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ISOMONODROMIC DEFORMATIONS: CONFLUENCE, REDUCTION AND QUANTISATION

ILIA GAIUR, MARTA MAZZOCCO, AND VLADIMIR RUBTSOV

Abstract. In this paper we study the isomonodromic deformations of systems of differential equations with poles of any order on the Riemann sphere as Hamiltonian flows on the product of co-adjoint orbits of the Takiff algebra (i.e. truncated current algebra). Our motivation is to produce confluent versions of the celebrated Knizhnik–Zamolodchikov equations and explain how their quasiclassical solution can be expressed via the isomonodromic τ-function. In order to achieve this, we study the confluence cascade of $r+1$ simple poles to give rise to a singularity of arbitrary Poincaré rank r as a Poisson morphism and explicitly compute the isomonodromic Hamiltonians.

Introduction

In this paper we study the theory of isomonodromic deformations for systems of differential equations with poles of any order on the Riemann sphere. Our initial motivation was to generalise an observation by Reshetikhin that the quasi–classical solution of the standard Knizhnik–Zamolodchikov equations (i.e. with simple poles) is expressed via the isomonodromic τ-function arising in the case of Fuchsian systems [57]. Along the way of pursuing the project of extending this to poles of any order, we have found a number of interesting results, some of which were already known as folklore (i.e. either done as very specific examples or not really proved in detail), others completely original.

The Knizhnik–Zamolodchikov (KZ) equations emerged in theoretical physics as the system of linear differential equations satisfied by the correlation functions in the two–dimensional Wess–Zumino–Witten model of conformal field theory associated to a genus 0 curve [46, 8]. In the case of $\mathfrak{g} = \mathfrak{gl}_m$, the KZ equations can be represented as a system of linear differential equations for a local section ψ of the trivial bundle $B \times U(\mathfrak{gl}_\mathfrak{m}(\mathbb{C}))^{\otimes n} \to B$ over the base B given by the configuration space of ordered n-uples of points in $\mathbb{C}$, namely $B := \{(u_1, \dots, u_n) \in \mathbb{C}^n | u_i \neq u_j \text{ for } i \neq j\}$:

$$d\psi = \sum_{i \neq j} \Pi^{ij} \frac{du_i - du_j}{u_i - u_j} \psi, \tag{1}$$

where $\Pi^{ij} \in \mathrm{End}(U(\mathfrak{gl}_m(\mathbb{C})^{\otimes n})$ is the extension of the non-degenerated symmetric tensor

$$\Pi \in \mathfrak{gl}_m(\mathbb{C}) \times \mathfrak{gl}_m(\mathbb{C}) = \mathrm{End}(\mathfrak{gl}_\mathrm{m}(\mathbb{C}))$$

acting by left multiplication on the $i-$th and $j-$th components of the tensor product $U(\mathfrak{gl}_m(\mathbb{C})^{\otimes n})$. Geometrically one can think about (1) as a flat Hitchin connection in geometric quantisation [35].

As proved by Reshetikhin in [57] (see also [30] where this result was explained in terms of passing from Shrödinger to Heisenberg representation), the KZ equations can be also viewed as deformation quantisation of the Schlesinger system [59] of non-linear differential equations

$$dA^{(i)} = \sum_{i \neq j} [A^{(i)}, A^{(j)}] \frac{du_i - du_j}{u_i - u_j}, \tag{2}$$

controlling the isomonodromic deformation of a Fuchsian system on $\mathbb{P}^1$,

$$\frac{dY}{d\lambda} = \sum_{i=1}^{n} \frac{A^{(i)}}{\lambda - u_i} Y, \tag{3}$$

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with $n+1$ simple poles $u_1,\dots,u_n,\infty$. These equations are multi–time non-autonomous Hamiltonian systems with Hamiltonians

$$H_i : B\times \mathfrak{gl}_m(\mathbb{C})^n \to \mathbb{C} \tag{4}$$

given by

$$H_i := \sum_{i\neq j} \frac{\mathrm{Tr}(A^{(i)}A^{(j)})}{u_i-u_j}.$$

Interestingly, if we treat the quantities $u_1\dots,u_n$ in the Hamiltonian as parameters rather than times, these Hamiltonians form a family of autonomous Poisson commuting Hamiltonians called *Gaudin Hamiltonians*. This simple observation has been key to several efforts to introduce specific examples of confluent analogues of KZ: by first introducing confluent analogues of Gaudin, then quantising them and finally generating the non-autonomous versions. Let us give a summary of our understanding of these results here below.

The main idea for the quantisation of the Gaudin Hamiltonians was based on the standard point of view that for any finite dimensional Lie algebra $\mathfrak{g}$, the universal enveloping algebra $U(\mathfrak{g})$ can be considered as a deformation of the symmetric algebra $S(\mathfrak{g})$ via the Poincaré-Birkhoff-Witt map. One then defines the quantum enveloping algebra as

$$U_\hbar(\mathfrak{g}) = T(\mathfrak{g})/(X\otimes Y - Y\otimes X - \hbar[X,Y]),\quad X,Y\in\mathfrak{g},$$

by naturally extending the symmetrisation map to the map $S(\mathfrak{g})^{\otimes n}\to U_\hbar(\mathfrak{g})^{\otimes n}$, the functions $\mathrm{Tr}(A^{(i)}A^{(j)})$ on $\mathfrak{g}^{\otimes n}$ are transformed to Π^{ij}.

To define a quantisation of the Gaudin Hamiltonians it is necessary to describe the Hilbert space of the quantum model as tensor product of some representations of $\mathfrak{g}^{\oplus n}$. The quantised Hamiltonians $\widehat{H}_i$ act on this Hilbert space and the quantum problem consists in finding their spectrum, matrix elements and so on. Formulated rigorously, the quantum Gaudin Hamiltonians generate a large commutative subalgebra in $U(\mathfrak{g})^{\otimes n}$ which can be easily completed to a maximal commutative subalgebra. This subalgebra is usually called *Gaudin* or *Bethe subalgebra*. The explicit formulae for the generators (namely the quantum Hamiltonians) were obtained in [51, 60].

In the case of $\mathfrak{g}=\mathfrak{gl}_\mathfrak{m}$, one can fix a co-vector $\mu\in\mathfrak{g}^*$ and using the standard basis of $\mathfrak{gl}_\mathfrak{m}$ one can re-write the quantised Gaudin Hamiltonians as

$$\widehat{H}_i = \sum_{j\neq i}\sum_{r,s=1}^m \frac{E_{rs}^{(i)}E_{sr}^{(j)}}{u_i-u_j} + \sum_{r,s=1}^m \mu(E_{rs})E_{sr}^{(i)}, \tag{5}$$

where $E_{rs}^{(i)}$ means E_{rs} (as the element of standard basis in $\mathfrak{gl}_\mathfrak{m}$) considering in the $i-$th tensor factor. We observe that even the case of *regular* $\mu\in\mathfrak{g}^*$ (i.e. semi-simple, when $\mu(E_{rs})=\mu_r\delta_{rs}$ with distinct $\mu_r\in\mathbb{C}$), the point ∞ is an order two pole. The case of semi–simple but not regular μ was treated in [28].

The autonomous Gaudin model (5) can be generalised in two directions: by allowing higher order singularities at the marked points $u_i\in\mathbb{C}$ thus giving rise to Gaudin models with irregular singularities in [29] or by taking an element $\mu\in\mathfrak{g}^*$ that is not semi-simple (i.e. has non-trivial Jordan blocks). These two approaches were unified in the classical and in the quantum cases in [61] where an analogue of the *bispectral dynamical duality* of [25] between the models was proved.

The next important step consisted in deforming the quantum Gaudin Hamiltonian to obtain KZ. This was done in the case of the A_n root system by de Concini and Procesi [21] and generalised to any Lie algebra in [50, 25]. More precisely, for any complex simple Lie algebra $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ and a corresponding root system $\Delta\subset\mathfrak{h}^*$, Millson and Toledano-Laredo [50] introduced the following *Casimir connection:*

$$\nabla_C\psi := d\psi - \frac{\hbar}{2\pi i}\sum_{\alpha\in\Delta} C_\alpha \frac{d\alpha}{\alpha}\psi, \tag{6}$$

where for every α one takes the principal embedding of $\mathfrak{sl}_2$ so that $C_\alpha = \frac{\langle\alpha,\alpha\rangle}{2}(e_\alpha f_\alpha + f_\alpha e_\alpha + \frac{1}{2}h_\alpha^2)$ is the Casimir in 3-dimensional subalgebra $\mathfrak{sl}_{2,\alpha}$ with respect to the restriction of the fixed non-degenerated $ad-$invariant bilinear form $\langle -,-\rangle$ on $\mathfrak{sl}_{2,\alpha}$ and $\hbar\in\mathbb{C}$. A special class of quantum connections with one irregular singularity of Poincaré rank 2 and several other simple poles appeared in [25] as dual to the standard KZ connection, and in [12] was re–obtained as quantisation of Dubrovin's system (without the skew-symmetry condition). Dubrovin system was then generalised to simply laced Dynkin diagrams in [11] and quantised in [56].

Confluent versions of the KZ equation, or in other words, KZ equations with irregular singular points of arbitrary Poincaré rank were obtained for $\mathfrak{sl}_2$ by Jimbo, Nagoya and Sun [39]. In [29] a class of quantum integrable systems generalising the Gaudin model was introduced by considering non-highest weight representations of any simple Lie algebra. These *Gaudin models with irregular singularities* are expected to give rise to *confluent KZ equations* as the corresponding differential equations on conformal blocks. Such KZ equations have not been explicitly written and one of the purposes of this paper is to do this.

In order to achieve our aim, we first needed to find explicit formulae for the isomonodromic Hamiltonians and to introduce a good set of Darboux coordinates. We have succeeded in doing this for a class of isomonodromic connections which behave well under confluence. Let us describe this class in some detail here. It is well known that the isomonodromic deformation equations in the case of higher order poles have a co-adjoint orbit interpretation on a loop algebra. In the case of the Painlevé equations, Harnad and Routhier [32] produced finite–dimensional parameterisations by introducing suitable truncations of the loop algebra. Korotkin and Samtleben [47] then conjectured the standard Lie–Poisson bracket on Takiff algebras (i.e. truncated current algebras) and later Boalch proved that indeed these brackets are preserved by the Jimbo-Miwa isomonodromic deformations [14]. In this paper, we unify these two approaches to study connections as elements of the product of co-adjoint orbits in the Takiff algebra. More precisely, we consider linear systems of ODEs with poles at $u_1, u_2, \dots, u_n, \infty$ of Poincaré rank $r_1, r_2, \dots, r_n, r_\infty$ respectively, in the form

$$\frac{dY}{d\lambda} = A(\lambda)Y, \quad A(\lambda) = \sum_{i=1}^{n}\sum_{k=0}^{r_i} \frac{A_k^{(i)}}{(\lambda - u_i)^{k+1}} + \sum_{k=1}^{r_\infty} A_k^{(\infty)} z^{k-1}, \tag{7}$$

where $A(\lambda)$ is an element of the phase space

$$M \simeq \hat{\mathcal{O}}^\star_{r_1} \times \hat{\mathcal{O}}^\star_{r_2} \times \dots \hat{\mathcal{O}}^\star_{r_n} \times \hat{\mathcal{O}}^\star_{r_\infty}, \tag{8}$$

where $\hat{\mathcal{O}}^\star_{r_i}$ stands for the co-adjoint orbit of the complex Lie group $\widehat{G}_{r_i}$ corresponding to the Takiff algebra of degree r_i, for $r_i > 0$, and for the standard Lie algebra $\mathfrak{g}$ co-adjoint orbit for $r_i = 0$.

Following the ideology of [3], in Theorem 2.6, we show how to obtain the standard Lie–Poisson bracket

$$\{A_k^{(i)} \underset{,}{\otimes} A_l^{(j)}\} = \begin{cases} -\delta_{ij}[\Pi, A_{k+l}^{(i)} \otimes \mathbb{I}] & k + l \le r_i \\ 0 & k + l > r_i. \end{cases} \tag{9}$$

on our phase space (8) as Marsden–Weinstein reduction of the Poisson structure on

$$\oplus_{i=1}^{n+1} (T^\star \mathfrak{gl}_m)^{r_i+1} = \oplus_{k=1}^{d} T^\star \mathfrak{gl}_m,$$

obtained by endowing each copy of $T^\star\mathfrak{gl}_m$ with the canonical symplectic structure $\mathrm{d}P \wedge \mathrm{d}Q$. Here $d = \sum_{i=1}^{n+1} r_i + n + 1$ denotes the degree of the divisor D of the connection (7). The Marsden–Weinstein reduction is obtained by the additional first integrals given by the moment maps of the inner group action by $\widehat{G}_{r_i}$ as in formulae (46).

These coordinates $(Q_1, P_1, \dots, Q_d, P_d)$, that we call *lifted Darboux coordinates,* were first introduced by Jimbo, Miwa, Mori and Sato in the case of linear systems of ODEs with n simple poles and possibly a Poincaré rank one pole at ∞ [38]. Harnad generalised these coordinates to allow rectangular $m_1 \times m_2$ matrices and used them to generalise Dubrovin duality [22] between two systems of linear ODEs: one of dimension m_1 and the other of dimension m_2 [31] and [63]. Similar coordinates were also introduced and partly used in the context of non-autonomous Hamiltonian description of Garnier-Painlevé differential systems by M. Babich and Derkachov [5, 6]. However in these latter works, the authors restricted to the case of rational parametrisation of co–adjoint orbits of $Gl_n(\mathbb{C})$ and other semi-simple Lie groups and did not consider loop algebras.

Interestingly, using the lifted Darboux coordinates, we can describe all possible isomonodromic systems with a fixed degree d of the divisor of the connection (7) as Marsden–Weinstein reductions of different inner group actions on the universal phase space $\oplus_{k=1}^{d} T^\star\mathfrak{gl}_m$. These reductions give rise to symplectic leaves of dimension $(r_1 + \dots + r_n + r_\infty + n)(m^2 - m)$. We explain how to produce the Darboux coordinates, which we call *intermediate Darboux coordinates,* on such symplectic leaves. In the case of the Jimbo-Miwa isomonodromic problems associated to the fifth, fourth, third and second Painlevé equations the degree is always $d = 4$, the intermediate symplectic leaves have always dimension 6 and are determined by the choice of 3 spectral invariants giving a total dimension 9 for the Poisson manifold. This is the dimension of the moduli space of $SL_2(\mathbb{C})$ connections with a given divisor D of degree 4 [48].

Remark 0.1. *The problem of extending the Riemann-Hilbert symplectomorphism between the de Rahm moduli space of meromorphic connections on a Riemann surface Σ with non-simple divisor (a divisor of points that can have multiplicity > 1) and the analogous of the Betti moduli space of representations of the fundamental group of Σ, namely with the cusped character variety introduced in* [15, 16] *is till open and is beyond the scope of the current paper. However, the Darboux coordinate description of the de Rahm moduli space achieved in this paper constitues an important first step towards that goal.*

Remark 0.2. *It is worth mentioning here that the phase space* (8) *is not a moduli space per se, however K. Hiroe and D. Yamakawa* [33] *showed that the sub–space of stable connections admits a nice quotient with respect to the diagonal action of $GL_m(\mathbb{C})$ on M:*

$$M' = \{A(\lambda)\in M | \sum_{i=1}^{n+1}\pi(A_0^{(i)})=0, \text{“stable”}\}/GL_m(\mathbb{C}),$$

where

$$\pi:\widehat{\mathfrak{g}}^*_{r_i}\to\mathfrak{gl}^*_m$$

is the moment map under the diagonal action of $GL_m(\mathbb{C})$ on M, thus assuring that M' is a smooth complex symplectic variety. The space M' can be regarded as a certain moduli space for meromorphic connections on $\mathcal{O}_{\mathbb{P}^1}^{\oplus m}$. Fix n distinct points $u_1,\dots,u_n\in\mathbb{P}^1$, and endow $\mathbb{P}^1$ with a coordinate z for which $z(u_i)\neq\infty$. The variable z_i can be identified with $\lambda-u_i$ and $\widehat{\mathfrak{g}}^*_{r_i}$ can be embedded in $\mathfrak{gl}_m(\mathbb{C}[z_i^{-1}])\frac{dz_i}{z_i}$ via trace-residue pairing. Then each $A(\lambda)\in M$ determines a meromorphic connection $d-A(\lambda)$ on on $\mathcal{O}_{\mathbb{P}^1}^{\oplus m}$, having poles at $u_1,\dots,u_n,\infty$. The condition $\sum_{i=1}^{n+1}\pi(A_0^{(i)})=0$ singles out the connections which have no residue at infinity.

Our next result is the classification of all linear Takiff algebra automorphisms that preserve the standard Lie–Poisson structure (9) on the phase space (8) (see Theorem 2.8 for a more articulated statement).

Theorem 0.3. *Consider two elements $A(\lambda)$ and $B(\lambda)$ of the phase space* (8)*, so that they both have the form* (7)*:*

$$A(\lambda)=\sum_{i=1}^{n}\sum_{k=0}^{r_i}\frac{A_k^{(i)}}{(\lambda-u_i)^{k+1}}+\sum_{k=1}^{r_\infty}A_k^{(\infty)}\lambda^{k-1},\quad B(\lambda)=\sum_{i=1}^{n}\sum_{k=0}^{r_i}\frac{B_k^{(i)}}{(\lambda-u_i)^{k+1}}+\sum_{k=1}^{r_\infty}B_k^{(\infty)}\lambda^{k-1}.$$

Assume that $A(\lambda)$ and $B(\lambda)$ are related by a linear automorphism of the Lie algebra

$$B_k^{(i)}=\sum_{l=0}^{r_i}T_{kl}^{(i)}A_l^{(i)},\quad\forall i=1,\dots,n,\infty,$$

for some scalar quantities $T_{kl}^{(i)}$. Then the Poisson condition

$$\{B_k^{(i)}\underset{,}{\otimes}B_l^{(j)}\}=\begin{cases}\delta_{ij}[B_{k+l}^{(i)}\otimes\mathbb{I},\Pi] & k+l\le r_i\\ 0 & k+l>r_i\end{cases}\iff\{A_k^{(i)}\underset{,}{\otimes}A_l^{(j)}\}=\begin{cases}\delta_{ij}[A_{k+l}^{(i)}\otimes\mathbb{I},\Pi], & k+l\le r_i\\ 0 & k+l>r_i\end{cases}$$

is satisfied if and only if the the following formulae are satisfied:

$$B_k^{(i)}=\sum_{j=k}^{r_i}A_j^{(i)}\mathcal{M}_{k,j}^{(r_i)}(t_1^{(i)},t_2^{(i)},\dots t_{r_i}^{(i)}),\tag{10}$$

where

$$\mathcal{M}_{k,j}^{(r_i)}=\sum_{w(\alpha)=j}^{|\alpha|=k}\frac{k!}{\alpha_1!\alpha_2!\dots\alpha_{r_i}!}\left(\prod_{l=1}^{r_i}(t_l^{(i)})^{\alpha_l}\right),\quad|\alpha|=\sum_{l=1}^{r_i}\alpha_l,\quad w(\alpha)=\sum_{l=1}^{r_i}l\cdot\alpha_l.\tag{11}$$

This result allows us to introduce extra (i.e. in addition to the positions of poles) deformation parameters $t_1^{(i)},\dots,t_{r_i}^{(i)}$, $i=1,\dots,n,\infty$ for any connection belonging to the phase space (8). In other words, we consider families of the form

$$A(\lambda)=\sum_{i=1}^{n}\sum_{k=0}^{r_i}\frac{B_k^{(i)}}{(\lambda-u_i)^{k+1}}+\sum_{k=1}^{r_\infty}B_k^{(\infty)}\lambda^{k-1}$$

where the elements $B_k^{(i)}$ contain explicitly the deformation parameters $t_1^{(i)},\dots,t_{r_i}^{(i)}$ as prescribed by formulae (10) and (11). The isomonodromic deformation equations will then impose a further implicit dependence of the matrices $A_k^{(i)}$ on the deformation parameters $t_1^{(i)},\dots,t_{r_i}^{(i)}$ and on the position of the poles $u_1,\dots,u_n$.

Remark 0.4. *Let us stress that the class of connections we consider in this paper are elements of the space (8). This class excludes some of the Jimbo-Miwa-Ueno connections. Indeed, our deformation parameters correspond to a subset of the Jimbo-Miwa-Ueno ones and this correspondence is* $1:1$ *only in the case of rank* $m=2$*. For example, the famous Dubrovin's system*

$$\frac{dY}{d\lambda}=\left(U+\frac{V}{\lambda}\right)Y,$$

where U *is a diagonal* $n\times n$ *matrix and* $V\in\mathfrak{so}_n$*, is not an element of* $\hat{\mathcal{O}}^\star_{r_1}\times\hat{\mathcal{O}}^\star_{r_\infty}$ *for some* r_1,r_∞ *because the diagonal elements of* U *are independent deformation parameters. Of course the isomonodromic deformation equations for* V *as a function of* $u_1,\dots,u_n$ *can be written as a flow on a co–adjoint orbit* $\mathcal{O}^\star$ *of the Lie algebra* $\mathfrak{so}_n$*, but not as equations for the whole connection* $U+\frac{V}{z}$ *on the product of two co–adjoint orbits as our theory dictates. To include the Dubrovin system (and indeed all of Jimbo-Miwa-Ueno deformation parameters) in our theory, one should either consider the extended coadjoint orbits introduced in* [13, 14] *or exploit the Laplace transform. In the latter setting, the confluence procedure distroys semi-simplicity, therefore it is a different process from the one considered by Cotti, Dubrovin and Guzzetti* [18, 19].

This is the correct framework to study confluence of two or more poles. Indeed, we show that the confluence cascade of $r+1$ simple poles at certain positions depending on $t_1^{(i)},\dots,t_{r_i}^{(i)}$ gives rise to an element of the phase space (8) which has a singularity of Poincaré rank r and depends on $t_1^{(i)},\dots,t_{r_i}^{(i)}$, $i=1,\dots,n,\infty$, as prescribed by formulae (10) and (11). The following theorem provides the inductive step to create the confluence cascade (we drop the index $^{(i)}$ for convenience).

Theorem 0.5. *Consider an* r*–parameter family of connections of the following form:*

$$A=\sum_{k=0}^{r}\frac{B_k(t_1,t_2\dots t_{r-1})}{(\lambda-u)^{k+1}}+\frac{C}{\lambda-v}+\textit{holomorphic terms},\tag{12}$$

where by holomorphic terms we mean terms holomorphic in $\lambda-u$ *and* $\lambda-v$*, and each* B_k *depends on the parameters* $t_1,\dots,t_r$ *as specified by* (10), (11)*. Assume*

$$v=u+\sum_{i=1}^{r}t_i\varepsilon^i=u+P_r(t,\varepsilon),\tag{13}$$

and that we have the following asymptotic expansions as $\varepsilon\to 0$

$$C\simeq\sum_{j=-r}^{\infty}W^{[j]}\varepsilon^j,\quad A_k\simeq-\sum_{l=1}^{r-k}\frac{W^{[-k-l]}}{\varepsilon^l}+A^{[k,0]}+\sum_{l=1}^{\infty}A^{[k,l]}\varepsilon^l,\tag{14}$$

for some matrices $W^{[-k-l]},A^{[k,l]}$*. Then the limit* $\varepsilon\to 0$ *the connection exists and is equal to*

$$\tilde{A}=\sum_{i=0}^{r+1}\frac{\tilde{B}_i(t_1,t_2\dots t_r,t_{r+1})}{(\lambda-u)^{i+1}}+\textit{holomorphic terms},$$

where $\tilde{B}_i$*'s are given by*

$$\tilde{B}_i(t_1\dots,t_{r+1})=\sum_{k=i}^{r}\tilde{A}_k\mathcal{M}_{i,k}^{(r+1)}(t_1\dots t_{r+1}),\quad \tilde{A}_k=\left\{\begin{array}{ll}W^{[-k]}+A^{[k,0]}, & k<r+1.\\ W^{[-r-1]}, & k=r+1.\end{array}\right.\tag{15}$$

We prove that the confluence procedure gives a Poisson morphism on the product of co-adjoint orbits and we calculate explicitly the confluent Hamiltonians, which define the correct isomonodromic deformations.

Theorem 0.6. *Let* u *be a pole of a connection* A *with Poincaré rank* r*, which is the result of confluence of* r *simple poles with the simple pole* u*. Then the confluent Hamiltonians* $H_1,\dots,H_r$ *which correspond* *to the times $t_1,\dots t_r$ are defined as follows:*

$$\begin{pmatrix} H_1\\ H_2\\ \dots\\ H_r\end{pmatrix} = \left(\mathcal{M}^{(r)}\right)^{-1}\begin{pmatrix} S_1^{(u)}\\ S_2^{(u)}\\ \dots\\ S_r^{(u)}\end{pmatrix}, \tag{16}$$

where

$$S_k^{(u)} = \frac{1}{2}\oint\limits_{\Gamma_u}(\lambda-u)^k \mathrm{Tr}\, A^2 \mathrm{d}\lambda \tag{17}$$

are spectral invariants of order i in u and the matrix $\mathcal{M}^{(r)}$ is given by (11). The Hamiltonian H_u corresponding to the time u is instead given by the standard formula

$$H_{u_i} = \frac{1}{2}\operatorname*{res}_{\lambda=u_i}\mathrm{Tr}\, A(\lambda)^2.$$

Remark 0.7. *It is well known that the isomonodromic deformation equations are Hamiltonian, namely that the flow is Hamiltonian with respect to the Jimbo-Miwa-Ueno deformation parameters, see for example* [27, 36, 62]. *In* [27]*, the isomonodromy equations have been described as non-integrable autonomous Hamiltonian systems. A symplectic fibre bundle whose base is the Jimbo-Miwa-Ueno deformation parameters space and the fibers are certain moduli spaces of unramified meromorphic connections was introduced in* [14]*. This approach was extended by D. Yamakawa for any reductive Lie algebra $\mathfrak{g}$* [64] *who removed some multiplicity restrictions and introduced a symplectic two-form on the fibration. Following the same geometric approach and Jimbo-Miwa-Ueno isomonodromic tau-function D. Yamakawa* [65] *has proven that the isomonodromy equations of Jimbo–Miwa–Ueno is a completely integrable non-autonomous Hamiltonian systems. He was also motivated by the quantisation theorem of Reshetikhin but he did not try to consider the quantisation of general isomonodromy equations*[1]*. Recently, Bertola and Korotkin have derived a new Hamiltonian formulation of the Schlesinger equations (i.e. for the Fuchsian case) in terms of the dynamical r–matrix structure.*

Remark 0.8. *The results of the theorem 0.5 still hold true for the autonomous systems which are obtained by the confluence procedure from the Gaudin system. It was shown by Yu. Chernyakov in* [17] *that the Poisson algebra which arises in the confluent elliptic and rational Gaudin systems coincides with the dual Takiff algebra equipped with the standard Lie–Poisson bracket (in* [17] *the author use the word"fission" instead of "confluence").*

One of the main theorems of our paper gives a general formula for the confluent KZ Hamiltonians with singularities of arbitrary Poincaré rank in any dimension.

Theorem 0.9. *Consider the differential operators:*

$$\nabla_{u_j} := \frac{\partial|}{\partial u_j} - \widehat{H}_{u_j}, \quad j=1,\dots,n \tag{18}$$

and

$$\nabla_k^{(i)} := \frac{\partial|}{\partial t_k^{(i)}} - \widehat{H}_k^{(i)}, \quad i=1,\dots,n,\infty, \quad k=1,\dots,r_i \tag{19}$$

where the Hamiltonians $\widehat{H}_{u_j}$ which correspond to the positions of the poles u_j, $j=1\dots,n$, and $\widehat{H}_1^{(i)},\dots,\widehat{H}_r^{(i)}$ which correspond to the times $t_1^{(i)},\dots t_{r_i}^{(i)}$, for $i=1,\dots,n,\infty$, are given by the following elements of the universal enveloping algebra $U\left(\mathfrak{g}_{r_1}\oplus\cdots\oplus\mathfrak{g}_{r_\infty}\right)$:

$$\hat{H}_{u_j} = \frac{1}{2}\operatorname*{res}_{\lambda=u_j}\mathrm{Tr}_0\hat{A}(\lambda)^2,$$

and

$$\mathcal{M}^{(r_i)}\begin{pmatrix}\hat{H}_1^{(i)}\\ \hat{H}_2^{(i)}\\ \dots\\ \hat{H}_{r_i}^{(i)}\end{pmatrix} = \begin{pmatrix}\hat{S}_1^{(u_i)}\\ \hat{S}_2^{(u_i)}\\ \dots\\ \hat{S}_{r_i}^{(u_i)}\end{pmatrix}, \quad \hat{S}_k^{(u_i)} = \frac{1}{2}\oint\limits_{\Gamma_{u_i}}(\lambda-u_i)^k\mathrm{Tr}_0\hat{A}(\lambda)^2\mathrm{d}\lambda,$$

[1]We are grateful to prof. M.Jimbo who has drown our attention and has sent a file of the paper [65]

where

$$\hat{A}(\lambda)=\sum_{i}^{n}\left(\sum_{j=0}^{r_i}\frac{\hat{B}_j^{(i)}\left(t_1^{(i)},t_2^{(i)}\dots t_{r_i}^{(i)}\right)}{(\lambda-u_i)^{j+1}}\right),$$

with $\hat{B}^{(i)}$'s given by

$$\hat{B}_j^{(i)}(t_1^{(i)},\dots t_{r_i}^{(i)})=\sum_{k=j}^{r}\hat{A}_k^{(i)}\mathcal{M}_{j,k}^{(r_i)}(t_1^{(i)},t_2^{(i)}\dots t_{r_i}^{(i)}),\quad \hat{A}_k=\sum_{\alpha}e_{\alpha}^{(0)}\otimes e_{\alpha}^{(i)}\otimes z_i^k,$$

and $e_\alpha^{(0)}$ corresponds to the quantisation of $\mathfrak{g}^*$ to $\mathfrak{g}$ while

$$e_\alpha^{(i)}=1\otimes\dots\otimes \underset{i}{e_\alpha}\otimes\dots\otimes 1.$$

Then the differential operators commute

$$[\nabla_{u_j},\nabla_{u_s}]=[\nabla_k^{(i)},\nabla_{u_s}]=[\nabla_k^{(i)},\nabla_l^{(a)}]=0,$$

$\forall\, j,s=1,\dots,n,\ i,a=1,\dots,n,\infty,\ k=1,\dots,r_i,\ l=1,\dots,r_a.$ *We call the system of differential equations*

$$\nabla_{u_j}\Psi=0,\quad \nabla_k^{(i)}\Psi=0,\qquad j=1,\dots,n,\ i=1,\dots,n,\infty,\ k=0,\dots,r_i,$$

confluent KZ equations.

Moreover, we express the isomonodromic Hamiltonians in terms of the lifted Darboux coordinates and show that the quasiclassical solutions of the confluent KZ equations is expressed via the isomonodromic τ-function.

