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Emptiness Formation in Polytropic Quantum Liquids

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Abstract. We study large deviations in interacting quantum liquids with the polytropic equation of state $P(\rho) \sim \rho^\gamma$, where ρ is density and P is pressure. By solving hydrodynamic equations in imaginary time we evaluate the instanton action and calculate the emptiness formation probability (EFP), the probability that no particle resides in a macroscopic interval of a given size. Analytic solutions are found for a certain infinite sequence of rational polytropic indexes γ and the result can be analytically continued to any value of $\gamma \geq 1$. Our findings agree with (and significantly expand on) previously known analytical and numerical results for EFP in quantum liquids. We also discuss interesting universal spacetime features of the instanton solution.

1. Introduction

Large deviations statistics in many-body systems has been a subject of the rapidly growing research efforts [1, 2, 3, 4] due to recent precision measurements of particle number fluctuations in ultra cold quantum gases [5, 6, 7]. Emptiness formation probability (EFP) is perhaps the most iconic and widely studied example of such large deviation. It is accessible through the Bethe ansatz [8] in certain integrable models [9, 10, 11, 12] and serves as the litmus test for validity of approximate non-perturbative techniques, such as the instanton calculus [13].

The EFP, $\mathcal{P}_{\rm EFP}(R)$, is the probability that no particles are found inside the space interval $[-R, R]$ in the ground state of e.g. a one-dimensional (1D) many-body system

$$\mathcal{P}_{\rm EFP}(R) = \prod_{i=1}^{N} \int_{|x_i| \geq R} dx_i \; |\Psi_{\rm GS}(x_1, x_2, ..., x_N)|^2. \tag{1}$$

Here $\Psi_{\rm GS}(x_1, x_2, ..., x_N)$ is the normalized ground state wave function of the N-particle system. Even in cases where $\Psi_{\rm GS}$ is known exactly (e.g. for free fermions, or through the Bethe Ansatz), it is still a formidable task to perform the multiple integrals over the restricted interval. The first discussion of such problem goes back to the random matrix theory (RMT) [14], where the probability that no eigenvalues are located within a certain energy interval was studied for different ensembles [15, 16].

Going beyond free fermions and random matrices, integrable spin-1/2 chains are probably the most studied systems in the context of EFP. The later is defined as the probability of measuring l aligned "up" spins in the ground state. Via the Jordan-Wigner transformation the problem becomes equivalent to the absence of quasiparticles on l consecutive sites [17] and EFP was found in terms of Fredholm determinants [9, 18, 17, 19]. The closed analytic expressions for EFP are available only in a few isolated cases in the parameter space [17, 20].

With few exceptions [21, 4] most studies have been focused on the asymptotic regime of large R, where EFP is exponentially small. This makes EFP suitable for semiclassical instanton approach, where $-\ln \mathcal{P}_{\rm EFP}(R)$ is given by classical action evaluated along a stationary trajectory of the imaginary time Euler-Lagrange equations [13]. Such trajectory is specified by imposing boundary conditions both in the distant "past" and "future", when the system is undisturbed, and at the observation time, when the rare fluctuation develops. Similar setup also shows up in studies of rare events in classical stochastic systems [22, 23, 24]. Even for classically integrable equations (such as eg. via inverse scattering technique) these problems are notoriously difficult to handle (for a very recent progress in this direction see Refs. [25, 26]). Although reasonably effective numerical methods has been developed [27, 28, 29], their applications still require significant time and computer resources.

In this paper, we focus on 1D polytropic liquid which is characterized by equation of state: $P(\rho) \sim \rho^\gamma$, where $P(\rho)$ is the pressure and the exponent γ is called the *polytropic index*. The value of γ is determined by the underlying microscopic model. For example, $\gamma = 3$ stands for non-interacting fermions and the corresponding analytic solution of the hydrodynamic equations was found by Abanov [30]. Weakly interacting bosons, described by $\gamma = 2$, were recently numerically studied in Ref. [31]. For the quasi-1D fermions, i.e. 3D fermions confined to 1D by a transverse harmonic trap, one finds $\gamma = 7/5$ [32]. Moreover, EFP in Calogero-Sutherland integrable model [33, 34, 35] was studied [36]. The leading term in its equation of state has the polytropic index $\gamma = 3$ [37, 38, 30, 39], conforming with the corresponding hydrodynamic solution ‡.

The goal of this work is to go beyond the above listed examples and find EFP in a 1D polytropic liquid with an arbitrary index. The classical hydrodynamics of polytropic liquids has been attracting attention of mathematicians since 1980's [41, 42, 43], when certain instances of classical integrability were discovered. Some techniques has been developed for initial condition problems [44, 45] However, the analytical closed form

‡ From the hydrodynamic perspective, the difference between the free fermions and the Calogero-Sutherland model is in renormalization of the sound velocity by the interaction parameter, λ, [40].

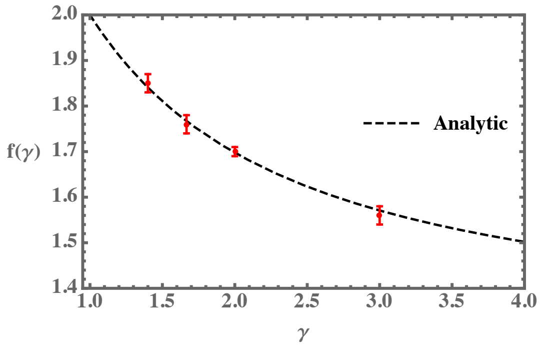

Figure 1. Universal function $f(\gamma)$. The black dashed line is Eq. (5), the red symbols are numerical results for $\gamma = 7/5, 5/3, 2, 3$. The numerical results for $\gamma = 2$ is taken from Ref. [31].

solution was achieved only for some values of γ [46]. Here we utilize these techniques to construct instanton solutions of EFP for an infinite sequence of rational indexes. This solution appears in a closed algebraic form, admitting a unique analytic continuation to an arbitrary value of $\gamma \geq 1$.

