ has size $|D| \leq 3n/4$, let $\mathcal{P} = (D_i)_{i \in [6]}$ be an equitable partition of D into six parts, and fix $y, z \in [n]$ distinct. Then $\mathbb{P}_D \left[r_{(y,z,\mathcal{P})}(\mathbf{H}) \geq n^2/10^{50} \right] \leq \mathbb{P}_D \left[\mathcal{E}_D^{(y,z)} \right]$.

Proof of Claim 7. Suppose that $H \in \mathcal{G}_D$ and that $r(H) \geq n^2/10^{50}$. By definition of $r(H)$ (see Definition 5.7), there is a collection \mathcal{B} of $n^2/10^{50}$ distinct (y, z, \mathcal{P}) -bridges such that for each $i \in [4]$, any $d_i \in D_i$, and any arc $e \in E_{d_i}(H)$ for which e does not have y nor z as an endvertex, we have that e is contained in at most one $B \in \mathcal{B}$. Note that for each $u \in [n] \setminus \{y, z\}$, we have for any $B \in \mathcal{B}$ which contains u , that B must contain an arc e incident to u with color in D_i for some $i \in [4]$ such that e is not incident to y nor z . Therefore u is contained in at most $4 \cdot 2 \cdot |D|/6 \leq n$ distinct $B \in \mathcal{B}$. For any color $d \in D_1$, any $B \in \mathcal{B}$ which uses the color d must be such that $N_d^+(y) \notin \{y, z\}$ and must contain the vertex $N_d^+(y)$, and thus the color d is used by at most n distinct $B \in \mathcal{B}$ (and similarly for $d \in D_i$ for all $i \in [6]$). Now, forming a collection \mathcal{B}' of (y, z) -bridging gadgets in H by deleting the arcs with colors in $D_5 \cup D_6$ for each $B \in \mathcal{B}$, it is clear that \mathcal{B}' witnesses that $H \in \mathcal{E}_D^{(y,z)}$. The claim follows. \blacksquare

Arbitrarily fix $c \in [n]$ and $D \subseteq [n]$ of size $|D| = n/10^6$, and let \mathcal{F} be the set of all possible color classes for a proper n -arc coloring of \vec{K}_n (more precisely, \mathcal{F} is the collection of all sets F of n arcs of \vec{K}_n such that every vertex of \vec{K}_n is the head of precisely one arc in F and the tail of precisely one arc in F). Observe that for a fixed equitable partition $\mathcal{P} = (D_i)_{i=1}^6$ of D into six parts, and for fixed distinct $y, z \in [n]$, by (5.1), the law of total probability, and Lemma 5.4,

$$\mathbb{P}_D \left[\widehat{\mathcal{Q}}_D \right] \leq \mathbb{P}_D \left[\overline{\mathcal{Q}}_D^1 \right] = \sum_{F \in \mathcal{F}} \mathbb{P}_D [\mathbf{F}_c = F] \mathbb{P}_D \left[\overline{\mathcal{Q}}_D^1 \mid \mathbf{F}_c = F \right] \leq \exp(-\Omega(n^2)). \quad (5.3)$$

Then the law of total probability, (5.3), and Lemma 5.9 give

$$\mathbb{P}_D \left[r_{(y,z,\mathcal{P})}(\mathbf{H}) \leq \frac{n^2}{10^{50}} \right] \leq \mathbb{P}_D \left[r_{(y,z,\mathcal{P})}(\mathbf{H}) \leq \frac{n^2}{10^{50}} \mid \widehat{Q}_D \right] + \mathbb{P}_D \left[\widehat{Q}_D \right] \leq \exp(-\Omega(n^2)). \quad (5.4)$$

By (5.4) and Claim 7, $\mathbb{P}_D \left[\overline{\mathcal{E}_D^{(y,z)}} \right] \leq \exp(-\Omega(n^2))$, so by a union bound we have $\mathbb{P}_D \left[\overline{\mathcal{E}_D} \right] \leq \exp(-\Omega(n^2))$. Then by Proposition 4.4, we have

$$\mathbb{P} \left[\overline{\mathcal{E}} \right] \leq \mathbb{P} \left[\overline{\mathcal{E}} \mid D \right] = \frac{\sum_{H \in \overline{\mathcal{E}_D}} \text{comp}(H)}{\sum_{H \in \mathcal{G}_D} \text{comp}(H)} \leq \mathbb{P}_D \left[\overline{\mathcal{E}_D} \right] \cdot \exp(O(n \log^2 n)) \leq \exp(-\Omega(n^2)),$$

which completes the proof of the lemma. ■

6 | ABSORPTION

The aim of this section is to show that if $G \in \Phi(\vec{K}_n)$ satisfies the conclusions of Lemmas 5.6 and 5.10, then G admits a small robustly rainbow-Hamiltonian subdigraph H (recall Definition 3.1), with arbitrarily chosen flexible sets of appropriate size. In Section 7, H will form the key “absorbing structure.”

Definition 6.1. Let A be a (v, c) -absorbing gadget with abutment vertices (x_4, x_5) , and let B be a (y, z) -bridging gadget (with all vertices retaining their notation as defined in Definitions 5.1 and 5.2). If $(y, z) = (x_4, x_5)$, $V(A) \cap V(B) = \{y, z\}$, and $\phi_A(A) \cap \phi_B(B) = \emptyset$, then we say B *bridges* A . In this case, we also say a collection (P_1, P_2, P_3, P_4) of properly arc-colored directed paths *completes* the pair (A, B) if:

- P_1 has tail x_2 and head x_3 .
- P_2 has tail w_1 and head w_4 .
- P_3 has tail w_5 and head w_2 .
- P_4 has tail w_3 and head w_6 .
- P_1, \dots, P_4 are mutually vertex-disjoint and the internal vertices of P_1, \dots, P_4 are disjoint from $V(A) \cup V(B)$.
- $\bigcup_{i=1}^4 P_i$ is rainbow and shares no color with $A \cup B$.