Theorem 0.10. *The semi-classical solution of the confluent Knizhnik–Zamolodchikov system evaluated along solutions of the classical isomonodromic Hamiltonians is given by the isomonodromic τ-function*

$$\mathrm{d}\ln(\tau)=\sum_i\left(H_{u_i}^{(i)}\mathrm{d}u_i+\sum_{k=1}^{r_i}H_k^{(i)}\mathrm{d}t_k^{(i)}\right).$$

The explicit form of the semi-classical solution is given by

$$\Psi(Q(t),t)\sim\tau^{\frac{i}{\hbar}}.\tag{20}$$

This statement was mentioned in [57] for the case of the standard KZ, namely with simple poles. We also discuss the quantisation of the reduced Darboux coordinates and provide the quantised reduced systems in some examples.

Remark 0.11. *Most of our results extend to the case of isomonodromic deformations for meromorphic connections on principal $G-$bundles over the Riemann sphere for any arbitrary complex reductive group G - this is for example the situation of the famous Fuji-Suzuki $D_{2n+2}^{(1)}-$higher Painlevé hierarchies and matrix Painlevé equations* [9]. *Only the results about the so-called "lifted Darboux coordinates coordinates" can't immediately be generalised to any connected complex reductive group case. However, we have decided to restrict to the $GL_m(\mathbb{C})$ case having in mind a wider audience. For the same reasons, we often do not use the language of sheaves and schemes. To extend our results to higher genus Riemann surfaces is instead a rather serious job. First, one can extend the "rational" truncated polynomial currents to their trigonometric and elliptic analogues and to define a proper pairing and basis. For $g=1$ case such job can be done probably using the results of* [54, 55, 24] *and we postpone to subsequent papers.*

This paper is organised as follows. In Section 1, we recall the case of Fuchisan connections, we discuss the lifted Darboux coordinates and the Marsden–Weinstein reduction to the phase space (8) in the case of $r_1=\dots=r_n=r_\infty=0$ and remind the Hamiltonian formulation. In Section 2, we collect facts about the Takiff algebras; we discuss the lifted Darboux coordinates at each separate pole for any choice of the Poincaré rank r and show how to obtain the standard Lie–Poisson bracket (9) as Marsden–Weinstein reduction of the Poisson structure on $\sum_{k=0}^{r}T^\star\mathfrak{gl}_m$. We also prove Theorem 0.3 and discuss the inner group action on the universal phase space. Finally, we show how to obtain the intermediate Darboux coordinates and discuss some examples. In Section 3, we discuss the isomonodromic deformations. In Section 4, we discuss the confluence procedure. We first carry out the confluence of two simple poles, explain how to obtain confluence cascades, prove Theorems

0.5 and 0.6. Is Section 5, we apply the theory to the case of the Painlevé equations. In Section 6, we deal with quantisation. We give a general formula for the confluent KZ equations with singularities of arbitrary Poincaré rank and prove Theorem 0.10.

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1. Fuchsian systems

In this section we summarize known facts about Fuchsian systems. The aim of this section is to review the Poisson and symplectic aspects of the deformation equations for connections over the $n+1$-holed sphere with simple poles at punctures. Starting from the linear system with simple poles at $\lambda = u_1, \dots, u_n, \infty$,

$$\frac{d}{d\lambda}\Psi = \sum_{i=1}^{n} \frac{A^{(i)}}{\lambda - u_i}\Psi, \quad \lambda \in \Sigma, \quad u_i \neq u_j \tag{21}$$

we consider the following matrix-valued 1-form

$$\Omega = (\mathrm{d}_u\Psi)\Psi^{-1}, \quad \mathrm{d}_u\Psi := \sum_i \partial_{u_i}\Psi \mathrm{d}u_i. \tag{22}$$

Since we consider only isomonodromic deformations, i.e. $\mathrm{d}M_i = 0$, the form Ω is a single-valued meromorphic 1-form with possible singularities at u_i's.

Using the local solutions of (21) in the neighbourhood of the poles u_i's and applying Liouville theorem this form may be written as

$$\Omega = -\sum_i \frac{A^{(i)}}{\lambda - u_i}\mathrm{d}u_i. \tag{23}$$

The compatibility condition for (21) and (22) (zero-curvature equation)

$$\mathrm{d}_u A - \frac{d}{d\lambda}\Omega + [A, \Omega] = 0, \tag{24}$$

gives the Schlesinger equations (2).

1.1. Phase space.

The Schlesinger equations are Hamiltonian, with natural phase space given by the direct product of co-adjoint orbits which are symplectic leaves of the standard Lie–Poisson bracket:

$$\left(A^{(1)}, A^{(2)}, \dots A^{(n)}\right) \in \mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \cdots \times \mathcal{O}_n^\star.$$

In case when $\mathfrak{g}$ is a Lie algebra with a non-degenerate bi-linear form (i.e. Killing form), we may identify the co-adjoint orbits with the adjoint orbits. The Poisson brackets then may be written as

$$\left\{A^{(i)} \underset{,}{\otimes} A^{(j)}\right\} = \delta_{ij}[\Pi, 1 \otimes A^{(i)}] \quad \Longleftrightarrow \quad \left\{A^{(i)}_\alpha, A^{(j)}_\beta\right\} = -\delta_{ij}\sum_\gamma \chi^\gamma_{\alpha\beta} A^{(i)}_\gamma \tag{25}$$

where the lower indices α, β and γ correspond to the Lie co-algebra basis, $\chi^\gamma_{\alpha\beta}$ are the structure constants of the Lie algebra and Ω is a quadratic Casimir element. In the case of $\mathfrak{gl}_m$ it acts as a permutation operator, i.e.

$$\Pi(A \otimes B)\Pi = B \otimes A.$$

In the case of a Lie algebra which allows orthogonal with respect to Killing form basis e_α, quadratic Casimir Π writes as

$$\Pi = \sum_\alpha e_\alpha \otimes e_\alpha.$$

Such bracket may be rewritten as an r-matrix bracket for the connection, i.e.

$$\{A(\lambda) \underset{,}{\otimes} A(\mu)\} = \left[\frac{\Pi}{\lambda - \mu}, A(z) \otimes \mathbb{I} + \mathbb{I} \otimes A(\mu)\right]. \tag{26}$$

The isomonodromic Hamiltonians for the Schlesinger equations are

$$H_i = \operatorname*{Res}_{\lambda=u_i} \operatorname{Tr} \frac{A(\lambda)^2}{2} = \sum_{j\neq i} \frac{\operatorname{Tr}(A^{(i)}A^{(j)})}{u_i - u_j}. \tag{27}$$

The Schlesinger equations can be reduced, for example in the case of $n = 3$ co-adjoint orbits in $\mathfrak{sl}_2$ they reduce to the Painlevé VI equation which is non-autonomous Hamiltonian system with 1 degree of freedom.

In the case of any number n of co-adjoint orbits, the fully reduced dimension can be computed using the spectral type technique introduced by Katz [43]. In case when $A^{(i)}$'s and $A^{(\infty)}$ are semi-simple, the spectral type approach gives the formula for the dimension of the fully reduced phase space as a function of the eigenvalues multiplicities of the residues [43]

$$N = 2 - (1-n)m^2 - \sum_{i=1}^{n}\sum_{j=1}^{l_i} (m_j^i)^2 - \sum_{j=1}^{l_\infty} (m_j^\infty)^2, \tag{28}$$

where l_i is a cardinality of the set of eigenvalues for the residue $A^{(i)}$ and m_j^i is the multiplicity of the j-th eigenvalue of the residue $A^{(i)}$. The fully reduced systems may be seen as a reduction with respect to the additional Fuchs condition:

$$\sum_{i=1}^{n} A^{(i)} = -A^{(\infty)}. \tag{29}$$

On the other hand this relation may be viewed as a moment map of the gauge group action via constant matrix (i.e. gauge doesn't depend in z) and A_∞ is a constant of motion for the Schlesinger equations.

From the point of view the symplectic reduction Katz formula may be rewritten in the following way

$$N = \sum_{i=1}^{n} \dim \mathcal{O}_i^\star - \dim G - \operatorname{stab} \mathcal{O}_\infty^\star, \tag{30}$$

where $\operatorname{stab} \mathcal{O}_\infty^\star$ is the dimension of the stabilizer for the Jordan form of the residue at ∞. When $A^{(\infty)}$ is the element of the co-adjoint orbit of the general form (non-degenerate), we have that $\operatorname{stab} \mathcal{O}_\infty^\star = \dim \mathfrak{h}$, so the formula simplifies to

$$N = \sum_{i=1}^{n} \dim \mathcal{O}_i^\star - \dim G - \dim \mathfrak{h}.$$

For example, in the case of the Painlevé VI equation we deal with the coadjoint orbits of the $\mathfrak{sl}_2(\mathbb{C})$. In the general situation formula (30) gives

$$N = 3 \cdot \dim \mathcal{O}_{\mathfrak{sl}_2} - \dim(SL_2) - \dim \mathfrak{h}_{\mathfrak{sl}_2} = 3\cdot 2 - 3 - 1 = 2,$$

which is exactly the dimension of the Painlevé VI equation. In some sense the multiplicity of the eigenvalues tells us that the Jordan form may be written as the tensor product of identity matrices of sizes corresponding to the the multiplicities. The stabilizer of such matrix is a set of the block diagonal matrices, so the dimension is greater then the dimension of the Cartan torus and finally we obtain the smaller phase space.

Our first goal is to describe this full reduction as a Hamiltonian reduction and a Marsden-Weinstein quotient. To this aim, we will need first to extend the phase space to $T^\star\mathfrak{gl}_m$ and show that the Darboux coordinates on this cotangent bundle reduced to the Kirillov-Kostant-Souriau form on the co-adjoint orbits. We will then discuss how the invariants of the co-adjoint orbits correspond to moment maps with respect to different Hamiltonian group actions on the extended phase space.

1.2. **Extended phase space and its Darboux coordinates.** In this subsection, we start by working locally, namely we restrict to the case of a single co-adjoint orbit $\mathcal{O}^\star$ of $\mathfrak{gl}_m$ and identify $\mathfrak{gl}_m^\star$ with $\mathfrak{gl}_m$ via Killing form. In the last part of this subsection we extend to the product of n co-adjoint orbits.

We consider $T^\star\mathfrak{gl}_m$ with the standard Darboux coordinates (Q, P) and the canonical symplectic structure:

$$\omega = \operatorname{Tr}\,(\mathrm{d}P \wedge \mathrm{d}Q) = \sum_{i,j} \mathrm{d}P_{ij} \wedge \mathrm{d}Q_{ji}. \tag{31}$$

Following [3, 1, 2], we explain how to obtain the standard Lie–Poisson bracket (25) on $\mathfrak{g}^\star$ as Marsden–Weinstein reduction of the Poisson structure on $T^\star\mathfrak{gl}_m$. There is a direct way to see this reduction by a straightforward computation (see [38]) that we resume in the next proposition:

Proposition 1.1. *Consider the canonical symplectic structure on $T^\star\mathfrak{gl}_m$:*

$$\omega = \mathrm{Tr}\,(\mathrm{d}P \wedge \mathrm{d}Q) = \sum_{i,j} \mathrm{d}P_{ij} \wedge \mathrm{d}Q_{ji}. \tag{32}$$

Let

$$A = QP, \tag{33}$$

where we use the ring structure of $\mathfrak{gl}_m$ to justify the multiplication of the Q and P. Then A satisfies the standard Lie–Poisson bracket (25) *for $\mathfrak{gl}_m$.*

Proof. The Poisson bracket which corresponds to the symplectic form in (32) may be written in the following way

$$\{P \underset{,}{\otimes} Q\} = \Pi, \quad \{P \underset{,}{\otimes} Q\} = 0$$

Inserting this relation to the bracket between A's we get

$$\begin{aligned}\{A \underset{,}{\otimes} A\} = \{QP \underset{,}{\otimes} QP\} &= (Q\otimes\mathbb{I})\{P \underset{,}{\otimes} Q\}(\mathbb{I}\otimes P) + (\mathbb{I}\otimes Q)\{Q \underset{,}{\otimes} P\}(P\otimes\mathbb{I}) = \\ &= (Q\otimes\mathbb{I})\Pi(\mathbb{I}\otimes P) - (\mathbb{I}\otimes Q)\Pi(P\otimes\mathbb{I}) = [\Pi, \mathbb{I}\otimes QP] = [\Pi, \mathbb{I}\otimes A]\end{aligned}$$

As we wanted to prove. □

Definition 1.2. We call $T^\star\mathfrak{gl}_m$ *extended phase space* and the canonical coordinates P, Q *lifted Darboux coordinates.*

To restrict ourself to the co-adjoint orbit we have to fix invariants of the co–adjoint actions, i.e. the Jordan form of matrix $QP = A$. Such procedure leads to some additional non-linear equations for the entries of Q and P, and there is no hope to derive the explicit symplectic structure on the co-adjoint orbit from such a perspective. Therefore, we follow the construction of [3] to obtain the co-adjoint orbits via Hamiltonian reduction.

The space $T^\star\mathfrak{gl}_m \simeq \mathfrak{gl}_m \times \mathfrak{gl}_m$ carries two natural commuting symplectic actions of GL_m which we call *inner and outer*:

$$g \underset{\text{inner}}{\times} (P, Q) = (gP, Qg^{-1}), \qquad h \underset{\text{outer}}{\times} (P, Q) = (Ph, h^{-1}Q), \qquad h, g \in GL_m. \tag{34}$$

Lemma 1.3. *These inner and outer actions are Hamiltonian with equivariant moment maps given by*

$$\begin{array}{ll}\mu_{\text{inner}}: & T^\star\mathfrak{gl}_m \to \mathfrak{gl}_m^\star \\ & (P,Q) \mapsto \Lambda = PQ\end{array} \qquad \begin{array}{ll}\mu_{\text{outer}}: & T^\star\mathfrak{gl}_m \to \mathfrak{gl}_m^\star \\ & (P,Q) \mapsto A = QP\end{array}. \tag{35}$$

Let us restrict to the open affine subset of $T^\star\mathfrak{gl}_m$ where at least one of the two matrices Q and P is invertible. For example Q. Then, resolving the moment map for Λ we obtain

$$P = \Lambda Q^{-1}, \quad A = QP = Q\Lambda Q^{-1}.$$

As a consequence, A and Λ belong to the same co-adjoint orbit.

Since the inner and outer actions commute, A is invariant under the inner action, while Λ is invariant under the outer action. Therefore we use the inner group action to fix Λ in a Jordan normal form without changing A. In other words, we take the Jordan normal form Λ_0 of A and select $\Lambda = \Lambda_0$. This gives

$$T^\star\mathfrak{gl}_m \underset{\Lambda_0}{/\!\!/} G = \mu_{\text{inner}}^{-1}(\Lambda_0)/G,$$

here we denote by $\underset{\Lambda_0}{/\!\!/}$ the quotient with respect to the inner action of GL_m on $T^\star\mathfrak{gl}_m$. We may resume these results in the following:

Lemma 1.4. *The map*

$$\begin{array}{ccc} T^\star\mathfrak{gl}_m \underset{\Lambda_0}{/\!\!/} G_{\text{inner}} & \to & \mathcal{O}^\star \\ (Q,P) & \mapsto & A := QP \end{array}$$

is a rational symplectomorphism and the Jordan normal form Λ_0 of A is given by

$$\Lambda_0 = PQ.$$

Remark 1.5. *When A is a full-rank matrix, both P and Q must be invertible. So we may embed (P,Q) into the group GL_m and P and Q can be seen as left and right eigenvector matrices for the matrix A. In the case when A may be diagonilized, the action of the Cartan torus (i.e. the stabilizer of Λ) leads to a well known fact from linear algebra - the eigenvectors are defined up to multiplication by non-zero constant. When A is not a full-rank matrix, we may choose Q to be an invertible matrix (so it may be viewed as an element of GL_n). Then the rank of P must equal to the rank of A. The the moment map Λ will inherit the rank of A automatically. Since P in this case not invertible, the reduced coordinates take the form*

$$P = \Lambda Q^{-1}, \quad A = Q\Lambda Q^{-1}, \quad \det\Lambda = \det A = \det P = 0.$$

This means that instead of considering $T^\star\mathfrak{gl}_m$ as lifted space, we could take $T^\star GL_m \ni (Q, \Lambda Q^{-1})$. Such consideration is closely related to the approach introduced in [10]*. However, this approach is not very useful for our purposes, since we wish to work with polynomial unreduced parametrisation, rather then rational.*

Remark 1.6. *In the case when we consider $\mathfrak{g}$ to be any reductive Lie algebra and $A \in \mathfrak{g}^*$, then we expect that Lemma 1.4 is still valid if we fix the value Λ of the moment map in $\mathfrak{g}^*$ and Q and P (or just Q in the case of degenerate orbit) as the elements from the corresponding Lie group G.*

Let us now consider the case of the product of many co-adjoint orbits. Since the Poisson brackets (25) are *local,* namely the residues at different marked points commute, the facts we summarised so far easily extend to this case. Indeed, we can apply the above construction to the co-adjoint orbit at each pole of the Fuchsian system (except ∞) and define:

$$A^{(i)} = Q_i P_i.$$

In this case we have that inner and outer actions can be lifted to the direct sum of n copies $T^\star\mathfrak{gl}_m$ in a natural way

$$g \underset{\text{inner}}{\times} (P_1, P_2, \dots P_n, Q_1, Q_2, \dots Q_n) = (g_1P_1, \dots g_nP_n, Q_1g_1^{-1}, \dots Q_ig_i^{-1}, \dots Q_ng_n^{-1}), \quad g \in \underset{n}{\times} GL_m$$

$$h \underset{\text{outer}}{\times} (P_1, P_2, \dots P_n, Q_1, Q_2, \dots Q_n) = (P_1h_1, \dots P_nh_n, h_1^{-1}Q_1, \dots h_i^{-1}Q_i, \dots h_n^{-1}Q_n), \quad h \in \underset{n}{\times} GL_m$$

and the lemma 1.3 repeats

Lemma 1.7. *These inner and outer actions are Hamiltonian with equivariant moment maps given by*

$$\mu_{\text{inner}}: \begin{array}{ccc} \underset{n}{\oplus} T^\star\mathfrak{gl}_m & \to & \underset{n}{\oplus}\mathfrak{gl}_m^\star \\ (P_1, \dots P_n; Q_1, \dots Q_n) & \to & (P_1Q_1, P_2Q_2, \dots P_nQ_n) \end{array}$$

$$\mu_{\text{outer}}: \begin{array}{ccc} \underset{n}{\oplus} T^\star\mathfrak{gl}_m & \to & \underset{n}{\oplus}\mathfrak{gl}_m^\star \\ (P_1, \dots P_n; Q_1, \dots Q_n) & \to & (Q_1P_1, Q_2P_2, \dots Q_nP_n) \end{array}$$

Proof. Let us prove it for the inner action only. The vector field generated by the group action (via element $\xi = (\xi_1, \xi_2, ...\xi_n) \in \oplus_n \mathfrak{gl}_m$ is given by

$$X_\xi(P_i, Q_i) = \frac{d}{dt}(e^{-t\xi_i}P_i, Q_ie^{t\xi_i})\Big|_{t=0} = (-\xi_iP_i, Q_i\xi_i) = \sum_{i=1}^n \Big(\sum_{k,j} -(\xi_iP_i)_{kj}\frac{\partial}{\partial P_{i_{kj}}} + (Q_i\xi_i)_{kj}\frac{\partial}{\partial Q_{i_{kj}}}\Big).$$

Inserting X_ξ into the symplectic form we obtain

$$\omega(X_\xi, \circ) = \sum_i^n \sum_{k,j}\Big[-(\xi_iP_i)_{kj}dQ_{i_{jk}} - (Q_i\xi_i)_{kj}dP_{i_{jk}}\Big] = -\sum_i^n \operatorname{Tr}(\xi_iP_idQ_i + Q_i\xi_idP_i) = -\sum_i^n d\operatorname{Tr}(\xi_iP_iQ_i),$$

so the corresponding Hamiltonian is

$$h_\xi(m) = \langle \mu(m), \xi\rangle = \sum_i^n \operatorname{Tr}(\xi_i\mu(m)_i) = \operatorname{Tr}(\xi_iP_iQ_i)$$

where $m = (P_1, P_2, \dots P_n, Q_1, \dots Q_n)$. So the moment map is given by

$$\mu(m) = (P_1Q_1, P_2Q_2, \dots P_iQ_i, \dots P_nQ_n),$$

which is equivariant

$$\mu(g\circ m) = \left(g_1^{-1}P_1Q_1g_1, g_2^{-1}P_2Q_2g_2, \dots g_i^{-1}P_iQ_ig_i, \dots g_n^{-1}P_nQ_ng_n\right) = g^{-1}\mu(m)g = \operatorname{Ad}^\star_{g^{-1}}(\mu(m))$$

□

Then the following result is a straightforward computation

Lemma 1.8. *A Hamiltonian system on the phase space*

$$\mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \dots \times \mathcal{O}_n^\star \ni \left(A^{(1)}, A^{(2)}, \dots A^{(n)}\right)$$

can be lifted up to the extended phase space

$$T^\star\mathfrak{gl}_m \times T^\star\mathfrak{gl}_m \times \dots \times T^\star\mathfrak{gl}_m \ni (Q_1, P_1, Q_2, P_2 \dots Q_n, P_n)$$

with additional first integrals given by the moment maps of the inner group action

$$\mu_{\text{inner}} := P_i Q_i = \Lambda^{(i)},$$

where the inner group action is given by

$$(g_1, g_2, \dots g_n) \underset{\text{inner}}{\times} (P_1, Q_1, P_2, Q_2, \dots P_n, Q_n) = (g_1 P_1, Q_1 g_1^{-1}, \dots g_i P_i, Q_i g_i^{-1}, \dots g_n P_n, Q_n g_n^{-1}).$$

Moreover, if $\Lambda_0^{(i)}$ is the Jordan normal form of $A^{(i)}$, we can fix $\Lambda^{(i)} = \Lambda_0^{(i)}$.

In particular, the Schlesinger Hamiltonians (27) can be lifted to the extended phase space $T^\star\mathfrak{gl}_m$ as follows

$$H_i = \sum_{j\neq i} \frac{\operatorname{Tr}(Q_i P_i Q_j P_j)}{u_i - u_j}, \tag{36}$$

and it can be checked directly that they Poisson commute with the moment maps of the inner group action.

1.3. **Outer group action and the gauge group.** We have seen that the inner group action allows us to restrict from $T^\star\mathfrak{gl}_m$ to $\mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \dots \times \mathcal{O}_n^\star$. Now we consider the *outer group action* that will allow us to reduce further. This is given by

$$(g_1, g_2, \dots g_n) \underset{\text{outer}}{\times} (P_1, Q_1, P_2, Q_2, \dots P_n, Q_n) = (P_1 g_1, g_1^{-1} Q_1, \dots P_i g_i, g_i^{-1} Q_i, \dots P_n g_n, g_n^{-1} Q_n)$$

and is also Hamiltonian (see Lemma 1.3).

Because inner and outer group actions commute, their moment maps Poisson commute too. However, the Schlesinger Hamiltonians are generally not invariant under outer action, unless the outer action is restricted to be a diagonal action, i.e.

$$g_1 = g_2 = \dots = g_n = g.$$

In this case, the outer action reduces to the standard GL_m-action on $\mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \dots \times \mathcal{O}_n^\star$, or equivalently to the constant gauge group action:

$$g \underset{\text{outer}}{\times} A = \sum \frac{g^{-1} A^{(i)} g}{z - u_i}.$$

The moment map of such diagonal action is

$$\sum_{i=1}^n Q_i P_i = \sum_{i=1}^n A^{(i)} = -A^{(\infty)}, \tag{37}$$

which is the Fuchs relation.

In order to describe the reduction procedure induced by the outer diagonal action in terms of the Marsden-Weinstein reduction, following Proposition 2.2.7 of [4] (see also [34]) we further extend the phase space by adding another copy of $T^\star\mathfrak{gl}_m$:

$$(P_1, Q_1 \dots P_n, Q_n; P_\infty, Q_\infty) \in \bigoplus_{i=1}^{n+1} T^\star\mathfrak{gl}_m, \quad \omega = \sum_{i=1}^n \operatorname{Tr} \mathrm{d}P_i \wedge \mathrm{d}Q_i + \operatorname{Tr} \mathrm{d}P_\infty \wedge \mathrm{d}Q_\infty, \tag{38}$$

with the outer group action of the form

$$g \underset{\text{outer}}{\times} (P_1, Q_1 \dots P_n, Q_n; P_\infty, Q_\infty) = (P_1 g, g^{-1} Q_1, \dots P_i g, g^{-1} Q_i, \dots P_n g, g^{-1} Q_n; P_\infty g, g^{-1} Q_\infty).$$

The corresponded extended space which is given by the reduction with respect to the inner group action takes form

$$\left(A^{(1)}, A^{(2)}, \dots A^{(n)}; A^{(\infty)}\right) \in \mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \dots \mathcal{O}_n^\star \times \mathcal{O}_\infty^\star.$$

The reduction with respect to the relation (37) on the extended phase space may be viewed as the Marsden-Weinstein quotient

$$\bigoplus_{i=1}^{n+1} T^\star \mathfrak{gl}_m \mathbin{/\!\!/} G = \mu^{-1}(0)/G, \quad \mu = \sum_{i=1}^{n} Q_i P_i + Q_\infty P_\infty.$$

which corresponds to the Fuchsian relation on the reduced with respect to the inner group action phase space.

Finally, the fully reduced phase space then has form

$$M \simeq \mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \dots \mathcal{O}_n^\star \times \mathcal{O}_\infty^\star \mathbin{/\!\!/} G \simeq \bigoplus_{i=1}^{n+1} \left(T^\star \mathfrak{gl}_m \mathbin{\underset{\Lambda^{(i)}}{/\!\!/}} G \right) \mathbin{/\!\!/} G.$$

Moreover, the Hamiltonians are the homogeneous polynomials in the lifted Darboux coordinates. Such dependence plays crucial role in the quantisation of the isomonodromic systems and we will discuss it in the next sections.

Our aim to extend this scheme for the isomonodromic problems with irregular singularities and to introduce a well defined confluence procedure which creates an irregular singularity of Poincaré rank r as a result of collision of $r+1$ simple poles. In the next section, we study the case of the irregular singularities along the same lines of the regular one.

2. Takiff algebras and associated symplectic manifolds.

It is well known that the isomonodromic deformation equations in the case of higher order poles also have a co-adjoint orbit interpretation on a loop algebra. In the case of the Painlevé equations, Harnad and Routhier [32] produced finite dimensional parameterisations that can be interpreted as introducing suitable truncations of the loop algebra. Korotkin and Samtleben then conjectured the standard Lie–Poisson structure on truncated loop algebras - also called Takiff algebras. In this section, we unify these two approaches and classify the linear Takiff algebra automorphisms that preserve the standard Lie–Poisson structure. As a consequence we obtain a general formula that prescribes the way to introduce independent deformation parameters in generic connections with poles of any Poincaré rank.

By introducing a suitable truncation, one obtains a Takiff algebra, that may be viewed as a Taylor part of a loop algebra quotiented by the ideal generated by z^{r_i} where r_i is the order of the pole at u_i and z is the local coordinate at u_i. For a general system with poles at $u_1, u_2, \dots, u_n, \infty$ of Poincaré rank $r_1, r_2, \dots, r_n, r_\infty$ respectively, the phase space is

$$M \simeq \hat{\mathcal{O}}_{r_1}^\star \times \hat{\mathcal{O}}_{r_2}^\star \times \dots \hat{\mathcal{O}}_{r_n}^\star \times \hat{\mathcal{O}}_{r_\infty}^\star \mathbin{/\!\!/} G_{\text{gauge}},$$

where $\hat{\mathcal{O}}_{r_i}^\star$ stands for the co-adjoint orbit of the Takiff algebra of degree r_i which is the Poincaré rank of the pole u_i. In this section we remind generalities about Takiff algebra and describe the Poisson structure on the co-adjoint orbits.

Moreover, we show the lifted Darboux parametrisation for the co-adjoint orbits of Takiff algebras. We show that the lifted space is always the same and the way to distinguish between different isomonodromic systems is the Hamiltonian group action we choose to obtain the reduced phase space.

In order to consider the Poisson structure of the irregular isomonodromic problems, we have to deal with more complicated co-adjoint orbits than in the Fuchsian case. In this section we describe the co-adjoint of the so-called Takiff algebras (which are also called truncated loop algebras). In Section 4 we will show that that such algebras naturally arise during the confluence procedure.

The Takiff algebra $\hat{\mathfrak{g}}_r$ of the Lie algebra $\mathfrak{g}$ is the Lie algebra of polynomials in some variable z of a fixed degree with the following Lie bracket

$$\left[\sum_{i=0}^{r} A_i z^i, \sum_{j=0}^{r} B_j z^j \right] = \sum_{i=0}^{r} \left(\sum_{j=0}^{i} [A_i, B_{i-j}] \right) z^i. \tag{39}$$

This algebra may be viewed as a double quotient of the loop algebra $\mathfrak{g}[[z]]$ as follows. Denote by $\mathfrak{g}[z]^+$ the subalgebra of the elements which has a finite limit when z goes to the origin. Then $\hat{\mathfrak{g}}_r$ is defined as

$$\hat{\mathfrak{g}}_r = \mathfrak{g}[z]^+/z^{r+1}\mathfrak{g}[z]^+, \quad \mathfrak{g}[z]^+ \simeq \mathfrak{g}[[z]] \Big/ \mathfrak{g}[z]^-, \quad \mathfrak{g}[z]^- = \left\{ f \in \mathfrak{g}[[z]] : \lim_{z\to\infty} f(z) = 0 \right\}.$$

Since that we will also call such algebras as the truncated loop algebras or truncated current algebras. The variable z is usually called spectral parameter and, as we will illustrate here below, it induces a grading on the Takiff algebra.