The sought instanton solution of the hydrodynamic equations of motion involves distortion of the density in the spatial region of the size of the emptiness, $\sim R$. This distortion persists for the time of the order R/v_s, where v_s is the hydrodynamic sound velocity in the liquid. Therefore the instanton action, equal to the negative logarithm of EFP, is expected to be proportional to R^2/v_s. These considerations motivate the scaling form of the leading EFP exponent:

$$\lim_{R\to\infty} \frac{-\ln \mathcal{P}_{\mathrm{EFP}}(R)}{R^2} = \frac{\rho_0}{\xi}\, f(\gamma), \tag{2}$$

where the equilibrium density of the 1D liquid ρ_0 and the quantum correlation length $\xi = \hbar/(mv_s)$ provide the correct dimensional prefactor in front of dimensionless function $f(\gamma)$. The speed of sound in this expression is determined from the equation of state by the thermodynamic relation,

$$mv_s^2 = \partial_\rho P(\rho)\Big|_{\rho=\rho_0}. \tag{3}$$

where $P(\rho)$ is the hydrodynamic pressure and m is mass of the particles. The polytropic equation of state with an exponent γ may thus be parametrized as

$$P(\rho) = \frac{mv_s^2}{\gamma\rho_0^{\gamma-1}}\,\rho^\gamma. \tag{4}$$

The analytic expression for the universal function $f(\gamma)$ in Eq. (2) is the main result of this paper. We found:

$$f(\gamma) = \frac{\pi\; 2^{\frac{\gamma-5}{\gamma-1}} \left[\Gamma\left(\frac{\gamma+1}{\gamma-1}\right)\right]^2}{\Gamma\left(\frac{3\gamma-1}{2\gamma-2}\right)\left[\Gamma\left(\frac{\gamma+1}{2\gamma-2}\right)\right]^3}. \tag{5}$$

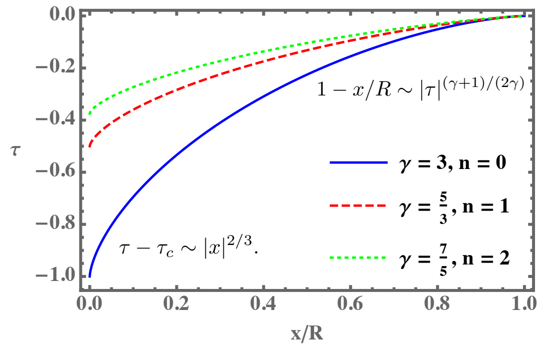

Figure 2. The edge of emptiness region for $\gamma = 3$ (blue solid line), $\gamma = 5/3$ (red dashed line) and $\gamma = 7/5$ (green dotted line), as given by Eqs. (63), (64) and (65). The asymptotic behaviors are given by Eq. (7) and (6).

where $\gamma \geq 1$. Figure 1 shows function $f(\gamma)$ along with numerical results for several values of γ. In particular, the free fermion point is given by $f(3) = \pi/2$ (and $mv_s = \hbar\pi\rho_0 = p_F$ - the Fermi momentum), which agrees with the RMT [15, 14] and hydrodynamic [30] results. For the weakly interacting bosons $f(2) = 16/(3\pi) \approx 1.698$, which agrees well with the numerical estimate $f(2) = 1.70(1)$ of Ref. [31]. Away from these points $f(\gamma)$ is a monotonically decreasing function with the asymptotic limits $f(1) = 2$ and $f(\infty) = 4/\pi$.

The appropriate instanton solution exhibits the empty interval which nucleates near $x = 0$ at some instance of the imaginary time $\tau = -\tau_c \propto R/v_s$. It then develops into the macroscopic interval $|x| \leq R$ at $\tau = 0$ and closes up again at $\tau = +\tau_c$ at $x = 0$. In the free fermion case, $\gamma = 3$, the emptiness region in (x, τ) plane is bounded by an astroid $(x/R)^{2/3} + (\tau/\tau_c)^{2/3} = 1$ [30]. For other γ's the exact analytical shape of the empty region is more complicated. We found that for $\tau \lesssim |\tau_c|$ and $|x| \ll R$, the emptiness nucleates in the same way as for the free fermions

$$\tau - \tau_c \sim |x|^{2/3}. \tag{6}$$

for any $\gamma \geq 1$. However, the other corner of the emptiness region at $|x| \lesssim R$ and $|\tau| \ll \tau_c$ is not universal and is described by a γ-dependent exponent:

$$R - x \sim |\tau|^{(\gamma+1)/(2\gamma)}. \tag{7}$$

The boundaries of the emptiness region on the (x, τ) plane for some specific values of γ are depicted in Fig. 2.

The remainder of the paper is organized as follows: in Section 2 we formulate the instanton approach for calculation of EFP for polytropic liquids and construct a systematic method to solve hydrodynamic equations. Analytic solutions for a sequence of rational γ's are constructed in Section 3. Conclusions and discussions are presented in Section 4. Some technical details are delegated to Appendices.

2. Instanton solution for Polytropic Liquids

2.1. Instanton Calculus

The hydrodynamic instanton approach to the emptiness formation, developed in Refs. [47, 48, 30, 49, 36], is justified in the regime of a macroscopic emptiness, $R \gg \rho_0^{-1}$. A state of the system is characterized by hydrodynamic degrees of freedom: the local particle density, $\rho(x,t)$, and the local current, $j(x,t)$. The two are constrained by the continuity equation,

$$\partial_t \rho + \partial_x j = 0. \tag{8}$$

The real time action, that yields proper hydrodynamic equations as its extremal conditions, is given by

$$S[\rho, j] = \iint dxdt \left[\frac{mj^2}{2\rho} - V(\rho)\right]. \tag{9}$$

It consists of the liquid's kinetic, $mj^2/(2\rho)$, and internal, $V(\rho)$, energy densities. The internal energy density is related to the pressure through the thermodynamic relation $P(\rho) = \rho\partial_\rho V(\rho) - V(\rho)$. For the polytropic liquid with the pressure, given by Eq. (4) with $\gamma > 1$, the internal energy density is thus

$$V(\rho) = \frac{mv_s^2\rho_0}{\gamma - 1}\left[\frac{1}{\gamma}\left(\frac{\rho}{\rho_0}\right)^\gamma - \frac{\rho}{\rho_0}\right]. \tag{10}$$

The linear in ρ term is $-\mu\rho$ with the chemical potential $\mu = mv_s^2/(\gamma-1)$. It fixes the average density to be ρ_0 through the condition $\partial_\rho V(\rho)|_{\rho=\rho_0} = 0$ and does not affect the equation of state. For $\gamma = 1$ one finds

$$V(\rho) = mv_s^2\rho\left[\log(\rho/\rho_0) - 1\right]. \tag{11}$$

Since we are only interested in the leading term ($-\ln \mathcal{P}_{\rm EFP} \sim R^2$) in EFP, we neglect higher gradient terms such as quantum pressure in the equation of state (10). Including gradient terms results in sub-leading contributions in $\xi/R \ll 1$.

