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A characterisation of elementary fibrations [☆]

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A B S T R A C T

In the categorical approach to logic proposed by Lawvere, which systematically uses adjoints to describe the logical operations, equality is presented in the form of a left adjoint to reindexing along diagonal arrows in the base. Taking advantage of the modular perspective provided by category theory, one can look at those Grothendieck fibrations which sustain just the structure of equality, the so-called elementary fibrations, aka fibrations with equality.
The present paper provides a characterisation of elementary fibrations which is a substantial generalisation of the one already available for faithful fibrations. The characterisation is based on a particular structure in the fibres which may be understood as proof-relevant equality predicates equipped with a principle of indiscernibility of identicals à la Leibniz. We exemplify this structure for several classes of fibrations, in particular, for fibrations used in the semantics of the identity type of Martin-Löf type theory. We close the paper discussing some fibrations related to Hofmann and Streicher's groupoid model of the identity type and showing that one of them is elementary.

1. Introduction

Grothendieck fibrations provide a unifying algebraic framework that underlies the treatment of various form of logics, such as first order logic, higher order logics and dependent type theories. The approach dates back to the seminal work of Bill Lawvere on functorial semantics, in particular his work on hyperdoctrines [17,18]. The structure consists of a functor $P : \mathcal{A} \longrightarrow \mathcal{T}$ which is a fibration (we recall the definition in Section 2); the base $\mathcal{T}$ is to be understood as the universe of discourse given by the sorts and the terms of the theory, while, for an object X in $\mathcal{T}$, the category $\mathcal{A}_X$ of objects and arrows over X presents the

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properties, the "attributes" in the words of Lawvere, of the object X and the relevant entailments between them. Logic, in a sense, appears via properties of the fibration which involve adjoints to basic functors such as "doubling the objects"

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\langle \mathrm{Id}_{\mathcal{A}}, \mathrm{Id}_{\mathcal{A}} \rangle"] \arrow[dr, "P"'] & & \mathcal{A} \times_{\mathcal{T}} \mathcal{A} \arrow[dl, "P \times_{\mathcal{T}} P"] \\ & \mathcal{T} & \end{tikzcd}$$

see also [14].

Some time after Lawvere proffers his structural view of logic, Per Martin-Löf puts forward the proposal of a dependent type theory where the constructions on types match very closely the construction on formulas, see [19–21]. Expressly, he refers to preliminary work of Dana Scott [27] on a tentative calculus of proofs as constructions; in turn Scott acknowledges the earlier attempts of Lawvere as they all address the idea to give propositions the same status as types (this idea indeed would later become known with similar, more fashionable locutions).

One of the many remarkable features of the Lawverian proposal for a categorical approach to logic is the systematic use of adjoint functors to describe the logical operations, in particular the realisation that equality comes in the form of a left adjoint to certain, structural, reindexing functors, see [18]. The counterpart in type theory appears in the literature some fifteen years later in the form of "propositional equality", see [20], but also [24] which calls it "intensional equality", often it is simply called "identity type". Roughly speaking, given a type T, the identity type for T can be understood as a family of "proofs of equality" between terms of type T, inductively generated by reflexivity with a fixed endpoint.

Quite remarkably, some fundamental work in the categorical semantics of type theory—the groupoid interpretation of Martin-Löf Type Theory of [13] and the interpretation of identity types as homotopies of [1]—brings about again intuitions about equality which featured prominently in [18]. This cannot be a surprise. The Lawverian approach intended to conjoin the geometric view and the abstract view of logic, in particular equality had to be as powerful as to admit that two homotopical paths could be considered, in fact argued as if they were, equal, as powerful as to admit that isomorphic structures could be considered, in fact transformed into, the same.

Taking advantage of the modular perspective provided by category theory, one can look at those Grothendieck fibrations which sustain just the structure of equality. These are called "fibrations with equality satisfying Frobenius" in [14, 3.4.1]. In some previous work [6] we concentrated on the particular case of *faithful* fibrations with equality satisfying Frobenius—which we renamed elementary doctrines referring to Lawvere's original terminology—and we shall follow suit and christen ***elementary fibration*** the notion defined in [14, 3.4.1].

Motivated by the characterisation obtained in [6], we present a characterisation of elementary fibrations that contributes to shed light on the relationship between the approaches to equality via category theory and via type theory. The characterisation is based on a structure which consists of a family of internal actions on the "attributes". Slightly more precisely the family consists, on each of the objects in the total category $\mathcal{A}$ of the fibration $P : \mathcal{A} \longrightarrow \mathcal{T}$, of an algebra map for a certain pointed endofunctor on $\mathcal{A}$ over the endofunctor on $\mathcal{T}$ which maps an object X to the product $X \times X$. However, we find it convenient to introduce such a structure in a more elementary way, by a gradual strengthening of weaker structures. This simplifies the comparison with the type-theoretic approach: as it will become clear, in type-theoretic terms this family of actions can be understood as a transport along a proof of equality.

The complete statement of our main result lists other equivalent characterisations of an elementary fibration and the proof builds on the well-known observation that existence of left adjoints to reindexing is equivalent to existence of cocartesian lifts. In the case of faithful fibrations, the characterisation reduces to the well-known characterisation of first-order equality as a reflexive and substitutive relation stable under products, see [6]. In that paper the authors produced a comonadic characterisation of elementary faithful fibrations and applied that construction to the elimination of imaginaries of Shelah, see [28,25]. That direct connection with classical model theory extended elimination of imaginaries also to non-classical theories. The parallel with the present situation suggests that there should be a comonadic characterization of general elementary fibrations and, if so, it would be interesting to see what elimination of imaginaries produces in the general, non-necessarily faithful case. This will appear in a subsequent paper in preparation.

We also apply the characterisation to discuss the relationship between elementary fibrations and fibrations coming from the homotopical semantics of identity types. Since the work by Awodey and Warren [1] and Gambino and Garner [9], weak factorisation systems, best known as part of Quillen model categories [26], have proved to provide a suitable framework to account for the inductive nature of identity types, see for instance [16,29,15,2]. Recall that a ***weak factorisation system*** on a category $\mathcal{C}$ consists of two classes of arrows $\mathcal{L}$ and $\mathcal{R}$ which contain all the isomorphisms and are closed under retracts, and such that each arrow in $\mathcal{C}$ factors as an arrow in $\mathcal{L}$ followed by an arrow in $\mathcal{R}$, and for each commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"'] & X \arrow[d, "r"] \\ B \arrow[r] & Y \end{tikzcd}$$

with $l \in \mathcal{L}$ and $r \in \mathcal{R}$ there is a diagonal filler $d : B \longrightarrow X$ making the two triangles commute. Note that, contrary to orthogonal factorisation systems, such a filler is not required to be unique. A weak factorisation system determines a fibration: the full subfibration of the codomain fibration on those arrows in the right class $\mathcal{R}$. When the factorisation system is orthogonal (plus some additional conditions) that fibration is always elementary. But, not surprisingly, for weak factorisation systems it is rarely so.

A weak factorisation system seems to lack the structure to isolate a suitable non-full subcategory structure on $\mathcal{R}$. It also seems to lack the structure to soundly interpret the rules of the identity type in its associated fibration. Indeed, what in a weak factorisation system is a property of an arrow, namely the existence of diagonal fillers, in type theory is structure on a type, namely a choice of a family of terms. This has made it impossible so far to interpret, for example, (one of) the substitution rules for the identity type. As shown by van den Berg and Garner [3], this gap can be overcome by imposing some algebraic conditions on a choice of fillers for just certain squares. Further work by Garner and others [8,11,10] identified a suitable framework to express these conditions in algebraic weak factorisation systems [12], see also [5].

The richer structure of algebraic weak factorisation systems, whose definition we recall in Section 5, produces also more structured fibrations. In particular, this is the case with the algebraic weak factorisation system on the category of small categories $\mathcal{C}at$ (and its full subcategory $\mathcal{G}pd$ on the groupoids) whose underlying weak factorisation system is the one of acyclic cofibrations and fibrations from the canonical, or "folk", model structure on $\mathcal{C}at$ (and $\mathcal{G}pd$). We prove that the fibration of algebras associated to the algebraic weak factorisation system is elementary. In the case of the algebraic weak factorisation system on $\mathcal{G}pd$, the associated full comprehension category is the Hofmann–Streicher groupoid model of Martin-Löf Type Theory from [13]. See [10,32] for the relation to the groupoid model, and [7] for a discussion of the enriched case.

There are two side results worth noticing: the first one is that we find models of dependent type theory where equality is given as an adjunction, but the identity types are not trivial. The second one is that the intensional model of type theory given by the groupoid interpretation can be reconnected to one of the original suggestions of Lawvere in [18].

In Section 2 we recall notations and results from the theory of fibrations which are necessary for the following sections. In Section 3 we introduce the structure of transporters in a fibration and prove some elementary properties of these. These are put to use in Section 4 which contains the main characterisation theorem. Finally, Section 5 contains applications to algebraic weak factorisation systems. In particular, we illustrate the connection between the groupoid hyperdoctrine of Lawvere and the groupoid model of Hofmann and Streicher.

2. Preliminaries

Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a functor. An arrow φ in $\mathcal{E}$ is said to be ***over*** an arrow f in $\mathcal{B}$ when $p(\varphi) = f$. For X in $\mathcal{B}$, the fibre $\mathcal{E}_X$ is the subcategory of $\mathcal{E}$ of arrows over id_X. In particular, an object A in $\mathcal{E}$ is said to be ***over*** X when $p(A) = X$, and an arrow $\varphi : A \longrightarrow B$ over id_X is often called ***vertical***.

Recall that an arrow $\varphi : A \longrightarrow B$ is ***cartesian*** if, for every $\chi : A' \longrightarrow B$ such that $p(\chi)$ factors through $p(\varphi)$ via an arrow $g : X' \longrightarrow X$, there is a unique $\psi : A' \longrightarrow A$ over g such that $\varphi\psi = \chi$, as in the left-hand diagram below. And an arrow $\theta : A \longrightarrow B$ is ***cocartesian*** if it satisfies the dual universal property of cartesian arrows depicted in the right-hand diagram below.

$$
\begin{tikzcd}
\mathcal{E} \arrow[d, "p"] \\
\mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[dr, dotted, "\psi"'] \arrow[drr, bend left, "\chi"] & & \\
& A \arrow[r, "\varphi"'] & B \\
X' \arrow[dr, "g"'] \arrow[drr, bend left] & & \\
& X \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\theta"] \arrow[drr, bend right, "\upsilon"'] & B \arrow[dr, dotted, "\omega"] & \\
& & B' \\
X \arrow[r] \arrow[drr, bend right] & Y \arrow[dr, "g"] & \\
& & Y'
\end{tikzcd}
$$

Once we fix an arrow $f : X \longrightarrow Y$ in $\mathcal{B}$ and an object B in $\mathcal{E}_Y$, a cartesian arrow $\varphi : A \longrightarrow B$ over f is uniquely determined up to isomorphism, *i.e.* if $\varphi' : A' \longrightarrow B$ is cartesian over f, then there is a unique vertical iso $\psi : A' \longrightarrow A$ such that $\varphi\psi = \varphi'$.

Clearly, every property of cartesian arrows applies dually to cocartesian arrows. So for an arrow $f : X \longrightarrow Y$ in $\mathcal{B}$ and an object A in $\mathcal{E}_X$, a cocartesian arrow $\theta : A \longrightarrow B$ over f is uniquely determined up to isomorphism.

In the following, we write cartesian arrows as $\longrightarrow$, and cocartesian arrows as $\longrightarrow\!\!\!\!\!+$.

A functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ is a ***fibration*** if, for every arrow $f : X \longrightarrow Y$ in $\mathcal{B}$ and for every object A in $\mathcal{E}_Y$, there is a ***cartesian lift*** of f into A, that is, an object f^*A and a cartesian arrow $f^{\Rsh A} : f^*A \longrightarrow A$ over f. A ***cleavage*** for the fibration p is a choice of a cartesian lift for each arrow $f : X \longrightarrow Y$ in $\mathcal{B}$ and object B in $\mathcal{E}_Y$, and a ***cloven fibration*** is a fibration equipped with a cleavage. In a cloven fibration, for every $f : X \longrightarrow Y$ in $\mathcal{B}$, there is a functor $f^* : \mathcal{E}_Y \longrightarrow \mathcal{E}_X$ called ***reindexing*** along f. Henceforth we assume that fibrations can be endowed with a cleavage.

2.1 Remark. It is well-known that, for the fibration $p : \mathcal{E} \longrightarrow \mathcal{B}$, an arrow $f : X \longrightarrow Y$ in $\mathcal{B}$ has cocartesian lifts if and only if the reindexing functor $f^* : \mathcal{E}_Y \longrightarrow \mathcal{E}_X$ has a left adjoint. The value of the left adjoint at an object A over X can be chosen as the codomain A' of a cocartesian lift $A \longrightarrow\!\!\!\!\!+ A'$ of $f : X \longrightarrow Y$ at A. Conversely, the cocartesian lift is given by the composition

$$A \xrightarrow{\ \eta_A\ } f^*(\exists_f(A)) \xrightarrow{\ f^{\Rsh \exists_f(A)}\ } \exists_f(A)$$

of the unit $\eta_A : A \longrightarrow f^*(\exists_f(A))$ of the adjunction $\exists_f \dashv f^*$ and the cartesian lift of f.

Fibrations are ubiquitous in mathematics and the list of examples is endless. Since our aim is to characterise those fibrations which encode a proof-relevant notion of equality, we choose the following classes of examples.

2.2 Examples. (a) A first important logical example is the fibration determined by the Lindenbaum–Tarski algebras of well-formed formulas of a theory $\mathscr{T}$ in the first order language $\mathscr{L}$ [23,22]. We believe it shows how fibrations provide the appropriate abstract mathematical structure of logic. The base category is the category $\mathcal{V}$ of lists of variables and term substitutions. Objects of $\mathcal{V}$ are lists of distinct variables $\vec{x} = (x_1, \ldots, x_n)$ while morphisms are lists of substitutions for variables $[\vec{t}/\vec{y}] : \vec{x} \longrightarrow \vec{y}$ where each term t_j in $\vec{t}$ is built in $\mathscr{L}$ on the variables $x_1, \ldots, x_n$; the composition

$$\vec{x} \xrightarrow{[\vec{t}/\vec{y}]} \vec{y} \xrightarrow{[\vec{s}/\vec{z}]} \vec{z}$$

is given by simultaneous substitutions

$$\vec{x} \xrightarrow{\left[s_1[\vec{t}/\vec{y}]/z_1, \ldots, s_k[\vec{t}/\vec{y}]/z_k\right]} \vec{z}\ .$$

The product of two objects $\vec{x}$ and $\vec{y}$ is given by a(ny) list $\vec{w}$ of as many distinct variables as the sum of the number of variables in $\vec{x}$ and of that in $\vec{y}$. Projections are given by substitution of the variables in $\vec{x}$ with the first in $\vec{w}$ and of the variables in $\vec{y}$ with the last in $\vec{w}$.