In the case when $\mathfrak{g}$ has invariant non-degenerate bi-linear form (Killing form), we may define the co-algebra $\hat{\mathfrak{g}}_r^\star$ in the following way

$$\hat{\mathfrak{g}}_r^\star = \mathfrak{g}[z]^-/z^{-(r+1)-1}\mathfrak{g}[z]^- = \left\{ A = \frac{A_r}{z^{r+1}} \cdots + \frac{A_0}{z} \,\middle|\, A_i \in \mathfrak{g} \right\}.$$

The pairing between $\hat{\mathfrak{g}}_r$ and $\hat{\mathfrak{g}}_r^\star$ is given by

$$\langle A, B\rangle = \oint_{S^1} \mathrm{Tr}\,(AB)dz = \sum_{i=1}^{r} \mathrm{Tr}\, A_i B_i. \tag{40}$$

Let us assume that the Lie algebra $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathrm{Span}\,\{X_1, \dots X_m\}\,, \quad [X_i, X_j] = C_{ij}^k X_k, \quad \langle X_i, X_j\rangle = \delta_{ij},$$

then for the truncated loop algebra $\hat{\mathfrak{g}}_r$ we have the following basis and structure equations

$$X_{\alpha,i} = X_i z^\alpha, \quad [X_{i,\alpha}, X_{j,\beta}] = \begin{cases} C_{ij}^k X_{k,\alpha+\beta}, & \alpha+\beta \le r \\ 0 & \alpha+\beta > r. \end{cases}$$

For the dual algebra $\mathfrak{g}_r^\star$ we use the following basis

$$X^{\alpha,i} = X^i z^{-\alpha-1}, \quad \langle X^i, X_j\rangle = \delta_{ij},$$

so the pairing for the truncated loop algebra is given by

$$\langle X^{i,\alpha}, X_{j,\beta}\rangle = \delta_{\alpha\beta}\langle X^i, X_j\rangle = \delta_{\alpha\beta}\delta_{ij}.$$

The details about Takiff algebras or truncated current algebras and it's standard Lie–Poisson bracket may be found in [26] (see part 2, chap. 4 §1). In the following sub-section we recall the essentials of this construction.

2.1. **Standard Lie–Poisson bracket for the Takiff algebras.** Let us remind the reader that the standard Lie–Poisson bracket on the dual Lie algebra $\mathfrak{g}^\star$ is given by

$$\{f,g\}(L) = -\langle L, [\mathrm{d}f(L), \mathrm{d}g(L)]\rangle, \quad f,g \in C^\infty(\mathfrak{g}^\star), \quad \mathrm{d}f(L), \mathrm{d}g(L) \in \mathfrak{g}.$$

The coadjoint orbits $\mathcal{O}^\star$ are symplectic leaves of the standard Lie–Poisson structure on $\mathfrak{g}^\star$. The vector fields on $\mathcal{O}^\star$ may be identified with the elements of Lie algebra $\mathfrak{g}$ and the symplectic form takes form

$$\omega_{\mathrm{KKS}}(X,Y)(L) = -\langle L, [X,Y]\rangle.$$

Following [26], we now describe the standard Lie–Poisson structure on the dual $\hat{\mathfrak{g}}_n^\star$ of the Takiff algebra. Let's consider the following element of the coadjoint orbit

$$A = \sum_{\alpha=1}^{r}\sum_i A_{\alpha,i}X^{\alpha,i} \in \hat{\mathfrak{g}}_n^\star,$$

The coefficients $A_{\alpha,i}$ are functions on the coadjoint orbit, with $\mathrm{d}A_{\alpha,i} = X_{i,\alpha}$ so that the standard Lie–Poisson bracket is given by

$$\{A_{\alpha,i}, A_{\beta,j}\} = -\langle A, [X_{i,\alpha}, X_{j,\beta}]\rangle = -\langle A, C_{ij}^k X_k z^{\alpha+\beta}\rangle = \begin{cases} -C_{ij}^k A_{\alpha+\beta,k}, & \alpha+\beta \le r \\ 0 & \alpha+\beta < r. \end{cases} \tag{41}$$

This is a graded Poisson structure of degree 1, and the Takiff co-algebra inherits the grading:

$$\hat{\mathfrak{g}}_r^\star := \bigoplus_{i=0}^{r} \hat{\mathfrak{g}}_r^{\star,i}, \quad \{\hat{\mathfrak{g}}_r^{\star,i}, \hat{\mathfrak{g}}_r^{\star,j}\} \subseteq \hat{\mathfrak{g}}_r^{\star,i+j},$$

where $\hat{\mathfrak{g}}_r^{\star,i} = \left\{ A = \frac{A_i}{z^{i+1}} \,\middle|\, A_i \in \mathfrak{g}^\star \right\}$. The same grading is induced to the co-adjoint orbit $\hat{\mathcal{O}}_r^\star$.

Remark 2.1. *Note that the degree of the grading is due to the choice of the pairing* (40) *in the loop algebra. If we had chosen a different measure, say* $\frac{\mathrm{d}z}{z^k}$*, then the degree would have been* k.

In the case when $\mathfrak{g}$ is $\mathfrak{gl}_\mathfrak{m}$ we have the following Poisson structure

$$\{(A_\alpha)_{ij},(A_\beta)_{kl}\} = \begin{cases} (A_{\alpha+\beta})_{il}\delta_{jk} - (A_{\alpha+\beta})_{kj}\delta_{il} & \alpha+\beta\leq r \\ 0 & \alpha+\beta > r, \end{cases} \tag{42}$$

which may be written in the r-matrix form

$$\{A_\alpha \underset{,}{\otimes} A_\beta\} = \begin{cases} -[\Pi, A_{\alpha+\beta}\otimes \mathbb{I}] & \alpha+\beta\leq r \\ 0 & \alpha+\beta > r. \end{cases} \tag{43}$$

As we said before, the co-adjoint orbits of the Takiff algebras form phase space of the isomonodromic deformations equations in case of irregular singularities. We may also see that Fuchsian case is just a case when all co-adjoint orbits are $\hat{\mathcal{O}}_0^\star \simeq \mathcal{O}^\star$.

2.2. **Lifted Darboux coordinates.** As seen in the previous section, the lifted Darboux coordinates for the co-adjoint orbits of an ordinary Lie algebra are given by a symplectic reduction from $T^\star\mathfrak{gl}_m$. We use the same idea for the truncated loop algebras. This follows ideas introduced in [20] to parametrize the space of the irregular Gaudin systems, which are autonomous version of the isomonodromic systems. However, Chervof and Talalaev didn't perform the explicit reduction and the parametrisation of the co-adjoint orbit of the truncated loop algebras. So in this section we provide detailed reduction procedure.

We start from the following space

$$\mathfrak{g}=\mathfrak{gl}_\mathfrak{m},\quad T^\star\hat{\mathfrak{g}}_r = \left\{(P,Q)\,\middle|\, P=\sum_{i=0}^r P_i z^i,\, Q=\sum_{i=0}^r Q_i z^{-i-1},\quad P_i,Q_i\in\mathfrak{gl}_m\right\}.$$

The symplectic form on $T^\star\hat{\mathfrak{g}}_n$ is given by the differential of the Liouville form:

$$\omega = \mathrm{d}\langle P,\mathrm{d}Q\rangle = \oint_{S^1}\mathrm{Tr}\,(\mathrm{d}P\wedge\mathrm{d}Q)\,dz = \mathrm{d}\sum_{i=0}^r \mathrm{Tr}\,(P_i\wedge \mathrm{d}Q_i)\,, \tag{44}$$

here d is a differential on the space of spectral parameter z, while d is a differential on a phase space.

Lemma 2.2. *The map*

$$\begin{array}{ccc} \bigoplus_{i=0}^r T^\star\mathfrak{gl}_m & \to & T^\star\hat{\mathfrak{g}}_r \\ (P_0,\dots,P_r,Q_0,\dots,Q_r) & \mapsto & (P,Q) \end{array}$$

is a symplectomorphism.

The proof of this result is a straightforward consequence of the fact that $T^\star\hat{\mathfrak{g}}_n$ and $\bigoplus_{i=1}^n T^\star\mathfrak{gl}_m$ are isomorphic as vector spaces and formula (44) shows that they are symplectomorphic to each other. However, we have enphasised this simple fact into a Lemma because $\bigoplus_{i=1}^n T^\star\mathfrak{gl}_m$ provides the ambient space for the confluence procedure.

We now want to construct the Lie group $\hat{G}_r$ of the Takiff algebra. Its elements are given by:

$$g(z) = g_0+\sum_{i=1}^r g_i z^i,\quad g_0\in GL_m,\quad g_i\in\mathfrak{gl}_m,$$

where, in order to be able to multiply both on the left and on the right, $\mathfrak{gl}_m$ is considered as a bi-module of GL_m. The group structure of $\hat{G}_n$ is given by GL_m multiplication mod z^n, i.e.

$$g(z)\cdot h(z) = g(z)h(z) \bmod z^{r+1} = g_0h_0 + \sum_{i=1}^r\left(\sum_{j=0}^i g_{i-j}h_j\right)z^i.$$

The inverse is given by

$$g^{-1} = g_0^{-1}\left[1+\sum_{i=1}^r g_0^{-1}g_{i+1}z^i\right]^{-1} = g_0^{-1}(1+\tilde{g}(z))^{-1} = g_0^{-1}\sum_{i=0}^\infty (-1)^i\tilde{g}(z)^i \bmod z^{r+1},$$

and the neutral element is given by the identity matrix. The induced inner and outer actions on $T^\star\hat{\mathfrak{g}}^r$ are given by

$$g\underset{\text{outer}}{\times}(P,Q) = \left([P\circ g]\bmod z^{r+1};\pi_-\left[g^{-1}\circ Q\right]\right) \tag{45}$$

$$g \underset{\text{inner}}{\times} (P,Q) = \left([g\circ P] \bmod z^{r+1}; \pi_-\left[Q\circ g^{-1}\right]\right) \tag{46}$$

where π_- is a projection to the Laurent part with respect to spectral parameter z, i.e.

$$\pi_-\left[\sum_{i=-\infty}^{\infty} T_i z^i\right] = \sum_{i=-\infty}^{-1} T_i z^i$$

Lemma 2.3. *Both inner and outer actions are Hamiltonian with the moment maps respectively*

$$\begin{array}{ll} \mu_{\text{inner}}: & T^\star\hat{\mathfrak{g}}_r \to \hat{\mathfrak{g}}_r^\star \\ & (P,Q)\mapsto \Lambda(z) = \pi_-[PQ] \end{array} \qquad \begin{array}{ll} \mu_{\text{outer}}: & T^\star\hat{\mathfrak{g}}_r \to \hat{\mathfrak{g}}_r^\star \\ & (P,Q)\mapsto A(z) = \pi_-[QP] \end{array}. \tag{47}$$

These two moment maps are dual in a sense of Adams–Harnad–Previato duality [3]. Since inner and outer group actions commute, $A(z)$ and $\Lambda(z)$ Poisson commute with respect to Poisson bracket induced by (44). As in the Fuchsian case, $A(z)$ is an element of the co-adjoint orbit for the truncated loop algebra. On the other hand, $\Lambda(z)$ becomes an invariant of the orbit after quotient via the inner group action.

This gives us the opportunity to generalise the statement of the Lemma 1.4 to the case of Takiff algebras:

Lemma 2.4. *The map*

$$\begin{array}{ccc} T^\star\hat{\mathfrak{g}}_r \underset{\Lambda_0}{/\!\!/} \hat{G}_r & \to & \hat{\mathcal{O}}_r^\star \\ (Q,P) & \mapsto & A(z) := \pi_-[QP] \end{array}$$

where $\underset{\Lambda_0}{/\!\!/}$ denotes the Hamiltonian reduction w.r.t. the inner action in which the moment map has value Λ_0, is a rational symplectomorphism and the Jordan normal form Λ_0 of A is given by

$$\Lambda_0(z) = \pi_-[PQ].$$

The explicit form of $A(z)$ is

$$A(z) = \frac{A_r}{z^{r+1}}\cdots + \frac{A_0}{z}, \quad A_k = \sum_{i=0}^{r-k}\chi_{i,i+k}, \quad \chi_{i,j} = Q_iP_j. \tag{48}$$

while $\Lambda_0(z)$ takes form

$$\Lambda_0(z) = \frac{\Lambda_r}{z^{r+1}}\cdots + \frac{\Lambda_0}{z}, \quad \Lambda_k = \sum_{i=0}^{r-k} P_{i+k}Q_i, \tag{49}$$

Remark 2.5. *According to Lemma 2.2, all co-adjoint orbits, i.e the ones of for the ordinary Lie algebras and the one for the Takiff algebras, are reductions of the same phase space. Systems with different orders of poles are obtained by different choices of the group realising the reduction: in the Fuchsian case we considered the action of the direct product of GL_m, while in the case of the Takiff algebra we use the inner action of $\hat{G}_m^r := GL_m[z]/z^{r+1}GL_m[z]$.*

The parametrisation (48) allows a nice combinatorial description which is presented on the Fig. 1.

Theorem 2.6. *The Poisson bracket induced by the Darboux coordinates Q_i, P_i to the space of matrices A_k, $k=0,\dots,r$ coincides with the graded Poisson structure (43).*

Proof. This statement is a straightforward corollary of the Lemma 2.4. However, here we prove id directly for the sake of clarity. The Poisson bracket on the elements χ_{ij} in (48) is given by

$$\begin{aligned}\{\chi_{ij} \underset{,}{\otimes} \chi_{kl}\} = \{Q_iP_j \underset{,}{\otimes} Q_kP_l\} &= \delta_{jk}(Q_i\otimes 1)\Omega(\mathbb{I}\otimes P_l) - \delta_{il}(\mathbb{I}\otimes Q_k)\Omega(P_j\otimes\mathbb{I}) = \\ &= \delta_{jk}(Q_iP_l\otimes\mathbb{I})\Omega - \delta_{il}\Omega(Q_kP_j\otimes\mathbb{I}) = \delta_{jk}(\chi_{il}\otimes\mathbb{I})\Omega - \delta_{il}\Omega(\chi_{kj}\otimes\mathbb{I})\end{aligned}$$

which is the same as

$$\left\{(\chi_{ij})_{\alpha\beta}, (\chi_{kl})_{\gamma\delta}\right\} = \delta_{jk}\delta_{\gamma\beta}(\chi_{il})_{\alpha\delta} - \delta_{il}\delta_{\alpha\delta}(\chi_{kj})_{\gamma\beta}.$$

By direct computation

$$\begin{aligned}\{A_k \underset{,}{\otimes} A_l\} = \sum_{i,j}\{\chi_{i,i+k}\underset{,}{\otimes}\chi_{j,j+l}\} &= \sum_{i,j}\delta_{j,i+k}(\chi_{i,j+l}\otimes\mathbb{I})\Pi - \delta_{i,j+l}\Pi(\chi_{j,i+k}\otimes\mathbb{I}) = \\ &= \sum_i(\chi_{i,i+k+l}\otimes\mathbb{I})\Pi - \sum_j\Pi(\chi_{j,j+k+l}\otimes\mathbb{I}) = -[\Pi, A_{k+l}\otimes\mathbb{I}]\end{aligned}$$

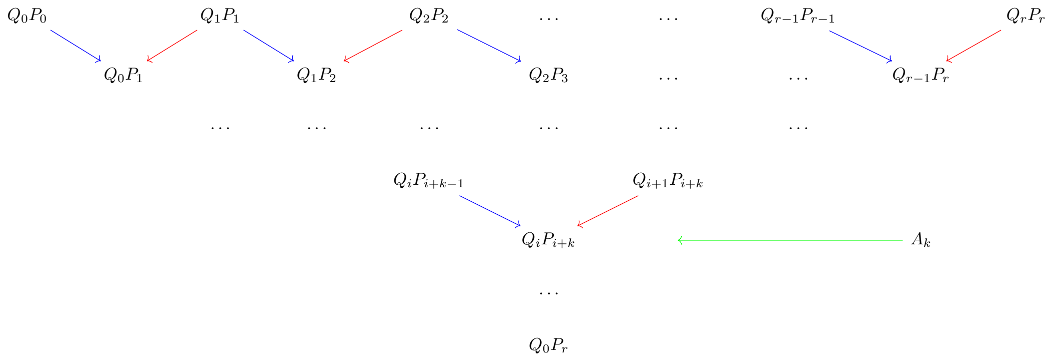

FIGURE 1. Lifted Darboux coordinates for the Takiff algebra of degree r. In this diagram we have $r+1$ rows, and we number them starting at the top with row 0, all the way down to row r. The sum of the elements in row k gives the coefficient A_k of the power of z^{-k-1}, the blue arrow follows each Q_i matrix from the formula above to the one below, while the red one follows P_i.

we obtain the proof of the statement. In case if $k+l>r$ the Poisson bracket shall automatically be zero. □

In the next lemma, we show that the quadratic Casimirs for the Takiff algebra are given by functions of the spectral invariants of the co-adjoint orbit:

Lemma 2.7. *For the Takiff algebra of degree r, the following quantities are Casimirs*

$$I_k = \operatorname*{res}_{z=0}\left(z^{r+k}\mathrm{Tr}\,A^2\right), \quad 0<k<r. \tag{50}$$

Proof. The fact that I_k are Casimirs may be checked by the direct computation, here we demonstrate it for $k=1$, since we use it in the further text. Explicitly I_1 writes as follows

$$I_1 = \sum_{j=0}^{r} \mathrm{Tr}\,A_jA_{r-j}.$$

The Poisson bracket with an arbitrary generator of the Poisson algebra defined via Lie-Poisson bracket for the Takiff algebra gives

$$\sum_{j=0}^{r}\{(A_i)_\alpha, \mathrm{Tr}\,A_jA_{r-j}\} = \sum_{j=0}^{r}[A_{i+j},A_{r-j}]_\alpha + \sum_{j=i}^{r}[A_{r-j+i},A_j]_\alpha =$$
$$= \sum_{l=i}^{r}[A_l,A_{r-l+i}]_\alpha + \sum_{j=i}^{r}[A_{r-j+i},A_j]_\alpha = 0.$$

In the same way we may prove that I_k are the Casimirs for $k>1$. □

2.3. **Poisson automorphisms of the Takiff algebra and independent deformation parameters.** In this subsectionwe describe the class of linear automorphisms of the Takiff algebra which preserve the Poisson bracket, namely linear maps

$$B_i = \sum_{j=0}^{r} T_{ij}A_j, \qquad T_{ij}\in\mathbb{C}, \quad i,j=0,\dots r, \tag{51}$$

such that

$$\{B_i \underset{,}{\otimes} B_j\} = [\Pi, \mathbb{I}\otimes B_{i+j}] \quad \Longleftrightarrow \quad \{A_i \underset{,}{\otimes} A_j\} = [\Pi, \mathbb{I}\otimes A_{i+j}] \tag{52}$$

In the next theorem we describe explicitly the constraints on the coefficients T_{ij} and show that they define an ideal in $\mathbb{C}[T_{00}\dots T_{rr}]$. We then give an explicit parameterisation of the quotient of $\mathbb{C}[T_{00}\dots T_{rr}]$ with respect to this ideal in terms of r parameters. $t_1,\dots,t_r$.

Theorem 2.8. *The Poisson condition (52) generates the ideal $\mathcal{P}$ in the ring $\mathbb{C}[T_{11}\dots T_{rr}]$ given by the equations*

$$\mathcal{P}=\begin{cases} T_{00}=1 & \\ T_{0k}=0, & k>0\\ T_{k0}=0, & k>0\\ T_{ik}=0, & k<i\\ T_{sl}=\sum\limits_{i,j>0}^{i+j=l} T_{pi}T_{mj} & \forall p,m>0:\, p+m=s.\end{cases}\tag{53}$$

Moreover we have the following ring isomorphism for the quotient

$$\mathcal{Q}:\quad \mathbb{C}[T_{00}\dots T_{rr}]/\mathcal{P}\to\mathbb{C}[t_1\dots t_r]\tag{54}$$

such that

$$T_{1i}=t_i,\quad T_{ki}=\frac{1}{i!}\frac{d^i}{d\varepsilon^i}P_r(t,\varepsilon)^k\Big|_{\varepsilon=0},\quad P_r(t,\varepsilon)=\sum_{i=1}^{r}\varepsilon^i t_i,\tag{55}$$

so that T_{ki} is just the coefficient of the ε^i term in the polynomial $P_r(t,\varepsilon)^k$.

Remark 2.9. *The equations which define the ideal $\mathcal{P}$ do not depend on the specific form of Π, i.e. on the structure constants of a Poisson bracket. Therefore, the classification of the automorphisms is a consequence of the grading structure and not a property of the specific Lie co-algebra.*

Proof. Assume the matrices A_i and B_i satisfy the Poisson relations (52) and prove the relations for the coefficients T_{ij}. Let us start from the relation for B_1

$$\{B_0\underset{,}{\otimes}B_0\}=[\Pi,\mathbb{I}\otimes B_0].\tag{56}$$

Substituting (51) in (56) and expanding, we obtain

$$\{B_0\underset{,}{\otimes}B_0\}=\sum_{i,j=0}^{r}T_{0i}T_{0j}\{A_i\underset{,}{\otimes}A_j\}=\sum_{k=0}^{r}\left(\sum_{i=0}^{k}T_{0i}T_{0,k-i}\right)[\Pi,\mathbb{I}\otimes A_k]=$$
$$=[\Pi,\mathbb{I}\otimes B_0]=\sum_{k=1}^{r}T_{0k}[\Pi,\mathbb{I}\otimes A_k].\tag{57}$$

This equation defines the system for the coefficients T_{0j}, which takes form

$$T_{00}T_{00}=T_{00},\quad 2T_{00}T_{0k}+\sum_{i=1}^{k-1}T_{0i}T_{0,k-i}=T_{0k},$$

that, by recursion, leads to the first set of equations which generate ideal $\mathcal{P}$:

$$T_{00}=1,\quad T_{0k}=0,\quad k>0.$$

The next statement we want to prove is that $T_{k0}=0$ for $k>1$. We use

$$\{B_1\underset{,}{\otimes}B_k\}=[\Pi,\mathbb{I}\otimes B_{k+1}]\qquad k=1,\dots,r.\tag{58}$$

Again, substituting (51) and expanding, we obtain

$$\sum_{i,j=0}^{r}T_{1i}T_{kj}[\Pi,\mathbb{I}\otimes A_{i+j}]=\sum_{j=0}^{r}T_{k+1,j}[\Pi,\mathbb{I}\otimes A_j],$$

and collecting all coefficients of $[\Pi,\mathbb{I}\otimes A_1]$, we have that

$$T_{k+1,0}=T_{10}T_{k0},$$

that is solved by

$$T_{k+1,0}=(T_{10})^k.$$

On the other hand substituting (51) in

$$\{B_1\underset{,}{\otimes}B_r\}=0,\tag{59}$$

we obtain

$$T_{10}T_{r0}=0=(T_{10})^r\quad\Rightarrow\quad T_{10}=0,$$

as we wanted. Now to demonstrate the statement that $T_{ik}=0$ for $k<i$ we use the relation

$$\{B_1 \underset{,}{\otimes} B_1\} = [\Pi, \mathbb{I}\otimes B_2]. \tag{60}$$

By substituting (51) we see that the left hand side of (60)

$$\{B_1 \underset{,}{\otimes} B_1\} = \sum_{i,j>2} T_{1i}T_{1j}[\Pi, A_{i+j}] = T_{11}T_{11}[\Pi, A_2] + \sum_{i=4}\kappa_i[\Pi, A_i], \tag{61}$$

does not contain terms in A_0, or A_1, it contains only one term that depends on A_2, given by $T_{11}T_{11}[\Pi, A_2]$ and all other terms depend on $A_3,\dots,A_r$. Expanding right hand side of (60) we get

$$[\Pi, \mathbb{I}\otimes B_2] = T_{21}[\Pi, \mathbb{I}\otimes A_1] + \sum_{i=2} T_{2i}[\Pi, \mathbb{I}\otimes A_i]. \tag{62}$$

Therefore, we obtain that $T_{21}=0$. Similarly, applying the $\{B_1\otimes\circ\}$ to $B_2\dots B_r$ and using the same approach we obtain that $T_{ik}=0$ for $k<i$. The last relation in (53) is obtained by imposing (52), substituting (51) and expanding as before, and then by imposing all other conditions we have obtained so far.

We now prove the second part of the Theorem. First of all, we observe that thanks to relations (53), the coefficients $t_j := T_{1j}$ for $j>0$ form a basis in the quotient ring $\mathcal{Q}: \quad \mathbb{C}[T_{00}\dots T_{rr}]/\mathcal{P}$. Then, because each T_{ik} must be given by a polynomial $P_k^{(i)}$of $t_1,\dots,t_r$, we just need to check the degree and the form of the coefficients. To this aim we use the last relation of (53) for T_{ij} by induction on j from i to r. We omit this computation as it is straightforward. □

In the next section we will see how such dependence on the parameters t_i's arises during the confluence procedure. In some sense, the irregular deformation parameters are just the deformation of the representation for the Takiff algebra.

Example 2.10. In order to give a taste of how the general elements of the Takiff co-algebra depend on the Poisson automorphism parameters t_i, we provide a few examples of low degree. We consider an element of the Takiff co-algebra as a polynomial in $\frac{1}{z}$. In the case of $\hat{\mathfrak{g}}_1^\star$ we have

$$B(z) = \frac{t_1 A_1}{z^2} + \frac{A_0}{z}. \tag{63}$$

Here this automoprhism just establish that the invariant space of the action of A_0 is defined up to multiplication by a constant, so in a sense this example is quite trivial. For the $\hat{\mathfrak{g}}_2^\star$ the general element writes as

$$B(z) = \frac{t_1^2 A_2}{z^3} + \frac{t_1A_1 + t_2A_2}{z^2} + \frac{A_0}{z}. \tag{64}$$

The next example is the case of $\hat{\mathfrak{g}}_3^\star$ and the element of the co-algebra writes as

$$B(z) = \frac{t_1^3 A_3}{z^4} + \frac{t_1^2A_2 + 2t_1t_2A_3}{z^3} + \frac{t_1A_1 + t_2A_2 + t_3A_3}{z^2} + \frac{A_0}{z}. \tag{65}$$

2.4. **Direct product of the co-adjoint orbits and outer group action.** To study systems with more than one pole, we will need to consider the symplectic space given by the direct product of different co-adjoint orbits of the Lie algebra for simple poles, or of the appropriate Takiff algebra for higher order poles. We use here a unified notation, in which we understand that for poles of order 1, the Poincaré rank is $r=0$ and $\hat{\mathfrak{g}}_0$ is $\mathfrak{g}$, $\hat{\mathcal{O}}_0^\star$ is $\mathcal{O}_i^\star$, $T^\star\hat{\mathfrak{g}}_0$ is $T^\star\mathfrak{g}$ and $\hat{G}_0$ is G. With this notation in mind, the symplectic space we consider is

$$\hat{\mathcal{O}}_{r_1}^\star \times \hat{\mathcal{O}}_{r_2}^\star \times \dots \hat{\mathcal{O}}_{r_n}^\star \times \hat{\mathcal{O}}_{r_\infty}^\star, \tag{66}$$

where we always assume to have a pole at infinity like in the Fuchsian case. This product of co-adjoint orbits may be viewed as the reduction of the following symplectic space

$$\bigoplus_{i=1}^{n} T^\star\hat{\mathfrak{g}}_{r_i} \simeq \bigoplus_{i=1}^{r_1+r_2+\dots+r_n+r_\infty} T^\star\mathfrak{gl}_m$$

with respect to the inner action of the following group

$$\mathcal{G}^{(n)} := \hat{G}_{r_1}\times\hat{G}_{r_2}\times\dots\times\hat{G}_{r_n}\times\hat{G}_{r_\infty}.$$

In this way, we obtain that

$$\hat{\mathcal{O}}^{\star}_{r_1}\times\hat{\mathcal{O}}^{\star}_{r_2}\times\dots\hat{\mathcal{O}}^{\star}_{r_n}\times\hat{\mathcal{O}}^{\star}_{r_\infty}=\left(\bigoplus_{i=1,\dots,n,\infty}T^{\star}\hat{\mathfrak{g}}_{r_i}\right)\underset{\otimes\Lambda_{r_i}}{/\!\!/}\mathcal{G}^{(n)}\simeq\left(\bigoplus_{i=1}^{r_1+\dots+r_n+r_\infty}T^{\star}\mathfrak{gl}_m\right)\underset{\otimes\Lambda_{r_i}}{/\!\!/}\mathcal{G}^{(n)}$$

where we denote by $\underset{\otimes\Lambda_{r_i}}{/\!\!/}$ the Hamiltonian reduction with respect to the inner action in which the value of the moment map is given by the product of values Λ_{r_i} of the inner moment map of each $\hat{G}_{r_i}$.

We now take into account the outer action on each co-adjoint orbit; similarly to the Fuchsian case, in order to have a well defined action on the whole connection, we again restrict to the diagonal case

$$g_1=g_2=\dots=g_n=g_\infty=g,$$

where g_i doesn't depend on the spectral parameter z_i. This constant diagonal action is a constant gauge group G action as in the Fuchsian case.

The moment map of this outer action takes form

$$\mu=\sum_{j=1,\dots,n,\infty}\sum_{i=0}^{r_j}Q_i^{(j)}P_i^{(j)},$$

which may be again seen as the sum of residues at poles. Finally, the fully reduced space takes form

$$M=\hat{\mathcal{O}}^{\star}_{r_1}\times\hat{\mathcal{O}}^{\star}_{r_2}\times\dots\hat{\mathcal{O}}^{\star}_{r_n}\times\hat{\mathcal{O}}^{\star}_{r_\infty}/\!\!/\mathcal{G}^{(n)}\simeq\left[\left(\bigoplus_{i=1}^{r_1+\dots+r_n+r_\infty}T^{\star}\mathfrak{gl}_m\right)\underset{\Lambda}{/\!\!/}\mathcal{G}^{(n)}\right]/\!\!/\mathcal{G}^{(n)}. \tag{67}$$

The quotient with respect to the diagonal outer action has the same effect as in the Fuchsian case - it specifies the residue at the infinity. However, differently from the Fuchsian case, where this was enough to fully characterise the Fuchsian singularity at infinity, here we have a pole of Poincare rank r at infinity, where the connection takes the form

$$A(\lambda)=\lambda^{r-1}A^{(\infty)}_{r+1}+\sum_{k=0}^{r-1}A^{(\infty)}_{k+2}\lambda^k+\text{regular terms at }\infty.$$

We may view the moment map as fixing the term $A_1^{(\infty)}$. In the next section we will study the isomonodomic deformations of irregular connections that are elements of the space (67).

2.5. **Fixing the spectral invariants. Reduction with respect to the inner action.** In this section we compute explicitly the reduced coordinates for the co-adjoint orbits of the quotient of Takiff algebras with respect to the inner group action on the lifted Darboux coordinates in the case of degrees 1, 2, 3 and 4 - this choice is motivated by the fact that in the Painlevé confluence scheme the maximal pole order we have is 4. However, the described procedure can be easily expanded for the Takiff algebra of any degree - we give a hint and some explanation in the discussion after the examples. In each example we give explicit results in the case of $\mathfrak{sl}_2$, since this is the case of the isomonodromic problems for the Painlevé equations. We also provide the coordinates in the diagonal gauge - the case when the leading term is diagonal. We do this because there is also the additional outer action of the group G which can be used to put one orbit in such form.