Variation of the action (9) over ρ and j with the continuity constraint, Eq. (8), yields classical Euler equation of the hydrodynamic flow [50]. The emptiness formation does *not* come from the dynamics of this equation, since the emptiness is a large quantum fluctuation (similar to tunneling), which is outside of the classically allowed region of the phase space. In the instanton approach, the path integral $\int \mathcal{D}\rho\, \mathcal{D}j\, e^{iS[\rho,j]/\hbar}\, \delta(\partial_t\rho + \partial_x j)$, with proper boundary conditions, determines the quantum transition amplitude. One has to deform the fields into the complex plane to go through a classically forbidden stationary configuration, corresponding to the emptiness. Such a deformed stationary point may be found at a purely imaginary current, j, and a real density, ρ (similarly to the quantum mechanics, where tunneling trajectories correspond to an imaginary momentum and a real coordinate). One notices then that redefining $j \to ij$ and

$t \to -i\tau$ (known as the Wick rotation), the stationary point equations may be brought to the purely real form. Finally, we pass to dimensionless coordinates and fields: $x \to Rx,\ \tau \to (R/v_s)\tau,\ \rho \to \rho_0\rho$ and $j \to (\rho_0 v_s)j$, to write the corresponding Eucledian action as

$$\frac{i}{\hbar}S[\rho,j] = -\frac{\rho_0 R^2}{\xi}\iint dx d\tau\left[\frac{j^2}{2\rho}+V(\rho)\right]; \tag{12}$$

$$V(\rho) = \frac{1}{\gamma-1}\left[\frac{\rho^\gamma}{\gamma}-\rho\right]. \tag{13}$$

The Planck constant is suppressed below. The internal energy density $V(\rho)$ changes sign (cf. Eq. (9)) similarly to the inverted potential in the tunneling problem. The dimensionless equations of motion in the imaginary time are

$$\partial_\tau\rho + \partial_x(\rho v) = 0; \tag{14}$$

$$\partial_\tau v + v\partial_x v = \rho^{\gamma-2}\partial_x\rho, \tag{15}$$

where the velocity field, $v(x,\tau)$, is defined as $v = j/\rho$. The fact that the dimensionless equations of motion and the boundary conditions (see below) depend only on γ and no other parameters, justifies the scaling form (2) of EFP.

One should notice that the right hand side of the Euler equation (15) contains the *negative* pressure. This is a consequence of the Wick rotation and the "wrong" sign in front of the internal energy in Eq. (12). As a result, the equilibrium configuration, $\rho = 1$, $j = 0$, is an *unstable* solution of Eqs. (14), (15). It spontaneosly develops deformations, which keep growing in (imaginary) time. Our goal is to find a very specific solution, which develops a deformation growing into the empty region, $\rho = 0$ for $|x| < 1$, at the observation time $\tau = 0$. The probability of such rare event is $\mathcal{P} \propto |e^{iS_{\rm inst}/\hbar}|^2$, where the classical action along the proper (i.e. instanton) trajectory, $iS_{\rm inst}$, cf. Eq. (12), is real and negative.

Apart from the emptiness formation, the polytropic hydrodynamic equations (14) and (15) appear in a variety of distinct fields across physics. From early studies of Cauchy (initial condition) problem of unstable media [51] and large-N limit of matrix model [52] to the recent works on large deviation in classical stochastic systems [53, 54, 55, 56].

2.2. Riemann Invariants and Hodograph Transformation

Following Ref. [46], Eqs. (14) and (15) may be reformulated to simplify their solution. First, one introduces the Riemann invariant, $\lambda(x,t)$, and the complex "velocity", $w(x,t)$, as

$$\lambda = v + i\tfrac{2}{\gamma-1}\rho^{\frac{\gamma-1}{2}}, \tag{16}$$

$$w = v + i\rho^{\frac{\gamma-1}{2}}. \tag{17}$$

In terms of these quantities the equations of motion (14), (15) acquire a more symmetric form:

$$\partial_\tau \lambda + w \partial_x \lambda = 0, \tag{18}$$
$$\partial_\tau \overline{\lambda} + \overline{w} \partial_x \overline{\lambda} = 0, \tag{19}$$

where $\overline{\lambda}$ and $\overline{w}$ are complex conjugates of λ and w. Notice that for $\gamma = 3$, $\lambda = w = v + i\rho$, which reduces the problem to finding an analytic function $\lambda(x,\tau)$ which can be done in a relatively straightforward way [30].

To proceed in the general case we employ the so-called Hodograph transformation (See Appendix A). The idea is to find $x(\lambda, \overline{\lambda})$ and $\tau(\lambda, \overline{\lambda})$ as functions of Riemann invariants. Under the Hodograph transformation Eqs. (18) and (19) become

$$\partial_{\overline{\lambda}} x - w \partial_{\overline{\lambda}} \tau = 0, \tag{20}$$
$$\partial_\lambda x - \overline{w} \partial_\lambda \tau = 0. \tag{21}$$

Now, one can solve these equations of motion by adopting the ansatz

$$x - w\tau = \partial_\lambda \mathcal{V}, \tag{22}$$
$$x - \bar{w}\tau = \partial_{\overline{\lambda}} \mathcal{V}, \tag{23}$$

where the real function $\mathcal{V}(\lambda, \bar{\lambda})$ depends only on λ and $\overline{\lambda}$. After substituting this ansatz into the equations of motion (20) and (21), one obtains the equation for the function $\mathcal{V}(\lambda, \bar{\lambda})$,

$$\partial_\lambda \partial_{\overline{\lambda}} \mathcal{V} = \frac{n}{\lambda - \overline{\lambda}} \left(\partial_\lambda \mathcal{V} - \partial_{\overline{\lambda}} \mathcal{V} \right), \tag{24}$$

where n is defined as

$$n = -\frac{1}{2}\frac{\gamma - 3}{\gamma - 1}; \qquad \gamma = \frac{2n+3}{2n+1}. \tag{25}$$

The Riemann invariant in terms of n is

$$\lambda = v + i(2n+1)\rho^{1/(2n+1)}. \tag{26}$$

The main idea of this mathematical manipulation is to map the original hydrodynamic equations, (14) and (15), onto an electrostatic-like problem of finding a solution for the 2D "potential" $\mathcal{V}$ in the complex λ plane. The emptiness condition only partially fixes the density at $\tau = 0$, i.e. $\rho(|x| < 1, \tau = 0) = 0$, but the velocity is left unspecified at $\tau = 0$. This seems to provide insufficient information to find the potential. However, in terms of Riemann invariants, both density and velocity are combined together. The boundary conditions for the density is actually also constraining the velocity as well by requiring analyticity of the potential in the plane of Riemann invariants.

The method of Riemann invariants has been known for a while, (see Ref. [44] and references there) and it is a powerful tool. It was deployed in e.g. recent studies of Bose liquid [57] and relativistic fluid [58]. It was noticed by Kamchatnov [46] that Eq. (24) admits a closed form analytic solution if n is a *non-negative integer*. Employing this approach along with the relation, found by Abanov [30], between EFP and the asymptotic behavior of the density at $x \to \infty$ and $\tau = 0$, the EFP may be calculated exactly for the discrete sequence of the rational polytropic indices, Eq. (25). Finally, this result allows for the unique analytic continuation to find EFP for any $\gamma \geq 1$.