Denote by $LT(\mathscr{L}, \mathscr{T})$ the category whose objects are pairs $(\vec{x}, A)$ where $\vec{x} = (x_1,\ldots,x_n)$ and where A is well-formed formulas of $\mathscr{L}$ with no more free variables than $x_1,\ldots,x_n$ and whose arrows $[\vec{t}/\vec{y}] : (\vec{x}, A) \longrightarrow (\vec{y}, B)$ are morphisms $[\vec{t}/\vec{y}] : \vec{x} \longrightarrow \vec{y}$ in $\mathcal{V}$ such that $A \vdash_{\mathscr{T}} B[\vec{t}/\vec{y}]$, *i.e.* $A \vdash B[\vec{t}/\vec{y}]$ is a provable consequence in $\mathscr{T}$; the composition is given by composition of $\mathcal{V}$ together with the cut rule in the logical calculus, while the identity on $(\vec{x}, A)$ is given by the logical rule $A \vdash_{\mathscr{T}} A$.

The first projection $\mathcal{L} : LT(\mathscr{L}, \mathscr{T}) \longrightarrow \mathcal{V}$ sending $[\vec{t}/\vec{y}] : (\vec{x}, A) \longrightarrow (\vec{y}, B)$ to $[\vec{t}/\vec{y}] : \vec{x} \longrightarrow \vec{y}$ is a fibration. Reindexing is given by substitution of terms in formulas.

(b) Given a category $\mathcal{C}$, let $\mathbf{Fam}(\mathcal{C})$ be the category whose objects are set-indexed families of objects in $\mathcal{C}$, *i.e.* pairs $(I, (A_i)_{i \in I})$ where I is a set and A_i is an object in $\mathcal{C}$, for $i \in I$, and an arrow from $(I, (A_i)_{i \in I})$ to $(I', (A'_j)_{j \in I'})$ is a pair (f, φ) where $f : I \longrightarrow I'$ is a function and $\varphi = (\varphi_i : A_i \longrightarrow A_{f(i)})_{i \in I}$ is a family of arrows in $\mathcal{C}$, see [14, 1.2.1].

Equivalently, an object of $\mathbf{Fam}(\mathcal{C})$ is a functor $A : I \longrightarrow \mathcal{C}$ where I is a set seen as a discrete category, and an arrow from A to $B : J \longrightarrow \mathcal{C}$ is a pair (f, φ) where $f : I \longrightarrow J$ is a function and $\varphi : A \dot{\longrightarrow} Bf$ is a natural transformation as in the diagram

$$\begin{tikzcd} I \arrow[d, "f"'] \arrow[dr, "A"] & \\ J \arrow[r, "B"'] & \mathcal{C}. \arrow[from=1-1, to=2-2, phantom, "\varphi\,\dot\downarrow"] \end{tikzcd}$$

Since I is discrete, all commutative diagrams for naturality are trivial.

The functor $\mathsf{Pr}_1 : \mathbf{Fam}(\mathcal{C}) \longrightarrow \mathcal{S}et$ that sends $(f, \varphi) : A \longrightarrow B$ to $f : I \longrightarrow J$ is a fibration. An arrow $(f, \mathrm{id}) : Bf \longrightarrow B$ is cartesian into B over $f : I \longrightarrow J$. Note that $\mathbf{Fam}(\mathit{1}) \equiv \mathcal{S}et$ and that the fibration $\mathsf{Pr}_1 : \mathbf{Fam}(\mathcal{C}) \longrightarrow \mathcal{S}et$ is isomorphic to $\mathbf{Fam}(!) : \mathbf{Fam}(\mathcal{C}) \longrightarrow \mathbf{Fam}(\mathit{1})$, where $! : \mathcal{C} \longrightarrow \mathit{1}$ is the unique functor.

(c) Let $\mathcal{F}$ be a full subcategory of $\mathcal{C}^{\mathbf{2}}$, so that an arrow $f : a \longrightarrow b$ in $\mathcal{F}$, where a and b are arrows in $\mathcal{C}$, is a commutative square

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ X \arrow[r, "f_0"] & Y. \end{tikzcd}$$

Assume that, for every $f : X \longrightarrow Y$ in $\mathcal{C}$ and $g : B \longrightarrow Y$ in $\mathcal{F}$, there is a pullback square

$$\begin{tikzcd} P \arrow[r] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and $g' \in \mathcal{F}$. In that case the composite

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook] \arrow[rr, bend right, "\mathsf{cod}\restriction_{\mathcal{F}}"'] & \mathcal{C}^{\mathbf{2}} \arrow[r, "\mathsf{cod}"] & \mathcal{C} \end{tikzcd}$$

is a fibration. Given $f : X \longrightarrow Y$ in $\mathcal{C}$, a cartesian lift into the object $g : B \longrightarrow Y$, where $g \in \mathcal{F}$, is a pullback square as the one above. Since $\mathcal{F}$ is full, in the following we shall often confuse it with its collection of objects, *i.e.* a collection of arrows of $\mathcal{C}$.

When $\mathcal{C}$ has pullbacks we can choose $\mathcal{F}$ even as $\mathcal{C}^{\mathbf{2}}$ itself. In the particular case of $\mathcal{C} := \mathcal{S}et$ the example in (b) come to be the same as the example in (c) since there is an equivalence

$$\begin{tikzcd} \mathbf{Fam}(\mathcal{S}et) \arrow[rr, equal] \arrow[dr, "\mathsf{Pr}_1"'] & & \mathcal{S}et^{\mathbf{2}} \arrow[dl, "\mathsf{cod}"] \\ & \mathcal{S}et & \end{tikzcd}$$

Recall that a fibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ ***has finite products*** if the base $\mathcal{B}$ has finite products as well as each fibre $\mathcal{E}_X$, and each reindexing functor preserves products. Equivalently, both $\mathcal{B}$ and $\mathcal{E}$ have finite products and p preserves them.

2.3 Notation. We do not require a functorial denotation for products; when we write 1 we refer to any terminal object in $\mathcal{B}$ and, similarly for objects X and Y in $\mathcal{B}$, when we write $X \times Y$, $\mathrm{pr}_1 : X \times Y \longrightarrow X$ and $\mathrm{pr}_2 : X \times Y \longrightarrow Y$, we refer to any diagram of binary products in $\mathcal{B}$. Universal arrows into a product induced by lists of arrows shall be denoted as $\langle f_1, \ldots, f_n\rangle$, but lists of projections $\langle \mathrm{pr}_{i_1}, \ldots, \mathrm{pr}_{i_k}\rangle$ will always be abbreviated as $\mathrm{pr}_{i_1,\ldots,i_k}$. In particular, as an object X is a product of length 1, sometimes we find it convenient to denote the identity on X as pr_1, the diagonal $X \longrightarrow X \times X$ as $\mathrm{pr}_{1,1}$ and the unique $X \longrightarrow 1$ as pr_0. As the notation is ambiguous, we shall always indicate domain and codomain of lists of projections and sometimes we may distinguish projections decorating some of them with a prime symbol.

We shall employ a similar notation for terminal objects, binary products and projections in a fibre $\mathcal{E}_X$, writing $\top_X$, $A \wedge_X B$, $\pi_1 : A \wedge_X B \longrightarrow A$ and $\pi_2 : A \wedge_X B \longrightarrow B$, respectively, and dropping the subscript in $\top_X$ and $\wedge_X$ when it is clear from the context. Given a third object C in $\mathcal{E}_X$ and two vertical arrows $\varphi_1 : C \longrightarrow A$ and $\varphi_2 : C \longrightarrow B$, we denote the induced arrow into $A \wedge B$ also as $\langle \varphi_1, \varphi_2\rangle$.

2.4 Examples. (a) The fibration $\mathcal{L} : LT(\mathscr{L}, \mathscr{T}) \longrightarrow \mathcal{V}$ defined in Example 2.2(a) has finite products. For a list of finite variables $\vec{x}$, a terminal object in $LT(\mathscr{L}, \mathscr{T})_{\vec{x}}$ is any true formula of $\mathscr{T}$ in the contexts $\vec{x}$. A product of A and B in $LT(\mathscr{L}, \mathscr{T})_{\vec{x}}$ is given by the conjunction $A \wedge B$.

(b) Consider the fibration $\mathsf{Pr}_1 : \mathbf{Fam}(\mathcal{C}) \longrightarrow \mathcal{S}et$ defined in Example 2.2(b) and suppose that $\mathcal{C}$ has finite products. Then the fibration Pr_1 has finite products. Indeed a product of the two families $A : I \longrightarrow \mathcal{C}$ and $B : I \longrightarrow \mathcal{C}$ in the fibre $\mathbf{Fam}(\mathcal{C})_I$ is the family $A \wedge B : I \longrightarrow \mathcal{C}$ where $(A \wedge B)_i$ is $A_i \times B_i$ with projections $(\mathrm{id}_I, \mathrm{pr}_1)$ and $(\mathrm{id}_I, \mathrm{pr}_2)$ where $(\mathrm{pr}_1)_i$ is the first projection $A_i \times B_i \longrightarrow A_i$. A terminal object in $\mathbf{Fam}(\mathcal{C})_I$ is given by the family $1 : I \longrightarrow \mathcal{C}$ which is constantly a chosen terminal object of $\mathcal{C}$.

(c) Assume that the base $\mathcal{C}$ of the fibration $\mathsf{cod}\restriction_{\mathcal{F}}$ defined in Example 2.2(c) has finite products. Then the fibration $\mathsf{cod}\restriction_{\mathcal{F}}$ has finite products, and in the fibres these are given by pullback of arrows in $\mathcal{F}$.

3. Transporters

This section presents a structure that will be useful in the characterisation in Theorem 4.7, providing along the way examples and some instrumental results.

3.1 Definition. Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration with finite products and consider an object X in $\mathcal{B}$. A ***loop*** on X consists of an object I_X over $X \times X$ and an arrow $\partial_X : \top_X \longrightarrow \mathrm{I}_X$ over $\mathrm{pr}_{1,1} : X \longrightarrow X \times X$. The fibration p ***has loops*** if it is equipped with a choice of a loop on every object X in $\mathcal{B}$. We shall also say that p is a ***fibration with loops***.

3.2 Notation. Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration with loops. For two objects X, Y in $\mathcal{B}$, we write $\mathrm{I}_X \boxtimes \mathrm{I}_Y$ for the vertex of the product span in $\mathcal{E}$ displayed below.

$$\begin{tikzcd}
 & & \mathrm{pr}_{1,3}{}^*\mathrm{I}_X \wedge \mathrm{pr}_{2,4}{}^*\mathrm{I}_Y \arrow[dll] \arrow[dl] \arrow[dr] \arrow[drr] & & \\
\mathrm{I}_X & \mathrm{pr}_{1,3}{}^*\mathrm{I}_X \arrow[l] & & \mathrm{pr}_{2,4}{}^*\mathrm{I}_Y \arrow[r] & \mathrm{I}_Y \\
X \times X & & X \times Y \times X \times Y \arrow[ll, "\mathrm{pr}_{1,3}"'] \arrow[rr, "\mathrm{pr}_{2,4}"] & & Y \times Y
\end{tikzcd}$$

As $\top_X \leftarrow \top_{X \times Y} \longrightarrow \top_Y$ is a product too, we also write $\partial_X \boxtimes \partial_Y$ for the unique arrow $\top_{X \times Y} \longrightarrow \mathrm{I}_X \boxtimes \mathrm{I}_Y$ induced by ∂_X and ∂_Y.

3.3 Definition. A fibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ with finite products ***has productive loops*** if

(i) it has loops, *i.e.* there is a loop $\top_X \xrightarrow{\partial_X} \mathrm{I}_X$ on every X in $\mathcal{B}$;
(ii) for every X and Y in $\mathcal{B}$, there is a vertical arrow $\chi_{X,Y} : \mathrm{I}_X \boxtimes \mathrm{I}_Y \longrightarrow \mathrm{I}_{X \times Y}$.

The fibration p ***has strictly productive loops*** if it has productive loops and moreover the following diagram commutes for every X and Y.

$$\begin{tikzcd}
 & \top_{X \times Y} \arrow[dl, "\partial_X \boxtimes \partial_Y"'] \arrow[dr, "\partial_{X \times Y}"] & \\
\mathrm{I}_X \boxtimes \mathrm{I}_Y \arrow[rr, "\chi_{X,Y}"] & & \mathrm{I}_{X \times Y}
\end{tikzcd}$$

3.4 Notation. Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration with loops. Given A over X, we find it convenient to write δ_A for the arrow $\langle \mathrm{pr}_{1,1}^{\Rsh(\mathrm{pr}_1{}^*A)}, \partial_X !_A \rangle : A \longrightarrow (\mathrm{pr}_1{}^*A) \wedge \mathrm{I}_X$. We shall also need a parametric version of it, as for instance in Definition 3.12. When A is an object over $Z \times X$ we write δ^Z_A for the arrow

$\langle \mathrm{pr}_{1,2,2}^{\ulcorner(\mathrm{pr}_{1,2}{}^*A)}, \partial'_X !_A\rangle : A \longrightarrow (\mathrm{pr}_{1,2}{}^*A) \wedge (\mathrm{pr}_{2,3}{}^*\mathrm{I}_X)$, where $\partial'_X : \top_{Z\times X} \longrightarrow \mathrm{pr}_{2,3}{}^*\mathrm{I}_X$ is the unique arrow over $\mathrm{pr}_{1,2,2}$ obtained reindexing ∂_X as shown in the diagram below.

$$
\begin{tikzcd}
\top_{Z\times X} \arrow[rr, "\partial'_X = \mathrm{pr}_{2,3}{}^*\partial_X"] \arrow[dr] & & \mathrm{pr}_{2,3}{}^*\mathrm{I}_X \arrow[dr] & \\
& \top_X \arrow[rr, "\partial_X"'] & & \mathrm{I}_X
\end{tikzcd}
$$

$$
\begin{tikzcd}
Z\times X \arrow[rr, "\mathrm{pr}_{1,2,2}"] \arrow[dr, "\mathrm{pr}_2"'] & & Z\times X\times X \arrow[dr, "\mathrm{pr}_{2,3}"] & \\
& X \arrow[rr, "\mathrm{pr}_{1,1}"'] & & X\times X
\end{tikzcd}
$$

We write Δ for the class of arrows of the form $\mathrm{pr}_{1,2,2} : Z \times X \longrightarrow Z \times X \times X$, for Z, X in $\mathcal{B}$. For every X in $\mathcal{B}$, we write Λ_X for the class of all arrows (isomorphic to one) of the form δ^Z_A in $\mathcal{E}$ defined above, for A over $Z \times X$ and Z in $\mathcal{B}$, and Λ for the union of all Λ_X for X in $\mathcal{B}$.