2.5.1. *First order pole. Takiff algebra of degree 1.* In this case Takiff algebra coincide with the ordinary Lie algebra. Parametrisation in such situation was obtained in works [5, 6]

2.5.2. *Second order pole. Takiff algebra of degree* 1. The Darboux parametrisation is given by

$$A(z)=\frac{Q_0P_1}{z^2}+\frac{Q_0P_0+Q_1P_1}{z},\quad \omega=\mathrm{d}\Theta,\quad \Theta=\mathrm{Tr}\,\left(P_1\mathrm{d}Q_1+P_0\mathrm{d}Q_0\right),$$

so that the extended phase space is of dimension $4m^2$. We now want to reduce this dimension by solving the moment map conditions

$$P_1Q_0=\Lambda_1,\quad P_0Q_0+P_1Q_1=\Lambda_0$$

w.r.t. P_0 and P_1. To do this, we only need to assume that Q_0 is invertible, namely $(Q_0,Q_1)\in\bigoplus Gl_m\times\mathfrak{gl}_m$. This inversion sends the Liouville form to

$$\theta=\mathrm{Tr}\,\left(\Lambda_1Q_0^{-1}\mathrm{d}Q_1+\Lambda_1Q_0^{-1}\mathrm{d}Q_0-\Lambda_1Q_0^{-1}Q_1Q_0^{-1}\mathrm{d}Q_0\right),$$

while A goes to

$$A(z)=\frac{Q_0\Lambda_1Q_0^{-1}}{z^2}+\frac{Q_0\Lambda_0Q_0^{-1}+[Q_1Q_0^{-1},Q_0\Lambda_1Q_0^{-1}]}{z}.$$

We now want to reduce the dimension by $2m$ via the torus action $Q_i\to Q_iD_i$, where D_i is a diagonal matrix, that fixes the invariants of the co-adjoint orbit Λ_0,Λ_1. To this aim, we find the Darboux coordinates $p_1,\dots p_{m(m-1)},q_1,\dots q_{m(m-1)}$ explicitly in such a way that

$$\Theta=\mathrm{Tr}\,\left(\Lambda_1Q_0^{-1}\mathrm{d}Q_1+\Lambda_0Q_0^{-1}\mathrm{d}Q_0-\Lambda_1Q_0^{-1}Q_1Q_0^{-1}\mathrm{d}Q_0\right)=\sum_{i=1}^{m(m-1)}p_i\mathrm{d}q_i. \tag{68}$$

The number of unknown functions also equals to $2m(m-1)$, due to the factorisation of the torus action (this is the truncated current algebra analog of the statement that the eigenvectors are defined up to multiplication of the diagonal matrix). There are many possible choices for the Darboux coordinats in this situation, our aim to find one of them; it is convenient to use the following change

$$L_1=Q_0^{-1}Q_1,$$

then Liouville form transforms to

$$\Theta=\mathrm{Tr}\,\left[\Lambda_1\mathrm{d}L_1+(\Lambda_0+[L_1,\Lambda_1])\,Q_0^{-1}\mathrm{d}Q_0\right].$$

The Liouville form is always defined up to a closed form. Since Λ_1 is an invariant of the co-adjoint orbit (i.e. is a constant) the term

$$\Lambda_1\mathrm{d}L_1=\mathrm{d}\,(\Lambda_1L_1)$$

is exact, so we may drop it. The equation for the differential form therefore simplifies to

$$\mathrm{Tr}\,\left[(\Lambda_0+[L_1,\Lambda_1])\,Q_0^{-1}\mathrm{d}Q_0\right]=\sum_{i=1}^{m(m-1)}p_i\mathrm{d}q_i,$$

which allows us to pick our Darboux coordinates in such a way that Q_0 depends only on $q_1,\dots q_{m(m-1)}$ (i.e.Q_0 is a section of a principal bundle over the Lagrangian sub-manifold), while the entries of L_1 are given by solution of $m(m-1)$ linear equations. For example we can we may take off-diagonal entries of Q_0 as the coordinates on the Lagrangian sub-manifold. By using the torus action, we can make the following choice of Q_0:

$$Q_0=\begin{pmatrix}1&q_1&\dots&\dots&q_{m-1}\\0&1&q_m&\dots&q_{2m-3}\\\vdots&0&\ddots&\ddots&\vdots\\0&\dots&0&1&q_{\frac{m(m-1)}{2}}\\0&\dots&\dots&0&1\end{pmatrix}\begin{pmatrix}1&0&\dots&\dots&0\\q_{\frac{m(m-1)}{2}+1}&1&0&\dots&0\\q_{\frac{m(m-1)}{2}+2}&q_{\frac{m(m-1)}{2}+3}&1&\ddots&0\\\vdots&\vdots&\ddots&\ddots&\vdots\\q_{(m-1)^2}&\dots&\dots&q_{m(m-1)}&1\end{pmatrix}.$$

For $\mathfrak{sl}_2$ we have

$$\Lambda_i=\begin{pmatrix}\theta_i&0\\0&-\theta_i\end{pmatrix},\quad Q_0=\begin{pmatrix}1&q_1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\q_2&1\end{pmatrix},\quad L_1=\frac{1}{2\theta_1}\begin{pmatrix}0&-p_2\\p_2q_2^2-2\theta_0q_2+p_1&0\end{pmatrix}$$

and A goes to

$$A(z)=\frac{2\theta_1}{z^2}\begin{pmatrix}q_1q_2+1/2&-(q_1q_2+1)\,q_1\\q_2&-q_1q_2-1/2\end{pmatrix}+$$
$$+\frac{1}{z}\begin{pmatrix}p_1q_1-q_2p_2+\theta_0&-p_1q_1^2+(2q_1q_2+1)\,p_2-2\theta_0q_1\\p_1&-p_1q_1+q_2p_2-\theta_0\end{pmatrix}. \tag{69}$$

If we take into account the outer action of SL_2, the leading term can be chosen in diagonal form and we have

$$Q_1^{-1}A(z)Q_1=\frac{\theta_1}{z^2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\frac{1}{z}\begin{pmatrix}\theta_0&p_2\\p_2q_2^2-2\,\theta_0q_2+p_1&-\theta_0\end{pmatrix}$$

2.5.3. *Third order pole. Takiff algebra of degree* 2. In this case, the parametrisation in terms of lifted Darboux coordinates is given by

$$A(z)=\frac{Q_0P_2}{z^3}+\frac{Q_0P_1+Q_1P_2}{z^2}+\frac{Q_0P_0+Q_1P_1+Q_2P_2}{z},$$

so that the extended phase space is of dimension $6m^2$. The moment map is given by the equations

$$P_2Q_0=\Lambda_2,\quad P_1Q_0+P_2Q_1=\Lambda_1,\quad P_0Q_0+P_1Q_1+P_2Q_2=\Lambda_0.$$

Here we again use the following change of variables

$$L_1=Q_0^{-1}Q_1,\quad L_2=Q_0^{-1}Q_2$$

that maps the Liouville form to

$$\Theta=\mathrm{Tr}\,\left(\Lambda_2\mathrm{d}L_2+\Lambda_1\mathrm{d}L_1-\Lambda_2L_1\mathrm{d}L_1+\left(\Lambda_0+[L_2,\Lambda_2]+[L_1,\Lambda_1-\Lambda_2L_1]\right)Q_0^{-1}\mathrm{d}Q_0\right).$$

As in the previous case, the first 2 terms are closed differential forms, so we can drop them. The dimension of the reduced phase space equals to $3m(m-1)=3N$ and we consider the following parametrisation

$$\mathrm{Tr}\,(-\Lambda_2L_1\mathrm{d}L_1)=\sum_{i=1}^{N/2}p_i\mathrm{d}q_i,\quad \mathrm{Tr}\,\left[\left(\Lambda_0+[L_2,\Lambda_2]+[L_1,\Lambda_1-\Lambda_2L_1]\right)Q_0^{-1}\mathrm{d}Q_0\right]=\sum_{i=N/2+1}^{3N/2}p_i\mathrm{d}q_i.$$

For simplicity, let us denote

$$\Theta_1=\mathrm{Tr}\,(-\Lambda_2L_1\mathrm{d}L_1),\quad \Theta_2=\mathrm{Tr}\,\left[\left(\Lambda_0+[L_2,\Lambda_2]+[L_1,\Lambda_1-\Lambda_2L_1]\right)Q_0^{-1}\mathrm{d}Q_0\right],$$

so that $\Theta=\Theta_1+\Theta_2$. Now if we will find the right parametrisation of L_1, we may put Q_0 to be a matrix which depends only on $q_{N/2+1},\dots q_{3N/2}$ (i.e. again Q_0 depends only on the coordinates of Lagrangian sub-manifold) and then obtain L_2 by solving a system of linear equations. In non-degenerate case when Λ_2 is a semi-simple matrix with distinct eigenvalues ζ_i we have

$$\Theta_1=\sum_{i<j}-\zeta_i(L_1)_{ij}\mathrm{d}(L_1)_{ji}-\zeta_j(L_1)_{ji}\mathrm{d}(L_1)_{ij}=$$
$$\sum_{i<j}(\zeta_i-\zeta_j)(L_1)_{ji}\mathrm{d}(L_1)_{ij}-\mathrm{d}(\zeta_i(L_1)_{ij}(L_1)_{ji})\simeq\sum_{i<j}(\zeta_i-\zeta_j)(L_1)_{ji}\mathrm{d}(L_2)_{ij}$$

and we see that a natural choice of the Darboux coordinates are the off-diagonal entries of L_1, such that

$$\{(L_1)_{ij},(L_1)_{kl}\}=\mathrm{sgn}(j-i)\delta_{kj}\delta_{li}(\zeta_i-\zeta_j).$$

In the case of $\mathfrak{sl}_2$ we have

$$\Lambda_2=\begin{pmatrix}\theta_2&0\\0&-\theta_2\end{pmatrix},\quad L_1=\begin{pmatrix}\dots&q_1\\\frac{p_1}{2\theta_2}&\dots\end{pmatrix}.$$

Here the diagonal part of L_1 is irrelevant, since it does not contribute to Θ_1,Θ_2 and generally may be be chosen to be zero by a torus action. Solving the linear equations for the Cartan form Θ_2 we obtain

$$\Lambda_i=\begin{pmatrix}\theta_i&0\\0&-\theta_i\end{pmatrix},\quad Q_0=\begin{pmatrix}1&q_1\\q_2&1\end{pmatrix}$$

$$L_1=\frac{1}{2\theta_2}\begin{pmatrix}0&(p_2q_2+p_3q_3-\theta_0)q_1-p_2+\frac{\theta_1}{\theta_2}p_3\\p_1-p_1q_1q_2+(p_3q_3-\theta_0)q_2-2\theta_1q_3&0\end{pmatrix}$$

Here we take in a slightly different form of Q_0 respect to in the previous example for the sake of obtaining a neater final formula. The matrix $A(z)$ takes form

$$A(z)=\frac{1}{z^3}\frac{1}{1-q_1q_2}\begin{pmatrix}\theta_2\,(q_1q_2+1)&-2\,\theta_2q_1\\2\,q_2\theta_2&-\theta_2\,(q_1q_2+1)\end{pmatrix}+$$
$$+\frac{1}{z^2}\frac{1}{1-q_1q_2}\begin{pmatrix}\theta_1q_1q_2+2\,\theta_2q_1q_3-q_2p_3+\theta_1&-2\,q_1^2q_3\theta_2-2\,\theta_1q_1+p_3\\-q_2^2p_3+2\,\theta_1q_2+2\,\theta_2q_3&-\theta_1q_1q_2-2\,\theta_2q_1q_3+q_2p_3-\theta_1\end{pmatrix}+$$
$$+\frac{1}{z}\begin{pmatrix}p_1q_1-q_2p_2-p_3q_3+\theta_0&-p_1{q_1}^2+p_3q_1q_3-\theta_0q_1+p_2\\-p_2{q_2}^2-p_3q_2q_3+\theta_0q_2+p_1&-p_1q_1+q_2p_2+p_3q_3-\theta_0\end{pmatrix}\tag{70}$$

The diagonal gauge gives

$$Q_0^{-1}A(z)Q_0 = \frac{1}{z^3}\begin{pmatrix} \theta_3 & 0 \\ 0 & -\theta_3 \end{pmatrix} + \frac{1}{z^2}\begin{pmatrix} \theta_1 & p_3 \\ 2\,\theta_2 q_3 & -\theta_1 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} -p_3q_3+\theta_0 & -p_2q_1q_2 - p_3q_1q_3 + \theta_0q_1 + p_2 \\ -p_1q_1q_2 + p_3q_2q_3 - \theta_0q_2 + p_1 & p_3q_3 - \theta_0 \end{pmatrix} \tag{71}$$

Choosing another LU parametrisation for Q_0, i.e.

$$Q_0 = \begin{pmatrix} 1 & q_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q_2 & 1 \end{pmatrix}$$

the diagonal gauged system takes form

$$Q_0^{-1}A(z)Q_0 = \frac{\theta_3}{z^3}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{z^2}\begin{pmatrix} \theta_2 & -2\theta_3 q_1 \\ p_1 & -\theta_2 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} q_1p_1+\theta_1 & p_3 \\ p_3q_3^2 + (-2q_1p_1 - 2\,\theta_1)\,q_3 + p_2 & -q_1p_1 - \theta_1 \end{pmatrix}.$$

2.5.4. *Fourth order pole. Takiff algebra of degree* 3. Here we provide only the result

$$\begin{aligned} Q_1^{-1}A(z)Q_1 &= \frac{\theta_4}{z^4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{z^3}\begin{pmatrix} \theta_3 & -2\theta_4 q_3 \\ 2\theta_4 q_4 & -\theta_3 \end{pmatrix} + \\ &+ \frac{1}{z^2}\begin{pmatrix} 2\theta_4q_3q_4 + \theta_2 & -\theta_4q_3^3q_4^2 + (\theta_3 - 4\theta_4)\,q_4q_3^2 - \theta_4q_3 + p_4 \\ -\theta_4q_3^2{q_4}^3 + (\theta_3 - 4\theta_4)\,q_4^2q_3 + (2\theta_3 - \theta_4)\,q_4 + p_3 & -2\theta_4q_3q_4 - \theta_2 \end{pmatrix} + \\ &+ \frac{1}{z}\begin{pmatrix} q_3p_3 - q_4p_4 + \theta_1 & p_2 \\ p_2q_2^2 - 2\,p_3q_2q_3 + 2\,p_4q_2q_4 - 2\,\theta_1q_2 + p_1 & -q_3p_3 + q_4p_4 - \theta_1 \end{pmatrix} \end{aligned} \tag{72}$$

Remark 2.11. *There is an interesting difference between poles of odd or even order. Indeed, when the order of pole is even $r+1=2k$, then the reduced phase space dimension is divisible by 4, and we have a kind of polarisation. Indeed, for poles of order $2k$ we locally write the connection as*

$$\frac{A_0}{z} + \dots \frac{A_{2k-1}}{z^{2k}},$$

and the matrices $A_k,\dots,A_{2k-1}$ form a Poisson commuting family of half the total dimension. Therefore they define a Lagrangian sub-manifold in the phase space. We can then assume that these matrices are parameterized by $Q_0,\dots,Q_{k-1},P_k,\dots P_{2k-1}$ only. This hints at a hidden quaternionic (hyper-Kähler) structure. In the case pole of odd order, we will still have that $A_{k+1},\dots,A_{2k-1}$ form a Poisson commuting family, but now this is not of half the dimension. In this case, we may expect an analog of Sasakian structure.

3. Isomonodromic deformations

Let us discuss an important consequence of Theorem 2.8. Suppose we consider a connection on the Riemann sphere with $n+1$ poles of Poincaré ranks $r_1,\dots,r_n,r_\infty$ and ask about how to deform it by keeping the monodromy data constant. To answer, we have to choose some independent deformation variables and then impose that all other quantities depend on those according to the isomonodromicity condition. When all poles are simple, their positions give us enough independent variables for generic isomonodromic deformations, because the number of the isomonodromic Hamiltonians equals half of the dimension of the space of accessory parameters. When higher order poles are present, their positions don't give enough independent variables. Theorem 2.8 allows us to introduce further $r-1$ independent variables for every singularity of Poincaré rank r, or in other words we have the following

Corollary 3.1. *The general element in the Takiff algebra co-adjoint orbit $\widehat{\mathcal{O}}_r^\star$ has the form*

$$A \sim \sum_{i=0}^{r} \frac{B_i(t_1,t_2\dots t_r)}{(\lambda-u)^{i+1}} + \dots, \tag{73}$$

with

$$B_i(t_1,t_2,\dots t_r) = \sum_{j=i}^{r} A_j \mathcal{M}_{i,j}^{(r)}(t_1,t_2,\dots t_r), \quad \mathcal{M}_{i,j}^{(r)} = \frac{1}{j!}\frac{d^j}{d\varepsilon^j}P_r(t,\varepsilon)^i\Big|_{\varepsilon=0}, \quad P_r(t,\varepsilon) = \sum_{i=1}^{r}\varepsilon^i t_i,$$

and the coefficients A_j satisfy the Takiff algebra Poisson bracket (43).

In this paper, we therefore consider the isomondoromic deformations of connections of the form

$$\frac{d}{d\lambda}\Psi = \sum_{i=0}^{n}\left(\sum_{j=0}^{r_i}\frac{B_j^{(i)}\left(t_1^{(i)}, t_2^{(i)}\dots t_{r_i-1}^{(i)}\right)}{(\lambda-u_i)^{j+1}} - \sum_{i=1}^{r_\infty}\lambda^{i-1}B_i^{(\infty)}\left(t_1^{(\infty)}, t_2^{(\infty)}\dots t_{r_\infty-1}^{(\infty)}\right)\right)\Psi, \tag{74}$$

where the deformation parameters are the locations of the poles $u_1\dots u_n$ and the coefficients of the Poisson Takiff algebra automorphisms $t_j^{(i)}$. The isomonodromic deformation condition means that the matrix differential one from

$$\Omega = \mathrm{d}_{u,t}\Psi\Psi^{-1} = \sum_{i=1}^{n}\left[\Omega_i^{(0)}\mathrm{d}u_i + \sum_{j=1}^{r_i-1}\Omega_i^{(j)}\mathrm{d}t_j^{(i)}\right], \tag{75}$$

is a single valued holomorphic one form on $\mathbb{CP}^1\setminus\{u_1\dots u_n\}$. In general, the explicit form of Ω may be obtained by studying the local solutions of the equation (74) as in the celebrated papers by Jimbo, Miwa [40].

In this paper we consider the general isomonodromic problem as a non-autonomous Hamiltonian system written on a suitable set of the co-adjoint orbits. Therefore, the zero curvature condition splits into a Lax equation that defines the dynamics on the co-adjoint orbits, and an additional relation between the partial derivative of Ω w.r.t. λ and the partial derivative of the connection with respect to deformation parameters

$$\frac{d}{dt_j^{(i)}}A - \frac{\partial}{\partial\lambda}\Omega_j^{(i)} + \left[A, \Omega_j^{(i)}\right] = \underbrace{\left(\frac{\partial}{\partial t_j^{(i)}}A - \frac{\partial}{\partial\lambda}\Omega_j^{(i)}\right)}_{0} + \underbrace{\left(\left(\frac{d}{dt_j^{(i)}} - \frac{\partial}{\partial t_j^{(i)}}\right)A + \left[A, \Omega_j^{(i)}\right]\right)}_{0} = 0.$$

Thanks to this, we may define deformation the one form Ω through the following formula:

$$\Omega_j^{(i)} = \int\frac{\partial A}{\partial t_j^{(i)}}d\lambda. \tag{76}$$

The matrix $\Omega_j^{(i)}$ is defined up to the addition of a matrix which does not depend on λ. Different choices of the gauge result in different constant terms - we will see how to fix this constant term in the examples (see for example section 5.5).

As mentioned before, the deformation parameters $t_1^{(i)},\dots t_{r_i}^{(i)}$, $i=1,\dots,n,\infty$ appear as the result of confluence and may be seen as avatars of the Schlesinger system deformation parameters we start with. If we consider the divisor of singularities (where we denote ∞ by u_{n+1})

$$D := \sum_{i+1}^{n+1}(r_i+1)u_i,$$

we see that the total number of deformation parameters we introduce is given via the degree of such divisor, i.e.

$$d = \underbrace{n+1}_{\#\text{ of singularities}} + \underbrace{\sum r_i}_{\#\text{ irregular times}}.$$

In this paper, the idea is that the number of deformation parameters doesn't change during the confluence procedure, or in other words d is fixed.

Here we want to answer an important question raised by Bertola and Harnad: what is the relation between our deformation parameters and the Jimbo-Miwa-Ueno ones? In [40], the number of deformation parameters depends on the degree of singularity divisor *as well as on the rank of the connection.* The number of Jimbo-Miwa deformation parameters is not preserved during the confluence cascade. Each coalescence leads to the appearance of additional $m-1$ parameters, where m is a rank of isomonodromic problem. Here we refer to the rank of Lie algebra which is dimension of the Cartan subalgebra $\mathfrak{h}$. Obviously in the case of $\mathfrak{sl}_2$ connection, this number equals to zero and the number of Jimbo-Miwa-Ueno coincides with ours.

Let's dwell on this case in more details to explain the relation. Consider a $\mathfrak{sl}_2$ connection with a pole of the Poincaré rank r, i.e.

$$A \underset{\lambda\simeq u}{\sim} \frac{B_r}{z^{r+1}} + \frac{B_{r-1}}{z^r} + \dots\frac{B_0}{z} + O(1) \quad \in \mathfrak{sl}_2,$$

where $z = \lambda - u$ is the local coordinate and the matrices B_k are linear combinations of the bare co-adjoint orbit coordinates A_j and contain our deformation parameters as specified in formula (10).

The Jimbo-Miwa-Ueno deformation parameters w_j are the exponents of asymptotic behaviour of the formal solution at the irregular pole:

$$\Psi \underset{\lambda\simeq u}{\sim} P(z)\left(\mathbb{I} + o(z)\right) z^{\Lambda_0} \exp\left[-\sum_{j=1}^{r} \frac{w_j}{j z^j}\sigma_3\right], \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These w_j can in fact be seen as the spectral invariants associated to the matrices B_k. Thanks to this fact, in the case of $\mathfrak{sl}_2$ there is a rational map which sends Jimbo-Miwa deformation parameters to the parameters obtained via coalescence.To obtain this map explicitly, we perform local diagonalisation at the pole $\lambda \sim u$ and we obtain the following correspondence between Jimbo-Miwa deformation parameters w_i and our t_j via

$$\begin{aligned}
&w_r = \theta_r t_1^r \\
&w_{r-1} = \theta_{r-1} t_1^{r-1} + (r-1)\theta_r t_1^{r-2} t_2 \\
&\dots \\
&w_k = \sum_{j=k}^{r} \theta_j \mathcal{M}_{k,j}^{(r)}(t_1, t_2 \dots, t_r) \\
&\dots \\
&w_1 = \sum_{i=1}^{r} \theta_i t_i.
\end{aligned}$$

Here θ's can be seen as the spectral invariants of matrices A_j, so we separate non-autonomous part (dependence on deformation parameters) and phase space symplectic leaf. Roughly speaking, this map is a map between 2 phase spaces

$$\hat{\mathfrak{g}}_r \to \hat{\mathcal{O}}_r \times \mathbb{C}^r,$$

which is not bi-rational - starting from the irregular point of Poincaré rank 2 we have to deal with square roots if when we write $t_1 \dots t_r$ via Jimbo-Miwa parameters w_j's.

Fpr higher rank, we may think about our times as a special subfamily of the Jimbo-Miwa isomonodromic deformations. The local solution writes as

$$\Psi \underset{\lambda\simeq u}{\sim} P(z)\left(\mathbb{I} + o(z)\right) z^{\Lambda_0} \exp\left[-\sum_{j=1}^{r} \frac{1}{j z^j} \begin{pmatrix} w_1^{(j)} & 0 & \dots & 0 \\ 0 & w_2^{(j)} & \dots & 0 \\ & \dots & \dots & \\ 0 & \dots & 0 & w_m^{(j)} \end{pmatrix}\right]$$

and $w_k^{(j)}$ are the deformation parameters. Then our deformation parameters are given by the special trajectory into the Jimbo-Miwa parameters which may be written as

$$\frac{w_k^{(j)}}{w_l^{(j)}} = \text{const},$$

and may be considered as the deformation along the projective line in a space of Jimbo-Miwa parameters.

In the next section we will see how the general form (74) of the isomonodromic problem with irregular singularities naturally arises during the confluence procedure.

4. Confluence procedure

4.1. **Coalescence of two simple poles.** Without loss of generality, we consider confluence of $u_n := v_1$ and $u_{n-1} := w$, which is given by the following change of deformation parameters

$$u_i = u_i, \quad i = 1 \dots n-1, \quad v_1 = w + \varepsilon t_1. \tag{77}$$

Taking the limit $\varepsilon \to 0$ the deformation parameter v_1 tends to w which is a coalescence. We rewrite matrix $A(\lambda)$ as

$$A(\lambda) = \sum_{i=1}^{n-2} \frac{A^{(i)}}{\lambda - u_i} + \frac{B}{\lambda - w} + \frac{C}{\lambda - w - \varepsilon t_1}, \quad B = A^{(n-1)}, \quad C = A^{(n)},$$

where B and C are introduced as a convenient notation to avoid too many indices. We want to assume some ε expansions for the matrices B and C in order that the limit of $A(\lambda)$ as $\varepsilon \mapsto 0$ is well defined and the resulting system has a double pole at w. To this aim, observe that by rewriting the last two terms in $A(\lambda)$ as

$$\frac{B}{\lambda-w}+\frac{C}{\lambda-w-\varepsilon t_1}=\frac{B}{\lambda-w}+\frac{1}{\lambda-w}C\left(1-\frac{\varepsilon t_1}{\lambda-w}\right)^{-1}$$

and expanding $\left(1-\frac{\varepsilon t_1}{\lambda-w}\right)^{-1}$ in ε we obtain

$$\frac{B}{\lambda-w}+\frac{C}{\lambda-w-\varepsilon t_1}\sim\frac{C+B}{\lambda-w}+\frac{\varepsilon t_1}{(\lambda-w)^2}C+O(\varepsilon^2).$$

In order to produce the second order pole we need two limits to exist

$$\lim_{\varepsilon\to0}(\varepsilon C):=A_1^{(n-1)}\neq0,\quad \lim_{\varepsilon\to0}(C+B):=A_0^{(n-1)},$$

Assuming that $A^{(i)}$'s, B and C may be expanded in the Laurent series in ε we obtain expansions

$$A^{(i)}=\tilde{A}^{(i)}+O(\varepsilon),\quad C=\frac{1}{\varepsilon}A_1^{(n-1)}+C_0+O(\varepsilon),\quad B=-\frac{1}{\varepsilon}A_1^{(n-1)}+B_0+O(\varepsilon),\quad C_0+B_0=A_0^{(n-1)}. \tag{78}$$

Note that we have called these limits $A_0^{(n-1)}$ and $A_1^{(n-1)}$ respectively to adhere to the notation of section 3.

In these hypotheses, we can take the limit as $\varepsilon\to0$ and define

$$\tilde{A}(\lambda):=\lim_{\varepsilon\to0}A(\lambda)=\sum_{i=1}^{n-2}\frac{\tilde{A}^{(i)}}{\lambda-\tilde{u}_i}+t_1\frac{A_1^{(n-1)}}{(\lambda-w)^2}+\frac{A_0^{(n-1)}}{\lambda-w} \tag{79}$$

Remark 4.1. *Observe that the number of deformation parameters has not changed after the confluence, $n-1$ of them have remained as positions of poles, but one of them has become part of the leading term at the second order pole - this is compatible with Theorem 2.8. Indeed, in the next Proposition 4.3 we will prove that the matrices $A_1^{(n-1)}$ and $A_0^{(n-1)}$ satisfy the Takiff algebra Poisson brackets. We will see that as we increase the Poincaré rank of the poles in the confluence procedure, more and more deformation parameters will appear in the numerators of pole expansions exaclty in the way predicted by Theorem 2.8.*

Now let us focus on the deformation equations. The change of variables (77) transforms the deformation 1-form (23) to

$$\Omega=-\sum_{i=1}^{n-2}\frac{A^{(i)}}{\lambda-u_i}\mathrm{d}u_i-\frac{A^{(n-1)}}{\lambda-w}\mathrm{d}w-\frac{A^{(n)}}{\lambda-w-\varepsilon t_1}(\mathrm{d}w+\varepsilon\mathrm{d}t_1).$$

Applying the expansion (78) we obtain

$$\tilde{\Omega}=\lim_{\varepsilon\to0}\Omega=-\sum_{i=1}^{n-2}\frac{\tilde{A}^{(i)}}{\lambda-u_i}\mathrm{d}u_i-\left(t_1\frac{A_1^{(n-1)}}{(\lambda-w)^2}+\frac{A_0^{(n-1)}}{\lambda-w}\right)\mathrm{d}w-\frac{A_1^{(n-1)}}{\lambda-w}\mathrm{d}t_1. \tag{80}$$

The obtained deformation 1-form coincides with the deformation form which can be constructed by considering the local expansions. The deformation one form Ω satisfies equation (76).

Definition 4.2. We call the process of taking the expansions (78) and the limits (79), (80), *1+1 confluence procedure.*

The considered structures - the connection A and the deformation one form Ω are linear in $A^{(i)}$'s so the $O(\varepsilon)$ terms vanish during the limiting procedure. Since the Poisson structure and the Schlesinger Hamiltonians are quadratic structures the limiting procedure becomes more complicated.

Proposition 4.3. *The 1+1 confluence procedure gives a Poisson morphism between the direct product of the co-adjoint orbits to the Lie algebra and the co-adjoint orbit of the Takiff algebra:*

$$\mathcal{O}_1^\star\times\mathcal{O}_2^\star\times\dots\mathcal{O}_n^\star\times\mathcal{O}_\infty^\star\xrightarrow{\text{confluence}}\mathcal{O}_1^\star\times\mathcal{O}_2^\star\times\dots\mathcal{O}_{n-2}^\star\times\hat{\mathcal{O}}_{2,n-1}^\star\times\mathcal{O}_\infty^\star.$$

Namely, if the matrices $A^{(i)}, B, C$ satisfy the standard Lie–Poisson brackets (25)*, then the matrices $\tilde{A}^{(i)}, A_0^{(n-1)}, A_1^{(n-1)}$ satisfy the Poisson algebra of the coefficients for the Takiff algebra* (43)*, i.e.*

$$\left\{\tilde{A}^{(i)}_\alpha,\tilde{A}^{(j)}_\beta\right\}=-\delta_{ij}\sum_\gamma \chi^\gamma_{\alpha\beta}\tilde{A}^{(i)}_\gamma,\quad \left\{\tilde{A}^{(i)}_\alpha,A^{(n-2)}_{0,\beta}\right\}=\left\{\tilde{A}^{(i)}_\alpha,A^{(n-2)}_{1,\beta}\right\}=0\quad i,j=1,\dots n-2,$$
$$\left\{A^{(n-2)}_{1,\alpha},A^{(n-2)}_{1,\beta}\right\}=0,\quad \left\{A^{(n-2)}_{1,\alpha},A^{(n-2)}_{0,\beta}\right\}=-\chi^\gamma_{\alpha\beta}A^{(n-2)}_{1,\gamma},\quad \left\{A^{(n-2)}_{0,\alpha},A^{(n-2)}_{0,\beta}\right\}=-\chi^\gamma_{\alpha\beta}\left(A^{(n-2)}_{0,\gamma}\right), \tag{81}$$

Proof. That Poisson structure (81) for the coefficients of the connection near the irregular singularity is given by Kirillov-Kostant-Souriau form for the co-adjoint orbit $\tilde{\mathcal{O}}^\star_2$ of the Takiff algebra $\mathfrak{g}_2 \backsimeq \mathfrak{g}[z]/(z^2\mathfrak{g}[z])$, where $\mathfrak{g}[z]$ is a Lie algebra of the polynomials with coefficients in $\mathfrak{g}$. Therefore, if we prove that (81), then the $1+1$ confluence procedure gives a Poisson morphism between the direct product of the co-adjoint orbits to the Lie algebra and the co-adjoint orbit of the Takiff algebra.