In this case, we say that $A^* := A \cup B \cup \bigcup_{i=1}^4 P_i$ is a (v, c) -*absorber* (see Figure 1), and we also define the following.

- The *initial vertex* of A^* is x_1 , and the *terminal vertex* of A^* is x_6 .
- The (v, c) -*absorbing path* in A^* is the directed path with arc set $\{x_1 v, vx_2, x_3x_4, x_5x_6\} \cup \bigcup_{i=1}^4 E(P_i)$.
- The (v, c) -*avoiding path* in A^* is the directed path with arc set $\{x_1x_2, x_3x_5, x_4x_6\} \cup \bigcup_{i=1}^4 E(P_i)$.

Observe that the (v, c) -absorbing path and (v, c) -avoiding path of a (v, c) -absorber satisfy the following key properties.

- (6.1) The initial (resp. terminal) vertex of a (v, c) -absorber is the tail (resp. head) of both the (v, c) -absorbing path and the (v, c) -avoiding path.
- (6.2) The (v, c) -absorbing path in a (v, c) -absorber contains all of the vertices.

- (6.3) The (v, c) -absorbing path in a (v, c) -absorber is rainbow and contains all of the colors.
 (6.4) The (v, c) -avoiding path in a (v, c) -absorber contains all of the vertices except v .
 (6.5) The (v, c) -avoiding path in a (v, c) -absorber is rainbow and contains all of the colors except c .

We now use (v, c) -absorbers to define a “ T -absorber” for a bipartite graph T , which will essentially form our absorbing structure in almost all $G \in \Phi(\vec{K}_n)$, for suitably chosen T . The role of T is to provide the “template” for which pairs (v, c) must provide a (v, c) -absorbing path to the rainbow directed Hamilton cycle we are building, and which pairs must provide a (v, c) -avoiding path.

Definition 6.2. Let T be a bipartite graph with bipartition (A, B) . A digraph H equipped with a proper arc-coloring ϕ is a T -absorber if the following holds.

- (i) There exist injections $f_V : A \rightarrow V(H)$ and $f_C : B \rightarrow \phi(H)$ such that for every $ab \in E(T)$, there is a unique (v, c) -absorber $A_{ab} \subseteq H$, where $v = f_V(a)$ and $c = f_C(b)$, satisfying the following.
- (a) For every $ab \in E(T)$, if $V(A_{ab}) \cap V(A_{a'b'}) \neq \emptyset$ for some $a'b' \in E(T)$ where $a'b' \neq ab$, then $a = a'$ and $V(A_{a,b}) \cap V(A_{a',b'}) = \{f_V(a)\}$.
 (b) For every $ab \in E(T)$, if $\phi(A_{ab}) \cap \phi(A_{a'b'}) \neq \emptyset$ for some $a'b' \in E(T)$ where $a'b' \neq ab$, then $b = b'$ and $\phi(A_{ab}) \cap \phi(A_{a'b'}) = \{f_C(b)\}$.
- (ii) There exist pairwise vertex-disjoint length-three paths $P_1, \dots, P_{|E(T)|-1}$, each contained in H , satisfying the following.
- (a) $\bigcup_{i=1}^{|E(T)|-1} P_i$ is rainbow and $\phi\left(\bigcup_{i=1}^{|E(T)|-1} P_i\right) \cap \phi\left(\bigcup_{e \in E(T)} A_e\right) = \emptyset$.
 (b) For some enumeration $(e_1, \dots, e_{|E(T)|})$ of $E(T)$, for each $i \in [|E(T)| - 1]$, the tail of P_i is the terminal vertex of A_{e_i} and the head of P_i is the initial vertex of $A_{e_{i+1}}$.
 (c) For each $i \in [|E(T)| - 1]$, P_i is internally vertex-disjoint from $\bigcup_{e \in E(T)} A_e$.
- (iii) Subject to (i) and (ii), H is minimal.

In this case, we say a vertex $f_V(a)$ for some $a \in A$ is a *root vertex* of H and a color $f_C(b)$ for some $b \in B$ is a *root color* of H . Moreover, we say the *initial vertex* of H is the initial vertex of A_{e_1} and the *terminal vertex* of H is the terminal vertex of $A_{e_{|E(T)|}}$.

Suppose that a bipartite graph T is robustly matchable (recall Definition 4.2) with respect to flexible sets A' and B' . The following lemma shows that a T -absorber is robustly rainbow-Hamiltonian (recall Definition 3.1) with respect to the root vertices and colors corresponding to A' and B' . This (together with Lemmas 5.6 and 5.10) reduces the task of finding such a subdigraph in almost all $G \in \Phi(\vec{K}_n)$ to the task of using large well-spread collections of (v, c) -absorbing gadgets and (y, z) -bridging gadgets to embed a T -absorber, for an appropriate robustly matchable T .

Lemma 6.3. Let $G \in \Phi(\vec{K}_n)$ with proper n -arc-coloring ϕ . Let T be a bipartite graph with bipartition (A, B) , let $H \subseteq G$ be a T -absorber, and let u and v be the initial and terminal vertices of H , respectively. Let $A' \subseteq A$ and $B' \subseteq B$, and let V' and C' be the set of root vertices and colors of H corresponding to A' and B' , respectively. If T is robustly matchable with respect to flexible sets A' and B' , then H is robustly rainbow-Hamiltonian with respect to flexible sets V' and C' and initial and terminal vertices u and v .

Proof. Let $X \subseteq V'$ and $Y \subseteq C'$ such that $|X| = |Y| \leq \min\{|V'|/2, |C'|/2\}$. It suffices to show that $H - X$ contains a rainbow directed Hamilton path which starts at u and ends at v , not containing

a color in Y . Since H is a T -absorber, by Definition 6.2(i), there exist injections $f_V : A \rightarrow V(H)$ and $f_C : B \rightarrow \phi(H)$ such that for every $ab \in E(T)$, there is a unique (v, c) -absorber $A_{ab} \subseteq H$, where $v := f_V(a)$ and $c := f_C(b)$, satisfying (i)(a) and (i)(b). By Definition 6.2(ii), there also exist pairwise vertex-disjoint length-three paths $P_1, \dots, P_{|E(T)|-1}$, each contained in H , satisfying (ii)(a), (ii)(b), and (ii)(c). Let $(e_1, \dots, e_{|E(T)|})$ be the enumeration of $E(T)$ guaranteed by (ii)(b).