3.5 Definition. Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration with finite products, X an object in $\mathcal{B}$ and $\partial_X : \top_X \longrightarrow \mathrm{I}_X$ a loop on X. Given A over X, a ***carrier for the loop*** ∂_X ***at*** A is an arrow $\mathrm{t}_A : (\mathrm{pr}_1{}^*A) \wedge \mathrm{I}_X \longrightarrow A$ over $\mathrm{pr}_2 : X \times X \longrightarrow X$. The carrier t_A is ***strict*** if $\mathrm{t}_A\delta_A = \mathrm{id}_A$.

3.6 Definition. Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration with finite products and X an object in $\mathcal{B}$. A ***transporter on*** X consists of

(i) a loop $\partial_X : \top_X \longrightarrow \mathrm{I}_X$ on X;
(ii) for every A over X, a carrier for ∂_X.

A transporter is ***strict*** if every carrier is strict.

A fibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ ***has*** (***strict***) ***transporters*** if it has a (strict) transporter on each X in $\mathcal{B}$.

A fibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ ***has*** (***strict***) ***productive transporters*** if

(i) it has (strict) transporters;
(ii) loops are (strictly) productive.

3.7 Remark. Strict productive transporters give a commutative diagram

$$
\begin{tikzcd}
& \top_{X\times Y} \arrow[dl, "\partial_X \boxtimes \partial_Y"'] \arrow[dr, "\partial_{X\times Y}"] & \\
\mathrm{I}_X \boxtimes \mathrm{I}_Y & & \mathrm{I}_{X\times Y} \arrow[ll, "\omega_{X,Y}"']
\end{tikzcd}
$$

for an arrow $\omega_{X,Y} : \mathrm{I}_{X\times Y} \longrightarrow \mathrm{I}_X \boxtimes \mathrm{I}_Y$. The reader will see this in the proof of (iv)$\Rightarrow$ (v) in Theorem 4.7.

3.8 Examples. (a) Consider a theory $\mathscr{T}$ over a first order language $\mathscr{L}$ with equality and take the fibration $\mathcal{L} : LT(\mathscr{L}, \mathscr{T}) \longrightarrow \mathcal{V}$ from Example 2.2(a). Suppose $\vec{x} = (x_0, ..., x_n)$ and let $(x_0, ..., x_n, x'_0, ..., x'_n)$ be $\vec{x} \times \vec{x}$. Let $\mathrm{I}_{\vec{x}}$ be the formula $(x_0 = y_0) \wedge ... \wedge (x_n = y_n)$ in $LT(\mathscr{L}, \mathscr{T})_{\vec{x}\times\vec{x}}$. Reflexivity of the equality predicate, *i.e.* the fact that $\vdash_{\mathscr{T}} (x_0 = x_0) \wedge ... \wedge (x_n = x_n)$ is provable in every theory $\mathscr{T}$ over $\mathscr{L}$, determines an arrow $\partial_{\vec{x}} : \top_{\vec{x}} \longrightarrow \mathrm{I}_{\vec{x}}$ in $LT(\mathscr{L}, \mathscr{T})$ and makes $\mathcal{L}$ a fibrations with strictly productive loops. Let A be a well formed formula of $\mathscr{L}$ in $LT(\mathscr{L},\mathscr{T})_{\vec{x}}$. Substitutivity of the equality predicate, *i.e.* the fact that $A \wedge (x_0 = x'_0) \wedge ... \wedge (x_n = x'_n) \vdash_{\mathscr{T}} A[x'_0/x_0, ..., x'_n/x_n]$ is provable in every theory $\mathscr{T}$ over $\mathscr{L}$, determines a strict carrier for the loop $\partial_{\vec{x}}$ at A.

(b) Consider the fibration $\mathsf{Pr}_1 : \mathbf{Fam}(\mathcal{C}) \longrightarrow \mathcal{S}et$ from Example 2.2(b). And suppose that $\mathcal{C}$ has a stable initial object, *i.e.* an initial object 0 such that $0 \times A \xrightarrow{\sim} 0$ for all A. Let 1 be a terminal object of $\mathcal{C}$, and consider the family $\mathrm{I}_X : X \times X \longrightarrow \mathcal{C}$ as the function that maps (a, b) to 1 if $a = b$ and to 0 otherwise. There are two natural transformations $\iota : \top_X \stackrel{\cdot}{\to} \mathrm{I}_X \mathrm{pr}_{1,1}$, whose component on $x \in X$ is the identity, and $\tau_A : (A\mathrm{pr}_1) \wedge \mathrm{I}_X \stackrel{\cdot}{\to} A\mathrm{pr}_2$ whose component on (x_1, x_2) is the identity on $A(x_1)$ if $x_1 = x_2$, and the unique arrow $0 \longrightarrow A(x_2)$ otherwise. The object I_X and arrows $(\mathrm{pr}_{1,1}, \iota), (\mathrm{pr}_2, \tau_A)$ for A over X, form a strict transporter for the set X. And these are productive as the components on (x_1, y_1, x_2, y_2) of $\mathrm{I}_X \boxtimes \mathrm{I}_Y$ and $\mathrm{I}_{X \times Y}$ are both initial or both terminal. Hence one may take the canonical iso as the component of $\chi_{X,Y}$ on (x_1, y_1, x_2, y_2).

(c) Let $\mathcal{C}$ be a category with finite products and suppose that $\mathcal{C}$ has a weak factorisation system $(\mathcal{L}, \mathcal{R})$ such that $\mathcal{C}$ has pullbacks of arrows in $\mathcal{R}$ along any arrow. It follows that arrows in the right class $\mathcal{R}$ satisfy the hypothesis of Example 2.2(c), so $\mathsf{cod}\restriction_{\mathcal{R}} : \mathcal{R} \longrightarrow \mathcal{C}$ is a fibration with products. If arrows in the left class $\mathcal{L}$ are stable under pullback along arrows in the right class $\mathcal{R}$, then every object X of $\mathcal{C}$ has a strict transporter defined as follows. A loop $\partial_X :\, = \langle \mathrm{r}_X, \mathrm{pr}_{1,1} \rangle$ is obtained factoring the diagonal $\mathrm{pr}_{1,1} : X \longrightarrow X \times X$ as

$$\begin{tikzcd}
X \arrow[r, "\mathrm{r}_X"] \arrow[d, "\mathrm{id}_X"'] & PX \arrow[d, "\mathrm{I}_X"] \\
X \arrow[r, "\mathrm{pr}_{1,1}"'] & X \times X
\end{tikzcd}$$

where r_X is in $\mathcal{L}$. For an arrow $a \in \mathcal{R}$, consider the following commutative diagram.

$$\begin{tikzcd}
A \arrow[rr] \arrow[rrrr, bend left, "\mathrm{id}_A"] \arrow[dr, "\mathrm{r}_a"] \arrow[ddr, bend right, "\mathrm{id}_A"'] & & X \arrow[dr, "\mathrm{r}_X"] \arrow[ddr, "\mathrm{id}_X"'] & & A \arrow[d, "a"] \\
& A \times_X PX \arrow[rr] \arrow[d] & & PX \arrow[r, "\mathrm{pr}_2\mathrm{I}_X"] \arrow[d, "\mathrm{pr}_1\mathrm{I}_X"] & X \\
& A \arrow[rr, "a"'] & & X &
\end{tikzcd}$$

Since the arrow r_a is a pullback of r_X along an arrow in $\mathcal{R}$, it is in $\mathcal{L}$. It follows by weak orthogonality that there is $t_a : A \times_X PX \longrightarrow A$ filling in the previous diagram

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}_A"] \arrow[d, "\mathrm{r}_a"'] & & A \arrow[d, "a"] \\
A \times_X PX \arrow[r] \arrow[urr, "t_a"] & PX \arrow[r, "\mathrm{pr}_2\mathrm{I}_X"] & X
\end{tikzcd}$$

A carrier at a is then (pr_2, t_a). Instances of this situation can be found in any Quillen model category where acyclic cofibrations are stable under pullback along fibrations, but also in Shulman's type-theoretic fibration categories [29] and Joyal's tribes [15].

Suppose now that the class $\mathcal{L}$ is stable under products in the sense that, for every object X in $\mathcal{C}$, the functor $(-) \times X : \mathcal{C} \longrightarrow \mathcal{C}$ maps $\mathcal{L}$ into $\mathcal{L}$. Then $\mathsf{cod}{\restriction}_{\mathcal{R}}$ has strict productive transporters. Indeed in this case $\partial_X \boxtimes \partial_Y = (\mathrm{pr}_{1,2,1,2}, \mathrm{r}_X \times \mathrm{r}_Y)$ and the arrow $\mathrm{r}_X \times \mathrm{r}_Y$ is in $\mathcal{L}$ as it factors as shown below.

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{r}_X \times Y"] \arrow[rr, bend right, "\mathrm{r}_X \times \mathrm{r}_Y"'] & PX \times Y \arrow[r, "PX \times \mathrm{r}_Y"] & PX \times PY \end{tikzcd}$$

And the rest of the argument is similar to the one in (d) below.

(d) Let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be a path category. It consists of two full subcategories $\mathcal{W}$ and $\mathcal{F}$ of $\mathcal{C}^{\mathbf{2}}$ closed under isomorphisms and satisfying some additional conditions, see [4]. In particular, the category $\mathcal{F}$ satisfies the hypothesis of Example 2.2(c), so $\mathsf{cod}{\restriction}_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{C}$ is a fibration with products. In the notation of [4], the arrow $(s, t) : PX \longrightarrow X \times X$ together with the arrow $r : X \longrightarrow PX$ and, for every $f \in \mathcal{F}$, a transport structure in the sense of [4, Def. 2.24], provides a (not necessarily strict) transporter for X. In the following, when dealing with a path category, we shall always try to stick to the notation in [4]; however we prefer to denote the arrow $r : X \longrightarrow PX$ as r_X. Since the arrows in $\mathcal{W}$ are stable under pullback along arrows in $\mathcal{F}$, see [4, Prop. 2.7], it follows that $\mathrm{r}_X \times \mathrm{r}_Y$ is in $\mathcal{W}$, as terminal arrows are in $\mathcal{F}$. Hence we obtain $\chi_{X,Y}$ with the required properties as the arrow (id, k), where $k : PX \times PY \longrightarrow P(X \times Y)$ is a lower filler in

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{r}_{X \times Y}"] \arrow[d, "\mathrm{r}_X \times \mathrm{r}_Y"'] & P(X \times Y) \arrow[d, "(s,t)"] \\ PX \times PY \arrow[r] & X \times Y \times X \times Y \end{tikzcd}$$

see [4, Lemma 2.9]. It follows that the fibration $\mathsf{cod}{\restriction}_{\mathcal{F}}$ has productive transporters. Note that these are not necessarily strict as the lower filler need not make the upper triangle commute.

3.9 Remark. Example 3.8(d) fits only momentarily in the framework that we are developing. This will become clear after Theorem 4.7, as our aim is to characterise elementary fibrations. This suggests the relevance of a weaker notion than elementary fibration, which we shall consider in future work.

3.10 Remark. The notion of fibration with strictly productive transporters should provide the basic structure to interpret the so-called "Indiscernability of identicals" in dependent type theory [31, Section 1.12]. This principle alone is strictly weaker than the full induction principle for identity types.

3.11 Remark. For X, Y in $\mathcal{B}$, we can rewrite $\partial_X \boxtimes \partial_Y : \top_{X \times Y} \longrightarrow \mathrm{I}_X \boxtimes \mathrm{I}_Y$ as the composite $\langle \partial'_X !, \iota \rangle \partial'_Y$ as shown in the commutative diagram below.

$$
\begin{tikzcd}
 & & & \top_X \arrow[r, "\partial_X"] & \mathrm{I}_X \\
 & & \top_{X\times Y\times Y} \arrow[ur] \arrow[r, dashed, "\partial'_X"] & \mathrm{pr}_{2,4}{}^*\mathrm{I}_X \arrow[ur] & \\
\top_Y \arrow[dr, "\partial_Y"'] & \top_{X\times Y} \arrow[l] \arrow[r, "\partial'_Y"] \arrow[dr, dashed, "\partial'_Y"'] & \mathrm{pr}_{1,3}{}^*\mathrm{I}_Y \arrow[u, "!"] \arrow[d, "\mathrm{id}"] \arrow[r, "\langle \partial'_X !, \iota\rangle"] & \mathrm{I}_X \boxtimes \mathrm{I}_Y \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] & \\
 & \mathrm{I}_Y & \mathrm{pr}_{2,3}{}^*\mathrm{I}_Y \arrow[l] \arrow[r, "\iota"] & \mathrm{pr}_{2,4}{}^*\mathrm{I}_Y &
\end{tikzcd}
$$

$$
\begin{tikzcd}
Y \arrow[dr, "\mathrm{pr}_{1,1}"'] & X \times Y \arrow[l, "\mathrm{pr}_2"'] \arrow[dr, "\mathrm{pr}_{1,2,2}"] & & X \arrow[r, "\mathrm{pr}_{1,1}"] & X \times X \\
 & Y \times Y & X \times Y \times Y \arrow[loop above, "\mathrm{id}"] \arrow[l, "\mathrm{pr}_{2,3}"'] \arrow[ur, "\mathrm{pr}_1"] \arrow[r, shift left, "\mathrm{pr}_{1,2,1,3}"] & X \times Y \times X \times Y \arrow[l, shift left, "\mathrm{pr}_{1,2,4}"] \arrow[ur, "\mathrm{pr}_{1,3}"'] \arrow[ll, bend left, "\mathrm{pr}_{2,4}"] &
\end{tikzcd}
$$

3.12 Definition. Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration with transporters. Let Z, X be in $\mathcal{B}$ and let A be over $Z \times X$ in $\mathcal{E}$. A ***parametrised carrier*** at A for the transporter on X is an arrow

$$\mathrm{t}^Z_A : (\mathrm{pr}_{1,2}{}^*A) \wedge (\mathrm{pr}_{2,3}{}^*\mathrm{I}_X) \longrightarrow A$$

over $\mathrm{pr}_{1,3} : Z \times X \times X \longrightarrow Z \times X$. We say that the parametrised carrier t^Z_A is ***strict*** if $\mathrm{t}^Z_A \delta^Z_A = \mathrm{id}_A$, where $\delta^Z_A : A \longrightarrow (\mathrm{pr}_{1,2}{}^*A) \wedge (\mathrm{pr}_{2,3}{}^*\mathrm{I}_X)$ is the arrow defined in Notation 3.4.