Let us prove (81). The first row relations are straightforward and we omit the proof. To prove the relations in the second row of (81), let us consider the Poisson relations (25) for B and C

$$\{C_\alpha,C_\beta\}=-\sum_\gamma\chi^\gamma_{\alpha\beta}C_\gamma,\quad \{B_\alpha,B_\beta\}=-\sum_\gamma\chi^\gamma_{\alpha\beta}B_\gamma,\quad \{C_\alpha,B_\beta\}=0.$$

Inserting the expansion (78) and expanding the Poisson relations in ε, we obtain

$$\begin{aligned}&\frac{1}{\varepsilon^2}\left\{A^{(n-1)}_{1,\alpha},A^{(n-1)}_{1,\beta}\right\}+\frac{1}{\varepsilon}\left(\left\{A^{(n-1)}_{1,\alpha},C_{0,\beta}\right\}+\left\{C_{0,\alpha},A^{(n-1)}_{1,\beta}\right\}\right)+\\&\quad+\{C_{0,\alpha},C_{0,\beta}\}+\left\{A^{(n-1)}_{1,\alpha},C_{1,\beta}\right\}+\left\{C_{1,\alpha},A^{(n-1)}_{1,\beta}\right\}=-\chi^\gamma_{\alpha\beta}\left(\frac{1}{\varepsilon}A^{(n-1)}_{1,\gamma}+C_{0,\gamma}\right)+o(\varepsilon)\end{aligned}$$

$$\begin{aligned}&\frac{1}{\varepsilon^2}\left\{A^{(n-1)}_{1,\alpha},A^{(n-1)}_{1,\beta}\right\}-\frac{1}{\varepsilon}\left(\left\{A^{(n-1)}_{1,\alpha},B_{0,\beta}\right\}+\left\{B_{0,\alpha},A^{(n-1)}_{1,\beta}\right\}\right)+\\&\quad+\{B_{0,\alpha},B_{0,\beta}\}-\left\{A^{(n-1)}_{1,\alpha},B_{1,\beta}\right\}-\left\{B_{1,\alpha},A^{(n-1)}_{1,\beta}\right\}=\chi^\gamma_{\alpha\beta}\left(\frac{1}{\varepsilon}A^{(n-1)}_{1,\gamma}-B_{0,\gamma}\right)+o(\varepsilon)\end{aligned}$$

$$\begin{aligned}&-\frac{1}{\varepsilon^2}\left\{A^{(n-1)}_{1,\alpha},A^{(n-1)}_{1,\beta}\right\}+\frac{1}{\varepsilon}\left(\left\{A^{(n-1)}_{1,\alpha},B_{0,\beta}\right\}-\left\{C_{0,\alpha},A^{(n-1)}_{1,\beta}\right\}\right)+\\&\quad+\{C_{0,\alpha},B_{0,\beta}\}+\left\{A^{(n-1)}_{1,\alpha},B_{1,\beta}\right\}-\left\{C_{1,\alpha},A^{(n-1)}_{1,\beta}\right\}=o(\varepsilon).\end{aligned}$$

Collecting different terms in ε, we obtain

$$\begin{aligned}\varepsilon^{-2}:&\quad\left\{A^{(n-1)}_{1,\alpha},A^{(n-1)}_{1,\beta}\right\}=0,\\ \varepsilon^{-1}:&\quad\left\{A^{(n-1)}_{1,\alpha},C_{0,\beta}\right\}+\left\{C_{0,\alpha},A^{(n-1)}_{1,\beta}\right\}=-\chi^\gamma_{\alpha\beta}A^{(n-1)}_{1,\gamma},\\ \varepsilon^{-1}:&\quad\left\{A^{(n-1)}_{1,\alpha},B_{0,\beta}\right\}+\left\{B_{0,\alpha},A^{(n-1)}_{1,\beta}\right\}=-\chi^\gamma_{\alpha\beta}A^{(n-1)}_{1,\gamma},\\ \varepsilon^{-1}:&\quad\left\{A^{(n-1)}_{1,\alpha},B_{0,\beta}\right\}-\left\{C_{0,\alpha},A^{(n-1)}_{1,\beta}\right\}=0,\\ \varepsilon^{0}:&\quad\{C_{0,\alpha},C_{0,\beta}\}+\left\{A^{(n-1)}_{1,\alpha},C_{1,\beta}\right\}+\left\{C_{1,\alpha},A^{(n-1)}_{1,\beta}\right\}=-\chi^\gamma_{\alpha\beta}C_{0,\gamma},\\ \varepsilon^{0}:&\quad\{B_{0,\alpha},B_{0,\beta}\}-\left\{A^{(n-1)}_{1,\alpha},B_{1,\beta}\right\}-\left\{B_{1,\alpha},A^{(n-1)}_{1,\beta}\right\}=-\chi^\gamma_{\alpha\beta}B_{0,\gamma}\\ \varepsilon^{0}:&\quad\{C_{0,\alpha},B_{0,\beta}\}+\left\{A^{(n-1)}_{1,\alpha},B_{1,\beta}\right\}-\left\{C_{1,\alpha},A^{(n-1)}_{1,\beta}\right\}=0.\end{aligned}\tag{82}$$

The term of order ε^{-2} in (82) proves the first relation in the second row of (81). Let us prove the second relation. Take the $1/\varepsilon$ term

$$\left\{A^{(n-2)}_{1,\alpha},B_{0,\beta}\right\}-\left\{C_{0,\alpha},A^{(n-2)}_{1,\beta}\right\}=0\quad\Longleftrightarrow\quad\left\{C_{0,\alpha},A^{(n-2)}_{1,\beta}\right\}=\left\{A^{(n-2)}_{1,\alpha},B_{0,\beta}\right\}$$

and put it in the Poisson relation between $A_1^{(n-1)}$ and C_0. We get

$$-\chi_{\alpha\beta}^{\gamma} A_{1,\gamma}^{(n-2)} = \left\{A_{1,\alpha}^{(n-2)}, C_{0,\beta}\right\} + \left\{C_{0,\alpha}, A_{1,\beta}^{(n-2)}\right\} = \left\{A_{1,\alpha}^{(n-2)}, C_{0,\beta}\right\} + \left\{A_{1,\alpha}^{(n-2)}, B_{0,\beta}\right\} =$$
$$= \boxed{\left\{A_{1,\alpha}^{(n-2)}, C_{0,\beta} + B_{0,\beta}\right\} = -\chi_{\alpha\beta}^{\gamma} A_{1,\gamma}^{(n-2)}}$$

which proves the second relation. Now let us compute the last Poisson bracket

$$\{C_{0,\alpha} + B_{0,\alpha}, C_{0,\beta} + B_{0,\beta}\} = \{C_{0,\alpha}, C_{0,\beta}\} + \{C_{0,\alpha}, B_{0,\beta}\} + \{B_{0,\alpha}, C_{0,\beta}\} + \{B_{0,\alpha}, B_{0,\beta}\}.$$

Using ε^0-terms from (82) for $\{C_{0,\alpha}, C_{0,\beta}\}$ and $\{B_{0,\alpha}, B_{0,\beta}\}$ we obtain

$$\{C_{0,\alpha} + B_{0,\alpha}, C_{0,\beta} + B_{0,\beta}\} = -\chi_{\alpha\beta}^{\gamma}(C_{0,\beta} + B_{0,\beta}) - \left\{A_{1,\alpha}^{(n-2)}, C_{1,\beta}\right\} - \left\{C_{1,\alpha}, A_{1,\beta}^{(n-2)}\right\} +$$
$$+ \left\{A_{1,\alpha}^{(n-2)}, B_{1,\beta}\right\} + \left\{B_{1,\alpha}, A_{1,\beta}^{(n-2)}\right\} + \{C_{0,\alpha}, B_{0,\beta}\} + \{B_{0,\alpha}, C_{0,\beta}\} \quad (83)$$

The last ε^0-term in (82) leads to the following relations

$$\{C_{0,\alpha}, B_{0,\beta}\} = \left\{C_{1,\alpha}, A_{1,\beta}^{(n-2)}\right\} - \left\{A_{1,\alpha}^{(n-2)}, B_{1,\beta}\right\}$$
$$\{B_{0,\alpha}, C_{0,\beta}\} = \left\{A_{1,\alpha}^{(n-2)}, C_{1,\beta}\right\} - \left\{B_{1,\alpha}, A_{1,\beta}^{(n-2)}\right\}$$

which cancel all terms in the right-hand side of (83) except the first term, so we obtain

$$\{C_{0,\alpha} + B_{0,\alpha}, C_{0,\beta} + B_{0,\beta}\} = -\chi_{\alpha\beta}^{\gamma}(C_{0,\gamma} + B_{0,\gamma}),$$

which concludes proof. □

Observe that the relations (82) contain more information than we need, and that one could actually try to come up with a Poisson algebra involving all coefficients B_k, C_k in the expansion (78). However we are only interested in the Poisson subalgebra generated by $A_1^{(n-1)}$, $A_0^{(n-1)} = C_0 + B_0$ and $\tilde{A}^{(i)}$ for $i = 1, \dots, n-2$. The main feature of this subalgebra is that it does not depend on a choice of a Poisson algebra for the coefficients B_k and C_k. We call this subalgebra Isomonodromic Poisson Algebra (IPA), since these are the only elements which survive in the isomonodromic problem after the confluence procedure.

Proposition 4.4. *The $1+1$ confluence procedure produces the isomonodromic Hamiltonians giving the zero curvature condition*

$$\mathrm{d}_u \tilde{A} - \frac{d}{d\lambda}\tilde{\Omega} + [\tilde{A}, \tilde{\Omega}] = 0$$

as equation of motion.

Proof. To prove this, we start from the extended symplectic form for the Schlesinger equations:

$$\omega_{\mathrm{KKS}} + \sum_{i=1}^{n} \mathrm{d}u_i \wedge \mathrm{d}H_i.$$

Here ω_{KKS} is the symplectic form which corresponds to the standard Lie–Poisson structure on the direct product of the co-adjoint orbits. Thanks to Proposition 4.3, the standard Lie–Poisson bracket tends to the Takiff algebra Poisson bracket, therefore ω_{KKS} tends to the corresponding symplectic form. Let us concentrate on the $\sum_{i=1}^{n} \mathrm{d}u_i \wedge \mathrm{d}H_i$ part. This part transforms to

$$\sum_{i=1}^{n} \mathrm{d}u_i \wedge \mathrm{d}H_i \to \sum_{i=1}^{n-2} \mathrm{d}u_i \wedge \mathrm{d}H_i + \mathrm{d}w \wedge \mathrm{d}\left(H_{n-1} + H_n\right) + \mathrm{d}t_1 \wedge \mathrm{d}\left(\varepsilon H_n\right).$$

Since we are working on a symplectic leaf of the standard Lie–Poisson bracket, the central elements, or Casimirs, can be considered as fixed scalars, i.e. the differential d acts on them as a zero. To find the Hamiltonians of the confluent dynamic we have to calculate the limit of the "time-dependent" part of the symplectic structure as ε goes to zero. In other words, we have to find

$$\mathrm{d}\tilde{H}_i := \lim_{\varepsilon\to 0} \mathrm{d}H_i, \quad \mathrm{d}\tilde{H}_{n-1} := \lim_{\varepsilon\to 0} \mathrm{d}(H_{n-1} + H_n), \quad \mathrm{d}\tilde{H}_n := \lim_{\varepsilon\to 0} \varepsilon \mathrm{d}H_n. \quad (84)$$

To compute these limits, we can treat the Hamiltonians up to addition of Casimirs. This allows us to use the Casimirs for regularizing parts of the Hamiltonains that are singular in ε parts of Hamiltonians.

Therefore all $=$ signs in the rest of the proof are intended as equal up to Casimirs. For $i < n-2$ we have

$$\tilde{H}_i := \lim_{\varepsilon\to 0} H_i = \sum_{j\neq i}^{n-2} \frac{\mathrm{Tr}\left(\tilde{A}^{(i)}\tilde{A}^{(j)}\right)}{u_i - u_j} + t_1 \frac{\mathrm{Tr}\left(\tilde{A}_1^{(n-1)}\tilde{A}^{(i)}\right)}{(u_i - w)^2} + \frac{\mathrm{Tr}\left(\tilde{A}_0^{(n-1)}\tilde{A}^{(i)}\right)}{u_i - w}, \tag{85}$$

for $i = n-1$ we have

$$\tilde{H}_{n-1} = \lim_{\varepsilon\to 0}(H_{n-1} + H_n) = \lim_{\varepsilon\to 0} \sum_{j<n-2} \mathrm{Tr}\,\tilde{A}^{(j)} \left(\frac{A^{(n-1)}}{w - u_j} + \frac{A^{(n)}}{w + \varepsilon t_1 - u_j}\right) =$$

$$= \sum_{j<n-1} \mathrm{Tr}\,\tilde{A}^{(j)} \left(\frac{\tilde{A}_0^{(n-1)}}{w - u_j} - t_1 \frac{\tilde{A}_1^{(n-1)}}{(w-u_j)^2}\right). \tag{86}$$

For $i = n$

$$\tilde{H}_n = \lim_{\varepsilon\to 0} \varepsilon H_n$$

Substituting coalescence expansions we get

$$\varepsilon H_n = \left[\sum_{j<n-2} \frac{\mathrm{Tr}\,\tilde{A}^{(j)} A_1^{(n-2)}}{w - u_j} + O(\varepsilon)\right] + \frac{\mathrm{Tr}\,A^{(n)}A^{(n-1)}}{t_1}. \tag{87}$$

The last term in (87) contains the $1/\varepsilon$ terms

$$\frac{\mathrm{Tr}\,A^{(n)}A^{(n-1)}}{\tilde{u}_n} = \frac{1}{\tilde{u}_n}\left(-\frac{1}{\varepsilon^2}\mathrm{Tr}\left(\tilde{A}_1^{(n-1)}\right)^2 + \frac{1}{\varepsilon}\mathrm{Tr}\left(\tilde{A}_1^{(n-1)}B_0 - C_0\tilde{A}_1^{(n-1)}\right) + \mathrm{Tr}\,(B_0C_0)\right) +$$
$$+ \frac{1}{\tilde{u}_n}\mathrm{Tr}\left(\tilde{A}_1^{(n-1)}B_1 - C_1\tilde{A}_1^{(n-1)}\right)$$

The $1/\varepsilon^2$ term is a Casimir of the Poisson structure associated with truncated loop algebra, so we may drop it.

Let us show that also the $1/\varepsilon$-term is a Casimir and that, after eliminating the Casimirs, $\epsilon H_n \to \tilde{H}_n + O(\varepsilon)$ where

$$\tilde{H}_n = \sum_{j<n-2} \frac{\mathrm{Tr}\,\tilde{A}^{(j)}A_1^{(n-2)}}{w - u_j} + \frac{1}{t_1}\frac{\mathrm{Tr}\left(\tilde{A}_0^{(n-1)}\right)^2}{2}. \tag{88}$$

To see this, let us remind that the Casimirs of the Poisson algebra in a Fuchsian case are $\mathrm{Tr}\,\left(A^{(i)}\right)^k$, so the function

$$\frac{1}{2}\mathrm{Tr}\left(A^{(n)} + A^{(n-1)}\right)^2$$

differs from the last term of (87)

$$\mathrm{Tr}\,A^{(n)}A^{(n-1)}.$$

by a Casimir. Since the Hamiltonians are defined up to the addition of a Casimir, we obtain

$$\varepsilon H_n = \sum_{j<n-2} \frac{\mathrm{Tr}\,\tilde{A}^{(j)}A_1^{(n-2)}}{w-u_j} + \frac{\mathrm{Tr}\left(A^{(n)} + A^{(n-1)}\right)^2}{2t_1} + O(\varepsilon) = \sum_{j<n-2} \frac{\mathrm{Tr}\,\tilde{A}^{(j)}A_1^{(n-2)}}{w-u_j} + \frac{1}{t_1}\frac{\mathrm{Tr}\left(\tilde{A}_0^{(n-1)}\right)^2}{2} + O(\varepsilon).$$

Taking the limit as $\varepsilon \to 0$ we obtain the Hamiltonian (88).

4.2. **Irregular singularities arising as confluence cascades.** In this section we consider an irregular singularity of arbitrary Poincaré rank r as the result of a confluence cascade of r simple poles $v_1, v_2 \dots v_r$ with some chosen simple pole u on the Riemann sphere. At the first step, we send v_1 to u and create second order pole as in the previous subsection. Then we do the same for v_2 - we collide it with the second order pole at u and create a pole of order 3. In such a way, we continue this procedure, so at the l-th step we collide simple pole v_l with the pole of order l at u to create a new pole of order $l+1$. Finally, a the final r-th step, we obtain a pole of order $r+1$, i.e. of Poincaré rank r. During this procedure we expect that the poles v_l's that disappear give rise to deformation parameters t_l's for the irregular isomonodromic problem[2]. Since the number of poles decreases during the confluence

[2]The confluence procedure is not symmetric in v_i, however different choices of the order of the coalescence cascade will lead to the action of the permutation group on t_l's, so we fix this ordering once for all.

procedure, these deformation parameters appear explicitly in the coefficients of the local expansion of the connection near the singularity u. In the sub-section 4.2.1, we prove Theorem 0.5 that tells us that this dependence is the one described in Corollary 3.1. Before attacking that proof, let us formalise the definition of confluence:

Definition 4.5. The limiting procedure described in the hypotheses pf Theorem 0.5 is called $r+1$ *confluence.*

Observe that as a result of the $1+1$ confluence in subsection 4.1 we obtain a connection of the form (12) with $r=2$. We can then apply the $1+2$ confluence to this and again obtain a connection of the form (12) with $r=3$ and so on. Therefore we can give the following recursive definition:

Definition 4.6. The procedure if applying Theorem 0.5 recursively r times is called *confluence cascade of* $r+1$ *simple poles* on the Riemann sphere.

As mentioned at the beginning of this section, the inductive hypothesis on the local form of the connection (12) is not restrictive. Indeed, we expect the local form of a connection with a pole of order r at u to be given by an element in the Takiff algebra co-adjoint orbit $\widehat{\mathcal{O}}^\star_r$ with some spectral parameter $z=\lambda-u$. However, if we want to keep the number of independent variables to be maximal, we need to introduce some extra variables t_i by hand in such a way that they can be treated as independent variables. In Corollary 3.1, we proved that the only way to do this is by taking precisely the form (12). Indeed, formula (11) corresponds to (55). Therefore, Theorem 0.5 implies the following result:

Theorem 4.7. *Assume that u is a singularity of Poincaré rank r obtained by the confluence of $r+1$ simple poles. Then the coefficients of the local expansion*

$$A(\lambda)\sim\sum_{i=0}^{r}\frac{B_i(t_1,\dots t_r)}{(\lambda-u)^{i+1}}+\dots,$$

take the form

$$B_i(t_1,t_2,\dots t_r)=\sum_{j=i}^{r}B^{[j]}\mathcal{M}^{(r)}_{i,j}(t_1,t_2,\dots t_r),$$

where

$$\mathcal{M}^{(r)}_{k,j}=\sum_{w(\alpha)=j}^{|\alpha|=k}\frac{k!}{\alpha_1!\alpha_2!\dots\alpha_r!}\left(\prod_{i=1}^{r}t_i^{\alpha_i}\right),\quad |\alpha|=\sum_{i=1}^{r}\alpha_i,\quad w(\alpha)=\sum_{i=1}^{r}i\cdot\alpha_i,\quad \mathcal{M}^{(r)}_{k>j}:=0,$$

and $B_i^{[j]}$'s hold the following Poisson relations

$$\left\{\left(B^{[k]}\right)_\alpha,\left(B^{[p]}\right)_\beta\right\}=\begin{cases}-\chi^\gamma_{\alpha\beta}\left(B^{[k+p]}\right)_\gamma & k+p\le r\\ 0 & k+p>r,\end{cases}\tag{89}$$

where $\chi^\gamma_{\alpha\beta}$ are the structure constants of the corresponding Lie algebra.

In this section we give an explicit description of the local coefficients B_i for the connection near irregular singularity of Poincaré rank r in terms of the generators of the Takiff co-algebra of degree r. It turns out that the $B_i(t_1,\dots t_{r-1})$ are the special linear combinations of the generators of the Takiff co-algebra $A_0,\dots A_r$ with coefficients in $\mathbb{C}[t_1,t_2,\dots t_r]$. Now we state a theorem we are going to prove in this subsection.

We want to underline here that the Poisson structure (89) give rise to the Takiff co-algebra Poisson structure on the coefficients of the local expansion, i.e

$$\{B_i(t_1,\dots t_r)_\alpha,B_j(t_1,\dots t_r)_\beta\}=-\sum_\gamma\chi^\gamma_{\alpha\beta}B_{i+j}(t_1,\dots t_r)_\gamma.\tag{90}$$

However, such Poisson structure on the coefficients of the local expansion is quite rough, the coefficients depend on the deformation parameters explicitly and it is important to understand this dependence when we do the deformation with respect to t_i's. On the other hand, Poisson structure (89) contains the information about explicit dependence on t_i's, moreover this Poisson structure compatible with more rough Poisson structure.

To motivate the constructions which appear in the statement of this theorem we introduce some preliminaries on the confluence procedure and algebraic structures which appear during coalescence before the proof.

4.2.1. *The algebra of the weighted monomials and associated polynomials.* The aim of this subsection is to collect some useful algebraic relations involving the coefficients $t_1,\dots,t_r$ that arise during the confluence procedure and describe the general elements of the Takiff co-algebra with respect to the Poisson automorphisms.

In order to prove Theorem 0.5, in the of a simple pole v_r with a pole w of Poincaré rank r we take the following expansion

$$v_r = w + P_r(t,\varepsilon) = w + \sum_{i=1}^{r} t_i\varepsilon^i.$$

The powers of the polynomial $P_r(t,\varepsilon)$ play a significant role since they appear in the following expansion

$$\frac{C}{\lambda - v_r} = \frac{C}{\lambda - w}\left(1 - \frac{P_r(t,\varepsilon)}{\lambda - w}\right)^{-1} = \frac{C}{\lambda - w}\left(1 + \frac{P_r(t,\varepsilon)}{\lambda - w} + \dots + \frac{P_r(t,\varepsilon)^j}{(\lambda - w)^j} + \dots + \frac{P_r(t,\varepsilon)^r}{(\lambda - w)^r}\right) + O(\varepsilon^{r+1}).$$

Each power of $P_r(t,\varepsilon)$ may be seen as a polynomial in ε with coefficients in $\mathbb{C}[t_1,t_2\dots t_r]$

$$P_r(t,\varepsilon)^r = t_1^r\varepsilon^r + O(\varepsilon^{r+1}).$$

Because the aim of the confluence is to create a pole of Poinceré rank $r+1$, we need the coefficeints $(\lambda - w)^{-r-2}$ to survive, therefore, we have to require C to be a Laurent polynomial in ε starting from the power $-r$. Taking this fact into account, it is important to understand how each power of $P_r(t,\varepsilon)$ expands via ε

$$P_r(t,\varepsilon)^i \mod \varepsilon^{r+1} = \sum_{j=i}^{r} \mathcal{M}^{(r)}_{i,k}(t_1,\dots t_r)\varepsilon^k = \mathcal{M}^{(r)}_{i,i}(t_1,\dots t_r)\varepsilon^i + \dots + \mathcal{M}^{(r)}_{i,r}(t_1,\dots t_r)\varepsilon^r,$$

where

$$\mathcal{M}^{(r)}_{i,k}(t_1,\dots t_r) = \frac{1}{k!}\frac{d}{d\varepsilon}P_r(t,\varepsilon)^i\Big|_{\varepsilon=0}.$$

The following simple Lemma calculates an explicit formula for $\mathcal{M}^{(r)}_{i,j}(t_1,\dots t_r)$ and gives some useful identites.

Lemma 4.8. *$\mathcal{M}^{(r)}_{i,k}$ is a homogeneous polynomial in $t_1\dots t_r$ of degree i for any k given by*

$$\mathcal{M}^{(r)}_{i,k}(t_1,\dots t_r) = \sum_{w(\alpha)=k}^{|\alpha|=i} \frac{i!}{\alpha_1!\alpha_2!\dots\alpha_r!}\left(\prod_{j=1}^{r} t_j^{\alpha_j}\right), \quad w(\alpha) = \sum_{j=1}^{r} j\alpha_j, \quad |\alpha| = \sum_{j=1}^{r}\alpha_j.$$

The polynomials $\mathcal{M}^{(r)}_{i,k}$ satisfy the following identities

$$\mathcal{M}^{(r+1)}_{i,k} = \mathcal{M}^{(r)}_{i,k}, \quad \forall k \le r, \qquad \mathcal{M}^{(r+1)}_{i,r+1} = t^i_{r+1} \tag{91}$$

and

$$\mathcal{M}^{(r)}_{j,k} = \sum_{p=0}^{k}\left[\mathcal{M}^{(r)}_{j-1,p}\cdot\mathcal{M}^{(r)}_{i,k-p} + \mathcal{M}^{(r)}_{j-i,k-p}\cdot\mathcal{M}^{(r)}_{i,p}\right], \quad \forall i \le j. \tag{92}$$

Note that the function $w(\alpha)$, that we call *weight,* can be given by the following formula

$$w\left(\prod_{i=1}^{n} t_i^{\alpha_i}\right) = (\alpha_1,\alpha_2,\dots\alpha_n)\begin{pmatrix}1\\2\\..\\n\end{pmatrix} = \sum_{k=1}^{n} k\alpha_k. \tag{93}$$

The weights may be seen as elements in the semi-group of homomorphism from the semi-group of monomials in the variables $t_1,\dots t_r$ to the $(\mathbb{Z}_{\ge 0},+)$, in fact:

$$w(\theta\cdot\eta) = w(\theta) + w(\eta).$$

Remark 4.9. *Instead of considering the polynomials $P_r(t_1,\dots t_r)$, we might consider formal power series*

$$P^{(\infty)}(\varepsilon,t) = \sum_{i=1}^{\infty} t_i\varepsilon^i,$$

and truncate all expansions at ε^{r+1}. The result will be the same, but such approach probably makes more clear the recursive nature of the confluence procedure. In similar way, the matrix $\mathcal{M}^{(r)}$ with *entries $\mathcal{M}^{(r)}_{i,k}$ given in (11) can be considered as as a sub-matrix of size $r\times r$ in the upper left corner of some infinite dimensional upper triangular "master" matrix $\mathcal{M}^{(\infty)}$ with entries given by*

$$\mathcal{M}^{(\infty)}_{j,r} = \frac{1}{r!}\frac{d^r}{d\varepsilon^r}P^{(\infty)}(\varepsilon,t)^j\Big|_{\varepsilon=0}.$$

Moreover introduced previously matrix $\mathcal{M}^{(r)}$ may be considered as the upper triangular matrix, with entries which are given in (11). Considering $\mathcal{M}$ in a such way is quite usefull, since it obvious that each $\mathcal{M}^{(r)}$ may be viewed

4.2.2. *Proof of the Theorems 4.7 and 0.5.* We use induction here to prove the theorem. Here we will start with the proof of the explicit dependence of the local expansion on t_i's and then we will handle the Poisson structure.

The statement of the theorem is true for the $r = 0,1$ (Fuchsian case and the pole of order 2), this was proven via examples we consider before. Now let the statement be true for the irregular singularity of Pincaré rank $r-1$. Adding another simple pole v_r, we consider the following connection

$$A = \sum_{i=0}^{r-1}\frac{B_i(t_1,t_2\dots t_{r-1})}{(\lambda-w)^{i+1}} + \frac{C}{\lambda-v_r}+\dots,\quad B_i = \sum_{j=i}^{r-1}B^{[j]}\mathcal{M}^{(r-1)}_{i,j},$$

where the dots denote regular terms in $(\lambda-w)$ and $(\lambda-v_1)$, with the following behaviour of v_r

$$v_r = w+\sum_{j=1}^{r}t_j\varepsilon^j,\quad C\simeq\sum_{j=-r}^{\infty}C^{[i]}\varepsilon^i.$$

Expanding A with respect to ε at r'th order we obtain

$$A = \frac{C^{[-r]}t_1^r}{(\lambda-w)^{r+1}}+\sum_{i=0}^{r-1}\frac{B_i+CP_r(t,\varepsilon)^i}{(\lambda-w)^{i+1}}+\dots.$$

Using the formula (91) the coefficients B_i expands via polynomials $\mathcal{M}^{(r)}_{i,j}$ which gives the following

$$B_i+CP_r(t,\varepsilon)^i = C^{[-r]}\mathcal{M}^{(r)}_{i,r}+\sum_{j=i}^{r-1}\left(B^{[j]}+\varepsilon^jC\right)\mathcal{M}^{(r)}_{i,j}.$$

Since the confluence procedure requires existence of the limit $\varepsilon\to 0$, the negative powers of ε should vanish, so we obtain expansions for the coefficients $A^{(r)}_j$ in the form

$$B^{[k]}\simeq-\sum_{m=-r}^{-(k+1)}\frac{C^{[m]}}{\varepsilon^{m+k}}+B^{[k,0]}+\sum_{l=1}^{\infty}B^{[k,l]}\varepsilon^l.$$

Using these expansions and taking the $\varepsilon\to 0$ limit we obtain

$$A = \sum_{i=1}^{r+1}\frac{\tilde{B}_i(t_1\dots t_r)}{(\lambda-u)^i}+\dots,$$

where $\tilde{B}_i$'s are given by (15), which finishes the proof of the first part of the theorem.

Finally we want to prove that the Poisson structure for the coefficient of the local expansion of the connection near irregular singularity, which arises after confluence procedure is a Poisson structure given by (89). Our IPA here is given by the formula (15).