Since V' and C' are the sets of root vertices and colors of H corresponding to A' and B' , respectively, $f_V^{-1}(X) \subseteq A'$ and $f_C^{-1}(Y) \subseteq B'$. Thus, since T is robustly matchable with respect to A' and B' , there exists a perfect matching M in $T - (f_V^{-1}(X) \cup f_C^{-1}(Y))$. For each $ab \in E(T)$, define a directed path P_{ab} as follows. If $ab \in M$, then let P_{ab} be the $(f_V(a), f_C(b))$ -absorbing path in A_{ab} , and otherwise let P_{ab} be the $(f_V(a), f_C(b))$ -avoiding path in A_{ab} .

Now let $P := \bigcup_{e \in E(T)} P_e \cup \bigcup_{i=1}^{|E(T)|-1} P_i$. We claim that P is a rainbow directed Hamilton path in $H - X$ which starts at u , ends at v , and does not contain a color in Y . To that end, we first show the following:

- (a) u has out-degree one and in-degree zero in P .
- (b) v has in-degree one and out-degree zero in P .
- (c) every $w \in V(P) \setminus \{u, v\}$ has in-degree and out-degree one in P .
- (d) $V(P) \cap X = \emptyset$.
- (e) $\phi(P) \cap Y = \emptyset$.
- (f) $V(H) \setminus X \subseteq V(P)$.

Indeed, (a) and (b) follow from (6.1), 6.2(i)(a), 6.2(ii)(b), and 6.2(ii)(c), and (c) follows from (6.1), (6.4), 6.2(i)(a), 6.2(ii)(b), and 6.2(ii)(c).

To prove (d), note that if $w \in X$, then $a := f_V^{-1}(w) \notin V(M)$. Thus w is not in the paths P_{ab} for $b \in N_T(a)$ by (6.4) as they are all $(w, f_C(b))$ -avoiding. Therefore (d) follows again by 6.2(i)(a), 6.2(ii)(b), and 6.2(ii)(c). The proof of (e) is the same, with (6.5) instead of (6.4), 6.2(i)(b) instead of 6.2(i)(a), and 6.2(ii)(a) instead of 6.2(ii)(b) and 6.2(ii)(c). To prove (f), first note that if $w \in V' \setminus X$, then there exists $ab \in M$, where $a := f_V^{-1}(w)$. Thus, $w \in V(P_{ab})$ by (6.2) since P_{ab} is $(w, f_C(b))$ -absorbing. In particular, $V' \setminus X \subseteq V(P)$. By (6.2) and (6.4), 6.2(iii) implies that $V(H) \setminus V' \subseteq V(P)$. Thus, $V(H) \setminus X \subseteq V(P)$, as desired.

By (6.1), 6.2(ii)(b), 6.2(ii)(c), and (a)–(c), P contains no cycle, so (a)–(e) imply that P is indeed a directed path in $H - X$, not containing a color in Y , which starts at u and ends at v , and (f) implies that P is Hamilton in $H - X$, as required. It remains to show that P is rainbow. By (6.3), (6.5), and 6.2(i)(b), $\bigcup_{e \in E(T)} P_e$ is rainbow, so 6.2(ii)(a) implies that P is rainbow, as required. ■

The following proposition implies that there are many short rainbow paths that are “well-spread” in all $G \in \Phi(\vec{K}_n)$. This enables us to embed these paths in any such G in a vertex- and color-disjoint way while constructing a T -absorber for suitably chosen T , and while absorbing the colors unused by the large rainbow directed path forests we find in Section 7.

Proposition 6.4. *Let $G \in \Phi(\vec{K}_n)$ with proper n -arc-coloring ϕ , let $V := V(G)$, and let $C := \phi(G)$. Let $u, v \in V$ such that $u \neq v$, and let $c \in C$. The following holds for n sufficiently large.*

- (i) *There are at least $n^2/3$ length-three directed rainbow paths in G with head v and tail u .*
- (ii) *If at most $n/2$ loops in G are colored c , then there are at least $n^2/5$ length-four directed rainbow paths in G with head v and tail u such that the second arc is colored c .*
- (iii) *For every $w \in V$, there are at most $2n$ length-three directed rainbow paths in G with head v and tail u that contain w as an internal vertex.*

- (iv) For every $w \in V$, there are at most $3n$ length-four directed rainbow paths in G with head v and tail u that contain w as an internal vertex such that the second arc is colored c .
- (v) For every $d \in C$, there are at most $3n$ length-three directed rainbow paths in G with head v and tail u that contain an arc colored d .
- (vi) For every $d \in C \setminus \{c\}$, there are at most $3n$ length-four directed rainbow paths in G with head v and tail u that contain an arc colored d such that the second arc is colored c .

Proof. First, for each vertex $w \in V$, we let $B_w := \{x \in V \setminus \{w, v\} : \phi(wx) = \phi(xv)\}$, we say w is *bad* if $|B_w| > n/2$, and we let $B \subseteq V$ be the set of bad vertices. We claim that there is at most one bad vertex; that is, $|B| \leq 1$. To that end, suppose for a contradiction that distinct vertices w and w' are bad. Since $|B_w| + |B_{w'}| > n$, we have $B_w \cap B_{w'} \neq \emptyset$. Thus, there exists some $x \in B_w \cap B_{w'}$, so $\phi(wx) = \phi(xv) = \phi(w'x)$, contradicting that ϕ is a proper arc-coloring.