3.13 Proposition. *Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration.*

(i) If p has productive transporters, then for every Z, X in $\mathcal{B}$, there is a parametrised carrier at every A over $Z \times X$.
(ii) If the productive transporters are strict, then so are the parametrised carriers.

Proof. (i) The arrow t^Z_A can be obtained as the composite

$$
\begin{tikzcd}
(\mathrm{pr}_{1,2}{}^*A) \wedge (\mathrm{pr}_{2,3}{}^*\mathrm{I}_X) \arrow[r, "\alpha \wedge \langle \partial'_Z !, \iota\rangle"] & (\mathrm{pr}_{1,2}{}^*A) \wedge (\mathrm{I}_Z \boxtimes \mathrm{I}_X) \arrow[d, "\mathrm{id} \wedge \chi_{Z,X}"'] & \\
 & (\mathrm{pr}_{1,2}{}^*A) \wedge \mathrm{I}_{Z\times X} \arrow[r, "\mathrm{t}_A"] & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
Z \times X \times X \arrow[r, "\mathrm{pr}_{1,2,1,3}"] & Z \times X \times Z \times X \arrow[r, "\mathrm{pr}_{3,4}"] & Z \times X
\end{tikzcd}
$$

where $\alpha : \mathrm{pr}_{1,2}{}^*A \longrightarrow \mathrm{pr}_{1,2}{}^*A$ and $\iota : \mathrm{pr}_{2,3}{}^*(\mathrm{I}_X) \longrightarrow \mathrm{pr}_{2,4}{}^*(\mathrm{I}_X)$ are cartesian over $\mathrm{pr}_{1,2,1,3}$ and $\partial'_Z : \top_{Z\times X\times X} \longrightarrow \mathrm{pr}_{1,3}{}^*\mathrm{I}_Z$ is the unique arrow over $\mathrm{pr}_{1,2,1,3}$ obtained reindexing ∂_Z along the pullback square below.

$$\begin{tikzcd}
Z \times X \times X \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_{1,2,1,3}"'] & Z \arrow[d, "\mathrm{pr}_{1,1}"] \\
Z \times X \times Z \times X \arrow[r, "\mathrm{pr}_{1,3}"'] & Z \times Z
\end{tikzcd}$$

(ii) Let $\partial'_X : \top_{Z\times X} \longrightarrow {\mathrm{pr}_{2,3}}^*\mathrm{I}_X$ be the arrow over $\mathrm{pr}_{1,2,2}$ obtained reindexing ∂_X along the pullback square below.

$$\begin{tikzcd}
Z \times X \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_{1,2,2}"'] & X \arrow[d, "\mathrm{pr}_{1,1}"] \\
Z \times X \times X \arrow[r, "\mathrm{pr}_{2,3}"'] & X \times X
\end{tikzcd}$$

By Remark 3.11, one has that $\langle \partial'_Z!, \iota\rangle \partial'_X = \partial_Z \boxtimes \partial_X$. It follows that

$$\begin{aligned}
\mathrm{t}^Z_A \delta^Z_A &= \mathrm{t}_A(\mathrm{id} \wedge \chi_{Z\times X})(\alpha \wedge \langle \partial'_Z!, \iota\rangle)\langle \mathrm{pr}_{1,2,2}^{\ulcorner {\mathrm{pr}_{1,2}}^* A}, \partial'_X!_A\rangle \\
&= \mathrm{t}_A\langle \mathrm{pr}_{1,2,1,2}^{\ulcorner {\mathrm{pr}_{1,2}}^* A}, \partial_{Z\times X}!_A\rangle \\
&= \mathrm{t}_A \delta_A \\
&= \mathrm{id}_A. \quad \square
\end{aligned}$$

4. Elementary fibrations

Recall from [14, 3.4.1] the following definition.

4.1 Definition. A fibration with products $p : \mathcal{E} \longrightarrow \mathcal{B}$ is ***elementary*** if, for every pair of objects Z and X in $\mathcal{B}$, reindexing along the parametrised diagonal $\mathrm{pr}_{1,2,2} : Z \times X \longrightarrow Z \times X \times X$ has a left adjoint $\exists_{Z,X} : \mathcal{E}_{Z\times X} \longrightarrow \mathcal{E}_{Z\times X\times X}$, and these satisfy:

Frobenius Reciprocity: for every A over $Z \times X$ and B over $Z \times X \times X$, the canonical arrow

$$\exists_{Z,X}({\mathrm{pr}_{1,2,2}}^* B \wedge A) \longrightarrow B \wedge \exists_{Z,X} A$$

is iso, and

Beck–Chevalley Condition: for every pullback square

$$\begin{tikzcd}
U \times X \arrow[r, "f \times X"] \arrow[d, "\mathrm{pr}_{1,2,2}"'] & V \times X \arrow[d, "\mathrm{pr}_{1,2,2}"] \\
U \times X \times X \arrow[r, "f \times X \times X"'] & V \times X \times X
\end{tikzcd}$$

and every A over $V \times X$, the canonical arrow

$$\exists_{U,X}(f \times X)^* A \longrightarrow (f \times X \times X)^* \exists_{V,X} A$$

is iso.

4.2 Examples. (a) Let $\mathscr{L}$ be a first order language with equality and let $\vec{x} = (x_0, ..., x_n)$ and $\vec{x} \times \vec{x} = (x_0, ..., x_n, x'_0, ..., x'_n)$. It is an easy exercise in first order logic to show that for every list of variables $\vec{y}$ for every well formed formula A with free variables in $\vec{y} \times \vec{x}$ and for every well formed formula B with free variables in $\vec{y} \times \vec{x} \times \vec{x}$ the two following rules

$$\frac{A \wedge (x_0 = x'_0) \wedge, ..., \wedge (x_n = x'_n) \vdash B}{A \vdash B[x_0/x'_0, ..., x_n/x'_n]} \qquad \frac{A \vdash B[x_0/x'_0, ..., x_n/x'_n]}{A \wedge (x_0 = x'_0) \wedge, ..., \wedge (x_n = x'_n) \vdash B}$$

are valid. Therefore for every theory $\mathscr{T}$ the assignment

$$\exists_{\vec{y},\vec{x}} : LT(\mathscr{L}, \mathscr{T})_{\vec{y}\times\vec{x}} \longrightarrow LT(\mathscr{L}, \mathscr{T})_{\vec{y}\times\vec{x}\times\vec{x}}$$

that maps a well formed formula A to the well formed formula $A \wedge (x_0 = x'_0) \wedge, ..., \wedge (x_n = x'_n)$, gives the desired left adjoint. The validity of Frobenius Reciprocity and of the Beck–Chevalley Condition is immediate. Whence for every theory $\mathscr{T}$ over a first order language $\mathscr{L}$ with equality the fibration $\mathcal{L} : LT(\mathscr{L}, \mathscr{T}) \longrightarrow \mathcal{V}$ presented in Example 2.2(a) is elementary.

(b) The fibration $\mathsf{Pr}_1 : \mathbf{Fam}(\mathcal{C}) \longrightarrow \mathcal{S}et$ of the Example 2.2(b) is elementary when $\mathcal{C}$ has finite products and a stable initial object. Indeed, let $\mathrm{I}_X : X \times X \longrightarrow \mathcal{C}$ be the family defined in Example 3.8. Then, for every $A : Z \times X \longrightarrow \mathcal{C}$, the family

$$Z \times X \times X \xrightarrow{\exists_{Z,X}(A)} \mathcal{C}$$

$$(x, a, b) \longmapsto A(x, a) \times \mathrm{I}_X(a, b)$$

determines the required left adjoint, see [14, 3.4.3 (iii)].

(c) When $\mathcal{C}$ has finite limits, the fibration cod from Example 2.2(c) is elementary.

Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a functor. For a class of arrows Θ in $\mathcal{B}$, say that an arrow φ in $\mathcal{E}$ is ***over*** Θ if $p(\varphi) \in \Theta$. Recall from [30] that an arrow $\varphi : A \longrightarrow B$ is ***locally epic with respect to*** p if, for every pair $\psi, \psi' : B \longrightarrow B'$ such that $p(\psi) = p(\psi')$, whenever $\psi\varphi = \psi'\varphi$ it is already the case that $\psi = \psi'$.

4.3 Remark.

(i) When p is a fibration, φ is locally epic with respect to $p$if and only if $\psi\varphi = \psi'\varphi$ implies $\psi = \psi'$ just for vertical arrows ψ and ψ'.
(ii) Every cocartesian arrow is locally epic with respect to p.
(iii) An arrow $\varphi : A \longrightarrow B$ that factors as a cocartesian arrow followed by a vertical v, is locally epic with respect to $p$if and only if v is locally epic with respect to p. Moreover, if p is a fibration, this happens if and only if v is epic in the fibre $\mathcal{E}_{p(B)}$.

Assume from now on that $p : \mathcal{E} \longrightarrow \mathcal{B}$ is a fibration. We need to introduce a few definitions that will be instrumental in formulating the main Theorem 4.7.

It is well-known that, whenever there is a commuting square in $\mathcal{E}$ with two opposite arrows cartesian and sitting over a pullback square in $\mathcal{B}$

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\varphi'"'] & A \arrow[d, "\varphi"] & & W' \arrow[r] \arrow[d] & W \arrow[d] \\ B' \arrow[r] & B \arrow[r, phantom, "\overset{p}{\longmapsto}"] & & Z' \arrow[r] & Z \end{tikzcd} \tag{1}$$

then the left-hand square in (1) is a pullback too.

We say that a class Φ of arrows in $\mathcal{E}$ is ***product-stable*** when, in every diagram (1) where the right-hand pullback is of the form

$$\begin{tikzcd} U \times Y \arrow[r, "g \times Y"] \arrow[d, "U \times f"'] & V \times Y \arrow[d, "V \times f"] \\ U \times X \arrow[r, "g \times X"'] & V \times X \end{tikzcd}$$

and φ is in Φ, also φ' is in Φ. In such a situation, we may say that φ' is a ***parametrised reindexing of*** φ ***along*** g.

Recall from Notation 3.4 that we write Δ for the class of arrows of parametrised diagonals, *i.e.* arrows of the form $\mathrm{pr}_{1,2,2} : Z \times X \longrightarrow Z \times X \times X$, in $\mathcal{B}$.

4.4 Lemma. *Suppose $\mathcal{B}$ has binary products and $p : \mathcal{E} \longrightarrow \mathcal{B}$ is a fibration with left adjoints to reindexing along arrows in Δ. Let Φ be the class of cocartesian lifts of arrows in Δ, which exist thanks to Remark 2.1. Then the following are equivalent:*

(i) *The class Φ is product-stable.*
(ii) *The left adjoints satisfy the Beck–Chevalley Condition.*

Proof. (i)$\Rightarrow$ (ii) Given $f : Y \longrightarrow Z$ and an object A over $Z \times X$, in the commutative diagram

$$\begin{tikzcd} (f \times X)^* A \arrow[r] \arrow[d, dashed] & A \arrow[d] \\ (f \times X \times X)^*(\exists_{Z,X} A) \arrow[r] & \exists_{Z,X} A \end{tikzcd}$$

the dotted arrow is cocartesian by (i). The statement follows by Remark 2.1.
(ii)$\Rightarrow$ (i) Consider a diagram like in (1) for f in Δ and φ cocartesian over it:

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\varphi'"'] & A \arrow[d, "\varphi"] & & U \times X \arrow[r, "g \times X"] \arrow[d, "\mathrm{pr}_{1,2,2}"'] & V \times X \arrow[d, "\mathrm{pr}_{1,2,2}"] \\ B' \arrow[r] & B \arrow[r, phantom, "\overset{p}{\longmapsto}"] & & U \times X \times X \arrow[r, "g \times X \times X"'] & V \times X \times X \end{tikzcd}$$

So, by Remark 2.1, it is the case that $B \cong \exists_{V,X}(A)$. Hence $B' \cong \exists_{U,X}(A')$ by (ii) which yields that also φ' is cocartesian, again by Remark 2.1. $\square$

We say that Φ is ***pairable*** if, for every $\varphi : A \longrightarrow B$ and every cartesian arrow $\psi : C' \longrightarrow C$ over $p(\varphi)$, the arrow $\varphi \wedge \psi \colon = \langle \varphi\pi_1, \psi\pi_2 \rangle : A \wedge C' \longrightarrow B \wedge C$ is in Φ whenever φ is.

$$\begin{tikzcd} A \arrow[d, "\varphi"'] & A \wedge C' \arrow[l] \arrow[r] \arrow[d, "\varphi \wedge \psi"] & C' \arrow[d, "\psi"] \\ B & B \wedge C \arrow[l] \arrow[r] & C \end{tikzcd}$$

4.5 Lemma. *Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration with finite products and suppose that it has left adjoints to reindexing along arrows in Δ. Let Φ be the class of cocartesian lifts of arrows in Δ, which exist thanks to Remark 2.1. Then the following are equivalent:*

(i) *The class Φ is pairable.*
(ii) *The left adjoints satisfy the Frobenius Reciprocity.*

Proof. For a parametrised diagonal $\mathrm{pr}_{1,2,2} : Z \times X \longrightarrow Z \times X \times X$, an object A over $Z \times X$, and an object B over $Z \times X \times X$, consider a commutative diagram

$$\begin{tikzcd}
A & A \wedge {\mathrm{pr}_{1,2,2}}^*(B) \arrow[l] \arrow[r] \arrow[d] & {\mathrm{pr}_{1,2,2}}^*(B) \arrow[d] \\
\exists_{Z,X}(A) \arrow[from=u, "\varphi"'] & \exists_{Z,X}(A) \wedge B \arrow[l] \arrow[r] & B
\end{tikzcd}$$

By Remark 2.1 the middle arrow is cocartesian if and only if $\exists_{Z,X}(A) \wedge B \cong \exists_{Z,X}(A \wedge {\mathrm{pr}_{1,2,2}}^*(B))$. Hence the statement follows. $\square$

Let $A \in \mathcal{E}_{Z\times X}$, let $\rho : A' \longrightarrow A$ and $\sigma : A \longrightarrow A'$ be cartesian arrows over $\mathrm{pr}_{1,2} : Z \times X \times X \longrightarrow Z \times X$ and $\mathrm{pr}_{1,2,2} : Z \times X \longrightarrow Z \times X \times X$, respectively, and let $\varphi : B \longrightarrow C$ be over $\mathrm{pr}_{1,2,2}$. Then we say that the arrow $\varphi \wedge \sigma : B \wedge A \longrightarrow C \wedge A'$ is a ***split pairing of*** φ ***with*** A.