Again we use the induction here. The statement is obvious in case when $r = 0$, and we have previously proved that it holds for $r = 1$. Using the previous results we consider the same coalescence

$$A = \sum_{i=0}^{r-1}\frac{B_i(t_1,t_2\dots t_{r-1})}{(\lambda-w)^{i+1}}+\frac{C}{\lambda-v_r}+\dots\sim\sum_{i=0}^{r}\frac{\tilde{B}_i}{(\lambda-w)^{i+1}}$$

where $\tilde{B}_i$ is given by (15).The expansions take the same form

$$C\sim\sum_{j=-r}^{\infty}C^{[i]}\varepsilon^i,\quad B^{[k]}\simeq-\sum_{m=-r}^{-(k+1)}\frac{C^{[m]}}{\varepsilon^{m+k}}+B^{[k,0]}+\sum_{l=1}^{\infty}B^{[k,l]}\varepsilon^l.$$

In order to get rid of the indices let us use the following notation

$$V = B^{[k]},\quad W = B^{[l]},\quad U = B^{[k+l]}.$$

In case if indices on the right hand sides are out of bound we assume that the values are zero. The Poisson relations are

$$\{V_\alpha, W_\beta\} = -\chi^\gamma_{\alpha\beta} U_\gamma, \quad \{C_\alpha, C_\beta\} = -\chi^\gamma_{\alpha\beta} C_\gamma, \quad \{C_\alpha, V_\beta\} = \{C_\alpha, W_\beta\} = \{C_\alpha, U_\beta\} = 0$$

and the expansions are

$$\begin{aligned}
C &= \frac{C^{[-r]}}{\varepsilon^r} + \frac{C^{[-r+1]}}{\varepsilon^{r-1}} + \cdots + \frac{C^{[-1]}}{\varepsilon} + C^{[0]} + \sum_{i=1}^{\infty} C^{[i]}\varepsilon^i \\
V &= -\frac{C^{[-r]}}{\varepsilon^{r-k}} - \frac{C^{[-r+1]}}{\varepsilon^{r-k-1}} - \cdots - \frac{C^{[-k+1]}}{\varepsilon} + V^{[0]} + \sum_{i=1}^{\infty} V^{[i]}\varepsilon^i \ldots \\
W &= -\frac{C^{[-r]}}{\varepsilon^{r-l}} - \frac{C^{[-r+1]}}{\varepsilon^{r-l-1}} - \cdots - \frac{C^{[-l+1]}}{\varepsilon} + W^{[0]} + \sum_{i=1}^{\infty} W^{[i]}\varepsilon^i \ldots \\
U &= -\frac{C^{[-r]}}{\varepsilon^{r-k-l}} - \frac{C^{[-r+1]}}{\varepsilon^{r-k-l-1}} - \cdots - \frac{C^{[-k-l+1]}}{\varepsilon} + U^{[0]} + \sum_{i=1}^{\infty} U^{[i]}\varepsilon^i.
\end{aligned}$$

Due to the confluence formula we want to prove that the following Poisson relation holds

$$\begin{aligned}
\left\{V^{[0]}_\alpha + C^{[-k]}_\alpha, W^{[0]}_\beta + C^{[-l]}_\beta\right\} = \left\{V^{[0]}_\alpha, W^{[0]}_\beta\right\} + \left\{V^{[0]}_\alpha, C^{[-l]}_\beta\right\} + \left\{C^{[-k]}_\alpha, W^{[0]}_\beta\right\} + \left\{C^{[-k]}_\alpha, C^{[-l]}_\beta\right\} = \\
= -\chi^\gamma_{\alpha\beta}\left(U^{[0]}_\gamma + C^{[-k-l]}_\gamma\right).
\end{aligned}$$

Taking the corresponding ε-terms of the expansions of the Poisson relations we get

$$\begin{array}{ll}
\operatorname*{Res}_{\varepsilon=0} \varepsilon^{0-1}\{V_\alpha, W_\beta\}: & \left\{V^{[0]}_\alpha, W^{[0]}_\beta\right\} - \underline{\sum_{i=1}^{r-l}\left\{V^{[i]}_\alpha, C^{[-i-l]}_\beta\right\}} - \underline{\sum_{i=1}^{r-k}\left\{C^{[-i-k]}_\alpha, W^{[i]}_\beta\right\}} = -\chi^\gamma_{\alpha\beta}U^{[0]}_\gamma, \\
\operatorname*{Res}_{\varepsilon=0} \varepsilon^{l-1}\{V_\alpha, C_\beta\}: & \left\{V^{[0]}_\alpha, C^{[-l]}_\beta\right\} + \underline{\sum_{i=1}^{r-l}\left\{V^{[i]}_\alpha, C^{[-i-l]}_\beta\right\}} - \underline{\sum_{i=1}^{r-k}\left\{C^{[-i-k]}_\alpha, C^{[i-l]}_\beta\right\}} = 0, \\
\operatorname*{Res}_{\varepsilon=0} \varepsilon^{k-1}\{C_\alpha, W_\beta\}: & \left\{C^{[-k]}_\alpha, W^{[0]}_\beta\right\} + \underline{\sum_{i=1}^{r-k}\left\{C^{[-i-k]}_\alpha, W^{[i]}_\beta\right\}} - \underline{\sum_{i=1}^{r-l}\left\{C^{[i-k]}_\alpha, C^{[-i-l]}_\beta\right\}} = 0 \\
\operatorname*{Res}_{\varepsilon=0} \varepsilon^{k+l}\frac{\{C_\alpha, C_\beta\}}{\varepsilon}: & \left\{C^{[-k]}_\alpha, C^{[-l]}_\beta\right\} + \underline{\sum_{i=1}^{r-k}\left\{C^{[-k-i]}_\alpha, C^{[i-l]}_\beta\right\}} + \underline{\sum_{i=1}^{r-l}\left\{C^{[-k+i]}_\alpha, C^{[-i-l]}_\beta\right\}} = \\
& = -\chi^\gamma_{\alpha\beta}C^{[-l-k]}_\gamma.
\end{array} \tag{94}$$

Finally, taking a sum of the relations in (94) we get the desired Poisson structure

$$\left\{V^{[0]}_\alpha, W^{[0]}_\beta\right\} + \left\{V^{[0]}_\alpha, C^{[-l]}_\beta\right\} + \left\{C^{[-k]}_\alpha, W^{[0]}_\beta\right\} + \left\{C^{[-k]}_\alpha, C^{[-l]}_\beta\right\} = -\chi^\gamma_{\alpha\beta}\left(U^{[0]}_\gamma + C^{[-l-k]}_\gamma\right)$$

4.3. **Confluent Hamiltonians.** As we saw before, in the case of the $1+1$ confluence, the confluent Hamiltonians can be obtained as the limits of some functions on a phase space - linear combinations of the initial Hamiltonians with coefficients depending on a small parameter ε. Moreover the procedure of taking this limit requires the introduction of some shifts by Casimirs, since the Hamiltonians are defined up to Casimir element of the Poisson algebra. Such Casimir normalisation may be exploited in the case of the confluence for the higher order poles, however, this procedure becomes very heavy. In this section, we calculate these limits using residue calculus. Let us start by explaining these limits of the Hamiltonians in the $1+1$ confluence procedure; we want to calculate limits in (84):

$$\mathrm{d}\tilde{H}_i := \lim_{\varepsilon\to 0}\mathrm{d}H_i, \quad \mathrm{d}\tilde{H}_{n-1} := \lim_{\varepsilon\to 0}\mathrm{d}(H_{n-1} + H_n), \quad \mathrm{d}\tilde{H}_n := \lim_{\varepsilon\to 0}\varepsilon\mathrm{d}H_n.$$

The Fuchsian Hamiltonians (27) can be written in the following form

$$H_{u_i} = \frac{1}{2}\oint_{\Gamma_{u_i}} \mathrm{Tr}\left(A^2\right)\mathrm{d}\lambda, \tag{95}$$

where Γ_{u_i} is a such contour that no singularities except u_i are inside of it. Since the matrix A admits finite limit as $\varepsilon \to 0$, the integrand has a finite limit. When $u_i \neq v_1, u$, we can always deform Γ_{u_i}

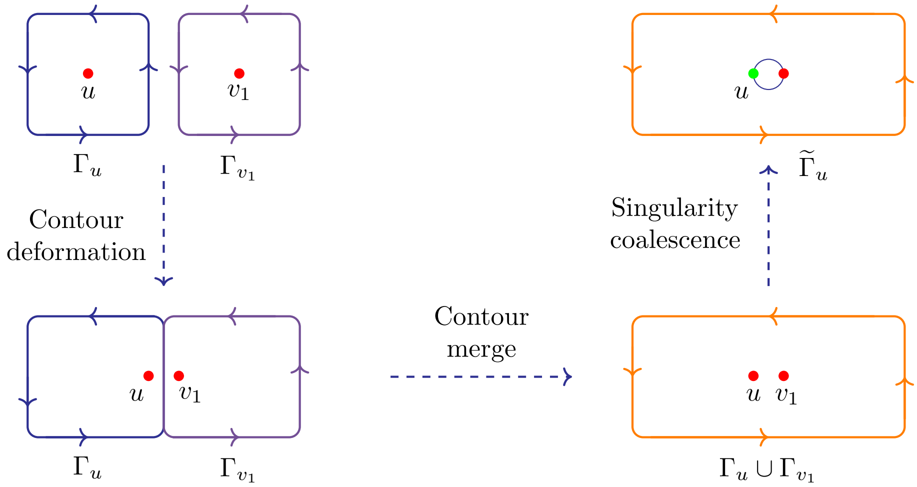

FIGURE 2. Poles and contours confluence procedure

in such a way that the coalescence of v_1 and u does not affect the contour of integration. This gives us opportunity to switch the limit and the integration operations, which gives the formula for the confluent Hamiltonian

$$\tilde{H}_{u_i} = \frac{1}{2} \oint\limits_{\Gamma_{u_i}} \mathrm{Tr}\, \left(\tilde{A}^2\right) \mathrm{d}\lambda, \qquad u_i \neq t_1, u. \tag{96}$$

Let us now deal with the limit of $H_w + H_1$. Because both contours Γ_w and Γ_1 will depend on ε, we cannot calculate the limits of H_w and H_1 separately. However, we can calculate the limit of the sum due to

$$H_u + H_{v_1} = \frac{1}{2} \oint\limits_{\Gamma_w} \mathrm{Tr}\, \left(A^2\right) \mathrm{d}\lambda + \frac{1}{2} \oint\limits_{\Gamma_{v_1}} \mathrm{Tr}\, \left(A^2\right) \mathrm{d}\lambda = \frac{1}{2} \oint\limits_{\Gamma_w \cup \Gamma_{v_1}} \mathrm{Tr}\, \left(A^2\right) \mathrm{d}\lambda,$$

where the last equality holds since the integrands in both integrals are the same and $\Gamma_u \cup \Gamma_{v_1}$ denotes the contour obtained by merging Γ_u and Γ_1 like in Fig. 2. Such contour may be deformed to the contour $\tilde{\Gamma}_u$, such that the coalescent singularities are located inside this contour and the confluence doesn't affect the contour itself. Using the same arguments as before we obtain that

$$\tilde{H}_u = \lim_{\varepsilon \to 0} \frac{1}{2} \oint\limits_{\Gamma_w \cup \Gamma_{v_1}} \mathrm{Tr}\, \left(A^2\right) \mathrm{d}\lambda, = \frac{1}{2} \oint\limits_{\Gamma_w} \mathrm{Tr}\, \left(\tilde{A}^2\right) \mathrm{d}\lambda. \tag{97}$$

In order to obtain $\tilde{H}_1$ we consider the following sum of Casimirs

$$\frac{1}{2} \oint\limits_{\Gamma_u} (\lambda - u) \mathrm{Tr}\, \left(A^2\right) \mathrm{d}\lambda + \frac{1}{2} \oint\limits_{\Gamma_{v_1}} (\lambda - v_1) \mathrm{Tr}\, \left(A^2\right) \mathrm{d}\lambda$$

which may be put to zero since the Hamiltonians are defined up to Casimirs. Expanding v_1 in ε we obtain

$$\frac{1}{2} \oint\limits_{\Gamma_u} (\lambda - u) \mathrm{Tr}\, \left(A^2\right) \mathrm{d}\lambda + \frac{1}{2} \oint\limits_{\Gamma_{v_1}} (\lambda - u) \mathrm{Tr}\, \left(A^2\right) \mathrm{d}\lambda - t_1 \varepsilon \frac{1}{2} \oint\limits_{\Gamma_{v_1}} \mathrm{Tr}\, \left(A^2\right) \mathrm{d}\lambda = \frac{1}{2} \oint\limits_{\Gamma_u \cup \Gamma_{v_1}} (\lambda - u) \mathrm{Tr}\, \left(A^2\right) \mathrm{d}\lambda - t_1 \varepsilon H_1 = 0.$$

The relation written above finally gives us

$$\tilde{H}_1 = \frac{1}{2\tilde{t}_1} \oint\limits_{\Gamma_u} (\lambda - u) \mathrm{Tr}\, \left(\tilde{A}^2\right) \mathrm{d}\lambda = \operatorname*{Res}_{\lambda = u} \left[\frac{(\lambda - u)}{t_1} \mathrm{Tr}\, \left(\frac{\tilde{A}^2}{2} \right) \right]. \tag{98}$$

Let us now add one more simple pole to w using the confluence, by a similar computation as above, we obtain the following isomonodromic Hamiltonians

$$H_{u_i} = \operatorname*{Res}_{\lambda=u_i} \mathrm{Tr}\left(\frac{A^2}{2}\right), \quad H_u = \operatorname*{Res}_{\lambda=u}\mathrm{Tr}\left(\frac{A^2}{2}\right)$$

$$H_1 = \operatorname*{Res}_{\lambda=u}\left(\left[\frac{(\lambda-u)}{t_1} - \frac{t_2(\lambda-u)^2}{t_1^3}\right]\mathrm{Tr}\left(\frac{A^2}{2}\right)\right), \quad H_2 = \operatorname*{Res}_{\lambda=u}\left[\frac{(\lambda-u)^2}{t_1^2}\mathrm{Tr}\left(\frac{A^2}{2}\right)\right].$$

The form of the Hamiltonians for t_1 and t_2 looks quite bizarre, but we may obtain them by solving the following linear system

$$\mathcal{M}^{(2)}\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} t_1 & t_2 \\ 0 & t_1^2 \end{pmatrix}\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad S_i = \frac{1}{2}\oint_{\Gamma_u}(\lambda-u)^i \mathrm{Tr}\, A^2 \mathrm{d}\lambda.$$

Here $\mathcal{M}^{(2)}$ is a matrix which entries were already introduced in (11).

We now prove Theorem 0.6.

Theorem. *Let u be a pole of a connection A with Poincaré rank r, which is the result of confluence of r simple poles with the simple pole u. Then the confluent Hamiltonians $H_1, \dots, H_r$ which correspond to the times $t_1, \dots t_r$ are defined as follows:*

$$\begin{pmatrix} H_1 \\ H_2 \\ \dots \\ H_r \end{pmatrix} = \left(\mathcal{M}^{(r)}\right)^{-1}\begin{pmatrix} S_1^{(u)} \\ S_2^{(u)} \\ \dots \\ S_r^{(u)} \end{pmatrix}, \tag{99}$$

where

$$S_k^{(u)} = \frac{1}{2}\oint_{\Gamma_u}(\lambda-u)^k \mathrm{Tr}\, A^2 \mathrm{d}\lambda \tag{100}$$

are spectral invariants of order i in u and the matrix $\mathcal{M}^{(r)}$ is given by (11). The Hamiltonian H_u corresponding to the time u is instead given by the standard formula

$$H_{u_i} = \frac{1}{2}\operatorname*{res}_{\lambda=u_i} \mathrm{Tr}\, A(\lambda)^2.$$

Proof. To prove this theorem we again use induction. We showed that it holds for $r = 2$ and it is trivial in case when $r = 1$. Now we want to prove that if the statement of the theorem holds for rank r it is also true for rank $r + 1$. The confluence expansion

$$v_{r+1} = u + P_{r+1}(t, \varepsilon) = u + \sum_{i=1}^{r+1} \varepsilon^i t_i$$

sends the extended symplectic form

$$\omega = \mathrm{d}t_1 \wedge \mathrm{d}H_1 + \dots + \mathrm{d}t_r \wedge \mathrm{d}H_r + \mathrm{d}u \wedge \mathrm{d}H_u + \mathrm{d}v_{r+1} \wedge \mathrm{d}H_{r+1} + \dots$$

where the last set of dots corresponds to the terms that are not changed in the confluence procedure, to

$$\mathrm{d}t_1 \wedge \mathrm{d}(H_1 + \varepsilon H_{r+1}) + \dots + \mathrm{d}t_r \wedge \mathrm{d}(H_r + \varepsilon^r H_{r+1}) + \mathrm{d}t_{r+1} \wedge \mathrm{d}(\varepsilon^{r+1} H_{r+1}) + \mathrm{d}u \wedge \mathrm{d}(H_u + H_{r+1}) + \dots =$$

that must be equal to

$$\mathrm{d}t_1 \wedge \mathrm{d}\tilde{H}_1 \dots \mathrm{d}t_r \wedge \mathrm{d}\tilde{H}_r + \mathrm{d}t_{r+1} \wedge \mathrm{d}\tilde{H}_{r+1} + \mathrm{d}u \wedge \mathrm{d}\tilde{H}_u + \dots.$$

Therefore, we must find the limits

$$\tilde{H}_u = \lim_{\varepsilon\to 0}\left(H_u + H_{r+1}\right), \quad \tilde{H}_k = \lim_{\varepsilon\to 0}\left(H_k + \varepsilon^k H_{r+1}\right), \quad k = 1 \dots r, \quad \tilde{H}_{r+1} = \lim_{\varepsilon\to 0}\varepsilon^{r+1} H_{r+1}$$

The first limit is quite simple and may be obtained via the union of contours which we already described before. To find the other limits, let us consider the relation

$$\oint_{\Gamma_u}(\lambda-u)^i \mathrm{Tr}\,\frac{A^2}{2}\mathrm{d}\lambda + \oint_{\Gamma_{r+1}}(\lambda-v_{r+1})^i \mathrm{Tr}\,\frac{A^2}{2}\mathrm{d}\lambda = S_i^{(u)} \mod (\text{Casimirs})$$

expanding v_{n+1} we obtain

$$\oint\limits_{\Gamma_u\cup\Gamma_{r+1}} (\lambda-u)^i \mathrm{Tr}\,\frac{A^2}{2}\mathrm{d}\lambda - \oint\limits_{\Gamma_{r+1}} \phi(\lambda)\mathrm{Tr}\,\frac{A^2}{2} = S_i^{(u)} \mod (\text{Casimirs})\,,$$

where $\phi(\lambda)$ is a holomorphic function inside Γ_{r+1} which is given by

$$\phi(\lambda) = (\lambda-u)^i - (\lambda-u-P_{r+1}(t,\varepsilon))^i.$$

Since $\phi(\lambda)$ has no zeros at v_{r+1} we have

$$\oint\limits_{\Gamma_{r+1}} \phi(\lambda)\mathrm{Tr}\,\frac{A^2}{2} = \phi(u+P_{r+1}(t,\varepsilon)) \oint\limits_{\Gamma_{r+1}} \mathrm{Tr}\,\frac{A^2}{2} = P_{r+1}(t,\varepsilon)^i \oint\limits_{\Gamma_{r+1}} \mathrm{Tr}\,\frac{A^2}{2} = P_{r+1}(t,\varepsilon)^i H_{r+1}.$$

Finally, we obtain the following identity:

$$\oint\limits_{\Gamma_u\cup\Gamma_{r+1}} (\lambda-u)^i \mathrm{Tr}\,\frac{A^2}{2}\mathrm{d}\lambda - P_{r+1}(t,\varepsilon)^i H_{r+1} = S_i^{(u)} \mod (\text{Casimirs})\,. \tag{101}$$

In the case $i=r+1$, $S_{r+1}^{(u)}$ is a Casimir due to the formula (50), therefore we have

$$\oint\limits_{\Gamma_u\cup\Gamma_{r+1}} (\lambda-u)^{r+1} \mathrm{Tr}\,\frac{A^2}{2}\mathrm{d}\lambda = P_{r+1}(t,\varepsilon)^{r+1} H_{r+1} \mod (\text{Casimirs})\,.$$

The left hand side of this identity has a finite limit when ε goes to 0. Indeed, since the contour contains both u and v_{r+1} the confluence procedure doesn't affect it and the only dependence in ε is in A. According to Theorem 0.5, A has a finite limit, the same has $\mathrm{Tr}\,A^2$, so we have that

$$\lim_{\varepsilon\to 0} \oint\limits_{\Gamma_u\cup\Gamma_{r+1}} (\lambda-u)^{r+1} \mathrm{Tr}\,\frac{A^2}{2}\mathrm{d}\lambda = \oint\limits_{\Gamma_u\cup\Gamma_{r+1}} (\lambda-u)^{r+1} \lim_{\varepsilon\to 0}\mathrm{Tr}\,\frac{A^2}{2}\mathrm{d}\lambda = \oint\limits_{\Gamma_u} (\lambda-u)^{r+1} \mathrm{Tr}\,\frac{\tilde{A}^2}{2}\mathrm{d}\lambda = \tilde{S}_{r+1}^{(u)},$$

where $\tilde{S}_{r+1}^{(u)}$ is a spectral invariant of the confluent system with connection $\tilde{A}$. Since after the confluence, the order of pole increases to $r+2$, such spectral invariant is not a Casimir for the confluent system. This means, that the limit of $P_{r+1}(t,\varepsilon)^{r+1}H_{r+1}$ up to Casimirs exists and equals to $\tilde{S}_{r+1}^{(u)}$. On the other hand we have

$$P_{r+1}(t,\varepsilon)^{r+1}H_{r+1} = \left(t_1^{r+1}\varepsilon^{r+1} + O(\varepsilon^{r+2})\right) H_{r+1},$$

and since the limit exists up to Casimirs we get that

$$H_{r+1} \mod (\text{Casimirs}) = \frac{\tilde{S}_{r+1}^{(u)}}{\varepsilon^{r+1}} + \sum_{i=-r}^{\infty} H_{r+1}^{[i]}\varepsilon^i,$$

so in principle H_{r+1} may have terms of lower order than $1/\varepsilon^{r+1}$, but these terms has to be Casimirs. Considering relations (101) for $i=1\dots r+1$ as a linear system we obtain

$$\begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \\ \tilde{S}_3 \\ \dots \\ \tilde{S}_{r+1} \end{pmatrix} - \mathcal{M}^{(r+1)} \begin{pmatrix} \varepsilon \\ \varepsilon^2 \\ \varepsilon^3 \\ \dots \\ \varepsilon^{r+1} \end{pmatrix} H_{r+1} = \begin{pmatrix} S_1 \\ S_2 \\ \dots \\ S_r \\ 0 \end{pmatrix} \mod (\text{Casimirs}) \tag{102}$$

where

$$\tilde{S}_i = \oint\limits_{\Gamma_u\cup\Gamma_{r+1}} (\lambda-u)^i \mathrm{Tr}\,\frac{A^2}{2}\mathrm{d}\lambda, \quad S_i = \oint\limits_{\Gamma_u} (\lambda-u)^i \mathrm{Tr}\,\frac{A^2}{2}\mathrm{d}\lambda$$

Note that the contours in the definition of $\tilde{S}_i$ are not affected by the confluence procedure. Using the same arguments as above, we compute the limits of these integrals, which are

$$\lim_{\varepsilon\to 0}\tilde{S}_i = \tilde{S}_i^{(u)} = \oint\limits_{\Gamma_u} (\lambda-u)^i \mathrm{Tr}\,\frac{\tilde{A}^2}{2}\mathrm{d}\lambda$$

where $\tilde{S}_i^{(u)}$ denote the spectral invariants of the confluent system with connection $\tilde{A}$.

The crucial point here is that the matrix $\mathcal{M}^{(r+1)}$ contains $\mathcal{M}^{(r)}$ as $r+1, r+1$ minor, i.e.

$$\mathcal{M}^{(r+1)} = \begin{pmatrix} & & & & t_{r+1} \\ & & & & \vdots \\ & \mathcal{M}^{(r)} & & & \\ & & & & \vdots \\ 0 & 0 & \dots & 0 & t_1^{r+1} \end{pmatrix}$$

Now let us consider the following matrix

$$\mathcal{C} = \begin{pmatrix} & & & & 0 \\ & & & & \vdots \\ & \left(\mathcal{M}^{(r)}\right)^{-1} & & & \\ & & & & \vdots \\ & & & & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

and let us act via $\mathcal{C}$ on the equation (102) from the left. Then we get

$$\mathcal{C}\tilde{S} = \begin{pmatrix} & & & & \vdots \\ & \mathbb{I}_r & & & \\ & & & & \vdots \\ 0 & 0 & \dots & 0 & t_1^{r+1} \end{pmatrix} \begin{pmatrix} \varepsilon H_{r+1} \\ \varepsilon^2 H_{r+1} \\ \dots \\ \varepsilon^n H_{r+1} \\ \varepsilon^{n+1} H_{r+1} \end{pmatrix} + \begin{pmatrix} H_1 \\ H_2 \\ \dots \\ H_r \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} & & & & \vdots \\ & \mathbb{I}_r & & & \\ & & & & \vdots \\ 0 & 0 & \dots & 0 & t_1^{r+1} \end{pmatrix} \begin{pmatrix} \varepsilon H_{r+1} + H_1 \\ \varepsilon^2 H_{r+1} + H_2 \\ \dots \\ \varepsilon^r H_{r+1} + H_r \\ \varepsilon^{r+1} H_{r+1} \end{pmatrix} = \mathcal{C}\mathcal{M}^{(r+1)} \begin{pmatrix} \varepsilon H_{r+1} + H_1 \\ \varepsilon^2 H_{r+1} + H_2 \\ \dots \\ \varepsilon^r H_{r+1} + H_r \\ \varepsilon^{n+1} H_{r+1} \end{pmatrix}.$$

In this way we have arranged the entries of equation (102) in such a way that the left hand side has a nice limit as ε goes to zero (the confluence of points is inside the contour of integration for $\tilde{S}_i$). On the right hand side we have the functions whose limits we want to find. Finally, multiplying by $\mathcal{C}^{-1}$ from the left we obtain

$$\mathcal{M}^{(r+1)} \begin{pmatrix} \tilde{H}_1 \\ \tilde{H}_2 \\ \dots \\ \tilde{H}_r \\ \tilde{H}_{r+1} \end{pmatrix} = \mathcal{M}^{(r+1)} \lim_{\varepsilon\to 0} \begin{pmatrix} \varepsilon H_{r+1} + H_1 \\ \varepsilon^2 H_{r+1} + H_2 \\ \dots \\ \varepsilon^r H_{r+1} + H_n \\ \varepsilon^{r+1} H_{r+1} \end{pmatrix} = \lim_{\varepsilon\to 0} \begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \\ \tilde{S}_3 \\ \dots \\ \tilde{S}_{r+1} \end{pmatrix} = \begin{pmatrix} \tilde{S}_1 \\ \tilde{S}_2 \\ \tilde{S}_3 \\ \dots \\ \tilde{S}_{r+1} \end{pmatrix}$$

which finishes the proof. □

Remark 4.10. *The matrix $\mathcal{M}^{(r)}$ is automatically upper triangular matrix, so it is quite easy to solve such a system for any reasonable n.*

In general we may consider Hamiltonians which are related to poles locations as a spectral invariant $S_0^{(u)}$. Then it is easy to extend formula from (99) as follows

$$\mathcal{N}^{(r)} \begin{pmatrix} H_0 \\ H_1 \\ \dots \\ H_r \end{pmatrix} = \begin{pmatrix} S_0^{(u)} \\ S_1^{(u)} \\ \dots \\ S_r^{(u)} \end{pmatrix}, \quad \mathcal{N}^{(r)} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \mathcal{M}^{(r)} & \\ 0 & & & \end{pmatrix}$$

4.4. **Examples of Hamiltonians.** We want to demonstrate obtained Hamiltonians in the previous section on a couple of examples. In order to see all the features of obtained Hamiltonians we consider connection with 3 simple poles at $0, 1$ and ∞, and one irregular singularity at some point u. The minimal example is

$$A(\lambda) = \frac{A^{(0)}}{\lambda} + \frac{A^{(1)}}{\lambda - 1} + \frac{A_0^{(u)}}{\lambda - u} + \frac{t_1 A_1^{(u)}}{(\lambda - u)^2}. \tag{103}$$

Explicit formulas for Hamiltonians are

$$H_u = \operatorname{Tr}\left[A_0^{(u)}\left(\frac{A^{(0)}}{u}+\frac{A^{(1)}}{u-1}\right)-t_1A_1^{(u)}\left(\frac{A^{(1)}}{(u-1)^2}+\frac{A^{(0)}}{u^2}\right)\right] \tag{104}$$

$$H_1 = \operatorname{Tr}\left[A_1^{(u)}\left(\frac{A^{(0)}}{u}+\frac{A^{(1)}}{u-1}\right)+\frac{A_0^{(u)}A_0^{(u)}}{2t_1}\right] \tag{105}$$

In lifted Darboux coordinates, Hamiltonians take form

$$\begin{aligned} H_u = \operatorname{Tr}\left(Q_0^{(u)}P_0^{(u)}+Q_1^{(u)}P_1^{(u)}\right)\left(\frac{Q^{(0)}P^{(0)}}{u}+\frac{Q^{(1)}P^{(1)}}{u-1}\right)- \\ -t_1Q_0^{(u)}P_1^{(u)}\left(\frac{Q^{(1)}P^{(1)}}{(u-1)^2}-\frac{Q^{(0)}P^{(0)}}{u^2}\right) \end{aligned} \tag{106}$$

$$H_1 = \operatorname{Tr}\frac{Q^{(0)}P^{(0)}Q_0^{(u)}P_1^{(u)}}{u}+\frac{Q^{(1)}P^{(1)}Q_0^{(u)}P_1^{(u)}}{u-1}+\frac{\left(Q_0^{(u)}P_0^{(u)}+Q_1^{(u)}P_1^{(u)}\right)^2}{2t_1}. \tag{107}$$

Hamiltonians obviously Poisson commute, moreover we may check that cross-derivative w.r.t. u and t_1 is zero, i.e.

$$\frac{\partial}{\partial t_1}H_u-\frac{\partial}{\partial u}H_1=0,$$

which tells us that τ-function

$$\begin{aligned} \mathrm{d}\ln\tau = \operatorname{Tr}\left(A_0^{(u)}\left(\frac{A^{(0)}}{u}+\frac{A^{(1)}}{u-1}\right)-t_1A_1^{(u)}\left(\frac{A^{(1)}}{(u-1)^2}-\frac{A^{(0)}}{u^2}\right)\right)\mathrm{d}u+ \\ +\operatorname{Tr}\left(A_1^{(u)}\left(\frac{A^{(0)}}{u}+\frac{A^{(1)}}{u-1}\right)+\frac{A_0^{(u)}A_0^{(u)}}{2t_1}\right)\mathrm{d}t_1 \end{aligned}$$

is defined correctly. If we go further and consider pole of Poincaré rank 2 at u connection takes form

$$A(\lambda)=\frac{A^{(0)}}{\lambda}+\frac{A^{(1)}}{\lambda-1}+\frac{A_0^{(u)}}{\lambda-u}+\frac{t_1A_1^{(u)}+t_2A_2^{(u)}}{(\lambda-u)^2}+\frac{t_1^2A_2^{(u)}}{(\lambda-u)^3}. \tag{108}$$

Hamiltonians then write as

$$\begin{aligned} H_u = \operatorname{Tr}\Bigg[A_0^{(u)}\left(\frac{A^{(0)}}{u}+\frac{A^{(1)}}{u-1}\right)-t_1A_1^{(u)}\left(\frac{A^{(0)}}{u^2}+\frac{A^{(1)}}{(u-1)^2}\right)+ \\ +A_2^{(u)}\left(\frac{t_1^2A^{(0)}}{u^3}+\frac{t_1^2A^{(1)}}{(u-1)^3}-\frac{t_2A^{(0)}}{u^2}-\frac{t_2A^{(1)}}{(u-1)^2}+\right)\Bigg] \end{aligned}$$

$$H_1 = \operatorname{Tr}\left[A_1^{(u)}\left(\frac{A^{(0)}}{u}+\frac{A^{(1)}}{u-1}\right)-t_1A_2^{(u)}\left(\frac{A^{(0)}}{u^2}+\frac{A^{(1)}}{(u-1)^2}\right)+\frac{A_0^{(u)}A_0^{(u)}}{2t_1}-t_2\frac{A_0^{(u)}A_1^{(u)}}{t_1^2}-t_2^2\frac{A_0^{(u)}A_2^{(u)}}{t_1^3}\right]$$

$$H_2 = \operatorname{Tr}\left[A_2^{(u)}\left(\frac{A^{(0)}}{u}+\frac{A^{(1)}}{u-1}\right)+\frac{A_0^{(u)}A_1^{(u)}}{t_1}+t_2\frac{A_0^{(u)}A_2^{(u)}}{t_1^2}\right].$$

As in the previous example cross-derivatives are zeros

$$\frac{\partial}{\partial t_1}H_u-\frac{\partial}{\partial u}H_1=\frac{\partial}{\partial t_2}H_u-\frac{\partial}{\partial u}H_2=\frac{\partial}{\partial t_1}H_2-\frac{\partial}{\partial t_2}H_1=0,$$

so the τ-function is defined correctly.