2.3. Instanton solution for integer n

Let's first examine the simplest case, $n = 0$, where the right hand side of Eq. (24) vanishes. Therefore $\mathcal{V}$ is given by the sum of two arbitrary analytic functions

$$\mathcal{V} = F_0(\lambda) + G_0(\overline{\lambda}), \qquad \text{for } n = 0. \tag{27}$$

For $n \neq 0$, the right hand side of Eq. (24) complicates the solution by introducing coupling between λ and $\overline{\lambda}$. The structure of Eq. (24) with integer n suggests to look for its solution in the form of a series expansion [46]

$$\mathcal{V} = \frac{F_0(\lambda) + G_0(\overline{\lambda})}{(\lambda - \overline{\lambda})^n} + \sum_{m=1}^{\infty} \mathcal{V}_m, \tag{28}$$

where $\{\mathcal{V}_m\}$ are to be determined order by order. By substituting it into Eq. (24), one finds

$$n(n-1)\frac{F_0(\lambda) + G_0(\overline{\lambda})}{(\lambda - \overline{\lambda})^{n+2}} + \sum_{m=1}^{\infty} \left[\partial_\lambda \partial_{\overline{\lambda}} \mathcal{V}_m - \frac{n}{\lambda - \overline{\lambda}}(\partial_\lambda \mathcal{V}_m - \partial_{\overline{\lambda}} \mathcal{V}_m)\right] = 0. \tag{29}$$

In order to cancel the term with $(\lambda - \overline{\lambda})^{-(n+2)}$, one requires $\mathcal{V}_1$ to have the form

$$\mathcal{V}_1 = a_1 \frac{F_1(\lambda) - G_1(\overline{\lambda})}{(\lambda - \overline{\lambda})^{n+1}}, \tag{30}$$

where a_1 is an overall coefficient and F_1 and G_1 are analytic functions. The requirement of cancellation of $(\lambda - \overline{\lambda})^{-(n+2)}$ term leads to the recurrence relation: $a_1 = -n(n-1)$, $F_0 = \partial_\lambda F_1$ and $G_0 = \partial_{\overline{\lambda}} G_1$. Now, Eq. (24) becomes

$$a_1(n+1)(n-2)\frac{F_1(\lambda) - G_1(\overline{\lambda})}{(\lambda - \overline{\lambda})^{n+3}} + \sum_{m=2}^{\infty} \left[\partial_\lambda \partial_{\overline{\lambda}} \mathcal{V}_m - \frac{n}{\lambda - \overline{\lambda}}(\partial_\lambda \mathcal{V}_m - \partial_{\overline{\lambda}} \mathcal{V}_m)\right] = 0, \tag{31}$$

where the term with $(\lambda - \overline{\lambda})^{-(n+3)}$ is left to be cancelled by a proper choice of $\mathcal{V}_2$. By repeating the corresponding cancellation procedure, subsequent $\{\mathcal{V}_m\}$ are recovered order by order.

The series expansion terminates if n is an integer. For example, for $n = 1$, $a_1 = 0$ and thus

$$\mathcal{V} = \frac{F_0(\lambda) + G_0(\overline{\lambda})}{\lambda - \overline{\lambda}}, \qquad \text{for } n = 1, \tag{32}$$

which can be verified by a direct substitution in Eq. (24). For a positive integer n, there are exactly n terms of series expansion in terms of $(\lambda - \overline{\lambda})$, as follow

$$\mathcal{V} = \frac{F_0(\lambda) + G_0(\overline{\lambda})}{(\lambda - \overline{\lambda})^n} + \sum_{m=1}^{n-1} a_m \frac{F_m(\lambda) + (-1)^m G_m(\overline{\lambda})}{(\lambda - \overline{\lambda})^{n+m}}, \tag{33}$$

where all the $\{F_m\}$ functions only depend on λ and all the $\{G_m\}$ functions only depend on $\overline{\lambda}$. The $\{a_m\}$ are the coefficients of series expansion. The recurrence relations for F_m and G_m functions are

$$F_{m-1} = \partial_\lambda F_m, \tag{34}$$

$$G_{m-1} = \partial_{\overline{\lambda}} G_m, \tag{35}$$

and for the coefficients

$$a_1 = -n(n-1), \tag{36}$$

$$a_m = -\frac{1}{m} a_{m-1}(n+m-1)(n-m). \tag{37}$$

The series terminates since $a_m = 0$ for $m \geq n$.

The specific form of $\{F_m\}$ and $\{G_m\}$ functions is to be determined from the boundary conditions. To find the boundary conditions for $\mathcal{V}$ one needs to go back to the original hydrodynamic variables and discuss the boundary conditions for the density and the velocity. For simplicity, let us focus on $x > 0$ in the following. Solutions at $x < 0$ can be obtained by spatial inversion: $\rho(x,\tau) = \rho(-x,\tau)$ and $v(x,\tau) = -v(x,\tau)$. The instanton solution evolves from a uniform state at a distant past, $\tau = -\infty$, to a state with the emptiness, i.e. zero density for $|x| < 1$, at the observation time, $\tau = 0$. At $\tau = 0$, the density diverges at $x = 1$ since the displaced particles accumulate on the emptiness boundary. In terms of Riemann invariants, $|\lambda| \to \infty$ at $x = 1$ and $\tau = 0$ and from (22) we have

$$\partial_\lambda \mathcal{V}\Big|_{|\lambda|\to\infty} = 1, \tag{38}$$

which fixes the boundary of emptiness $x = 1$. Far away from the emptiness, the density decays to the average density and the velocity approaches zero. In general, the density decays like $(\rho - 1) \propto 1/x^2$ as $x \to \infty$ [30]. In terms of Riemann invariants, $\lambda \to i\frac{2}{\gamma-1} = i(2n+1)$ as $x \to \infty$. Substituting this condition into Eq. (22), one finds the other boundary condition:

$$\partial_\lambda \mathcal{V}\Big|_{\lambda\to i(2n+1)} \sim \frac{1}{\sqrt{\lambda^2 + (2n+1)^2}}, \tag{39}$$

where the square root divergence on the right hand side comes from the $1/x^2$ behavior of the density. The requirement of zero density is encoded in the boundary conditions (38) and (39). Indeed, they imply that the solution has a branch point at $x = 1$ for $\tau = 0$ so that λ becomes a purely real function at $x < 1, \tau = 0$ corresponding to $\rho = 0$.