4.6 Remark. We can provide a more explicit description of the class of arrows Λ_X introduced in Notation 3.4. Recall that, for X in $\mathcal{B}$, the class Λ_X consists of all arrows isomorphic to one of the form $\delta^Z_A : A \longrightarrow ({\mathrm{pr}_{1,2}}^*A) \wedge ({\mathrm{pr}_{2,3}}^*\mathrm{I}_X)$ for Z in $\mathcal{B}$ and $A \in \mathcal{E}_{Z\times X}$. Using the terminology introduced in this section, these are the arrows that are obtained from the loop ∂_X by applying a parametrised reindexing followed by a split pairing. More explicitly, the arrows in Λ_X are precisely those $\psi : A \longrightarrow B$ over Δ that fit in a commutative diagram

$$\begin{tikzcd}
 & & \top_X \arrow[rr, "\partial_X"] & & \mathrm{I}_X & \\
 & \top_{Z\times X} \arrow[ur] \arrow[rr] & & {\mathrm{pr}_{2,3}}^*\mathrm{I}_X \arrow[ur] & & \\
{} \arrow[ddd, "p"'] & A \arrow[u, "!"] \arrow[rr, "\psi"] \arrow[d, "\mathrm{id}"'] & & B \arrow[u, "\pi_2"'] \arrow[d, "\pi_1"] & & \\
 & A \arrow[rr] \arrow[rrrr, bend right, "\mathrm{id}"'] & & {\mathrm{pr}_{1,2}}^*A \arrow[rr] & & A \\
 & & X \arrow[rr, "\mathrm{pr}_{1,1}"'] & & X \times X & \\
{} & Z \times X \arrow[ur, "\mathrm{pr}_2"] \arrow[rr, "\mathrm{pr}_{1,2,2}"] \arrow[rrrr, bend right, "\mathrm{id}"'] & & Z \times X \times X \arrow[ur, "\mathrm{pr}_{2,3}"] \arrow[rr, "\mathrm{pr}_{1,2}"] & & Z \times X
\end{tikzcd} \tag{2}$$

where the two arrows π_1, π_2 exhibit B as a fibred product of ${\mathrm{pr}_{1,2}}^*A$ and ${\mathrm{pr}_{2,3}}^*\mathrm{I}_X$ over $Z \times X \times X$.

In particular, it follows that the class Λ_X is closed under parametrised reindexing along any arrow in $\mathcal{B}$, and it is stable under split pairing with any object in $\mathcal{E}_{Z\times X}$ for Z in $\mathcal{B}$.

Note that the definition of split pairing can be given in more generality, replacing the section-retraction pair $\mathrm{pr}_{1,1}, \mathrm{pr}_1$ on X with an arbitrary section-retraction pair s, r on X. Accordingly, it is also possible to define a class of arrows similarly to Λ_X, but taking an arbitrary section-retraction pair s, r on X and starting from any arrow φ in $\mathcal{E}$ over s. In particular, let Λ'_X be the class of arrows obtained starting with the pair $\mathrm{pr}_{1,1}, \mathrm{pr}_2$ and the arrow ∂_X. Then $\Lambda'_X = \Lambda_X$.

We are at last in a position to state the main result of the paper.

4.7 Theorem. *Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a fibration with products. The following are equivalent:*

(i) *The fibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ is elementary.*

(ii) a. *Every arrow in Δ has all cocartesian lifts.*
  b. *The cocartesian arrows over Δ are product-stable and pairable.*

(iii) a. *For every object X in $\mathcal{B}$ there are an object I_X over $X \times X$ and a cocartesian arrow $\partial_X : \top_X \longrightarrow \mathrm{I}_X$ over $\mathrm{pr}_{1,1} : X \longrightarrow X \times X$.*
  b. *The cocartesian arrows over Δ are product-stable and pairable.*

(iv) a. *The fibration p has strict productive transporters.*
  b. *Every arrow in Λ is locally epic with respect to pover Δ.*

(v) a. *For every X in $\mathcal{B}$ there are an object I_X over $X \times X$ and an arrow $\partial_X : \top_X \longrightarrow \mathrm{I}_X$ over $\mathrm{pr}_{1,1} : X \longrightarrow X \times X$.*
  b. *The arrows in Λ are cocartesian over Δ.*

(vi) *For every X in $\mathcal{B}$ there is an object I_X over $X \times X$ such that, for every $Z, X \in \mathcal{B}$ and every A over $Z \times X$, the assignment*

$$A \longmapsto ({\mathrm{pr}_{1,2}}^*A) \wedge ({\mathrm{pr}_{2,3}}^*\mathrm{I}_X)$$

gives rise to a left adjoint to the reindexing functor ${\mathrm{pr}_{1,2,2}}^* : \mathcal{E}_{Z\times X\times X} \longrightarrow \mathcal{E}_{Z\times X}$.

Proof. (i)$\Leftrightarrow$ (ii) By Remark 2.1, the equivalence follows from Lemma 4.4 and Lemma 4.5.
(ii)$\Rightarrow$ (iii) The object I_X over $X \times X$ and the arrow ∂_X are obtained taking a cocartesian lift of $\mathrm{pr}_{1,1} : X \longrightarrow X \times X$ from $\top_X$.
(iii)$\Rightarrow$ (iv) We begin proving condition (iv).b. By Remark 4.6 arrows in Λ are obtained from some loop ∂_X first by parametrised reindexing and then with a split pairing, as in diagram (2). Since ∂_X is a cocartesian arrow over Δ and these are product-stable and pairable, arrows in Λ are cocartesian, in particular locally epic with respect to p.

Now, to prove condition (iv).a, first we construct a strict transporter on an object X. For this, it is enough to construct a carrier t_A for A over X. Note that the arrow δ_A from Notation 3.4, being in Λ, is cocartesian. The universal property of δ_A yields a unique arrow t_A as in the diagram

$$\begin{tikzcd} A \arrow[rr, "\delta_A"] \arrow[dr, "\mathrm{id}_A"'] & & ({\mathrm{pr}_1}^*A) \wedge \mathrm{I}_X \arrow[dl, dashed, "\mathrm{t}_A"] \\ & A & \end{tikzcd} \qquad \overset{p}{\longmapsto} \qquad \begin{tikzcd} X \arrow[rr, "\mathrm{pr}_{1,1}"] \arrow[dr, "\mathrm{pr}_1"'] & & X \times X \arrow[dl, "\mathrm{pr}_2"] \\ & X & \end{tikzcd}$$

Finally, to prove that this choice of transporters is productive, let $X, Y \in \mathcal{B}$. We can rewrite the arrow $\partial_X \boxtimes \partial_Y : \top_{X\times Y} \longrightarrow \mathrm{I}_X \boxtimes \mathrm{I}_Y$ as the composite $\langle \partial'_X !, \iota\rangle \partial'_Y$ with the notation in Remark 3.11. The diagram therein also shows that ∂'_Y and $\langle \partial'_X !, \iota\rangle$ are in Λ_Y and Λ_X, respectively. It follows that each is cocartesian, thus so is $\partial_X \boxtimes \partial_Y$. Its universal property applied to $\partial_{X\times Y}$ then yields the required $\chi_{X,Y}$.

(iv)⇒ (v) There is only condition (v).b to prove: we need to show that, given X in $\mathcal{B}$, arrows in Λ_X are cocartesian. These are the arrows δ^Z_A, for Z in $\mathcal{B}$ and $A \in \mathcal{E}_{Z\times X}$, introduced in Notation 3.4 and described more explicitly in Remark 4.6. Let $\varphi : A \longrightarrow B$ be an arrow over $\mathrm{pr}_{1,2,2} : Z \times X \longrightarrow Z \times X \times X$ and consider the following diagram

$$
\begin{tikzcd}
A \arrow[rr, "\delta^Z_A"] \arrow[d, "\widehat{\varphi}"'] \arrow[ddr, bend left, "\varphi"] & & (\mathrm{pr}_{1,2}{}^*A) \wedge (\mathrm{pr}_{2,3}{}^*\mathrm{I}_X) \arrow[d, "(\mathrm{pr}_{1,2}{}^*\widehat{\varphi}) \wedge \mathrm{id}"] & \\
\mathrm{pr}_{1,2,2}{}^*B \arrow[rr, "\delta^Z_{(\mathrm{pr}_{1,2,2}{}^*B)}"] \arrow[dr, "\beta"'] & & (\mathrm{pr}'_{1,2,2}{}^*B) \wedge (\mathrm{pr}_{2,3}{}^*\mathrm{I}_X) \arrow[dr, "\gamma"] & \\
& B & & (\mathrm{pr}_{1,2,3}{}^*B) \wedge (\mathrm{pr}_{3,4}{}^*\mathrm{I}_X) \arrow[ll, "\mathrm{t}^{Z\times X}_B"']
\end{tikzcd}
\tag{3}
$$

$$
\begin{tikzcd}
Z \times X \arrow[rr, "\mathrm{pr}_{1,2,2}"] \arrow[dr, "\mathrm{pr}_{1,2,2}"'] & & Z \times X \times X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\mathrm{pr}_{1,2,2,3}"] & \\
& Z \times X \times X & & Z \times X \times X \times X, \arrow[ll, "\mathrm{pr}_{1,2,4}"]
\end{tikzcd}
$$

where $\mathrm{pr}'_{1,2,2} = \mathrm{pr}_{1,2,2}\mathrm{pr}_{1,2} : Z \times X \times X \longrightarrow Z \times X \longrightarrow Z \times X \times X$, the arrow $\mathrm{t}^{Z\times X}_B$ is a strict parametrised carrier at B obtained by Proposition 3.13, β and γ are cartesian and $\widehat{\varphi}$ is the unique vertical arrow such that $\beta\widehat{\varphi} = \varphi$. We need to show that the vertical arrow $\mathrm{t}^B_{Z\times X}\gamma(\mathrm{pr}_{1,2}{}^*\widehat{\varphi} \wedge \mathrm{id})$ precomposed with δ^Z_A equals φ, and that it is the unique vertical such.

Since δ^Z_A is locally epic with respect to p, it follows that there is at most one vertical arrow $\varphi' : (\mathrm{pr}_{1,2}{}^*A) \wedge (\mathrm{pr}_{2,3}{}^*\mathrm{I}_X) \longrightarrow B$ such that $\varphi'\delta^Z_A = \varphi$. Therefore it is enough to show that diagram (3) commutes. To this aim we only need to show that

$$
\mathrm{t}^{Z\times X}_B \gamma \delta^Z_{(\mathrm{pr}_{1,2,2}{}^*B)} = \mathrm{t}^{Z\times X}_B \delta^{Z\times X}_B \beta = \beta
$$

which follows from $\mathrm{t}^{Z\times X}_B \delta^{Z\times X}_B = \mathrm{id}_B$ and commutativity of the diagram below, which holds by definition of the arrows involved.

$$\begin{tikzcd}
& \top_X \arrow[rrr, "\partial_X"] & & & \mathrm{I}_X \\
\top_{Z\times X} \arrow[ur] \arrow[rrr, "{\mathrm{pr}_{2,3}}^*\partial_X"] \arrow[dr] & & & {\mathrm{pr}_{2,3}}^*\mathrm{I}_X \arrow[ur] \arrow[dr] & \\
& \top_{Z\times X\times X} \arrow[uu] \arrow[rrr, "{\mathrm{pr}_{3,4}}^*\partial_X"] & & & {\mathrm{pr}_{3,4}}^*\mathrm{I}_X \arrow[uu] \\
{\mathrm{pr}_{1,2,2}}^*B \arrow[uu] \arrow[rrr, "\delta^Z_{({\mathrm{pr}_{1,2,2}}^*B)}"] \arrow[dr, "\beta"] \arrow[ddrrr, bend right] & & & ({\mathrm{pr}'_{1,2,2}}^*B)\wedge({\mathrm{pr}_{2,3}}^*\mathrm{I}_X) \arrow[uu] \arrow[dr, "\gamma"] \arrow[dd] & \\
& B \arrow[uu] \arrow[rrr, "\delta_B^{Z\times X\times X}"] \arrow[ddrrr, bend right] & & & ({\mathrm{pr}_{1,2,3}}^*B)\wedge({\mathrm{pr}_{3,4}}^*\mathrm{I}_X) \arrow[uu] \arrow[dd] \\
& & & {\mathrm{pr}_{1,2,2}}^*B \arrow[dr] & \\
& & & & {\mathrm{pr}_{1,2,3}}^*B
\end{tikzcd}$$

$$\begin{tikzcd}
& X \arrow[rrr, "\mathrm{pr}_{1,1}"] & & & X\times X \\
Z\times X \arrow[ur, "\mathrm{pr}_2"] \arrow[rrr, "\mathrm{pr}_{1,2,2}"'] \arrow[dr, "\mathrm{pr}_{1,2,2}"'] & & & Z\times X\times X \arrow[ur, "\mathrm{pr}_{2,3}"] \arrow[dr, "\mathrm{pr}_{1,2,2,3}"] & \\
& Z\times X\times X \arrow[uu, "\mathrm{pr}_3"'] \arrow[rrr, "\mathrm{pr}_{1,2,3,3}"'] & & & Z\times X\times X\times X \arrow[uu, "\mathrm{pr}_{3,4}"']
\end{tikzcd}$$

(v)$\Leftrightarrow$ (vi) This is just an instance of the equivalence in Remark 2.1.
(vi)$\Rightarrow$ (i) It is straightforward to verify that the left adjoints specified in (vi) satisfy the Beck–Chevalley condition for pullback squares of the form

$$\begin{tikzcd}
U\times X \arrow[r, "{\langle f,\mathrm{pr}_2\rangle}"] \arrow[d, "\mathrm{pr}_{1,2,2}"'] & V\times X \arrow[d, "\mathrm{pr}_{1,2,2}"] \\
U\times X\times X \arrow[r, "{\langle f,\mathrm{pr}_{2,3}\rangle}"'] & V\times X\times X
\end{tikzcd} \tag{4}$$

where $f : U\times X \longrightarrow V$, that is, the canonical arrow $\exists_{U,X}\langle f,\mathrm{pr}_2\rangle^* A \longrightarrow \langle f,\mathrm{pr}_{2,3}\rangle^* \exists_{V,X}(A)$ is iso.