The first example corresponds to the confluence of the Garnier system – isomonodromic deformation of the connection on the 5 punctured sphere. Reduction of this example for $\mathfrak{sl}_2$ has to give equation form the list introduced in [44, 45] and corresponds to so-called fourth-order Painlevé equations. The second example has to be more complicated – Hamiltonian reduction then gives Hamiltonian system with 3 degrees of freedom, which corresponds to the higher order Painlevé equations.

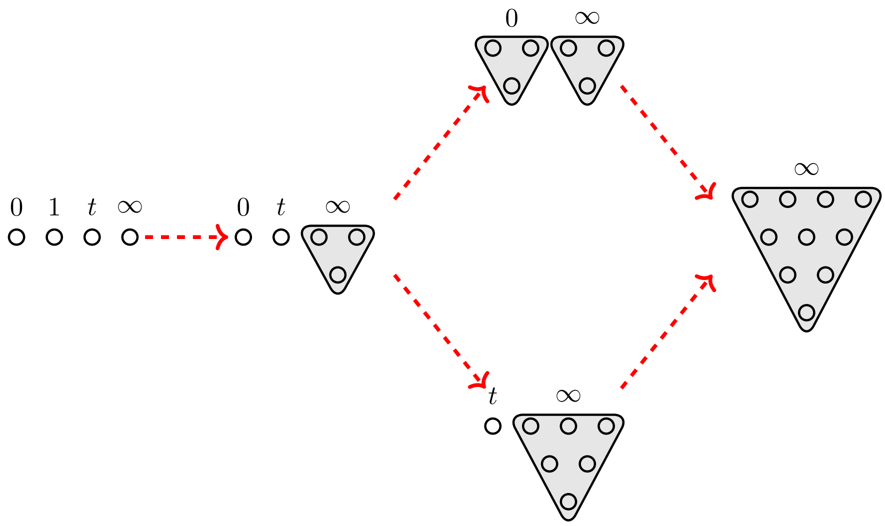

FIGURE 3. Confluence scheme for Painlevé equations. Each triangle at this diagram corresponds to the Takiff algebra Darboux coordinates which was introduced at the Fig. 1.

4.5. **Confluence of the higher order poles.** The confluence of two poles w_1 and w_2 of Poincaré rank r_1 and r_2 respectively can be treated a the confluence of r_1+r_2+2 simple poles. Indeed, we have seen at the beginning of section 5.2 that the generic connection with a Poincaré rank r singularity can be obtained as confluence of $r+1$ simple poles. Therefore, a connection with two poles w_1 and w_2 of Poincaré rank r_1 and r_2 respectively is obtained by coalescing r_1+1 and r_2+1 simple poles. To confluence w_1 with w_2 is therefore the same as coalescing r_1+r_2+2 simple poles.

5. THE NON-RAMIFIED PAINLEVÉ EQUATIONS

5.1. **General scheme.** The confluence diagram of the Darboux parametrisations in the case of rank 2 non-ramified connections with 4 points is given at Fig. 3. We start by illustrating the general scheme of reduction which works for any rank.

The Hamiltonians of the isomonodromic problem with irregular singularities of the Poincaré rank r_i at point u_i, allow additional symmetries with respect to the inner action (choice of the spectral invariants at each singularity) and outer action (gauge group action). Using the Darboux parametrisation of the co-adjoint for the $\mathfrak{sl}_2$–Takiff algebras, we automatically fix spectral invariants, i.e. reduce with respect to the inner action. The only symmetry which still needs to be taken into account is the gauge freedom which leads to the fully reduced phase space. In all the examples of this section, we write down Darboux coordinates with partly resolved gauge group moment map by diagonalizing the leading terms at one of the irregular singularities. We do it automatically by writing diagonal gauge intermediate Darboux coordinates for the Takiff algebra co-adjoint orbit. Therefore, the number of the intermediate coordinates in all examples is 4 and not 6 (because we have eliminated 2 coordinates by diagonalisation). Such coordinates are in correspondence with the Darboux coordinates which were used in [32]. In order to reduce to the smallest dimension of the system (namely 2), we have to always reduce with respect to the stabilizer action, which in all examples equivalent to the additional action of the Cartan torus, since we consider only unramified situation, which corresponds to integer Katz index. The ramified situation will be considered in the next work of the first author.

5.2. **Painlevé V.** The Isomonodromic problem takes form

$$\begin{aligned} \tfrac{d}{d\lambda}\Psi &= \left(\tfrac{A^{(0)}}{\lambda}+\tfrac{A^{(t)}}{\lambda-t}+B_1\right)\Psi \\ \tfrac{d}{dt}\Psi &= -\tfrac{A^{(t)}}{\lambda-t}\Psi. \end{aligned} \tag{109}$$

Deformation equations are

$$\frac{d}{dt}A^{(0)}=\frac{1}{t}[A^{(t)},A^{(0)}],\quad \frac{d}{dt}A^{(t)}=\frac{1}{t}[A^{(0)},A^{(t)}]+[B_1,A^{(t)}],\quad \frac{d}{dt}B_1=0. \tag{110}$$

The Poisson algebra is given by

$$\left\{A^{(i)} \underset{,}{\otimes} A^{(i)}\right\} = [\Pi, \mathbb{I} \otimes A^{(i)}], \quad \left\{A^{(0)} \underset{,}{\otimes} A^{(t)}\right\} = \left\{A^{(0)} \underset{,}{\otimes} B_1\right\} = \left\{A^{(t)} \underset{,}{\otimes} B_1\right\} = \left\{B_1 \underset{,}{\otimes} B_1\right\} = 0 \quad (111)$$

Isomonodromic Hamiltonian writes as

$$H_V = \underset{\lambda=t}{\operatorname{res}} \operatorname{Tr}\left(\frac{A(\lambda)^2}{2}\right) = \operatorname{Tr}\left(A^{(t)} B_1 + \frac{1}{t} A^{(t)} A^{(0)}\right). \quad (112)$$

In the $\mathfrak{sl}_2$ case Darboux parametrisation of the elements of the coadjoint orbit takes form

$$A^{(0)} = \begin{pmatrix} p_0 q_0 - \theta_0 & -(p_0 q_0 - 2\theta_0) p_0 \\ q_0 & -p_0 q_0 + \theta_0 \end{pmatrix}, \quad A^{(t)} = \begin{pmatrix} p_t q_t - \theta_t & -(p_t q_t - 2\theta_t) p_t \\ q_t & -p_t q_t + \theta_t \end{pmatrix},$$

with the symplectic form

$$\omega = \mathrm{d}p_t \wedge \mathrm{d}q_t + \mathrm{d}p_0 \wedge \mathrm{d}q_0.$$

Using gauge freedom, we set a constant matrix B to be diagonal

$$B_1 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

In such parametrisation Hamiltonian takes form

$$H = \underset{\lambda=t}{\operatorname{res}} \operatorname{Tr}\left(\frac{A(\lambda)^2}{2}\right) = 4k(p_t q_t - \theta_t) - \frac{2}{t}(q_t q_0 (p_t - p_0)^2 - 2(q_0 \theta_t - q_t \theta_0)(p_t - p_0) - 2\theta_0 \theta_t) =$$

$$= 4k(p_t q_t - \theta_t) - \frac{2 q_t q_0}{t}\left(p_t - p_0 - \frac{\theta_t}{q_t} + \frac{\theta_0}{q_0}\right)^2 + \frac{2}{t}\left(\theta_t^2 \frac{q_0}{q_t} + \theta_0^2 \frac{q_t}{q_0}\right).$$

This Hamiltonian is invariant under the following rescaling

$$p_i \to p_i \alpha, \quad q_i \to \frac{q_i}{\alpha}$$

which is the same as the gauge $SL(2)$ action via diagonal matrix. The moment map is

$$q_0 p_0 + q_t p_t.$$

The change of the coordinates

$$I = q_0 p_0 + q_t p_t, \quad \phi = \ln(q_0), \quad u = -\frac{q_t}{q_0}, \quad v = p_t q_0,$$

is a canonical transformation. Resolving it with respect to the q's and p's we obtain

$$q_0 = e^{\varphi}, \quad q_t = -e^{\varphi} u, \quad p_0 = e^{-\varphi}(I + uv) \quad p_t = e^{-\varphi} v,$$

and the symplectic form goes to

$$\omega = \mathrm{d}p_t \wedge \mathrm{d}q_t + \mathrm{d}p_0 \wedge \mathrm{d}q_0 = \mathrm{d}I \wedge \mathrm{d}\varphi + \mathrm{d}v \wedge \mathrm{d}u.$$

The Hamiltonian in these coordinates writes as

$$H = -4k(uv + \theta_t) + 2\frac{u}{t}\left(v - I - uv + \frac{\theta_t}{u} + \theta_0\right)^2 - \frac{2}{t}\left(\theta_t^2 \frac{1}{u} + \theta_0^2 u\right)$$

and it is obvious that I and φ are the part of the action-angle variables, so we may decrease degrees of freedom by 1 and consider the following Hamiltonian system

$$H = -4k(uv + \theta_t) + 2\frac{u}{t}\left(v - a - uv + \frac{\theta_t}{u} + \theta_0\right)^2 - \frac{2}{t}\left(\theta_t^2 \frac{1}{u} + \theta_0^2 u\right), \quad \omega = \mathrm{d}v \wedge \mathrm{d}u, \quad a = \mathrm{const}.$$

The equations of motion take form

$$\dot{u} = \frac{\partial H}{\partial v} = -4ku + \frac{4u}{t}(1 - u)\left(v - a - uv + \frac{\theta_t}{u} + \theta_0\right)$$

$$\dot{v} = -\frac{\partial H}{\partial u} = 4kv - \frac{2}{t}\left((v - uv - a + \frac{\theta_t}{u} + \theta_0)^2 - 2u(v - uv - a + \frac{\theta_t}{u} + \theta_0)\left(v + \frac{\theta_t}{u^2}\right) + \frac{\theta_t^2}{u^2} - \theta_0^2\right).$$

Writing second order ODE for u we obtain

$$\frac{d^2 u}{dt^2} = \left(\frac{1}{u-1} + \frac{1}{2u}\right)\left(\frac{du}{dt}\right)^2 - \frac{1}{t}\frac{du}{dt} + 8\theta_0 \frac{(u-1)^2}{t^2}\left(u - \left(\frac{\theta_t}{\theta_0}\right)^2 \frac{1}{u}\right) + 4k(4(a - \theta_0 - \theta_t) - 1)\frac{u}{t} - 8k^2 \frac{u(u+1)}{u-1}$$

which is the Gambier's form of the Painlevé V equations and the constants are given by

$$\theta_0 = \frac{\alpha}{8}, \quad \theta_t^2 = -\frac{\alpha\beta}{64}, \quad k^2 = -\frac{\delta}{8}, \quad 4k(4(a-\theta_0-\theta_t)-1) = \gamma.$$

The following canonical transformation

$$u = \frac{x}{x-1}, \quad v = -((x-1)y + a - 2\theta_0)(x-1), \quad \mathrm{d}v \wedge \mathrm{d}u = \mathrm{d}y \wedge \mathrm{d}x,$$

send H to the following form

$$tH = 2x(x-1)y^2 + 4(ktx(x-1) + x(\theta_t - \theta_0) - \theta_t)y + 4(xkt(a-2\theta_0) - \theta_t(kt - \theta_0))$$

which was introduced in [42]. The example of the Painlevé V equation as a system written on the co-adjoint orbits of the Takiff algebra was recently studied by [41] in more details.

5.3. **Painlevé IV.** The connectionis

$$A(\lambda) = \frac{A^{(t)}}{\lambda - t} - B_1 - B_2\lambda \tag{113}$$

and the deformation one-form is

$$\Omega = -\frac{A^{(t)}}{\lambda - t}\mathrm{d}t. \tag{114}$$

Deformation equations are

$$\dot{A}^{(t)} = [A^{(t)}, B_1 + B_2 t], \quad \dot{B}_1 = [B_2, A^{(t)}], \quad \dot{B}_2 = 0.$$

The Poisson structure is

$$\{A^{(t)} \underset{,}{\otimes} A^{(t)}\} = [\Pi, 1 \otimes C], \quad \{B_1 \underset{,}{\otimes} B_1\} = [\Pi, 1 \otimes B_2], \quad \{B_1 \underset{,}{\otimes} B_2\} = \{B_2 \underset{,}{\otimes} B_2\} = 0.$$

Hamiltonian writes as

$$H = \operatorname*{res}_{\lambda=t} \mathrm{Tr}\, \frac{A^2}{2} = -\mathrm{Tr}\left(A^{(t)}B_1 + tA^{(t)}B_2\right). \tag{115}$$

Since B_3 is a constant of motion the same holds for the transition matrix to the eigenbasis to B_3. This allows us to consider the gauge, which is equal to this transition matrix without changing the Poisson structure of $A^{(t)}$. In the case of $\mathfrak{sl}_2$ we have

$$A^{(t)} = \begin{pmatrix} p_t q_t - \theta_t & -(p_t q_t - 2\theta_t)\, p_t \\ q_t & -p_t q_t + \theta_t \end{pmatrix}, \quad -B_2\lambda - B_1 = -\lambda\theta_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \theta_2 & -2\theta_3 q_3 \\ p_3 & -\theta_2 \end{pmatrix} \tag{116}$$

The Hamiltonian writes as

$$H = (p_t q_t - 2\theta_t)\, p_t p_3 - 2\,(p_t q_t - \theta_t)\,(t\theta_3 + \theta_2) + 2\theta_3 q_3 q_t. \tag{117}$$

Since B_3 is a diagonal matrix (has no Jordan blocks) the stabilizer is the Cartan torus of SL_2,i.e.

$$S = \begin{pmatrix} h & 0 \\ 0 & 1/h \end{pmatrix}$$

The additional action of the stabilizer of B_3 leads to the following action on the reduced phase space

$$q_t \to \frac{q_t}{h^2}, \quad p_t \to h^2 p_t, \quad q_3 \to h^2 q_3, \quad p_3 \to \frac{p_3}{h^2},$$

which is Hamiltonian with the following moment map

$$I = q_3 p_3 - q_t p_t.$$

Using the symplectic change of coordinates

$$q_3 = e^{\phi}, \quad q_t = e^{-\phi}u, \quad p_3 = e^{-\phi}(I + uv), \quad p_t = e^{\phi}v, \quad \mathrm{d}p_3 \wedge \mathrm{d}q_3 + \mathrm{d}p_t \wedge \mathrm{d}q_t = \mathrm{d}I \wedge \mathrm{d}\phi + \mathrm{d}v \wedge \mathrm{d}u \tag{118}$$

and fixing the level set of moment map $I = I_0 = \text{const}$ we reduce to the system with one degree of freedom

$$H = (uv - 2\theta_t)\, v\, (uv + I_0) - 2\,(uv - \theta_t)\,(t\theta_3 + \theta_2) + 2\theta_3 u. \tag{119}$$

Finally, using the change of variables

$$u = x(xy - I_0), \quad v = \frac{1}{x}, \quad \mathrm{d}v \wedge \mathrm{d}u = \mathrm{d}y \wedge \mathrm{d}x$$

sends Hamiltonian to the Okamoto form of P_{IV}

$$H = 2yx^2 + \left(\theta_3 y^2 + (-2t\theta_3 - 2\,\theta_2)\, y - 2I_0\right) x + (-I_0\theta_3 - 2\theta_3\theta_t)\, y \tag{120}$$

Taking

$$\theta_3 = -1, \quad \theta_2 = 0, \quad I_0 = -\theta_0, \quad \theta_t = -\frac{1}{2}(\theta_\infty + \theta_0)$$

we obtain the P_{IV} Hamiltonian

$$H = 2yx^2 - (y^2 + 2ty + 2\theta_0)x + \theta_\infty y.$$

5.4. **Painlevé III.** The connection takes form

$$A = \frac{B_0}{\lambda} + t\frac{B_1}{\lambda^2} + C \tag{121}$$

with deformation one-form

$$\Omega = -\frac{B_1}{\lambda}\mathrm{d}t. \tag{122}$$

Poisson algebra is

$$\{C \underset{,}{\otimes} C\} = \{C \underset{,}{\otimes} B_{0,1}\} = \{B_1 \underset{,}{\otimes} B_1\} = 0, \quad \{B_0 \underset{,}{\otimes} B_0\} = [\Pi, 1 \otimes B_0], \quad \{B_0 \underset{,}{\otimes} B_1\} = [\Pi, 1 \otimes B_1] \tag{123}$$

Hamiltonian is given by

$$H = \frac{1}{2}\operatorname*{res}_{\lambda=0} \mathrm{Tr}\, \frac{\lambda}{t} A^2 = \mathrm{Tr}\left(CB_1 + \frac{B_0^2}{2t}\right). \tag{124}$$

In the case of $\mathfrak{sl}_2$, choosing the gauge such that C is diagonal we have the following Darboux parametrisation

$$B_0 = \begin{pmatrix} p_1q_1 - p_2q_2 + \theta_1 & -p_1q_1^2 + (2q_1q_2 + 1)\,p_2 - 2\theta_1 q_1 \\ p_1 & -p_1q_1 + p_2q_2 - \theta_1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 2q_1q_2\theta_2 + \theta_2 & -2\theta_2\,(q_1q_2 + 1)\,q_1 \\ 2\theta_2 q_2 & -2q_1q_2\theta_2 - \theta_2 \end{pmatrix}, \quad C = \begin{pmatrix} \theta_3 & 0 \\ 0 & -\theta_3 \end{pmatrix}. \tag{125}$$

Hamiltonian writes as

$$tH = p_2^2q_2^2 + 4t\theta_2\theta_3 q_1q_2 - 2\theta_1 p_2q_2 + p_1p_2. \tag{126}$$

The stabilizer of C (SL_2 torus) action gives integral of motion

$$I = q_1p_1 - q_2p_2.$$

In order to reduce the degrees of freedom we use change of variables

$$q_1 = e^{\phi}, \quad q_2 = -e^{-\phi}u, \quad p_1 = e^{-\phi}(I + uv), \quad p_2 = -e^{\phi}v, \quad \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2 = \mathrm{d}I \wedge \mathrm{d}\phi + \mathrm{d}v \wedge \mathrm{d}u$$

which leads to the following Hamiltonian

$$tH = v^2u^2 - \left(v^2 + 2\theta_1 v + 4t\theta_2\theta_3\right)u - I_0 v \tag{127}$$

where I_0 is a level set of the first integral I. Obtained Hamiltonian corresponds to the Painlevé III equation of type D_6 after some choice of constants. To obtain degenerations to D_7 and D_8 we have to consider nilpotent orbits.

5.5. **Painlevé II. Jimbo-Miwa.** The connection takes form

$$A(\lambda) = \frac{B_3}{\lambda^4} + \frac{B_2}{\lambda^3} + \frac{B_1 + tB_3}{\lambda^2} + \frac{B_0}{\lambda}. \tag{128}$$

Deformation one form is

$$\Omega = -\frac{B_3}{\lambda}\mathrm{d}t \tag{129}$$

Deformation equations are

$$\frac{d}{dt}B_3 = [B_2, B_3], \quad \frac{d}{dt}B_2 = [B_1, B_3], \quad \frac{d}{dt}B_1 = [B_0 - tB_2, B_3], \quad \frac{d}{dt}B_0 = 0. \tag{130}$$

The Poisson structure is given by

$$\left\{B_i \underset{,}{\otimes} B_j\right\} = [\Pi, \mathbb{I} \otimes B_{i+j-1}] \tag{131}$$

Hamiltonian takes form

$$H = \operatorname*{res}_{\lambda=0} \mathrm{Tr}\, \lambda^3 \frac{A(\lambda)^2}{2} = \mathrm{Tr}\left(\frac{B_1^2}{2} + B_0B_2 + tB_1B_3\right), \tag{132}$$

here we drop $\operatorname{Tr} B_3^2$ part since it is a Casimir. Since we assume that for Painlevé II there is no singularity at ∞, the value of the gauge group moment map should be put to zero, i.e.

$$B_0 = 0. \tag{133}$$

Such reduction, has to be viewed as a Hamiltonian reduction written on the co-adjoint orbit of the Takiff algebra $\hat{\mathfrak{g}}_3$, so we have to change not only Hamiltonian, but also the Poisson structure. However, usually the second Painlevé equation isomonodromic problem writes in chart where the only singularity is at ∞. The connection takes form

$$A(\lambda) = B_3\lambda^2 + B_2\lambda + B_3 t + B_1. \tag{134}$$

Here we already resolve the gauge group moment map, by setting residue at ∞ to be zero. The deformation one-form then my be written as

$$\Omega = (B_3\lambda + B_2)\,\mathrm{d}t.$$

The deformation equations are

$$\dot{B}_3 = 0, \quad \dot{B}_2 = [B_3, B_1], \quad \dot{B}_1 = t[B_2, B_3] + [B_2, B_1].$$

The deformation equations are Hamiltonian, with Hamiltonian written as

$$H = \operatorname*{res}_{\lambda=0} \operatorname{Tr} \frac{A^2}{2\lambda} = \operatorname{Tr}\left(\frac{B_1^2}{2} + tB_1B_3\right). \tag{135}$$

To obtain Painlevé II equation we consider the $\mathfrak{sl}_2$ case. Darboux parametrisation is given by

$$B_3 = \begin{pmatrix} \theta_4 & 0 \\ 0 & -\theta_4 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \theta_3 & -2\theta_4 q_3 \\ 2\theta_4 q_4 & -\theta_3 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 2\theta_4 q_3 q_4 + \theta_2 & -\theta_4 q_3^3 q_4^2 + (\theta_3 - 4\theta_4)\, q_4 q_3^2 - \theta_4 q_3 + p_4 \\ -\theta_4 q_3^2 {q_4}^3 + (\theta_3 - 4\theta_4)\, q_4^2 q_3 + (2\theta_3 - \theta_4)\, q_4 + p_3 & -2\theta_4 q_3 q_4 - \theta_2 \end{pmatrix}.$$

The Hamiltonian takes form

$$\begin{aligned} H = -(2\theta_4 q_3 q_4 + \theta_2)^2 - 2t(2\theta_4 q_3 q_4 + \theta_2)\theta_4 - ((\theta_3 - 4\theta_4) q_4 q_3^2 - \theta_4 q_3 + p_4 \\ - \theta_4 q_3^3 q_4^2)((\theta_3 - 4\theta_4) q_4^2 q_3 + (2\theta_3 - \theta_4) q_4 + p_3 - \theta_4 q_3^2 q_4^3). \end{aligned} \tag{136}$$

The action of stabilizer of B_4 gives us the moment map

$$I = p_3 q_3 - p_4 q_4$$

which gives us the following change of variables $(p_3, p_4, q_3, q_4) \to (I, v, \phi, u)$

$$p_3 = e^{-\phi}(I + uv), \quad p_4 = e^{\phi} v, \quad q_3 = e^{\phi}, \quad q_4 = e^{-\phi} u, \quad \mathrm{d}p_3 \wedge \mathrm{d}q_3 + \mathrm{d}p_4 \wedge \mathrm{d}q_4 = \mathrm{d}v \wedge \mathrm{d}u + \mathrm{d}I \wedge \mathrm{d}\phi.$$

The Hamiltonian then writes as

$$H = -(2\theta_4 u + \theta_2)^2 - 2t(2\theta_4 u + \theta_2)\theta_4 - (v - \theta_4 u^2 + (\theta_3 - 4\theta_4)u - \theta_4)(uv + (\theta_3 - 4\theta_4)u^2 + (2\theta_3 - \theta_4)u + I - \theta_4 u^3)$$

The change of variable

$$v = w + \frac{1}{2u}(2\theta_4 u^3 - 2u^2\theta_3 + 8\theta_4 u^2 - 2u\theta_3 + 2\theta_4 u - I), \quad w = -\frac{p}{q}, \quad u = -\frac{q^2}{2}$$

gives us

$$H = \frac{p^2}{2} - \theta_4^2 q^4 + \left(2\theta_4^2 t + 2\theta_2\theta_4 - \frac{\theta_3^2}{2}\right) q^2 - \frac{I^2}{2q^2}$$

which is the Hamiltonian of P_{34} equation, which is equivalent to Painlevé II in case when $I = 0$.

Remark 5.1. *The isomonodromic problem with connection matrix* (134) *corresponds to the non-autonomous version of the famous Nahm top which first appeared in* [53]. *Treating the variable t as a constant, we obtain an integrable system with Lax matrix* (134) *which is gauge equivalent to the Lax matrix for the Nahm equation. This gives the explicit Hamiltonian formulation of the Nahm equation in terms of the coadjoint orbits of the Takiff algebras. This should coincide with the Hamiltonian formalism for the Nahm equations introduced in* [58].

6. Quantisation

In this section we give a general formula for the confluent KZ equations with singularities of arbitrary Poincaré rank in any dimension. Moreover, we use the lifted Darboux coordinates in order to generalise an observation by Reshetikhin that the quasiclassical solution of the standard KZ equations (i.e. with simple poles) is expressed via the isomonodromic τ-function [57]. Here we propose an easy proof which is valid for any Poincaré configuration of the singularities on a Riemann sphere. Firstly we review a Reshetikhin approach for the quantum isomonodromic problems and then produce our proof which is based on the generalisation of an observation by Malgrange [49].

Throughout this section we work with the canonical quantisation of the linear Poisson brackets that prescribes the standard correspondence principle

$$\{f,g\} \longrightarrow [\hat{f},\hat{g}] = i\hbar\widehat{\{f,g\}}, \tag{137}$$

where the symbol $\hat{}$ denotes the quantum operator, i.e. $\hat{f}$ is the quantum operator corresponding to the classical function f. In the case of a semi-simple Lie algebra, $\hbar$ can be written via the dual Coxeter number and the level. Here we ignore this fact and we focus on the $\mathfrak{gl}_m$ case, so we simply replace the Poisson bracket by the commutator.

More accurately, one can speak about the so–called *Rees deformation* that assigns to a filtered vector space $R = \cup_i R_i$ a canonical deformation of its associated graded algebra $\mathrm{gr}(R)$ over the affine line $\mathbb{A}_1$ considered as the spectrum $\mathrm{Spec}(\mathbb{C}[\hbar])$ of the polynomials $\mathbb{C}[\hbar]$. The fiber at the point $\hbar$ is isomorphic to R if $\hbar \neq 0$ and to $\mathrm{gr}(R)$ for $\hbar = 0$. The corresponding $\mathbb{C}[\hbar]$–module here is the direct sum $\oplus_i R_i$ on which $\hbar$ acts by mapping each R_i to R_{i+1} [23]. In our case the Rees construction gives a one-parameter family of algebras $U_\hbar(\mathfrak{g})$, with the associated graded algebra $U_0(\mathfrak{g})$ being the symmetric algebra $S(\mathfrak{g})$. The $\hbar$ deformation re-scales the bracket by $\hbar$, so that the $\hbar$ linear terms define the standard Poisson bracket on $S(\mathfrak{g})$.

6.1. Finite-dimensional representation.

In this sub-section, we recall the basic ideas at the basis of Reshetikhin's approach to quantum isomonodromic problems for Fuchsian systems and the adapt it to the irregular case. We fix $\hbar = 1$ for simplicity.

In the case of the Fuchsian system we are dealing with canonical quantisation of the direct product of the co-algebras $\mathfrak{g}^\star$. The quantisation functor sends the functions on the phase space of the classical system to the differential operators which act on some Hilbert space in a way that (137) holds. In principle, a choice of finite dimensional representation may be seen as a choice of the special subspace of the Hilbert space of functions on which the algebra of quantum operators acts. However, we may avoid such complicated construction of finite dimensional representation when the classical Poisson algebra is given by a linear Poisson bracket. Indeed, for $\mathfrak{g}^\star$ the standard Lie–Poisson bracket endows the space of functions with the structure of a Lie algebra so that the structure constants of this Poisson algebra are identified with the structure constants of the Lie algebra $\mathfrak{g}$.