Now we prove (i). Since $|B| \leq 1$, there are at least $(n-3)(n/2-5)$ choices of an ordered pair (w_1, w_2) where $w_1 \in V \setminus (B \cup \{u, v\})$ and $w_2 \in V \setminus (B_{w_1} \cup \{u, v, w_1\})$ such that $\phi(uw_1) \notin \{\phi(w_1w_2), \phi(w_2v)\}$. For each such pair, there is a distinct directed path $P_{w_1, w_2} := uw_1w_2v$ in G . Since $\phi(uw_1) \notin \{\phi(w_1w_2), \phi(w_2v)\}$ and $w_2 \notin B_{w_1}$, P_{w_1, w_2} is rainbow. Therefore there are at least $n^2/3$ directed length-three rainbow paths in G with head v and tail u , as desired.

Now we prove (ii). Let $X := \{w \in V : \phi(wv) = c\}$; by assumption, $|X| \leq n/2$. Thus, since $|B| \leq 1$, there are at least $n/2 - 6$ choices of an ordered pair (w_1, w_2) where $w_2 \in V \setminus (X \cup B \cup \{u, v\})$, $w_1 \in V \setminus \{u, v, w_2\}$, $\phi(w_1w_2) = c$, and $\phi(uw_1) \neq c$. For each such pair, there are at least $n/2 - 8$ choices of a vertex $w_3 \in V \setminus (B_{w_2} \cup \{u, v, w_1, w_2\})$ such that $\{\phi(w_2w_3), \phi(w_3v)\} \cap \{\phi(uw_1), c\} = \emptyset$. For each such choice of (w_1, w_2, w_3) , there is a distinct directed length-four rainbow path $P_{w_1, w_2, w_3} := uw_1w_2w_3v$ with head v and tail u such that the second arc is colored c . Therefore there are at least $(n/2 - 7)(n/2 - 8) \geq n^2/5$ such paths, as desired.

The proofs of (iii)–(vi) are similar, so we only provide a complete proof of (vi). Fix $d \in C \setminus \{c\}$, and let $\mathcal{P}_{(vi)}$ be the set of length-four rainbow directed paths in G with head v and tail u that contain an arc colored d such that the second arc is colored c . Partition $\mathcal{P}_{(vi)}$ into sets $\mathcal{P}_{(vi),1}, \dots, \mathcal{P}_{(vi),4}$ such that for $i \in [4]$, a path $P \in \mathcal{P}_{(vi),i}$ if the arc colored d is the i th arc of P . Since $d \neq c$, $\mathcal{P}_{(vi),2} = \emptyset$. Every path in $\mathcal{P}_{(vi),1}$ is uniquely determined by the vertex in the path adjacent to v , every path in $\mathcal{P}_{(vi),3}$ is uniquely determined by an ordered pair (w_2, w_3) such that $\phi(w_2w_3) = d$, and every path in $\mathcal{P}_{(vi),4}$ is uniquely determined by an ordered pair (w_1, w_2) such that $\phi(w_1w_2) = c$. Therefore $|\mathcal{P}_{(vi)}| \leq 3n$, as desired.

We conclude by outlining the necessary changes to the proof of (vi) to obtain proofs of (iii)–(v). Define $\mathcal{P}_{(iii)}$, $\mathcal{P}_{(iv)}$, and $\mathcal{P}_{(v)}$ in an analogous way. Partition $\mathcal{P}_{(iii)}$ and $\mathcal{P}_{(iv)}$ based on the position of w in each path, and partition $\mathcal{P}_{(v)}$ based on the position of the arc colored d . Finally, show that each part contains at most n paths. \blacksquare

For any $m = \omega(1)$ (in Section 7, we will set $m := \lfloor n/\log^3 n \rfloor$, and we assume n to be sufficiently large) we have by Lemma 4.3 that there exists a 256-regular 2RMBG($7m, 2m$), say T (recall Definition 4.2). In the following lemma, we show that if $m < n/\log n$, then we may greedily embed a T -absorber in any $G \in \Phi(\vec{K}_n)$ that satisfies the conclusions of Lemmas 5.6 and 5.10, by choosing each absorber successively in three steps: for each edge vc of T , we first embed a (v, c) -absorbing gadget A , then choose a bridging gadget B that bridges A (recall Definition 6.1), and finally use Proposition 6.4 to embed the extra short rainbow paths required to complete the (v, c) -absorber and connect it to the previously embedded absorber.

Lemma 6.5. *Let $G \in \Phi(\vec{K}_n)$ with proper n -arc-coloring ϕ , let $V := V(G)$, and let $C := \phi(G)$. Let $m < n/\log n$, let T be a 256-regular 2RMBG($7m, 2m$), let $U \subseteq V$ and $D \subseteq C$ such that $|U| = |D| = 7m$, and let $V' \subseteq U$ and $C' \subseteq D$ such that $|V'| = |C'| = 2m$. For n sufficiently large, if*

- for all $v \in V$ and $c \in C$, G contains a well-spread collection $\mathcal{A}_{v,c}$ of at least $n^2/2^{100}$ (v, c) -absorbing gadgets and
- for all distinct $y, z \in V$, G contains a well-spread collection $\mathcal{B}_{y,z}$ of at least $n^2/10^{50}$ (y, z) -bridging gadgets,

then G contains a T -absorber H rooted on vertices U and colors D such that V' and C' are the sets of root vertices and colors of H corresponding to the flexible sets of T .