With the boundary condition (38) and (39) and the recurrence relations (34) and (35) one can construct all the $\{F_m\}$ and $\{G_m\}$ functions as long as the last terms F_{n-1} and G_{n-1} are specified. For the special case, $n = 0$,

$$F_0 = \sqrt{\lambda^2 + 1}, \qquad G_0 = \overline{F}_0, \tag{40}$$

where the fact that $\mathcal{V}$ is a real function is employed. For a positive integer n, the series of $\mathcal{V}$ terminates at the term with F_{n-1}, which must be taken as

$$F_{n-1} = \frac{\lambda}{n!}\left[\lambda^2 + (2n+1)^2\right]^{\frac{2n-1}{2}}, \tag{41}$$

$$G_{n-1} = (-1)^n \overline{F}_{n-1}, \tag{42}$$

where the coefficient $1/n!$ results from Eq. (38). Finally, one may find all $\{F_m\}$ functions, $\{G_m\}$ functions and coefficient $\{a_m\}$ from the recurrence relations, Eqs. (34),(35),(36),(37).

2.4. Emptiness formation probability

We are now at the position to calculate EFP. The semiclassical transition amplitude is given by $e^{iS_{\text{inst}}(R)}$. Since EFP is a probability of the fluctuation with respect to the ground state, we need to normalize this amplitude by dividing it by e^{iS_0}, where S_0 is the action evaluated at the static ground state solution, $\rho = \rho_0$ and $j = 0$. This results in the EFP of the form

$$-\ln \mathcal{P}_{\text{EFP}}(R) = 2\text{Im}\left[S_{\text{inst}}(R) - S_0\right] = \frac{\rho_0 R^2}{\xi} f(n), \tag{43}$$

where the second equality used the rescaled action Eq. (12) and $f(n)$ is a function depending on n only.

Following Ref. [30], one may connect EFP with the asymptotic behavior of the density at large x at $\tau = 0$. We rederive this relation in Appendix B and here only quote the result

$$i\partial_{\rho_0}\left[S_{\text{inst}} - S_0\right] = \frac{\pi R^2 \alpha}{\xi}, \tag{44}$$

where α is a coefficient in the following generic asymptotic expansion of the density, $\rho(x \to \infty, \tau = 0)$,

$$\rho(x, 0) = 1 + \frac{\alpha}{x^2} + \mathcal{O}\left(\frac{1}{x^4}\right). \tag{45}$$

For polytropic liquids, the correlation length depends on the average density as $1/\xi \propto \rho_0^{(\gamma-1)/2} = \rho_0^{1/(2n+1)}$. After integrating over ρ_0, one finds

$$i\left[S_{\text{inst}}(R) - S_0\right] = \frac{\pi \rho_0 R^2}{\xi} \frac{2n+1}{2n+2} \alpha. \tag{46}$$

To determine the n dependence of the coefficient α, we employ the instanton solution for positive integer n. According to Eq. (22), (41) and (42), the solution for any n can be explicitly written down. The leading term at $x \to \infty$ comes solely from $\partial_\lambda F_0(\lambda) = \partial_\lambda^n F_{n-1}(\lambda)$ due to the boundary condition (39) and the recurrence relation (34). This way one arrives at

$$x - w\tau = \frac{1}{(\lambda - \overline{\lambda})^n} \partial_\lambda^n \left\{ \frac{\lambda}{n!} \left[\lambda^2 + (2n+1)^2 \right]^{\frac{2n-1}{2}} \right\} + ..., \tag{47}$$

where the sub-leading terms are omitted. To satisfy the boundary condition (39) and thus generate the $[\lambda^2 + (2n+1)^2]^{-1/2}$ term, all n derivatives should act on the square bracket term in this expression. As a result one finds

$$x - w\tau = \frac{(2n-1)!!}{n!} \frac{\lambda^{n+1}}{(\lambda - \overline{\lambda})^n \sqrt{\lambda^2 + (2n+1)^2}} + ... \tag{48}$$

At $\tau = 0$, take the limit: $x \to \infty$, $\rho \to 1$, $\lambda \to i(2n+1)\rho^{1/(2n+1)}$ and $\overline{\lambda} \to -i(2n+1)\rho^{1/(2n+1)}$, as in Eq. (26); and $[\lambda^2 + (2n+1)^2] = (2n+1)^2 \times (1 - \rho^{2/(2n+1)}) \approx -2(2n+1)(\rho - 1)$. Therefore

$$x \approx \frac{(2n+1)!!}{2^n n!} \frac{1}{\sqrt{2(2n+1)(\rho - 1)}}. \tag{49}$$

Accordingly, the coefficient α in Eq. (45) is given by

$$\alpha = \frac{1}{2(2n+1)} \left[\frac{(2n+1)!!}{2^n n!} \right]^2, \tag{50}$$

and EFP is found with the help of Eqs. (43) and (46) as

$$-\ln \mathcal{P}_{\text{EFP}}(R) = \frac{\rho_0 R^2}{\xi} \frac{\pi}{2n+2} \left[\frac{(2n+1)!!}{2^n n!} \right]^2. \tag{51}$$

Finally, the function $f(n)$ may be written using the gamma-function representation of the factorials:

$$f(n) = \frac{\pi \Gamma^2(2n+2)}{2^{4n+1} \Gamma(n+2) \Gamma^3(n+1)}. \tag{52}$$

This expression can be extended to real n due to the uniqueness of the analytic continuation of the gamma function according to Bohr-Mollerup theorem [59]. Converting n into γ with the help of Eq. (25), one arrives at Eq. (5).

To illustrate the validity of Eq. (44), we numerically calculate $f(n)$ by substituting the instanton solutions for $n = 0$, 1 and 2 into Eq. (12) and performing Monte Carlo integration. The numerical results are summarized in Table. 1.

	$n=0$ ($\gamma=3$)	$n=1$ ($\gamma=5/3$)	$n=2$ ($\gamma=7/5$)
$f(n)$	1.56 ± 0.02	1.76 ± 0.02	1.85 ± 0.02
Eq.(52)	$\pi/2\approx1.571$	$9\pi/16\approx1.767$	$75\pi/128\approx1.841$

Table 1. Numerical value of function $f(n)$ for $n=0,1,2$.

3. Shape of the empty region

Having found the systematic way to construct analytic solutions for polytropic fluid, we now explicitly write them down for $n=0,1$ and 2. The $n=0$ case corresponds to the free fermions ($\gamma=3$) and is the simplest case of the polytropic fluid since the Riemann invariants λ equals to its complex "velocity" w. In other words, Eq. (18) and (19) become a pair of complex conjugate Hopf equations whose closed form solution is a generic analytic function [30]. This property can be understood from Eq. (24), where the right hand side vanishes at $n=0$ and $\mathcal{V}(\lambda,\overline{\lambda})$ consists of two analytic functions: one depends only on λ and the other – only on $\overline{\lambda}$. Therefore,

$$\mathcal{V}=F_0+G_0, \qquad F_0=\sqrt{\lambda^2+1}, \tag{53}$$

where $G_0=\overline{F}_0$, since $\mathcal{V}$ is real. For $n=1$ the corresponding solution is

$$\mathcal{V}=\frac{F_0+G_0}{\lambda-\overline{\lambda}}, \qquad F_0=\lambda\sqrt{\lambda^2+9}, \tag{54}$$

where $G_0=-\overline{F}_0$. For $n=2$:

$$\mathcal{V}=\frac{F_0+G_0}{(\lambda-\overline{\lambda})^2}-2\frac{F_1-G_1}{(\lambda-\overline{\lambda})^3}, \qquad F_1=\frac{\lambda}{2}(\lambda^2+25)^{3/2}, \tag{55}$$

where $F_0=\partial_\lambda F_1$ and $G_{0,1}=\overline{F}_{0,1}$.