In general, the quantisation procedure for the phase space of the Fuchsian system may be viewed as a map from

$$\underbrace{\mathfrak{g}^\star \times \mathfrak{g}^\star \times \cdots \times \mathfrak{g}^\star}_{n}$$

to the differential operators which act on the tensor product of Hilbert spaces $\mathcal{H}_i$:

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n.$$

However, the isomonodromic nature of the Hamiltonian systems we consider gives additional information which may be used to define a quantum problem in a uniform way. Following [39], we quantise the connection that becomes the generating function for the quantum Hamiltonians. Considering the connection as a matrix whose entries are functions on the $\mathfrak{g}^\star \times \mathfrak{g}^\star \times \cdots \times \mathfrak{g}^\star$, we obtain a following quantisation for the Fuchsian case:

$$\hat{A}(\lambda) = \sum_{i=1}^{n} \frac{\hat{A}^{(i)}}{\lambda - u_i}, \quad \hat{A}^{(i)} = \sum_\alpha e_\alpha^{(0)} \otimes e_\alpha^{(i)}, \quad e_\alpha^{(i)} = 1 \otimes \cdots \otimes \underset{i}{e_\alpha} \otimes \cdots \otimes 1,$$

where $e_\alpha^{(i)}$, $i = 1, \dots, n$ is a basis of representation we choose for a quantisation and the first $e_\alpha^{(0)}$ corresponds to auxiliary space $\mathcal{H}_0$ given by the connection. The Schlesinger Hamiltonians then transform to

$$\hat{H}_i = \sum_{j\neq i} \frac{\mathrm{Tr}^{(0)}(\hat{A}^{(i)}\hat{A}^{(j)})}{u_i - u_j}, \tag{138}$$

where $\mathrm{Tr}^{(0)}$ is a trace in the auxiliary space $\mathcal{H}_0$. The quantum Schlesinger Hamiltonians $\hat{H}_i$ are the solutions for the classical Yang-Baxter equations and may be written as

$$\hat{H}_i = \sum_{j\neq i} \frac{r_{ij}}{u_i - u_j}, \tag{139}$$

where r_{ij} is a solution of the classical Yang-Baxter equation

$$[r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{ik}, r_{jk}] = 0.$$

The corresponding set of Schrödinger equations are called Knizhnik–Zamolodchikov equations and take form

$$\nabla_i \Psi = \left(\frac{\partial}{\partial u_i} - \sum_{j\neq i} \frac{r_{ij}}{u_i - u_j} \right) \Psi = 0.$$

Moreover, the Knizhnik–Zamolodchikov operators commute, i.e.

$$[\nabla_i, \nabla_j] = 0 \quad \Longleftrightarrow \quad \frac{\partial}{\partial u_i}\hat{H}_j = \frac{\partial}{\partial u_j}\hat{H}_i, \quad [\hat{H}_i, \hat{H}_j] = 0.$$

Reproducing the same scheme for the Takiff co-algebras, we obtain the quantisation map that acts by replacing the co-algebra with the Lie algebra

$$\hat{\mathfrak{g}}^\star_{r_1} \times \hat{\mathfrak{g}}^\star_{r_2} \times \dots \hat{\mathfrak{g}}^\star_{r_n} \times \hat{\mathfrak{g}}^\star_{r_\infty} \quad \longrightarrow \quad \hat{\mathfrak{g}}_{r_1} \otimes \dots \hat{\mathfrak{g}}_{r_n} \otimes \hat{\mathfrak{g}}_{r_\infty}. \tag{140}$$

The quantum connection then takes the form

$$\hat{A}(\lambda) = \sum_i^n \left(\sum_{j=0}^{r_i} \frac{\hat{B}^{(i)}_j \left(t^{(i)}_1, t^{(i)}_2 \dots t^{(i)}_{r_i}\right)}{(\lambda - u_i)^{j+1}} \right),$$

where $\hat{B}^{(i)}$'s are given by

$$\hat{B}^{(i)}_j(t^{(i)}_1, \dots t^{(i)}_{r_i}) = \sum_{k=j}^{r} \hat{A}^{(i)}_k \mathcal{M}^{(r_i)}_{j,k}(t^{(i)}_1, t^{(i)}_2 \dots t^{(i)}_{r_i}), \quad \hat{A}^{(i)}_k = \sum_\alpha e^{(0)}_\alpha \otimes e^{(i)}_\alpha \otimes z^k_i, \quad e^{(i)}_\alpha = 1\otimes \dots \otimes \underset{i}{e_\alpha} \otimes \dots \otimes 1.$$

The Hamiltonians which correspond to the position of poles are given as in the Fuchsian case, i.e.

$$\hat{H}_{u_i} = \frac{1}{2} \operatorname*{res}_{\lambda=u_i} \mathrm{Tr}_0 \hat{A}(\lambda)^2,$$

where Tr_0 is the tarce in the 0-th space, so we now have to choose a quantum ordering, for example lexigraphical ordering. The irregular Hamiltonians have to be calculated according to the Theorem 0.6 at each irregular singularity changing Tr by Tr_0. Again we will choose a quantum ordering. Thus, we obtain that the irregular Hamiltonians are given by

$$\mathcal{M}^{(r_i)} \begin{pmatrix} \hat{H}^{(i)}_1 \\ \hat{H}^{(i)}_2 \\ \dots \\ \hat{H}^{(i)}_{r_i} \end{pmatrix} = \begin{pmatrix} \hat{S}^{(u_i)}_1 \\ \hat{S}^{(u_i)}_2 \\ \dots \\ \hat{S}^{(u_i)}_{r_i} \end{pmatrix}, \quad \hat{S}^{(u_i)}_k = \frac{1}{2} \oint_{\Gamma_{u_i}} (\lambda - u_i)^k \mathrm{Tr}_0 \hat{A}^2 \mathrm{d}\lambda$$

at the point u_i with the Poincare rank r_i. To prove Theorem 0.9 we need to show that the confluent KZ gives a quantum integrable system, namely that the differential operators defined in (18), (19)

$$\nabla_{u_j} := \frac{\partial|}{\partial u_j} - \widehat{H}_{u_j}, \quad j = 1, \dots, n$$

$$\nabla^{(i)}_k := \frac{\partial|}{\partial t^{(i)}_k} - \widehat{H}^{(i)}_k, \quad i = 1, \dots, n, \infty, \quad k = 1, \dots, r_i$$

commute. This is a simple consequence of the fact that in the quantisation process the derivatives remain commutative, i.e. for example $[\frac{\partial|}{\partial u_j}, \frac{\partial|}{\partial t^{(i)}_k}] = 0$, and that the quantum Hamiltonians are linear combinations of the quantum Gaudin spectral invariants $\hat{S}^{(u_i)}_k$, $k = 0, \dots, r_i$, which commute as proved in [51].

We have to mention that for the Fuchsian times the isomonodromic Hamiltonian depends on each phase space $\mathfrak{g}_{r_i}$ linearly – which means that it may be written as

$$\hat{H}_{u_i} \in \hat{\mathfrak{g}}_{r_1} \otimes \dots \hat{\mathfrak{g}}_{r_n} \otimes \hat{\mathfrak{g}}_{r_\infty} \subset U\left(\hat{\mathfrak{g}}_{r_1} \oplus \dots \oplus \hat{\mathfrak{g}}_{r_\infty}\right).$$

In the case of irregular times Hamiltonians becomes more complicated – there are quadratic terms which contains elements from the same space and in general we have that

$$\hat{H}_k^{(i)} \in U(\hat{\mathfrak{g}}_{r_1}) \otimes \dots U(\hat{\mathfrak{g}}_{r_n}) \otimes U(\hat{\mathfrak{g}}_{r_\infty}).$$

The problem of the explicit form of the Hamiltonians introduced in this paper has to deal with the $U(\hat{g}_{r_i})$ representation theory, which is rather complicated. In order to avoid this representational theoretic problems, we write down the quantum Hamiltonians for the irregular isomonodromic deformations using intermediate Darboux coordinates. For the classical examples of the Painlevé equations, we provide invariant subspaces for these Hamiltonians. These subspaces give finite dimensional representations for the Hamiltonians which are the quantum reduction of the irregular Hamiltonians introduced in this section.

6.2. **Intermediate Quantum Hamiltonians for Painlevé equations.** In this subsection we write quantum Hamiltonians for the Painlevé equations written in Darboux coordinates before the reduction with respect to the gauge group action. In the case of Painlevé VI the gauge group action is not taken into account. For other cases, we partly resolve the gauge group action by diagonalizing the leading term, but we do not finish reduction by ignoring additional Cartan torus action. Since that in the Painlevé VI example the number of coordinates for $\mathfrak{sl}_2$ for 4 punctures is 6 while in other examples the number of intermidiate coordinates is 4 (2 moments + 2 positions). Since we are dealing with Darboux coordinates, the quantisation process becomes fairly straightforward. In this subsection, we show that for each of the non-ramified Painlevé differential equations, there is a choice of quantisation such that the quantum operator acts nicely on the space of homogeneous polynomials. More precisely, we show that the quantum Hamiltonians invariant subspaces are the homogeneous polynomials in several variables (3 for Painlevé VI and 2 for others) with fixed degree. In this section we keep $\hbar$ explicit as that makes it clearer how to extract semi-classical limits.

6.2.1. *Painlevé VI.* For the $\mathfrak{sl}_2$ Fuchsian system we have that the Hamiltonians in the intermediate coordinates take form

$$\begin{aligned} H_i = \sum_{j\neq i} \frac{h_{ij}}{u_i - u_j}, \quad h_{ij} = 2p_i p_j q_i q_j - p_i^2 q_i q_j - p_j^2 q_i q_j - \\ 2\theta_j p_i q_i - 2\theta_i p_j q_j + 2\theta_i p_i q_j + 2\theta_j p_j q_i + 2\theta_i\theta_j \end{aligned} \tag{141}$$

The quantisation problem is not trivial since we have to choose the ordering for the mixed parts of Hamiltonian. There are three standard ways of the ordering, which are given by

$$:\widehat{p_i}\widehat{q_j}:=:\widehat{q_j}\widehat{p_i}:=\widehat{q_j}\widehat{p_i}+\delta_{ij}\varepsilon^{(i)}, \quad \varepsilon^{(i)} = \begin{cases} 0, & \text{left} \\ i\hbar, & \text{right} \\ \frac{i\hbar}{2}, & \text{Weyl} \end{cases}$$

This leads to the following forms of Hamiltonians

$$\hat{H}_i = \sum_{j\neq i} \frac{\hat{h}_{ij}}{u_i - u_j}$$

where

$$\begin{aligned} \hat{h}_{ij} = 2\hat{q}_i\hat{q}_j\hat{p}_i\hat{p}_j - \hat{q}_i\hat{q}_j\hat{p}_i^2 - \hat{q}_i\hat{q}_j\hat{p}_j^2 - 2(\theta_j - \varepsilon^{(j)})\hat{q}_i\hat{p}_i - 2(\theta_i - \varepsilon^{(i)})\hat{q}_j\hat{p}_j + 2(\theta_i - \varepsilon^{(i)})\hat{q}_j\hat{p}_i + \\ +2(\theta_j - \varepsilon^{(j)})\hat{q}_i\hat{p}_j + 2(\theta_i - \varepsilon^{(i)})(\theta_j - \varepsilon^{(j)}) \end{aligned}.$$

Here we see that different ordering leads to the different shifts of the local monodromies $\theta_i \to \theta_i - \varepsilon^{(i)}$. Since that we may consider left ordering without loss of generality, first of all because different ordering shifts the constants and also this shift is of the order $\hbar$.

The most remarkable property is that Hamiltonians $\hat{H}_i$ leave invariant the space of homogeneous polynomials of q_i with fixed degree in the following choice of the quantisation $\hat{p}_i = -i\hbar\frac{\partial}{\partial x_i}\cdot$, $\hat{q}_i = x_i\cdot$. So we may look for a solutions for the set of quantum Schrodinger equations

$$i\hbar\partial_{u_i}\Psi = \hat{H}_i\Psi \tag{142}$$

in the following form

$$\Psi^{(n)} = \sum_{|\alpha|=n} w_\alpha(u_1,..,u_i..,u_m)\prod_{i=1}^{m} x_i^{\alpha_k}, \quad |\alpha| = \sum_{i=1}^{m}\alpha_i$$

which will lead to the non-autonomous linear system of ODE for the $w_\alpha(\mathbf{u})$-s. The resulting equations in fact are KZ equations, since the equations for w_α inherit singularities of $\hat{H}_i$. Let's consider vector

$$W^{(n)} = \begin{pmatrix} w_{\alpha_1} \\ w_{\alpha_2} \\ .. \\ .. \\ w_{\alpha_N} \end{pmatrix}$$

where α_i are the distinct partitions of n with height m (with zero entries). Then $W^{(n)}$ satisfies the equations

$$i\hbar\frac{\partial}{\partial u_i}W^{(n)} - \sum_{j\neq i}\frac{M_n^{(i,j)}}{u_i-u_j}W^{(n)} = 0$$

where $M_n^{(i,j)}$ is action of $\hat{h}_{ij}$ on homogeneous polynomials of degree n. These equations are Knizhnik–Zamolodchikov–type equations.

In the case of the Painlevé VI equation we deal with 4-punctured sphere $0, 1, t, \infty$. The quantum Hamiltonian then writes as

$$\begin{aligned}\hat{H} =& \frac{1}{t}\left(2\hat{q}_1\hat{q}_2\hat{p}_1\hat{p}_2 - \hat{q}_1\hat{q}_2\hat{p}_1^2 - \hat{q}_1\hat{q}_2\hat{p}_2^2 - 2\theta_2\hat{q}_1\hat{p}_1 - 2\theta_1\hat{q}_2\hat{p}_2 + 2\theta_1\hat{q}_2\hat{p}_1 + +2\theta_2\hat{q}_1\hat{p}_2 + 2\theta_1\theta_2\right) + \\ &\frac{1}{t-1}\left(2\hat{q}_1\hat{q}_3\hat{p}_1\hat{p}_3 - \hat{q}_1\hat{q}_3\hat{p}_1^2 - \hat{q}_1\hat{q}_3\hat{p}_3^2 - 2\theta_3\hat{q}_1\hat{p}_1 - 2\theta_1\hat{q}_3\hat{p}_3 + 2\theta_1\hat{q}_3\hat{p}_1 + +2\theta_3\hat{q}_1\hat{p}_3 + 2\theta_1\theta_3\right)\end{aligned} \tag{143}$$

Let's consider simple case where $|\alpha| = 1$. Substitution of the following function

$$\Psi^{(1)} = w_1x_1 + w_2x_2 + w_3x_3$$

into Schrodinger equation (142) gives the following system

$$\begin{aligned} i\hbar\frac{\mathrm{d}}{\mathrm{d}t}w_1 &= \frac{2i\hbar\theta_2(w_2-w_1)+2\theta_1\theta_2w_1}{t} + \frac{2i\hbar\theta_3(w_3-w_1)+2\theta_1\theta_3w_1}{t-1} \\ i\hbar\frac{\mathrm{d}}{\mathrm{d}t}w_2 &= \frac{2i\hbar\theta_1(w_1-w_2)+2\theta_1\theta_2w_2}{t} + \frac{2\theta_1\theta_3w_2}{t-1} \\ i\hbar\frac{\mathrm{d}}{\mathrm{d}t}w_3 &= \frac{2\theta_1\theta_2w_3}{t} + \frac{2i\hbar\theta_1(w_1-w_3)+2w_3\theta_1\theta_3}{t-1}\end{aligned} \tag{144}$$

which solution is given by the hypergeometric function in the follwing way

$$\begin{aligned} w_1 =& C_1t^{-\frac{2i}{\hbar}\theta_1\theta_2}(t-1)^{-\frac{2i}{\hbar}\theta_1\theta_3} + C_2t^{-\frac{2i}{\hbar}\theta_1\theta_2}(t-1)^{-\frac{2i}{\hbar}\theta_1\theta_3-2(\theta_1+\theta_3)}{}_2F_1(2\theta_2,-2\theta_3+1;2(\theta_1+\theta_2)+1;t) \\ &C_3t^{-\frac{2i}{\hbar}\theta_1\theta_2-2(\theta_1+\theta_2)}(t-1)^{-\frac{2i}{\hbar}\theta_1\theta_3-2(\theta_1+\theta_3)}{}_2F_1(-2\theta_1,-2(\theta_1+\theta_2+\theta_3)+1;-2(\theta_1+\theta_2)+1;t).\end{aligned}$$

6.2.2. *Painlevé V.* Hamiltonian in the intermediate coordinates is given by

$$tH = 2t\theta_\infty q_1p_1 - q_0q_1p_0^2 + 2q_0q_1p_0p_1 - q_0q_1p_1^2 + 2\theta_0q_1p_0 + 2\theta_1q_0p_1 + 2\theta_0\theta_1 \tag{145}$$

Using the same argument as in the previous case, we consider left ordering. Moreover, we see that if quantise in the following way

$$\hat{q}_i = x_i\cdot, \quad \hat{p}_i = -i\hbar\frac{\partial}{\partial x_i} \tag{146}$$

the space of homogeneous polynomials in x_0 and x_1 is invariant under the action of the Hamiltonian. Considering example of the degree 2

$$\Psi^{(2)} = w_1x_1^2 + w_2x_0^2 + w_3x_0x_1$$

we get the following system of ordinary differential equations for coefficients

$$
\begin{aligned}
it\hbar\frac{d}{dt}w_1 &= -4it\theta_\infty\hbar w_1 - 2\,i\theta_0\hbar w_3 \\
it\hbar\frac{d}{dt}w_2 &= -2i\theta_1\hbar w_3 \\
it\hbar\frac{d}{dt}w_3 &= 2\hbar^2(w_1+w_2) - 2i\theta_\infty t\hbar w_3 - 4\theta_1 i\hbar w_1 - 4\theta_0 i\hbar w_2\theta_0 - 2\hbar^2 w_3
\end{aligned}
\tag{147}
$$

6.2.3. *Painlevé IV.* Hamiltonian in the intermidiate coordinates takes form

$$H = q_t p_t^2 p_3 - 2\theta_t p_t p_3 - 2\left(p_t q_t - \theta_t\right)\left(t\theta_3 + \theta_2\right) + 2\theta_3 q_3 q_t. \tag{148}$$

In general, choice of the Lagrangian submanifold for quantisation procedure defines the properties of the quantum Hamiltonian. Here quantum Hamiltonian will not preserve homogeneous polynomials if choose standard quantisation (146). However, choice of the Lagrangian submanifold is irrelevant when we deal with Darboux coordinates and corresponds to the integral transformation on the quantum level. If we choose the following quantisation

$$\hat{q}_3 = x\cdot, \quad \hat{p}_3 = \hbar\frac{\partial}{\partial x}, \quad \hat{q}_t = \hbar\frac{\partial}{\partial y}, \quad \hat{p}_t = y\cdot$$

quantum Hamiltonian will preserve degree of homogeneous polynomials. Moreover choice of the ordering shifts monodromy parameter θ_t by $\hbar$-small values. Hamiltonian writes as

$$\hat{H} = y^2\frac{\partial^2}{\partial x\partial y} - 2\theta_t y\frac{\partial}{\partial x} - 2\left(t\theta_3+\theta_2\right)\left(y\frac{\partial}{\partial y} - \theta_t\right) + 2\theta_3 x\frac{\partial}{\partial y} \tag{149}$$

Writing down system for second order polynomial wave function

$$\Psi^{(2)} = w_1 x^2 + w_2 y^2 + w_3 xy,$$

we obtain system

$$\frac{i\hbar}{2}\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} -(t\theta_3+\theta_2)\theta_t & 0 & -\theta_3 \\ & -(t\theta_3+\theta_2)\theta_t & 2\theta_t - 1 \\ 2\theta_t & -2\theta_3 & -(t\theta_3+\theta_2)\theta_t \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \tag{150}$$

which may be solved via exponents.

6.2.4. *Painlevé III.* Hamiltonian is

$$tH = p_2^2 q_2^2 + 4t\theta_2\theta_3 q_1 q_2 - 2\theta_1 p_2 q_2 + p_1 p_2.$$

quantisation is

$$\hat{q}_1 = x\cdot, \quad \hat{p}_1 = i\hbar\frac{\partial}{\partial x}, \quad \hat{q}_2 = i\hbar\frac{\partial}{\partial y}, \quad \hat{p}_2 = y\cdot$$

Quantum Hamiltonian (up to $\hbar$ shifts of θ_1) takes form

$$\hat{H} = y^2\frac{\partial^2}{\partial y^2} - 2\theta_1 y\frac{\partial}{\partial y} + 4t\theta_2 x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}. \tag{151}$$

Writing down system for second order polynomial wave function

$$\Psi^{(2)} = w_1 x^2 + w_2 y^2 + w_3 xy,$$

we obtain system

$$i\hbar\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4t \\ 0 & 2-4\theta_1 & 1 \\ 2 & 8t & -2\theta_1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \tag{152}$$

6.2.5. *Painlevé II.* Intermediate Darboux coordinates Hamiltonian is

$$H = (q_3^5 q_4^5 + 8q_3^4 q_4^4 + 18q_3^3 q_4^3 + 12q_3^2 q_4^2 + (4t+1)q_3 q_4)\theta_4^2 + \\ + (-2q_3^4 q_4^4 - 10q_3^3 q_4^3 - 10q_3^2 q_4^2 - 2q_3 q_4)\theta_3\theta_4 - \\ - (p_3 q_3^3 q_4^2 - p_4 q_3^2 q_4^3 - 4p_3 q_3^2 q_4 - 4p_4 q_3 q_4^2 + 4q_3 q_4 \theta_2 + 2t\theta_2 - p_3 q_3 - p_4 q_4)\theta_4 + \\ + (q_3^3 q_4^3 + 2q_3^2 q_4^2)\theta_3^2 + (p_3 q_3^2 q_4 + p_4 q_3 q_4^2 + 2p_4 q_4)\theta_3 + p_3 p_4 \quad (153)$$

Choice of the following quantisation

$$\hat{q}_3 = -i\hbar\frac{\partial}{\partial x}, \quad \hat{p}_3 = x\cdot, \quad \hat{q}_4 = y\cdot, \quad \hat{p}_4 = -i\hbar\frac{\partial}{\partial y}$$

leads to the invariance of degree of homogeneous polynomial with respect to action. Indeed, all the monomials in q_3, p_3, q_4 and p_4 are such that after quatisation Hamiltonian goes to the operator which consists of operators with the same number of derivatives and multiplications in each member. We do not provide explicit form of quantum Hamiltonian and the action on the eigenspaces since the calculation is straightforward but the answer is too long.

Remark 6.1. *In this section, we consider deformation quantisation of the intermediate Darboux coordinates. This means that the quantised Hamiltonians are elements of the Weyl algebra in two variables $\mathbb{W}[x,y] = \mathbb{C}[x,\partial_x,y,\partial_y]/\langle[\partial_x,x]=1,[\partial_y,y]=1\rangle$. However, we know that the Hamiltonian we quantise allows additional symmetry, which lifts to an additional vector field $\hat{I}$ which commutes with quantum Hamiltonian vector field. For example, in the case of the Painlevé III, the quantum Hamiltonian* (151) *commutes with*

$$\hat{I} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

By restricting to the eigenfunctions of $\hat{I}$ with some chosen eigenvalue I_0, we produce quantum Hamiltonian reduction, which is simply given by the quotient of the algebra $\mathbb{W}[x,y]/\langle\hat{I}-I_0\rangle$. As a result we obtain the following quantum Hamiltonian

$$\hat{H}_{\mathrm{III}} = q^2\frac{\partial}{\partial q} + \left(-q^2 - 2q\theta_1 + 4t\theta_2\right)\frac{\partial}{\partial q} + I_0 q, \quad q = \frac{y}{x},$$

which is just the Dirac quantisation of the Hamiltonian for the Painlevé III equation (127). *Such reduction may be performed for all examples, the resulting quantum Hamiltonians coincide with the quantum Hamiltonians introduced in* [39, 52] *up to change of variables and ordering.*

In order to extend the Reshetikhin theorem for the irregular singularity it is useful to work with the lifted coordinates. In the next subsection, we give a simple proof of this extended theorem for singularities of any type

6.3. **Semi-classical solution of the confluent Knizhnik–Zamolodchikov equation.** In this section we discuss the semi-classical solutions of the confluent Knizhnik–Zamolodchikov equations in terms of the isomonodromic tau function. In this subsection, we use only the lifted Darboux coordinates and quantise them according to (137)

$$[\widehat{P}_{i_{ab}}, \widehat{Q}_{j_{cd}}] = i\hbar\,\delta_{ij}\delta_{cb}\delta_{ad}. \quad (154)$$

Such quantisation leads to the infinite dimensional representation of the isomonodromic Hamiltonians as differential operators on a Hilbert space of functions depending on some coordinates $x_{j_{ab}}$, $j = 1\dots d, a, b = 1\dots m$ and the isomonodromic times. In particular we put

$$\widehat{Q}_{j_{ab}} = x_{j_{ab}}\cdot, \quad \widehat{P}_{i_{cd}} = \hbar\frac{\partial}{\partial x_{i_{dc}}}.$$

To study the semi-classical solutions Ψ_S, we use the following standard quantum mechanical formula

$$\Psi \sim \Psi_S := \exp\left(\frac{i}{\hbar}\mathcal{S}\right),$$

where $\mathcal{S}$ is the classical action functional which explicitly depends on entries of classical variables Q and the isomonodromic times. The dependence of $\mathcal{S}$ on P is implicit, since

$$P_{i_{kl}} = \frac{\partial\mathcal{S}}{\partial Q_{i_{lk}}}.$$

In this section we prove Theorem 0.10, namely that $\Psi_{\mathcal{S}}$ evaluated along solutions of the classical system may be written as the isomonodromic τ-function. This statement already appeared in [57] for the Knizhnik–Zamolodchikov equations with Fuchsian singularities. However, our approach works also for irregular systems.

Proof of Theorem 0.10. The semi-classical solution by definition is given by

$$\Psi_{\mathcal{S}} = \exp\left(\frac{i}{\hbar}\mathcal{S}\right),$$

where $\mathcal{S}$ is a classical action functional. In our case, we have a Hamiltonian system with Hamiltonians $H^{(i)}_{u_i}$ and $H^{(i)}_1, \dots, H^{(i)}_{r_i}$ for $i = 1, \dots, n$ and Darboux coordinates $P_1, P_2 \dots P_d, Q_1, Q_2 \dots Q_d$ the action functional satisfies the following relation

$$\mathrm{d}\mathcal{S} = \sum_{j=1}^{d} P_j \mathrm{d}Q_j - \sum_i \left(H_{u_i}\mathrm{d}u_i + \sum_{k=1}^{r_i} H^{(i)}_k \mathrm{d}t^{(i)}_k\right) = \sum_{j=1}^{d} P_j \mathrm{d}Q_j - \mathrm{d}\ln(\tau), \tag{155}$$

along the solutions of the system. It is easy to see that the logarithmic differential of the τ function is already in the definition of the action functional:

Lemma 6.2. *(Malgrange* [49]*) If the Hamiltonians are homogeneous polynomials of degree two in $P_1, \dots, P_d$, then along solutions one has*

$$\mathrm{d}\mathcal{S} = \sum_i \left(H^{(i)}_{u_i}\mathrm{d}u_i + \sum_{k=1}^{r_i} H^{(i)}_k \mathrm{d}t^{(i)}_k\right). \tag{156}$$

Proof. Evaluating the first term in (155) along the solutions of the isomonodromic deformation equations, we obtain

$$\sum_j \mathrm{Tr}(P_j \mathrm{d}Q_j) = \sum_j \mathrm{Tr}\left(P_j \sum_l \left(\frac{\mathrm{d}Q_j}{\mathrm{d}u_l}\mathrm{d}u_l + \sum_{k=1}^{r_l} \frac{\mathrm{d}Q_j}{\mathrm{d}t^{(l)}_k}\mathrm{d}t^{(l)}_k\right)\right) =$$
$$= \sum_j \mathrm{Tr}\left(P_j \sum_l \left(\frac{\partial H_{u_l}}{\partial P_j}\mathrm{d}u_l + \sum_{k=1}^{r_l} \frac{\partial H^{(l)}_k}{\partial P_j}\mathrm{d}t^{(l)}_k\right)\right).$$

Using the fact that the Hamiltonians are homogeneous of degree two in $P_1, \dots, P_d$, we obtain that

$$\sum_j \mathrm{Tr}\left(P_j \sum_l \left(\frac{\partial H_{u_l}}{\partial P_j}\mathrm{d}u_l + \sum_{k=1}^{r_l} \frac{\partial H^{(l)}_k}{\partial P_j}\mathrm{d}t^{(l)}_k\right)\right) = 2\sum_i \left(H^{(i)}_{u_i}\mathrm{d}u_i + \sum_{k=1}^{r_i} H^{(i)}_k \mathrm{d}t^{(i)}_k\right).$$

which leads to the statement of the lemma. □

According to the previous lemma which works for any homogeneous polynomial Hamiltonians we get close to the proof of the theorem for the general isomonodromic Hamiltonians. In the case of the Fuchsian isomonodromic deformation are given by (36)

$$H_i = \sum_{j\neq i} \frac{\mathrm{Tr}(Q_i P_i Q_j P_j)}{u_i - u_j},$$

which are definitely homogeneous of degree 2 in the entries of matrices $P_1, P_2 \dots P_n$. This provides a simple proof that the semi-classical solution is a τ-function in the Fuchsian case. The same holds for the irregular singularities - indeed, the irregular Hamiltonians are given by the quadratic spectral invariants, i.e.

$$H = \sum_{\alpha,\beta}\sum_{i,j} C^{\alpha,\beta}_{i,j}\mathrm{Tr}\left(A^{(\alpha)}_i A^{(\beta)}_k\right), \tag{157}$$

where α, β are the indices of the singular points, while i and j are the indices which correspond to the coefficients of local expansion near singularity and $C^{\alpha,\beta}_{i,j}$ are coefficients which can be explicitly computed by using the formulas from section 4. Thanks to Lemma 2.3, all the terms $\mathrm{Tr}\left(A^{(\alpha)}_i A^{(\beta)}_k\right)$ are homogeneous polynomials of degree 2 in the variables P_i (as well as homogeneous polynomials of degree 2 in the Q_i). This fact allows us to apply Lemma 6.2 to conclude. □

Observe that this proof depends on the coordinates we use to quantise. In general, the property of semi-classical solution to be a power of an isomonodromic τ-function breaks for the reduced systems. On the classical side this phenomenon is a straightforward statement that reduced Hamiltonians are not hoimogeneous in moments or coordinates. This can be seen on the Painlevé II example - in the fully reduced coordinates Hamiltonian writes as

$$H = \frac{p^2}{2} - \frac{1}{2}\left(q^2 + \frac{t}{2}\right) - \theta q$$

while the action along solution writes as

$$\mathrm{d}\mathcal{S} = p\mathrm{d}q - H\mathrm{d}t = \left[p\frac{\partial q}{\partial t} - H\right]\mathrm{d}t = \left[p^2 - H\right]\mathrm{d}t = \left[\frac{p^2}{2} + \frac{1}{2}\left(q^2 + \frac{t}{2}\right) + \theta q\right]\mathrm{d}t \neq H\mathrm{d}t$$

The classical action now differs from τ-function by some function depends on time. This deviation from the classical action functional was investigated in the paper by Its and Prokhorov [37] for the classical Painlevé equations written in fully reduced coordinates. From the quantum point of view reduction is a restriction to the eigenspace of the Casimir operator which provides partially separation of variables in the quantum problem. By passing to the less number of coordinates the parts which were depending on the lifted coordinates vanish, so the structure of solution changes rapidly. However, despite the fact that the theorem doesn't work in the reduced case we still see the avatars of this statement since τ-function still enters quasiclassical solution in some way, see paper [37] and formula (2.27) in [66].

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