Proof. Denote the bipartition of T by (A, B) , let $A' \subseteq A$ and $B' \subseteq B$ be the flexible sets of T , and enumerate the edges of T as $e_1, \dots, e_{|E(T)|}$. Let $f_V : A \rightarrow U$ and $f_C : B \rightarrow D$ be bijections, chosen such that $f_V(A') = V'$ and $f_C(B') = C'$. For each $j \in [|E(T)|]$, we inductively choose

- (i) a (v, c) -absorbing gadget A_j in G , where $ab := e_j$, $v := f_V(a)$, and $c := f_V(b)$,
- (ii) a (y_j, z_j) -bridging gadget B_j in G , where (y_j, z_j) is the pair of abutment vertices of A_j , which bridges A_j , and
- (iii) length-three rainbow directed paths $P_{j,1}, \dots, P_{j,5}$ in G , such that $P_{j,1}, \dots, P_{j,4}$ complete the pair (A_j, B_j) to a (v, c) -absorber A_j^* ,

such that the following holds:

- (a)_j $V(A_j) \cap U = \{f_V(a)\}$, $V(A_j) \cap V(P_{\ell,5}) = \emptyset$ for all $\ell < j$, and if $V(A_j) \cap V(A_\ell^*) \neq \emptyset$ for some $\ell < j$, then $V(A_j) \cap V(A_\ell^*) = \{f_V(a)\}$, where $a \in e_j \cap e_\ell \subseteq A$;
- (b)_j $\phi(A_j) \cap D = \{f_C(b)\}$, $\phi(A_j) \cap \phi(P_{\ell,5}) = \emptyset$ for all $\ell < j$, and if $\phi(A_j) \cap \phi(A_\ell^*) \neq \emptyset$ for some $\ell < j$, then $\phi(A_j) \cap \phi(A_\ell^*) = \{f_C(b)\}$, where $b \in e_j \cap e_\ell \subseteq B$;
- (c)_j $V(B_j) \cap V(A_\ell^* \cup P_{\ell,5}) = \emptyset$ for all $\ell < j$, and $V(B_j) \cap U = \emptyset$;
- (d)_j $\phi(B_j) \cap \phi(A_\ell^* \cup P_{\ell,5}) = \emptyset$ for all $\ell < j$, and $\phi(B_j) \cap D = \emptyset$;
- (e)_j $V(P_{j,k}) \cap V(A_\ell^* \cup P_{\ell,5}) = \emptyset$ for all $k \in [4]$ and $\ell < j$, and $V(P_{j,k}) \cap U = \emptyset$ for all $k \in [5]$;
- (f)_j $\phi(P_{j,k}) \cap \phi(A_\ell^* \cup P_{\ell,5}) = \emptyset$ for all $k \in [4]$ and $\ell < j$, and $\phi(P_{j,k}) \cap D = \emptyset$ for all $k \in [5]$;
- (g)_j $\phi(P_{j,5}) \cap \phi(A_\ell^*) = \emptyset$ for all $\ell \leq j$, and $\phi(P_{j,5}) \cap \phi(P_{\ell,5}) = \emptyset$ for all $\ell < j$;
- (h)_j if $j > 1$, then the tail of $P_{j,5}$ is the terminal vertex of A_{j-1}^* , the head of $P_{j,5}$ is the initial vertex of A_j^* , $P_{j,5}$ is internally vertex-disjoint from A_ℓ^* for all $\ell \leq j$, $V(P_{j,5}) \cap V(P_{\ell,5}) = \emptyset$ for all $\ell < j$, and if $j = 1$, then $P_{j,5} = \emptyset$.

To that end, we let $i \in [|E(T)|]$ and assume A_j, B_j , and $P_{j,1}, \dots, P_{j,5}$ satisfying (a)_j–(h)_j have been chosen for $j < i$, and we show that we can indeed choose A_j, B_j , and $P_{j,1}, \dots, P_{j,5}$ according to (i)–(iii) satisfying (a)_j–(h)_j for $j = i$. Let $U' := U \cup \bigcup_{\ell < j} V(A_\ell^* \cup P_{\ell,5})$, and let $D' := D \cup \bigcup_{\ell < j} \phi(A_\ell^* \cup P_{\ell,5})$. For every $\ell < j$, we have by (i)–(iii) that $|V(A_\ell^* \cup P_{\ell,5})|, |\phi(A_\ell^* \cup P_{\ell,5})| \leq 24$. Thus, since T is 256-regular and $m < n/\log n$,

$$|U'|, |D'| \leq 7m + 24j \leq 43\,008m < 43\,008n/\log n. \quad (6.6)$$

First we show that we can choose a (v, c) -absorbing gadget A_j according to (i) satisfying (a)_j and (b)_j. By assumption, G contains a well-spread collection $\mathcal{A}_{v,c}$ of $n^2/2^{100}$ (v, c) -absorbing gadgets. For each $u \in U'$, let $\mathcal{A}_u := \{A^{\text{gdgt}} \in \mathcal{A}_{v,c} : u \in V(A^{\text{gdgt}})\}$, and for each $d \in D'$, let $\mathcal{A}_d := \{A^{\text{gdgt}} \in \mathcal{A}_{v,c} : d \in \phi(A^{\text{gdgt}})\}$. Let $\mathcal{A}'_{v,c} := \mathcal{A}_{v,c} \setminus (\bigcup_{u \in U' \setminus \{v\}} \mathcal{A}_u \cup \bigcup_{d \in D' \setminus \{c\}} \mathcal{A}_d)$. Since $\mathcal{A}_{v,c}$ is well-spread,

$|\mathcal{A}_u|, |\mathcal{A}_d| \leq n$ for every $u \in U' \setminus \{v\}$ and $d \in D' \setminus \{d\}$, so by (6.6), $|\mathcal{A}'_{v,c}| \geq |\mathcal{A}_{v,c}| - 86\,016n^2 / \log n > 0$. In particular, there exists $A_j \in \mathcal{A}'_{v,c}$. By construction of $\mathcal{A}'_{v,c}$, A_j satisfies (a)_j and (b)_j, as desired.

Now let (y_j, z_j) be the abutment vertices of A_j . By a similar argument, we can choose a (y_j, z_j) -bridging gadget B_j according to (ii) satisfying (c)_j and (d)_j.

Finally, we show that we can choose $P_{j,1}, \dots, P_{j,5}$ according to (iii) satisfying (e)_j–(h)_j. The argument is again similar, using (i), (iii), and (v) of Proposition 6.4 instead of the existence of a well-spread collection of gadgets, so we omit the proof.