The Beck–Chevalley condition in Definition 4.1 follows as a particular case. Frobenius reciprocity follows using the Beck–Chevalley condition for the pullback square

$$\begin{tikzcd}
Z\times X \arrow[rr, "\mathrm{pr}_{1,2,2}"] \arrow[d, "\mathrm{pr}_{1,2,2}"'] & & Z\times X\times X \arrow[d, "\mathrm{pr}_{1,2,3,3}"] \\
Z\times X\times X \arrow[r, "\mathrm{pr}_{1,2,2,3}"'] & Z\times X\times X\times X \arrow[r, "\sim", "\mathrm{pr}_{1,2,4,3}"'] & Z\times X\times X\times X
\end{tikzcd}$$

and observing that isomorphisms of the form ${\mathrm{pr}_{1,3,2}}^* : \mathcal{E}_{V\times Y\times Y} \longrightarrow \mathcal{E}_{V\times Y\times Y}$ preserve the left adjoints in the sense that ${\mathrm{pr}_{1,3,2}}^*\exists_{V,Y} \cong \exists_{V,Y}$, since $\mathrm{pr}_{1,3,2}\mathrm{pr}_{1,2,2} = \mathrm{pr}_{1,2,2}$. $\square$

4.8 Remark. As it follows from the above proof of (vi)⇒ (i) in Theorem 4.7, the left adjoints in an elementary fibrations turn out to satisfy the Beck–Chevalley condition for pullback squares of the form (4), which are more general than those in Definition 4.1.

4.9 Remark. Using condition (iii) in Theorem 4.7, or condition 4.7(v).b, and Remark 3.11 we easily see that, in an elementary fibration, the arrows $\partial_X \boxtimes \partial_Y$ and $\partial_{X\times Y}$ are both cocartesian. It follows that the canonical arrow $\omega_{X,Y} : \mathrm{I}_{X\times Y} \longrightarrow \mathrm{I}_X \boxtimes \mathrm{I}_Y$ of Remark 3.7 is the inverse to the arrow $\chi_{X\times Y}$ of Definition 3.3.

4.10 Remark. Since faithful fibrations are equivalent to indexed posets, the equivalence between condition (i) and condition (iv) in Theorem 4.7 gives Proposition 2.4 of [6].

4.11 Proposition. *If $p : \mathcal{E} \longrightarrow \mathcal{B}$ is an elementary fibration, $\mathcal{A}$ is a category with finite products, and $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor which preserves finite products, then the fibration $F^*p : F^*\mathcal{E} \longrightarrow \mathcal{A}$ is also elementary.*

Proof. Since F preserves finite products, the fibration F^*p has finite products. To see that F^*p is elementary, we apply Theorem 4.7(iv). For V an object in $\mathcal{A}$, a transporter on $F(V)$ for the fibration p is also a transporter on V for the fibration F^*p since the diagram

$$FV \xleftarrow{F(\mathrm{pr}_1)} F(V\times V) \xrightarrow{F(\mathrm{pr}_2)} FV$$

is a product in $\mathcal{B}$. Because of this, also condition (iv).b for F^*p follows immediately from the same condition for p. □

5. Elementary fibrations and algebraic weak factorisation system

Consider the fibration $\mathsf{cod}\restriction_{\mathcal{R}} : \mathcal{R} \longrightarrow \mathcal{C}$ from Example 2.2(c) associated to the subcategory $\mathcal{R}$ of a weak factorisation system $(\mathcal{L}, \mathcal{R})$. As we saw in Examples 3.8(c), the fibration $\mathsf{cod}\restriction_{\mathcal{R}}$ has strict productive transporters when the base category $\mathcal{C}$ has finite products, $\mathcal{L}$ is closed under products and closed under pullbacks along arrows in $\mathcal{R}$. In this case, for every X in $\mathcal{C}$ and $(f_0, f_1) \in \Lambda_X$, the arrow f_1 is in $\mathcal{L}$. Indeed, r_X is in $\mathcal{L}$ by construction and, for every $\delta^Z_A = (\mathrm{pr}_{1,2,2}, \langle \mathrm{id}_A, \mathrm{r}_X\mathrm{pr}_2 a\rangle)$, the arrow $\langle \mathrm{id}_A, \mathrm{r}_X\mathrm{pr}_2 a\rangle$ is a pullback along an arrow in $\mathcal{R}$ of a product of r_X as in the diagram

$$\begin{tikzcd}
A \arrow[r, "\langle \mathrm{id}_A, \mathrm{r}_X\mathrm{pr}_2 a\rangle"'] \arrow[d, "a"'] \arrow[rrr, bend left, "\mathrm{id}_A"] & A\times_X PX \arrow[rr] \arrow[d] & & A \arrow[d, "a"] \\
Z\times X \arrow[r, "Z\times \mathrm{r}_X"] \arrow[rrr, bend right, "\mathrm{id}_{Z\times X}"'] & Z\times PX \arrow[r] & Z\times X\times X \arrow[r, "\mathrm{pr}_{1,2}"] & Z\times X
\end{tikzcd}$$

where the right-hand square is a pullback.

5.1 Lemma. *Let $\mathcal{F}$ be a full subcategory of $\mathcal{C}^{\mathbf{2}}$ closed under pullbacks. Given an arrow in $\mathcal{F}$*

$$\begin{tikzcd} A \arrow[r, "f_1"'] \arrow[d, "a"'] & B \arrow[d, "b"] \\ X \arrow[r, "f_0"'] & Z, \end{tikzcd}$$

the following are equivalent:

(i) *The arrow (f_0, f_1) is locally epic with respect to $\mathsf{cod}\restriction_{\mathcal{F}}$.*
(ii) *Every left lifting problem for f_1 against arrows in $\mathcal{F}$ has at most one solution.*

Proof. (i)⇒(ii) It is enough to show that a lifting problem of the form

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f_1"'] & C \arrow[d, "c"] \\ B \arrow[r, "\mathrm{id}_B"'] & B \end{tikzcd}$$

for $c \in \mathcal{F}$ has at most one solution. Let then $g, g' : B \longrightarrow C$ be two diagonal fillers. They fit in the diagram below which commutes except for the two parallel arrows.

$$\begin{tikzcd} A \arrow[r, "f_1"'] \arrow[d, "a"'] \arrow[rr, bend left, "h"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] \arrow[d, "b"'] & C \arrow[d, "bc"] \\ X \arrow[r, "f_0"'] & Y \arrow[r, "\mathrm{id}_Y"'] & Y \end{tikzcd}$$

But, since by hypothesis (f_0, f_1) is locally epic with respect to $\mathsf{cod}\restriction_{\mathcal{F}}$, also $g = g'$.
(ii)⇒(i) Let $(k, g), (k, g') : b \longrightarrow c$ be such that $(f_0, f_1)(k, g) = (f_0, f_1)(k, g')$. Then the commutative diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f_1"'] & C \arrow[d, "c"] \\ B \arrow[r, "kb"'] \arrow[ur, shift left, "g"] \arrow[ur, shift right, "g'"'] & Z \end{tikzcd}$$

exhibits g and g' as solutions to a lifting problem for f_1. It follows that $g = g'$. □

Combining Lemma 5.1 with Theorem 4.7 we immediately obtain a sufficient condition for the fibration $\mathsf{cod}\restriction_{\mathcal{R}}$ to be elementary.

5.2 Corollary. *Let $(\mathcal{L}, \mathcal{R})$ be a weak factorisation system on a category $\mathcal{C}$ with finite products such that $\mathcal{L}$ is closed under products and under pullbacks along arrows in $\mathcal{R}$. If $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system i.e. diagonal fillers are unique, then the fibration $\mathsf{cod}\restriction_{\mathcal{R}}$ is elementary.*

If the factorisation system $(\mathcal{L}, \mathcal{R})$ is not orthogonal, the fibration $\mathsf{cod}{\restriction}_{\mathcal{R}}$ is not elementary in general. An instance of this situation is obtained in Proposition 5.9 taking as $\mathcal{R}$ the class of isofibrations that can be equipped with a cleavage. Those isofibrations appear as the right class of maps in one of the two weak factorisation systems which contribute to define the canonical, or "folk", Quillen model structure on $\mathcal{C}at$, whose weak equivalences are equivalences of categories. Nevertheless, the category of split cloven isofibrations and morphisms that preserve the cleavage strictly, fibred over $\mathcal{C}at$ or $\mathcal{G}pd$, is elementary, as we show in Proposition 5.4.

The additional algebraic structure of a cleavage is conveniently expressed using the theory of algebraic weak factorisation systems. Recall from [12], see also [5], that an ***algebraic weak factorisation system*** (L, M, R) on a category $\mathcal{C}$ consists of functors $M : \mathcal{C}^{\mathbf{2}} \longrightarrow \mathcal{C}$, $R : \mathcal{C}^{\mathbf{2}} \longrightarrow \mathcal{C}^{\mathbf{2}}$, and $L : \mathcal{C}^{\mathbf{2}} \longrightarrow \mathcal{C}^{\mathbf{2}}$ giving rise to a functorial factorisation

$$\begin{tikzcd} A \arrow[r, "Lf"] \arrow[rr, bend right, "f"'] & Mf \arrow[r, "Rf"] & B, \end{tikzcd}$$

and suitable monad (η, μ) and comonad (ϵ, Δ) structures on R and L respectively, together with a distributive law between them. Let R-**Alg** be the category of algebras for the monad on R and let R-**Map** be the category of algebras for the pointed endofunctor on R. Similarly, let L-**Coalg** and L-**Map** be the categories of coalgebras for the comonad on L and the pointed endofunctor on L, respectively. When $\mathcal{C}$ has finite limits, the two forgetful functors

$$\begin{tikzcd} R\text{-}\mathbf{Alg} \arrow[r, "U'"] \arrow[dr, "\mathsf{S}"'] & R\text{-}\mathbf{Map} \arrow[r, "U"] \arrow[d, "\mathsf{N}"] & \mathcal{C}^{\mathbf{2}} \arrow[dl, "\mathsf{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

are homomorphisms of fibrations with finite products. Let $\mathcal{A}$ and $\mathcal{R}$ denote their full images in $\mathcal{C}^{\mathbf{2}}$, respectively.

$$\begin{tikzcd} R\text{-}\mathbf{Alg} \arrow[rr] \arrow[ddr, "U'"] \arrow[dddrr, bend right, "\mathsf{S}"'] & & \mathcal{A} \arrow[drr, hook] \arrow[ddr, hook] \arrow[ddd] & & \\ & & & & \mathcal{C}^{\mathbf{2}} \arrow[ddll, bend left, "\mathsf{cod}"] \\ & R\text{-}\mathbf{Map} \arrow[rr] \arrow[dr, "\mathsf{N}"] & & \mathcal{R} \arrow[ur, hook] \arrow[dl] & \\ & & \mathcal{C} & & \end{tikzcd}$$

5.3 Remark. The category $\mathcal{R}$ is the closure of $\mathcal{A}$ under retracts. In fact, it is straightforward to check that this holds for every monad. By duality, the full image $\mathcal{L}$ of L-**Map** in $\mathcal{C}^{\mathbf{2}}$ is the closure under retract of the full image of L-**Coalg**.

As discussed in [5, 2.7] and [10, Proposition 2.3], the underlying arrow of an (R, η)-algebra has the right lifting property against any arrow that can be given an (L, ϵ)-algebra structure. As the dual holds as well, the pair $(\mathcal{L}, \mathcal{R})$ forms a weak factorisation system, the ***underlying weak factorisation system*** of (L, M, R).

The category $\mathcal{C}at$ of small categories admits an algebraic weak factorisation system $(\mathsf{L}, \mathsf{M}, \mathsf{R})$ such that the fibrations $\mathsf{S} : \mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{C}at$ and $\mathsf{N} : \mathsf{R}\text{-}\mathbf{Map} \longrightarrow \mathcal{C}at$ are equivalent to the fibrations of split cloven isofibrations and of normal cloven isofibrations, respectively, with arrows the commutative squares that preserve the cleavage strictly. We recall the construction below, after few remarks. Section 4 of [10] describes the same algebraic weak factorisation system but restricted to the category Gpd of small groupoids.

The underlying weak factorisation systems $(\mathcal{L}, \mathcal{R})$ of the algebraic weak factorisation system $(\mathsf{L}, \mathsf{M}, \mathsf{R})$ is equivalent to the (acyclic cofibrations, fibrations) weak factorisation system of the canonical Quillen model structures on Cat that we mentioned earlier, in the sense that the category $\mathcal{R}$ is equivalent to the full subcategory of $\mathit{Cat}^{\mathbf{2}}$ on the isofibrations that can be equipped with a cleavage. Indeed, $\mathcal{R}$ is the full image of R-**Map**, and any cleavage can be turned into a normal one.

The full image $\mathcal{G}$ of R-**Alg** in $\mathit{Gpd}^{\mathbf{2}}$, instead, is equivalent to the fibration that underlies Hofmann and Streicher's groupoid model of Martin-Löf type theory as described in [13]. This is the fibration whose fibre over a small groupoid $\mathbb{A}$ is the category whose objects are functors $\mathbb{A} \longrightarrow \mathit{Gpd}$, and whose morphisms are lax 2-natural transformations, regarding $\mathbb{A}$ as a 2-discrete 2-category. Gambino and Larrea [10] have identified suitable conditions on an algebraic weak factorisation system (L, M, R) that make the fibration of (R, η)-algebras into a model of Martin-Löf type theory with identity types. As shown in [10], these conditions are met by the algebraic weak factorisation system $(\mathsf{L}, \mathsf{M}, \mathsf{R})$ on Gpd, thus yielding a version of the groupoid model with normal isofibrations instead of split ones. More recently, the results in [10] have been lifted to the fibration R-**Alg** of algebras for the monad of an algebraic weak factorisation system [32], thus providing an algebraic presentation of Hofmann and Streicher's groupoid model.