The imaginary time evolution of the density for these three cases is shown in Fig. 3. It is determined by numerically solving algebraic Eqs. (22) and (23) with the above explicit expressions for $\mathcal{V}$. The fluid evolves from the uniform density into the emptiness profile in Fig. 3. There is an empty region in the (x,τ) plane, where the density is zero. To find the boundary of this region we note that for $\rho=0$ the Riemann invariants are degenerate, $\lambda=\bar{\lambda}=v$ and our solution represents an equation for $v(x,t)$. For $n=0,1,2$ we have

$$x-v\tau=\frac{v}{\sqrt{v^2+1}}, \tag{56}$$

$$x-v\tau=\frac{3v}{2\sqrt{v^2+9}}-\frac{v^3}{2(v^2+9)^{3/2}}, \tag{57}$$

$$x-v\tau=\frac{15v}{8\sqrt{v^2+25}}-\frac{5v^3}{4(v^2+25)^{3/2}}+\frac{3v^5}{8(v^2+25)^{5/2}}. \tag{58}$$

Generally speaking the velocity v is a multi-valued function of coordinate x parametrised by τ. The ends of the emptiness interval for a given τ are real values $x=x_\pm(\tau)$ where

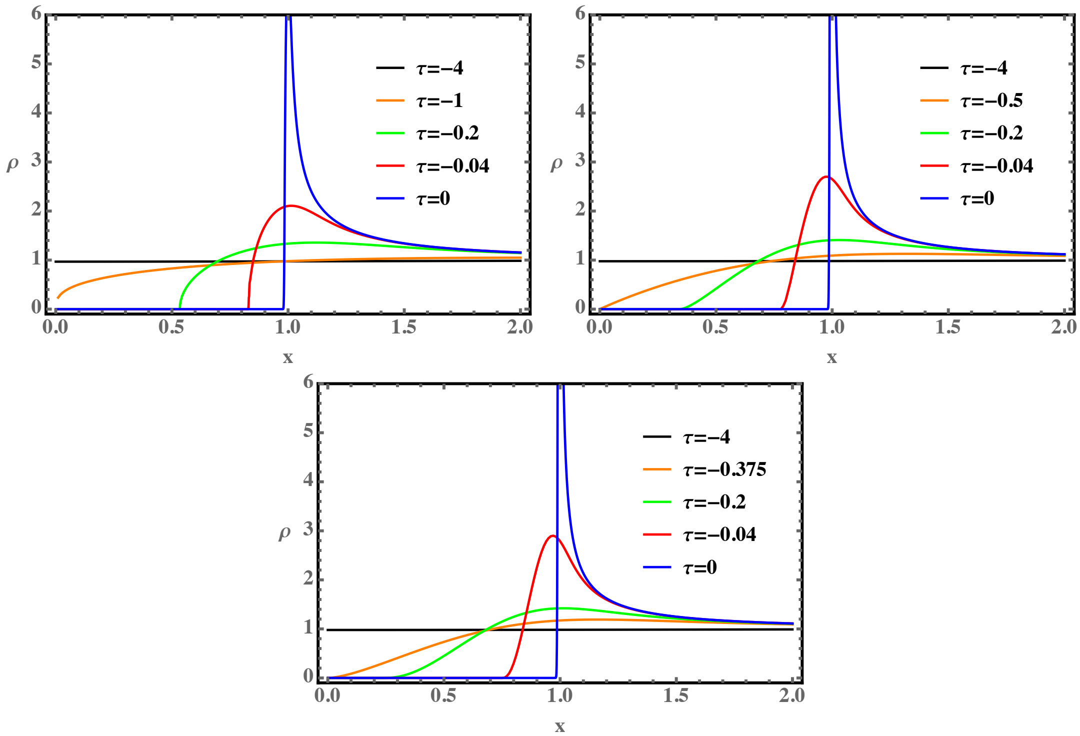

Figure 3. Time evolution of the density $\rho(x,\tau)$ for $n = 0$ (upper left), $n = 1$ (upper right) and $n = 2$ (bottom). The density evolves from the uniform value, $\rho = 1$, at large negative τ towards the emptiness within $|x| < 1$ at $\tau = 0$.

two branches of v meet. In other words $x_\pm$ are maximum and minimum value of x as function of v at given τ obtained from Eqs. (56),(57),(58). Such real points exist only within the interval $-\tau_c < \tau < \tau_c$ (e.g., $\tau_c = 1$ for $n = 0$). Outside this interval the branch points move away from the real axis of x.

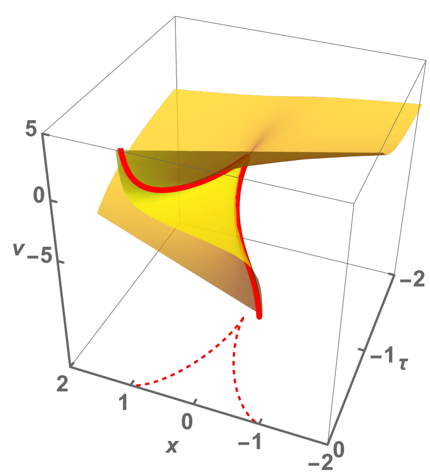

Figure 4. A 2D surface of Eq. (56). The red (solid) curve on the surface shows the local maximum (minimum) value of x which indicates the boundary of the empty region. After $\tau = -1$, v is a multi-valued function. The red (dashed) curve shows the projection onto (x, τ) plane is described by Eq. (63).

To visualize this construction the $n = 0$ case (56) is plotted in Fig. 4. The red solid curve on the surface in Fig. 4 are the collection of these local minima and maxima. The emptiness boundary is the projection of the red solid curve onto the space-time plane along the v axis, depicted by the red dashed curve in Fig. 4. For other values of n the emptiness boundary is obtained by the same procedure and is plotted in Fig. 2.