To complete the proof, we show that $H := \bigcup_{i=1}^{|E(T)|} A_i^* \cup \bigcup_{i=1}^{|E(T)|-1} P_{i+1,5}$ is a T -absorber rooted on vertices U and colors D . Since (a)_j–(f)_j hold for every $j \in [|E(T)|]$, H satisfies 6.2(i), and since (g)_j and (h)_j hold for every $j \in [|E(T)|]$, H satisfies 6.2(ii). Clearly H is minimal with respect to these properties, so H also satisfies 6.2(iii). Thus, H is a T -absorber rooted on U and D , as desired. Moreover, by the choice of f_V and f_C , V' and C' are the sets of root vertices and colors of H corresponding to the flexible sets of T , as desired. ■

7 | PROOF OF THEOREM 1.6

In this section, we use the results we have obtained thus far to prove Theorem 1.6. We begin by arguing that a “lower-quasirandomness” condition in $G \in \Phi(\vec{K}_n)$ (which holds in almost all $G \in \Phi(\vec{K}_n)$ by Theorem 4.7) is enough to ensure the existence of many rainbow directed path forests spanning all but a small arbitrary set of vertices, avoiding a small arbitrary forbidden set of colors, and having few components. We remark that the method we use to count the rainbow directed path forests is inspired by the method used by Kwan (see the proof of [23, Lemma 5.5]) to count large matchings in random Steiner triple systems.

Definition 7.1. Let $G \in \Phi(\vec{K}_n)$ with proper n -arc-coloring ϕ , let $V := V(G)$, and let $C := \phi(G)$. We say that G is *lower-quasirandom* if for all (not necessarily distinct) sets $U_1, U_2 \subseteq V$ and $D \subseteq C$, we have that $e_{G,D}(U_1, U_2) \geq |U_1||U_2||D|/n - n^{5/3}$.

Lemma 7.2. Let $G \in \Phi(\vec{K}_n)$ with proper n -arc-coloring ϕ , let $V := V(G)$, and let $C := \phi(G)$. Let $U \subseteq V$ and let $D \subseteq C$ be equal-sized sets of size at most $n/\log^2 n$. If G is lower-quasirandom, then there are at least $((1 - o(1))n/e^2)^n$ spanning rainbow directed path forests Q of $G - U$ such that $\phi(Q) \cap D = \emptyset$ and Q has at most $n^{9/10}$ components.

Proof. Throughout the proof, we implicitly assume n is sufficiently large for certain inequalities to hold. We say a rainbow directed path forest in G is *valid* if it has no vertices in U and no colors in D . Let $n' := n - |U| - \lfloor n^{9/10} \rfloor$, and let $n'' := n - |U|$. If Q is a spanning directed path forest of $G - U$, then the number of components of Q is equal to $|V \setminus U| - |E(Q)|$, so Q has at most $n^{9/10}$ components if and only if it has at least n' arcs. Thus, it suffices to count the number of valid rainbow directed path forests in G that have n' arcs. To that end, we first count the number of *ordered* sequences of arcs $(e_1, \dots, e_{n'})$ such that $\bigcup_{i=1}^{n'} e_i$ is a valid rainbow directed path forest in G . We claim that for every $j \in [n']$, if e_1, \dots, e_{j-1} are arcs in G such that $\bigcup_{i=1}^{j-1} e_i$ is a valid rainbow directed path forest, then there are at least $(1 - o(1))(n'' - (j - 1))^3 / n$ choices of an arc $e_j \in E(G)$ such that $\bigcup_{i=1}^j e_i$ is also a valid rainbow directed path forest. Let V_H be the set of vertices $u \in V \setminus U$ such that u has no out-neighbor in $\bigcup_{i=1}^{j-1} e_i$, let V_T be the set of vertices $u \in V \setminus U$ such that u has no in-neighbor in $\bigcup_{i=1}^{j-1} e_i$, and let

$D' := (C \setminus D) \setminus \bigcup_{i=1}^{j-1} \phi(e_i)$. Since G is lower-quasirandom, since $|V_H| = |V_T| = |D'| = n - |U| - (j-1)$, and since $j \leq n' = n'' - \lfloor n^{9/10} \rfloor$, we have that

$$e_{G,D'}(V_H, V_T) \geq \frac{(n - |U| - (j-1))^3}{n} - n^{5/3} \geq (1 - o(1)) \frac{(n'' - (j-1))^3}{n}. \tag{7.1}$$

The spanning path forest in $G - U$ with edge set $\{e_1, \dots, e_{j-1}\}$ has $k := n'' - (j-1)$ components, which we denote P_1, \dots, P_k . For every $i \in [k]$, there is a unique arc $f_i \in E(G)$ (whose head is the tail of P_i and whose tail is the head of P_i) such that $P_i \cup f_i$ is a directed cycle. Let $F := \{f_1, \dots, f_k\}$, and let $e_j \in E_{G,D'}(V_H, V_T) \setminus F$. By the choice of V_H, V_T, D' , and F , we have that $\bigcup_{i=1}^j e_i$ is rainbow, has maximum in-degree and out-degree one, and contains no cycle. Hence, it is a valid rainbow directed path forest, as required. By (7.1), $|E_{G,D'}(V_H, V_T) \setminus F| \geq (1 - o(1)) \frac{(n'' - (j-1))^3}{n}$, so the claim follows.