None of the three fibrations on $\mathcal{G}$, $\mathcal{R}$ and R-**Map**, over Cat or Gpd, is elementary, as we show in Corollary 5.7 and Proposition 5.9. Instead, the fibration on R-**Alg**, over Cat or Gpd, is shown to be elementary in Proposition 5.4.

The algebraic weak factorisation $(\mathsf{L}, \mathsf{M}, \mathsf{R})$ on Cat is constructed as follows. For a functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ between small categories, $\mathsf{M}F$ is the category whose objects are pairs $(A, x : B \xrightarrow{\sim} FA)$ where A is an object in $\mathbb{A}$ and x is an iso in $\mathbb{B}$, and whose arrows $(b, a) : (A, x : B \xrightarrow{\sim} FA) \longrightarrow (A', x' : B' \xrightarrow{\sim} FA')$ are pairs of an arrow $b : B \longrightarrow B'$ in $\mathbb{B}$ and an arrow $a : A \longrightarrow A'$ in $\mathbb{A}$ such that the square

$$\begin{tikzcd} B \arrow[r, "x"] \arrow[d, "b"'] & FA \arrow[d, "Fa"] \\ B' \arrow[r, "x'"] & FA' \end{tikzcd}$$

commutes. Denote $\mathsf{i}_{\mathbb{B}} : \mathrm{Iso}(\mathbb{B}) \longrightarrow \mathbb{B}^{\mathbf{2}}$ the embedding of the full subcategory $\mathrm{Iso}(\mathbb{B})$ of $\mathbb{B}^{\mathbf{2}}$ on the isos. Write

$$\begin{tikzcd} \mathrm{Iso}(\mathbb{B}) \arrow[rr, "\mathsf{i}_{\mathbb{B}}"] \arrow[dr, shift left=2, "\mathsf{c}_{\mathbb{B}}"] \arrow[dr, shift right=2, "\mathsf{d}_{\mathbb{B}}"'] & & \mathbb{B}^{\mathbf{2}} \arrow[dl, shift right=2, "\mathsf{dom}"'] \arrow[dl, shift left=2, "\mathsf{cod}"] \\ & \mathbb{B} \arrow[ul, "\mathsf{r}_{\mathbb{B}}" description] \arrow[ur, "\mathsf{id}" description] & \end{tikzcd}$$

the restrictions to $\mathrm{Iso}(\mathbb{B})$ of the three structural functors. Note that $\mathsf{M}F$ appears in the pullback of categories and functors

$$\begin{tikzcd} \mathsf{M}F \arrow[r, "F'"] \arrow[d, "\mathsf{c}'_{\mathbb{B}}"'] & \mathrm{Iso}(\mathbb{B}) \arrow[d, "\mathsf{c}_{\mathbb{B}}"] \\ \mathbb{A} \arrow[r, "F"] & \mathbb{B} \end{tikzcd} \tag{5}$$

and the functorial factorisation is obtained directly from it:

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[dr, "\mathsf{L}F"] \arrow[ddr, bend right, "\mathrm{Id}_{\mathbb{A}}"'] & \mathbb{B} \arrow[dr, "\mathsf{r}_{\mathbb{B}}"] \arrow[drr, bend left, "\mathrm{Id}_{\mathbb{B}}"] & & \\
& \mathsf{M}F \arrow[r, "F'"] \arrow[d, "\mathsf{c}'_{\mathbb{B}}"'] \arrow[rr, bend right, "\mathsf{R}F"'] & \mathrm{Iso}(\mathbb{B}) \arrow[r, "\mathsf{d}_{\mathbb{B}}"] \arrow[d, "\mathsf{c}_{\mathbb{B}}"'] & \mathbb{B} \\
& \mathbb{A} \arrow[r, "F"] & \mathbb{B}. &
\end{tikzcd}
$$

The factorisation extends to an algebraic weak factorisation system on ***Cat***: for the comonad the component at F of the counit is

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "\mathrm{Id}_{\mathbb{A}}"] \arrow[d, "\mathsf{L}F"'] & \mathbb{A} \arrow[d, "F"] \\
\mathsf{M}F \arrow[r, "\mathsf{R}F"] & \mathbb{B}
\end{tikzcd}
$$

while (the bottom component of) that of the comultiplication $\mathsf{L} \dot{\rightarrow} \mathsf{LL}$ is

$$\Delta_F(A, x : B \overset{\sim}{\rightarrow} FA) = (A, (x, \mathrm{id}_A) : (A, (A, x : B \overset{\sim}{\rightarrow} FA)) \overset{\sim}{\rightarrow} (\mathsf{L}F)A)$$

—from here onward we leave out the definition of a functor on arrows when it is obvious. The component at F of the unit of the monad is

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "\mathsf{L}F"] \arrow[d, "F"'] & \mathsf{M}F \arrow[d, "\mathsf{R}F"] \\
\mathbb{B} \arrow[r, "\mathrm{Id}_{\mathbb{B}}"] & \mathbb{B}
\end{tikzcd}
$$

and (the top component of) that of the multiplication $\mathsf{RR} \dot{\rightarrow} \mathsf{R}$ is

$$\mu_F((A, x : B \overset{\sim}{\rightarrow} FA), x' : B' \overset{\sim}{\rightarrow} B) = (A, xx' : B' \overset{\sim}{\rightarrow} FA).$$

The required distributive law follows from the identities

$$(\mathsf{RL}F) \circ \Delta_F = \mathrm{id}_{\mathsf{M}F} = \mu_F \circ (\mathsf{LR}F),$$

see [5, 2.2].

5.4 Proposition. *The fibration* $\mathsf{S} : \mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \boldsymbol{Cat}$ *is elementary.*

Proof. We shall make good use Theorem 4.7 checking that the fibration S verifies condition (iv). To construct a transporter on the small category $\mathbb{B}$ consider first the functor $\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle : \mathrm{Iso}(\mathbb{B}) \longrightarrow \mathbb{B} \times \mathbb{B}$ together with the structure map $\mathsf{s}_{\mathbb{B}}$ defined by[2]

[2] This choice provides a stable functorial choice of path objects in the sense of [10, Definition 2.8].

$$(y : B_2 \xrightarrow{\sim} B_1, (b_1, b_2) : (B_1', B_2') \xrightarrow{\sim} (B_1, B_2)) \longmapsto B_2' \xrightarrow{b_1^{-1} y b_2} B_1' \quad .$$

$$\begin{tikzcd}
\mathsf{M}\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle \arrow[rr, "\mathsf{s}_{\mathbb{B}}"] \arrow[d, "\mathsf{R}\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle"'] & & \mathrm{Iso}(\mathbb{B}) \arrow[d, "\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle"] \\
\mathbb{B} \times \mathbb{B} \arrow[rr, "\mathrm{Id}_{\mathbb{B}\times\mathbb{B}}"] & & \mathbb{B} \times \mathbb{B}
\end{tikzcd}$$

Then to provide a loop on $\mathbb{B}$ it is enough to show that the pair $(\mathrm{pr}_{1,1}, \mathsf{r}_{\mathbb{B}})$ is a morphism from the algebra $(\mathrm{Id}_{\mathbb{B}}, \mathsf{R}\mathrm{Id}_{\mathbb{B}})$ to the algebra $(\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}})$, which is an easy diagram chase in

$$\begin{tikzcd}
\mathsf{M}\mathrm{Id}_{\mathbb{B}} \arrow[r, "\mathsf{M}(\mathrm{pr}_{1,1}, \mathsf{r}_{\mathbb{B}})"] \arrow[d, "\mathsf{R}\mathrm{Id}_{\mathbb{B}}"'] & \mathsf{M}\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle \arrow[d, "\mathsf{s}_{\mathbb{B}}"] \\
\mathbb{B} \arrow[r, "\mathsf{r}_{\mathbb{B}}"] & \mathrm{Iso}(\mathbb{B}).
\end{tikzcd}$$

The construction of carriers is postponed to Lemma 5.6. But note that, once carriers are determined, transporters will be strictly productive as the iso

$$\langle \mathrm{Iso}(\mathrm{pr}_1), \mathrm{Iso}(\mathrm{pr}_2) \rangle : \mathrm{Iso}(\mathbb{B} \times \mathbb{C}) \cong \mathrm{Iso}(\mathbb{B}) \times \mathrm{Iso}(\mathbb{C})$$

is clearly a morphism of algebras.

Finally, to see that morphisms in Λ are locally epic with respect to S, consider an algebra $(\mathbb{A} \xrightarrow{F} \mathbb{I} \times \mathbb{B}, S)$; write $D : \mathbb{A} \times_{\mathbb{B}} \mathrm{Iso}(\mathbb{B}) \longrightarrow \mathbb{I} \times \mathbb{B} \times \mathbb{B}$ for the underlying functor of $({\mathrm{pr}_{1,2}}^* F) \wedge ({\mathrm{pr}_{2,3}}^* \langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle)$ and let $T : \mathsf{M}D \longrightarrow \mathbb{A} \times_{\mathbb{B}} \mathrm{Iso}(\mathbb{B})$ be its structure map, which maps an object $((A, x), (i, b_1, b_2) : (I, B_1, B_2) \xrightarrow{\sim} (FA, B))$ to $(S(A, (i, b_1)), b_1^{-1} x b_2)$. Note that there is a functor $K : \mathbb{A} \times_{\mathbb{B}} \mathrm{Iso}(\mathbb{B}) \longrightarrow \mathsf{M}(\mathrm{pr}_{1,2,2} F)$ mapping an iso $x : B \xrightarrow{\sim} \mathrm{pr}_2 FA$ to

$$(A, (\mathrm{id}_{FA}, x) : (FA, B) \xrightarrow{\sim} \mathrm{pr}_{1,2,2} FA)$$

and that the composite $\mathsf{M}(\mathrm{Id}_{\mathbb{I}\times\mathbb{B}\times\mathbb{B}}, \langle \mathrm{Id}_{\mathbb{A}}, \mathsf{r}_{\mathbb{B}} \mathrm{pr}_2 F \rangle) K : \mathbb{A} \times_{\mathbb{B}} \mathrm{Iso}(\mathbb{B}) \longrightarrow \mathsf{M}D$, is a section of the algebra structure map. Then for every vertical morphism $G : ({\mathrm{pr}_{1,2}}^* F) \wedge ({\mathrm{pr}_{2,3}}^* \langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle) \longrightarrow (F, S)$, it is the case that

$$G = GT\mathsf{M}(\mathrm{Id}_{\mathbb{I}\times\mathbb{B}\times\mathbb{B}}, \langle \mathrm{Id}_{\mathbb{A}}, \mathsf{r}_{\mathbb{B}} \mathrm{pr}_2 F \rangle) K = S\mathsf{M}(\mathrm{Id}_{\mathbb{I}\times\mathbb{B}\times\mathbb{B}}, G\langle \mathrm{Id}_{\mathbb{A}}, \mathsf{r}_{\mathbb{B}} \mathrm{pr}_2 F \rangle) K.$$

As $\delta_F^{\mathbb{I}} = (\mathrm{pr}_{1,2,2}, \langle \mathrm{Id}_{\mathbb{A}}, \mathsf{r}_{\mathbb{B}} \mathrm{pr}_2 F \rangle)$, algebra morphisms out of $(\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}})$ are determined by their precomposition with $\delta_F^{\mathbb{I}}$. □

5.5 Corollary. *The fibration* $\mathsf{S} : \mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{G}pd$ *is elementary.*

Proof. The algebraic weak factorisation system structure on $\mathcal{G}pd$ is obtained pulling back that on $\mathcal{C}at$ along the embedding $\mathcal{G}pd \longrightarrow \mathcal{C}at$. It follows that $\mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{G}pd$ is a change of base of $\mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{C}at$ along the embedding $\mathcal{G}pd \longrightarrow \mathcal{C}at$. Hence $\mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{G}pd$ is elementary by Proposition 4.11. □

5.6 Lemma. *Given an algebra* $(F : \mathbb{A} \longrightarrow \mathbb{B}, S : \mathsf{M}F \longrightarrow \mathbb{A})$ *in* $\mathsf{R}\text{-}\mathbf{Alg}$*, there is exactly one carrier for the loop* $(\mathrm{pr}_{1,1}, \mathsf{r}_{\mathbb{B}}) : (\mathrm{Id}_{\mathbb{B}}, \mathsf{R}\mathrm{Id}_{\mathbb{B}}) \longrightarrow (\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}})$ *and it is the case that* $(\mathrm{pr}_2, S) : {\mathrm{pr}_1}^*(F, S) \wedge (\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}}) \longrightarrow F.$

Proof. We may assume, without loss of generality, that the underlying functor of the algebra $\mathrm{pr}_1{}^*(F, S) \wedge (\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle, \mathsf{s}_{\mathbb{B}})$ is the diagonal $D : \mathsf{M}F \longrightarrow \mathbb{B} \times \mathbb{B}$ in the pullback of categories and functors

$$\begin{tikzcd} \mathsf{M}F \arrow[r, "F'"] \arrow[d, "{\langle \mathsf{c}'_{\mathbb{B}}, \mathsf{R}F \rangle}"'] \arrow[dr, "D"'] & \mathrm{Iso}(\mathbb{B}) \arrow[d, "{\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle}"] \\ \mathbb{A} \times \mathbb{B} \arrow[r, "F \times \mathrm{Id}_{\mathbb{B}}"'] & \mathbb{B} \times \mathbb{B} \end{tikzcd}$$

with the notation of diagram (5). The structure map $S_D : \mathsf{M}D \longrightarrow \mathsf{M}F$ is induced by those on F and $\langle \mathsf{c}_{\mathbb{B}}, \mathsf{d}_{\mathbb{B}} \rangle$ and maps a pair $(A, x : B \xrightarrow{\sim} FA), (b_1, b_2) : (B_1, B_2) \xrightarrow{\sim} (FA, B)$ to the pair $S(A, b_1), b_1^{-1} x b_2 : B_2 \xrightarrow{\sim} B_1$. A functor $T : \mathsf{M}F \longrightarrow \mathbb{A}$ in the second component of the carrier has to fit in the commutative diagram

$$\begin{tikzcd} \mathsf{M}F \arrow[r, "T"] \arrow[d, "D"'] & \mathbb{A} \arrow[d, "F"] \\ \mathbb{B} \times \mathbb{B} \arrow[r, "\mathrm{pr}_2"] & \mathbb{B} \end{tikzcd}$$

and, since it has to be a homomorphism of algebras, the following diagram must commute

$$\begin{tikzcd} \mathsf{M}D \arrow[r, "{\mathsf{M}(\mathrm{pr}_2, T)}"] \arrow[d, "S_D"'] & \mathsf{M}F \arrow[d, "S"] \\ \mathsf{M}F \arrow[r, "T"] & \mathbb{A}. \end{tikzcd} \tag{6}$$