Let's examine the time around the beginning of the emptiness formation, where one can do small v expansion for Eq. (56), (57) and (58). For any n the solutions can be approximated by a cubic polynomial:

$$x \approx (\tau - \tau_c)v - b_n v^3, \tag{59}$$

where b_n is a positive constant depending on n. From Eq. (59) it follows that v is multi-valued and x has local maximum and minimum for $\tau > \tau_c$. The power law behavior of the emptiness boundary is determined from the position of the local minimum and maximum of x on (x, τ) plane. According to Eq. (59), one finds

$$\tau - \tau_c \propto |x|^{2/3}. \tag{60}$$

This explains the universal power law exponent 2/3 at the start of the emptiness as a consequence of an underlying cubic equation. This scenario of emptiness formation can be described as a cusp catastrophe in the catastrophe theory [60], where the cusp catastrophe is classified as A_3 group and the cusp has the exponent 2/3 [61]. Moreover, the Burgers' equation features the same exponent 2/3 in the shock wave formation [62].

Focusing now on the vicinity of $\tau = 0$ and point $x = 1$, one can perform large v expansion for Eq. (56), (57) and (58), since v diverges there. This leads to:

$$\tau \approx \frac{x-1}{v} + \frac{c_n}{v^{2n+3}}, \tag{61}$$

where c_n is a positive constant depending on n. The linear term in $1/v$ shows that the empty region terminates at $x = 1$. Unlike Eq. (59), the highest power of the approximate polynomial equation is now depending on n and this leads to the n-dependent power law of the emptiness boundary. One can again solve for the position of the local maximum and minimum of τ on (x, τ) to find

$$1 - x \propto |\tau|^{(2n+2)/(2n+3)}. \tag{62}$$

Although Eq. (62) is based on discrete value of n, we expect that the solution is deformed continuously with the polytropic index γ leading to Eq. (7). The power law exponent is linear when $\gamma \to 1$ and square root when $\gamma \to \infty$.

Since the analytic instanton solutions are available for non-negative integer n, we are only able to write down the analytic expressions of the whole emptiness boundary for these n. Here we show the results for the emptiness boundary for $n = 0, 1$ and 2

$$x = (1 - \tau^{2/3})^{3/2}, \tag{63}$$

$$x = \left[1 - (2\tau)^{2/5}\right]^{3/2} \left[1 + \frac{3}{2}(2\tau)^{2/5}\right], \tag{64}$$

$$x = \left[1 - \left(\frac{8\tau}{3}\right)^{2/7}\right]^{3/2} \left[1 + \frac{3}{2}\left(\frac{8\tau}{3}\right)^{2/7} + \frac{15}{8}\left(\frac{8\tau}{3}\right)^{4/7}\right]. \tag{65}$$

These results are summarized in Fig. 2. Equation (63) is called the astroid curve: $x^{2/3} + \tau^{2/3} = 1$. For a general n we have a family of astroid-like curves, like Eqs. (64), (65).

Having established the shape of the empty region, we now look at the behavior of the density profile immediately outside of the empty region. The density grows from zero as a positive power of the distance from the boundary when $\tau \neq 0$ and diverges at $\tau = 0$, Fig. 3. The emptiness boundary is the branch point of the Riemann invariants λ and $\overline{\lambda}$. The exponent of the power law can be determined from the degree of this branch point. Denoting the boundary of the empty region as $x_0 = x_0(\tau)$, Eqs. (63)–(65), one finds

$$\rho \propto (x - x_0)^{1/(\gamma-1)}, \qquad x > x_0, \tag{66}$$

for $\tau \neq 0$ and

$$\rho \propto (x - 1)^{-2/(\gamma+1)}, \qquad x > 1, \tag{67}$$

for $\tau = 0$.

4. Conclusions and Discussion

Perhaps the most interesting application (besides previously well established fermion case with $\gamma = 3$) is the weakly interacting Bose gas. Within the Gross-Pitaevskii approximation, its internal energy may be written as [64] (in dimensionless form)

$$V(\rho) = \frac{1}{2}\rho^2 - \rho + \frac{\xi^2}{R^2}\frac{(\partial_x \rho)^2}{8\rho}, \tag{68}$$

where the last term constitutes the so-called quantum pressure. In the limit of the large emptiness, $R \gg \xi$, it is clearly sub-leading and the weakly interacting Bose gas is well approximated by the polytropic expression (13) with $\gamma = 2$. It is interesting to note that it corresponds to the non-integer value $n = 1/2$, which does not allow for an analytic solution of the hydrodynamic equations. Nevertheless we are able to deduce EFP through the analytic continuation procedure, resulting in

$$-\ln \mathcal{P}_{\mathrm{EFP}}^{\mathrm{Bose}}(R) = \frac{\rho_0 R^2}{\xi}\frac{16}{3\pi} + \mathcal{O}\left(\log(R/\xi)\right), \tag{69}$$

where $\xi \gg \rho_0^{-1}$ depends on the interaction strength. The logarithmic correction originates from the quantum pressure term in Eq. (68). This should be compared with the free fermion-RMT result [65, 66] for $\gamma = 3$:

$$-\ln \mathcal{P}_{\rm EFP}^{\rm Fermi}(R) = \frac{\rho_0 R^2}{\xi}\frac{\pi}{2} + \frac{1}{4}\log(R/\xi), \tag{70}$$

where $\xi = (\pi\rho_0)^{-1}$. Our calculations were performed in the regime $\xi, \rho_0^{-1} \ll R$ and give the coefficient of the leading term for all $\gamma \geq 1$. The coefficient of the first sub-leading term, $\propto \rho_0 R$, is not known in general. In few examples [17, 19, 63] where it is known analytically or numerically in models with local interactions, including Eqs. (69), (70, it appears to be zero. It is thus tempting to conjecture that the first non-vanishing correction to our result (2) is of the form $\mathcal{O}\big(\log(R/\xi)\big)$.

It is worth noticing that Eq. (69) and (70) are the two limiting cases of the integrable Lieb-Liniger model [67], for which the ground state wave function, $\Psi_{\rm GS}(x_1, x_2, ..., x_N)$, is known explicitly through the Bethe Ansatz. The Bose and Fermi polytropic liquids represent its weak and strong interaction limits, correspondingly. While the correlation length, ξ, is known for any interaction strength, the coefficient multiplying $\rho_0 R^2/\xi$ so far is only available in the two extreme limits. For intermediate interactions the equation of state of the Lieb-Liniger model is not polytropic. In fact, it interpolates between $\gamma = 3$ at small density and $\gamma = 2$ at large density. An appropriate solution of the hydrodynamic equations is not known for such equation of state. Nothing is known about the coefficient in front of the logarithmic correction, besides Tonks fermion limit, Eq. (70), either.

In weakly interacting systems with $v_{\rm s} \ll v_F \equiv \pi\rho_0/m$, there is another parametrically wide regime: $\rho_0^{-1} \ll R \ll \xi$. In this regime the leading contribution is linear in R, while quadratic term is subleading: $-\ln \mathcal{P}_{\rm EFP}(R) = 2\rho_0 R + \mathcal{O}\big(\rho_0 R^2/\xi\big)$ [31], The coefficient 2 in the leading term here reflects property of non-interacting Bose gas. The interaction-dependent sub-leading term in this expression was not evaluated analytically to the best our knowledge.