Therefore, the number of ordered sequences of arcs $e_1, \dots, e_{n'} \in E(G)$ such that $\bigcup_{i=1}^{n'} e_i$ is a valid rainbow directed path forest is at least

$$\prod_{j=1}^{n'} (1 - o(1)) \frac{(n'' - (j-1))^3}{n} = \left((1 - o(1)) \frac{(n'')^3}{n} \right)^{n'} \exp \left(3 \sum_{j=1}^{n'} \log \left(1 - \frac{j-1}{n''} \right) \right). \tag{7.2}$$

Since $|U| \leq n/\log^2 n$, we have $n' \geq n - 2n/\log^2 n$ and $(n'')^3/n \geq (1 - 3/\log^2 n)n^2$. Hence,

$$\left(\frac{(n'')^3}{n} \right)^{n'} \geq \left(\left(1 - \frac{3}{\log^2 n} \right) n^2 \right)^{n'} n^{-4n'/\log^2 n} \geq ((1 - o(1))n^2)^n. \tag{7.3}$$

Since the function $x \mapsto \log(1 - x)$ is negative and monotonically decreasing for $x \in [0, 1)$,

$$\sum_{j=1}^{n'} \frac{1}{n''} \log \left(1 - \frac{j-1}{n''} \right) \geq \int_0^{n'/n''} \log(1 - x) dx \geq \int_0^1 \log x dx = -1. \tag{7.4}$$

By substituting (7.3) and (7.4) into the right side of (7.2), the expression in (7.2) is at least

$$((1 - o(1))n^2)^n e^{-3n''} \geq \left((1 - o(1)) \frac{n^2}{e^3} \right)^n. \tag{7.5}$$

By Stirling's approximation, $n! = (1 + O(1/n))\sqrt{2\pi n}(n/e)^n \leq ((1 + o(1))n/e)^n$. Hence, since (7.5) provides a lower bound on the number of ordered sequences of edges $e_1, \dots, e_{n'} \in E(G)$ such that $\bigcup_{i=1}^{n'} e_i$ is a valid rainbow directed path forest, the total number of valid rainbow directed path forests in G with at most $n^{9/10}$ components is at least

$$\left((1 - o(1)) \frac{n^2}{e^3} \right)^n / n! \geq \left((1 - o(1)) \frac{n}{e^2} \right)^n,$$

as desired. ■

We now have all the tools we need to prove Theorem 1.6.

Proof of Theorem 1.6. Let $G \in \Phi(\vec{K}_n)$ with proper n -arc-coloring ϕ , let $V := V(G)$, and let $C := \phi(G)$. By Lemma 4.6, Theorem 4.7, and Lemmas 5.6 and 5.10, it suffices to show that if

- (7.6) for every $c \in C$, at most $n/2$ loops in G are colored c ,
- (7.7) G is lower-quasirandom,
- (7.8) G contains a well-spread collection of at least $n^2/2^{100}$ (v, c) -absorbing gadgets for every $v \in V$ and $c \in C$, and
- (7.9) G contains a well-spread collection of at least $n^2/10^{50}$ (y, z) -bridging gadgets for every $y, z \in V$,

then G contains at least $((1 - o(1))n/e^2)^n$ rainbow directed Hamilton cycles.

Let $m = \lfloor n/\log^3 n \rfloor$, and let T be a 256-regular $2RMBG(7m, 2m)$ (which exists by Lemma 4.3). We build a T -absorber H in G using Lemma 6.5, but first we need the following claim to choose the roots of H that will correspond to the flexible sets of T .

Claim 8. There exist $V' \subseteq V$ and $C' \subseteq C$ such that $|V'| = |C'| = 2m$ and for every $u, v \in V$ such that $u \neq v$ and for every $c \in C$, there are at least $n^{99/50}$ directed paths P in G such that

- (i) P is rainbow and has length four.
- (ii) P has head v and tail u .
- (iii) The second arc of P is colored c .
- (iv) $V(P) \setminus \{u, v\} \subseteq V'$.
- (v) $\phi(P) \setminus \{c\} \subseteq C'$.

Proof of Claim 8. Let $p := (2m + n^{9/10})/n$, and let $U' \subseteq V$ and $D' \subseteq C$ be chosen randomly by including every $v \in V$ in U' and every $c \in C$ in D' independently with probability p . We claim that the following holds with high probability:

- (a) $|U'|, |D'| = pn \pm n^{4/5}$.
- (b) For every $u, v \in V$ such that $u \neq v$ and for every $c \in C$, there are at least $p^6 n^2/10$ directed paths P in G satisfying (i)–(iii), such that $V(P) \setminus \{u, v\} \subseteq U'$ and $\phi(P) \setminus \{c\} \subseteq D'$.

Indeed, (a) follows from a standard application of the Chernoff bound. To prove (b), we use McDiarmid's inequality. To that end, fix $u, v \in V$ distinct and $c \in C$. We let f denote the random variable counting the number of paths satisfying (b). Note that f is determined by the independent binomial random variables $\{X_w : w \in V\} \cup \{X_d : d \in C\}$, where X_w indicates if $w \in U'$ and X_d indicates if $d \in D'$. By (7.6) and Proposition 6.4(ii), there are at least $n^2/5$ paths satisfying (i), (ii), and (iii), and each such path satisfies $V(P) \setminus \{u, v\} \subseteq U'$ with probability p^3 and $\phi(P) \setminus \{c\} \subseteq D'$ with probability p^3 , independently. Hence, $\mathbb{E}[f] \geq p^6 n^2/5$. For each $w \in V$, by Proposition 6.4(iv), X_w affects f by at most $3n$, and for each $d \in C$, by Proposition 6.4(vi), X_d affects f by at most $3n$. Therefore by McDiarmid's inequality (Theorem 4.1) applied with $t := \mathbb{E}[f]/2$, there are at least $p^6 n^2/10$ paths satisfying (b) with probability at least $1 - \exp(-\Omega(p^{12}n))$. Hence, by a union bound, (b) holds for every $u, v \in V$ with $u \neq v$ and every $c \in C$ with high probability, as claimed.