Moreover, the strictness condition imposes that the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[d, "\mathsf{L}F"'] \arrow[dr, "\mathrm{Id}_{\mathbb{A}}"] & \\ \mathsf{M}F \arrow[r, "T"] & \mathbb{A} \end{tikzcd} \tag{7}$$

commutes. Note also that there is a functor $H : \mathsf{M}(\mathsf{R}F) \longrightarrow \mathsf{M}D$ such that $S_D H = \mu_F$ and $\mathsf{M}(\mathrm{pr}_2, T)H = \mathsf{M}(\mathrm{Id}_{\mathbb{B}}, T)$. An object of $\mathsf{M}(\mathsf{R}F)$ consists of a pair $(A, x : B \xrightarrow{\sim} FA)$ together with $x' : B' \xrightarrow{\sim} B$, and the functor H maps it to the pair consisting of (A, x) itself and (id_{FA}, x'). Precomposing diagram (6) with H and using (7) together with a triangular identity for the monad, the commutative diagram

$$\begin{tikzcd} \mathsf{M}F \arrow[dr, "{\mathsf{M}(\mathrm{Id}_{\mathbb{B}}, \mathsf{L}F)}"] \arrow[drr, bend left, "\mathrm{Id}_{\mathsf{M}F}"] \arrow[ddr, bend right, "\mathrm{Id}_{\mathsf{M}F}"'] & & \\ & \mathsf{M}(\mathsf{R}F) \arrow[r, "{\mathsf{M}(\mathrm{Id}_{\mathbb{B}}, T)}"] \arrow[d, "\mu_F"] & \mathsf{M}F \arrow[d, "S"] \\ & \mathsf{M}F \arrow[r, "T"] & \mathbb{A}, \end{tikzcd}$$

shows that the only possible choice for T is the structural functor $S : \mathsf{M}F \longrightarrow \mathbb{A}$, and it is straightforward to see that choice makes diagrams (6) and (7) commute. $\square$

5.7 Corollary. *The fibrations* $\mathsf{N} : \mathsf{R}\text{-}\mathbf{Map} \longrightarrow \mathcal{C}at$ *and* $\mathsf{N} : \mathsf{R}\text{-}\mathbf{Map} \longrightarrow \mathcal{G}pd$ *are not elementary.*

Proof. We prove the statement for the fibration over $\mathcal{C}at$, but the same argument applies to the one over $\mathcal{G}pd$.

The forgetful fibered functor $U' : \mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathsf{R}\text{-}\mathbf{Map}$ takes loops to loops and, since it preserves finite products, the fibration $\mathsf{N} : \mathsf{R}\text{-}\mathbf{Map} \longrightarrow \mathcal{C}at$ has a choice of strictly productive loops induced by that of S given in Proposition 5.4. However, by Lemma 5.6, a carrier for an algebra in $\mathsf{R}\text{-}\mathbf{Map}$ exists if and only if the algebra is an algebra for the monad.

Suppose now that $\mathsf{N} : \mathsf{R}\text{-}\mathbf{Map} \longrightarrow \mathcal{C}at$ is elementary. In particular, for every $\mathbb{B}$ there are $\iota_{\mathbb{B}} : I_{\mathbb{B}} \longrightarrow \mathbb{B} \times \mathbb{B}$ in $\mathsf{R}\text{-}\mathbf{Map}$ and $\delta_{\mathbb{B}} : \mathbb{B} \longrightarrow I_{\mathbb{B}}$ such that $(\mathrm{pr}_{1,1}, \delta_{\mathbb{B}}) : \mathrm{Id}_{\mathbb{B}} \longrightarrow \iota_{\mathbb{B}}$ is cocartesian with respect to N. It follows that the loop on $\mathbb{B}$ in N is a retract of the loop from S in the sense that there are functors σ and ρ such that the diagram of functors

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "\mathrm{Id}"] \arrow[d, "\delta_{\mathbb{B}}"'] & \mathbb{B} \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathsf{r}_{\mathbb{B}}"'] & \mathbb{B} \arrow[d, "\delta_{\mathbb{B}}"] \\
I_{\mathbb{B}} \arrow[r, "\sigma"] \arrow[d, "\iota_{\mathbb{B}}"'] & \mathrm{Iso}(\mathbb{B}) \arrow[r, "\rho"] \arrow[d] & I_{\mathbb{B}} \arrow[d, "\iota_{\mathbb{B}}"] \\
\mathbb{B} \times \mathbb{B} \arrow[r, "\mathrm{Id}"] & \mathbb{B} \times \mathbb{B} \arrow[r, "\mathrm{Id}"] & \mathbb{B} \times \mathbb{B}
\end{tikzcd} \tag{8}$$

commutes and the rows compose to identities. The functor ρ is obtained as a diagonal filler using the coalgebra and algebra structure of $\mathsf{r}_{\mathbb{B}}$ and $\iota_{\mathbb{B}}$, respectively, as in [5, 2.4]. The functor σ is obtained by cocartesianess of $(\mathrm{pr}_{1,1}, \delta_{\mathbb{B}})$. Note that $(\mathrm{Id}, \sigma) : \iota_{\mathbb{B}} \longrightarrow \langle \mathsf{c}, \mathsf{d} \rangle$ is an arrow in $\mathsf{R}\text{-}\mathbf{Map}$ by construction.

But the existence of such a retraction would make $\iota_{\mathbb{B}}$ into an algebra for the monad. Indeed, let $j : M\iota_{\mathbb{B}} \longrightarrow I_{\mathbb{B}}$ be the algebra map of $\iota_{\mathbb{B}}$. We need to show that the front face in the diagram of functors below commutes.

$$\begin{tikzcd}
\mathsf{MR}\langle \mathsf{c}, \mathsf{d} \rangle \arrow[rr, "\mathsf{M}(\mathrm{Id}, \mathsf{s}_{\mathbb{B}})"] \arrow[dd, "\mu_{\langle \mathsf{c}, \mathsf{d} \rangle}"'] & & \mathsf{M}\langle \mathsf{c}, \mathsf{d} \rangle \arrow[dd, "\mathsf{s}_{\mathbb{B}}"] & \\
& \mathsf{MR}\iota_{\mathbb{B}} \arrow[ul, "\mathsf{M}(\mathrm{Id}, \mathsf{M}(\mathrm{Id}, \sigma))"'] \arrow[rr, "\mathsf{M}(\mathrm{Id}, j)"] \arrow[dd, "\mu_{\iota_{\mathbb{B}}}"] & & \mathsf{M}\iota_{\mathbb{B}} \arrow[ul, "\mathsf{M}(\mathrm{Id}, \sigma)"'] \arrow[dd, "j"] \\
\mathsf{M}\langle \mathsf{c}, \mathsf{d} \rangle \arrow[rr, "\mathsf{s}_{\mathbb{B}}"] & & \mathrm{Iso}(\mathbb{B}) & \\
& \mathsf{M}\iota_{\mathbb{B}} \arrow[ul, "\mathsf{M}(\mathrm{Id}, \sigma)"] \arrow[rr, "j"'] & & I_{\mathbb{B}} \arrow[ul, "\sigma"']
\end{tikzcd}$$

The back face commutes since $\mathsf{s}_{\mathbb{B}}$, which we constructed in Proposition 5.4, is an algebra for the monad. The bottom, top and right-hand faces commute since (Id, σ) is a morphism of algebras. Finally, the left-hand face commutes by naturality of μ. Thus $\sigma j \mathsf{M}(\mathrm{Id}, j) = \sigma j \mu_{\iota_{\mathbb{B}}}$. It follows that the front face commutes as required since σ has a left inverse ρ.

As the fibred forgetful functor $U' : \mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathsf{R}\text{-}\mathbf{Map}$ is full and faithful, it reflects cocartesian arrows. Therefore $\delta_{\mathbb{B}}$ and $\mathsf{r}_{\mathbb{B}}$ are both cocartesian over $\mathrm{pr}_{1,1}$ in $\mathsf{R}\text{-}\mathbf{Alg}$ and, in turn, isomorphic in $\mathsf{R}\text{-}\mathbf{Map}$. But this would provide carries in $\mathsf{R}\text{-}\mathbf{Map}$ for the loop given by $\mathsf{r}_{\mathbb{B}}$, in contradiction with Lemma 5.6. □

5.8 Remark. Note that the argument in the proof of Proposition 5.4 which shows that arrows in Λ are locally epic with respect to $\mathsf{S} : \mathsf{R}\text{-}\mathbf{Alg} \longrightarrow \mathcal{C}at$can be repeated to show that the same arrows are locally epic with respect to $\mathsf{N} : \mathsf{R}\text{-}\mathbf{Map} \longrightarrow \mathcal{C}at$.

According to Corollary 5.7 and Remark 5.8, the fibration N of algebras for the pointed endofunctor, over $\mathcal{C}at$ or $\mathcal{G}pd$, fails to be elementary because it lacks carriers for all those isofibrations whose cleavage is not split.

The underlying weak factorisation system $(\mathcal{L}, \mathcal{R})$ of $(\mathsf{L}, \mathsf{M}, \mathsf{R})$ satisfies the conditions in Example 3.8(c). It follows that the fibration $\mathsf{cod}{\restriction}_{\mathcal{R}}$ over $\mathcal{C}at$ (or $\mathcal{G}pd$) can be provided with a choice of strictly productive transporters. In particular, the choice of loops is induced by that one in $\mathsf{R}\text{-}\mathbf{Alg}$ and, similarly, a carrier on an algebra (F, S) is given by the algebra map $S : \mathsf{M}F \longrightarrow \mathbb{A}$ itself. Contrary to the case of $\mathsf{R}\text{-}\mathbf{Map}$, this is possible since S is not required to be a morphism of algebras, *i.e.* to make diagram (6) in the proof of Lemma 5.6 commute. But, for the very same reason, this choice of carriers is not necessarily unique any more. Indeed, as we show in Proposition 5.9 these fibrations fail to be elementary because there are groupoids for which the loop induced by $\mathsf{R}\text{-}\mathbf{Alg}$ is not locally epic and, still, that is the only possible choice (up to isomorphism). The same argument applies to the full image $\mathcal{G}$ of the fibration $\mathsf{R}\text{-}\mathbf{Alg}$ of algebras for the monad, as well as to the same fibrations restricted to $\mathcal{G}pd$.

5.9 Proposition. *The fibrations* $\mathsf{cod}{\restriction}_{\mathcal{R}}$ *and* $\mathsf{cod}{\restriction}_{\mathcal{G}}$*, over* $\mathcal{C}at$ *or* $\mathcal{G}pd$*, are not elementary.*

Proof. We prove the statement for $\mathsf{cod}{\restriction}_{\mathcal{R}}$ over $\mathcal{C}at$, the same argument applies in the other cases. We begin as in the proof of Corollary 5.7: the fibration $\mathsf{cod}{\restriction}_{\mathcal{R}}$ inherits a choice of (strictly productive) loops from S and, assuming that $\mathsf{cod}{\restriction}_{\mathcal{R}}$ is elementary, we conclude that the "elementary" choice of loops is a retract of the one induced by S. The only difference up to here is that now the functor ρ in diagram (8) is obtained as a diagonal filler from $\mathsf{r}_{\mathbb{B}} \in \mathcal{L}$ and $\iota_{\mathbb{B}} \in \mathcal{R}$, which hold since $\mathsf{r}_{\mathbb{B}}$ is a (cofree) L-coalgebra and by the assumption that $\mathsf{cod}{\restriction}_{\mathcal{R}}$ is elementary, respectively.

Using the notation in (8), the proposition follows once we provide a groupoid $\mathbb{B}$ such that $I_{\mathbb{B}}$ is in fact isomorphic to $\mathrm{Iso}(\mathbb{B})$ over $\mathbb{B} \times \mathbb{B}$, but $\mathsf{r}_{\mathbb{B}}$ is not locally epic. Let K_n denote the clique on n vertices, seen as a groupoid (in fact, an equivalence relation) and consider $\mathbb{B} :\ = K_2 + K_2$. Thus $\mathbb{B} \times \mathbb{B} \cong K_4 + K_4 + K_4 + K_4$, $\mathrm{Iso}(\mathbb{B}) \cong K_4 + K_4$ and $\langle \mathsf{cod}, \mathsf{dom} \rangle$ is isomorphic to the inclusion $[\iota_1; \iota_4] : K_4 + K_4 \longrightarrow K_4 + K_4 + K_4 + K_4$. Being a subcategory of $K_4 + K_4$, the groupoid $I_{\mathbb{B}}$ has two connected components too, say $I_{\mathbb{B}} = I_1 + I_2$, and σ is of the form $\sigma_1 + \sigma_2$ where $\sigma_i : I_i \longrightarrow K_4$. We already know that σ_1 and σ_2 are faithful and injective on objects. To show that they are both isomorphisms it is enough to show that they are full and surjective on objects. Note first that σ_1 and σ_2 are both isofibrations since $[\iota_1; \iota_4]\sigma = \iota_{\mathbb{B}}$ is. The claim that σ is an isomorphism thus follows from the following two simple facts about isofibrations φ between equivalence relations:

1. If φ is injective on objects, it is also full.
2. If the codomain of φ is connected and its domain is non-empty, then φ is surjective on objects.

To see that $(\mathrm{pr}_{1,1}, \mathsf{r}_{\mathbb{B}})$ is not locally epic in $\mathcal{R}$, it is enough to consider one of the two connected components of $\mathrm{Iso}(\mathbb{B})$. Denote the vertices of K_4 as labelled by the isos in K_2:

$$\begin{tikzcd}
\mathrm{id}_0 \arrow[r] \arrow[d] \arrow[dr] & x \arrow[d] \arrow[dl] \\
x^{-1} \arrow[r] & \mathrm{id}_1
\end{tikzcd}$$

where we draw only some arrows generating K_4. Clearly, the functor $\mathsf{r}_{\mathbb{B}}$ maps K_2 onto the diagonal (x, x) from id_0 to id_1. Let φ be the automorphism of K_4 that swaps x and x^{-1} and fixes the rest. Then φ agrees on the diagonal (x, x) with the identity Id_{K_4}, but it is clearly not equal to it. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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