Now we fix a choice of U' and D' satisfying both (a) and (b) simultaneously. By (a), $|U'|, |D'| \geq 2m$, so there exists $V' \subseteq U'$ and $C' \subseteq D'$ such that $|V'| = |C'| = 2m$, as required. Moreover, by (a), $|U' \setminus V'|, |D' \setminus C'| \leq 2n^{9/10}$. Thus, by Proposition 6.4(iv) and (vi), (b) implies that for every $u, v \in V$ distinct and $c \in C$, there are at least $p^6 n^2/10 - 2(2n^{9/10})(3n) \geq n^{99/50}$ directed paths satisfying (i)–(v), as desired. ■

Now let $U \subseteq V$ and $D \subseteq C$ such that $|U| = |D| = 7m$, $V' \subseteq U$, and $C' \subseteq D$. By Lemma 6.5, (7.8), and (7.9), there is a T -absorber H in G rooted on U and D such that V' and C' are the sets of

root vertices and colors of H corresponding to the flexible sets of T . By Lemma 6.3, H is robustly rainbow-Hamiltonian with respect to flexible sets V' and C' and initial and terminal vertices t and h , where t is the initial vertex of H and h is the terminal vertex of H . Note that

$$|V(H)| = 22|E(T)| - 2 + |U| \quad \text{and} \quad |\phi(H)| = 22|E(T)| - 3 + |D|. \tag{7.10}$$

Claim 9. If Q is a spanning rainbow path forest in $G - V(H)$, sharing no color with H , with at most $n^{9/10}$ components, then G has a rainbow directed Hamilton cycle F such that $F - V(H) = Q$.

Proof of Claim 9. Let P_1, \dots, P_k be the components of Q , and for each $i \in [k]$, let h_i be the head of P_i , and let t_i be the tail of P_i . Let t_{k+1} denote the initial vertex of H , and let h_0 denote the terminal vertex of H . That is, $t_{k+1} := t$ and $h_0 := h$. Let $c_1, \dots, c_{k'}$ be an enumeration of the colors in $C \setminus (\phi(H) \cup \phi(Q))$. Since Q is rainbow, $|\phi(Q)| = |E(Q)| = |V \setminus V(H)| - k$, so by (7.10), $k' = k + 1$.

By Claim 8, for each $j \in [k + 1]$, there is a collection \mathcal{P}_j of at least $n^{99/50}$ directed paths satisfying (i)–(v) where $u = h_{j-1}$, $v = t_j$, and $c = c_j$. For each $j \in [k + 1]$, we inductively choose a path $P'_j \in \mathcal{P}_j$ such that

- (a)_j $\phi(P'_j) \cap \phi(P'_\ell) = \emptyset$ for every $\ell < j$.
- (b)_j $V(P'_j) \cap V(P'_\ell) \cap V' = \emptyset$ for every $\ell < j$.

To that end, we let $i \in [k + 1]$ and assume $P'_j \in \mathcal{P}_j$ has been chosen to satisfy (a)_j and (b)_j for each $j < i$, and we show that we can indeed choose such a P'_j for $j = i$. Let $U_j := V' \cap \bigcup_{\ell=1}^{j-1} V(P'_\ell)$, and let $D_j := D' \cap \bigcup_{\ell=1}^{j-1} \phi(P'_\ell)$. For each $u \in U_j$, let $\mathcal{P}_u := \{P \in \mathcal{P}_j : u \in V(P)\}$, and for each $d \in D_j$, let $\mathcal{P}_d := \{P \in \mathcal{P}_j : d \in \phi(P)\}$. Let $\mathcal{P}'_j := \mathcal{P}_j \setminus \left(\bigcup_{u \in U_j} \mathcal{P}_u \cup \bigcup_{d \in D_j} \mathcal{P}_d \right)$. By Proposition 6.4(iv), $|\mathcal{P}_u| \leq 3n$ for each $u \in U_j$, and by Proposition 6.4(vi), $|\mathcal{P}_d| \leq 3n$ for each $d \in D_j$. Since P'_ℓ has length four for each $\ell < j$, we have $|U_j|, |D_j| \leq 3k$. Hence, $|\mathcal{P}'_j| \geq n^{99/50} - (6k)(3n) > 0$. In particular, there exists $P'_j \in \mathcal{P}'_j$. By construction of \mathcal{P}'_j , P'_j satisfies (a)_j and (b)_j, as desired.

Since Q shares no vertices or colors with H , for each $j \in [k + 1]$, since the internal vertices of P'_j are in $V' \subseteq V(H)$ and since $\phi(P'_j) \setminus D' = \{c_j\}$, it follows that $P'_j \cup P_j$ is a rainbow directed path with tail h_{j-1} and head h_j . Moreover, by induction, using (a)_j and (b)_j, if $j \leq k$, then $\bigcup_{\ell=1}^j (P'_\ell \cup P_\ell)$ is a rainbow directed path with tail h_0 and head h_j , and in particular, $P^*_1 := P'_{k+1} \cup \bigcup_{\ell=1}^k (P'_\ell \cup P_\ell)$ is a rainbow directed path with tail h_0 and head t_{k+1} .

Let $X := V' \cap \bigcup_{j=1}^{k+1} V(P'_j)$, and let $Y := D' \cap \bigcup_{j=1}^{k+1} \phi(P'_j)$. Note that $|X|, |Y| \leq 3(k + 1) \leq m$. Therefore, since H is robustly rainbow-Hamiltonian, there exists a rainbow directed Hamilton path P^*_2 in $H - X$ with tail t_{k+1} and head h_0 , not containing a color in Y . Now $F := P^*_1 \cup P^*_2$ is a rainbow directed Hamilton cycle in G , and $F - V(H) = Q$, as desired. ■

By Lemma 7.2 and (7.10), there is a collection \mathcal{Q} of at least $((1 - o(1))n/e^2)^n$ spanning rainbow directed path forests in $G - V(H)$ that share no colors with $\phi(H)$ and have at most $n^{9/10}$ components. By Claim 9, for every $Q \in \mathcal{Q}$, there is a rainbow directed Hamilton cycle F_Q such that $F - V(H) = Q$. Therefore $\{F_Q : Q \in \mathcal{Q}\}$ is a collection of at least $((1 - o(1))n/e^2)^n$ distinct rainbow directed Hamilton cycles in G , as desired. ■

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