There is an intriguing relation between imaginary time hydrodynamic for $n = 0$, $\gamma = 3$ and the density of states (DOS) in disordered superconductors with magnetic impurities [68, 69, 70]. The superconducting gap closes gradually with increasing concentration of the magnetic impurities. It turns out that the energy dependence of DOS is identical to spatial profile of the density in the top panel of Fig. 3, with magnetic impurity concentration playing the role of time. The details of this relation and its possible generalizations to $n > 0$ are discussed in Appendix C.

Another possibility to observe the spacetime shape of the emptiness arises naturally in the context of the well known mapping of (1+1)D quantum field theory onto 2D statistical models. The instanton solution of the former corresponds to stationary configuration dominating some statistical mechanics models such as random tilings and crystal surfaces, subject to proper boundary conditions [71, 72]. In fact, there is a one to one correspondence between random tilings and crystal surface heights on the one hand and world lines of free fermions in imaginary time on the other hand, which was established via transfer matrix representation of the partition function of these statistical models. It is interesting to find statistical models with coarse grained properties described by polytropic equation of state with $\gamma \neq 3$.

To conclude, we have developed a systematic way to construct analytic emptiness formation solution of hydrodynamic equations for polytropic liquids with the polytropic index $\gamma = (2n+3)/(2n+1)$, where n is a non-negative integer. We evaluate the EFP and analytically continue the result to access EFP in polytropic liquid with an arbitrary $\gamma \geq 1$. In particular, it yields a novel result for weakly interacting bosons, which may be experimentally verified in cold atom systems.

5. Acknowledgments

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Appendix A. Hodograph Transformation

We want to go from spacetime (x, τ) to Riemann invariants $(\lambda, \overline{\lambda})$ as new coordinates. Following chain rules, partial derivatives with respect to λ and $\overline{\lambda}$ can be expressed in terms of x and τ

$$\begin{pmatrix} \partial_\lambda \\ \partial_{\overline{\lambda}} \end{pmatrix} = \begin{pmatrix} \partial_\lambda x & \partial_\lambda \tau \\ \partial_{\overline{\lambda}} x & \partial_{\overline{\lambda}} \tau \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_\tau \end{pmatrix}. \tag{A.1}$$

Let's invert this equation,

$$\begin{pmatrix} \partial_x \\ \partial_\tau \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \partial_{\overline{\lambda}} \tau & -\partial_\lambda \tau \\ -\partial_{\overline{\lambda}} x & \partial_\lambda x \end{pmatrix} \begin{pmatrix} \partial_\lambda \\ \partial_{\overline{\lambda}} \end{pmatrix}, \tag{A.2}$$

$$J = \partial_\lambda x \partial_{\overline{\lambda}} \tau - \partial_\lambda \tau \partial_{\overline{\lambda}} x, \tag{A.3}$$

where the Jacobian J and is assumed to be nonzero. This is called Hodograph transformation. With Eq. (A.2), Eq. (18) and (19) become

$$\partial_{\overline{\lambda}} x - w \partial_{\overline{\lambda}} \tau = 0, \tag{A.4}$$

$$\partial_\lambda x - \overline{w} \partial_\lambda \tau = 0. \tag{A.5}$$

These are the Eq. (20) and (21).

Appendix B. Relation between EFP and the density asymptotic

The following derivation is based on unpublished notes of A. Abanov. The exponent of EFP is related to the instanton action

$$-\ln \mathcal{P}_{\mathrm{EFP}} = 2\mathrm{Im}\left[S_{\mathrm{inst}} - S_0\right]. \tag{B.1}$$

The factor of two in front the action allows to extend imaginary time integration to run from $-\infty$ to $+\infty$. The solutions at $\tau > 0$ is determined from time reversal symmetry: $\rho(x,\tau) = \rho(x,-\tau)$ and $j(x,\tau) = -j(x,-\tau)$. The action can be formulated as a space-imaginary time integral

$$2i(S_{\text{inst}} - S_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx d\tau \left[\frac{mj^2}{2\rho} + V(\rho) - V(\rho_0)\right], \tag{B.2}$$

where we consider the polytropic equation of state $V(\rho)$ from Eq. (10). By performing the variation of action with respect to ρ, j and ρ_0, one gets

$$2i\delta(S_{\text{inst}} - S_0) = \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx d\tau \left\{\frac{mj}{\rho}\delta j + \left[-\frac{mj^2}{2\rho^2} + \partial_\rho V(\rho)\right]\delta\rho + \left[\partial_{\rho_0} V(\rho) - \partial_{\rho_0} V(\rho_0)\right]\delta\rho_0\right\}, \tag{B.3}$$

However ρ, j and ρ_0 are not independent of each other. They are constrained by the continuity relation. Introducing the displacement field u, as

$$\rho = \rho_0 + \partial_x u, \qquad j = -\partial_\tau u, \tag{B.4}$$

allows to automatically resolve the continuity constraint. The variation of action now takes the form

$$2i\delta(S_{\text{inst}} - S_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx d\tau \left\{-\frac{mj}{\rho}\partial_\tau \delta u + \left[-\frac{mj^2}{2\rho^2} + \partial_\rho V(\rho)\right]\partial_x \delta u \right. \\ \left. + \left[-\frac{mj^2}{2\rho^2} + \partial_\rho V(\rho) + \partial_{\rho_0} V(\rho) - \partial_{\rho_0} V(\rho_0)\right]\delta\rho_0\right\}. \tag{B.5}$$

After integration by parts, one arrives

$$2i\delta(S_{\text{inst}} - S_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx d\tau \left\{\left[\partial_\tau\left(\frac{mj}{\rho}\right) - \partial_x\left(-\frac{mj^2}{2\rho^2} + \partial_\rho V(\rho)\right)\right]\delta u \right. \\ \left. + \left[-\frac{mj^2}{2\rho^2} + \partial_\rho V(\rho) + \partial_{\rho_0} V(\rho) - \partial_{\rho_0} V(\rho_0)\right]\delta\rho_0\right\}, \tag{B.6}$$

where the first square bracket is zero on the equation of motion (15), with the velocity field $v = j/\rho$. The boundary term is discarded since it vanishes at infinity. Only the second square bracket contributes to the variation of the action. Then one substitutes the equation of state (10) into integral and notices the sound velocity depending on ρ_0: $v_s \propto \rho_0^{(\gamma-1)/2}$. By rescaling the variables to dimensionless coordinates and fields: $x \to Rx$, $\tau \to R\xi m\tau$, $\rho \to \rho_0\rho$ and $j \to (\rho_0/m\xi)j$, the action taken derivative with respect to average density is

$$2i\partial_{\rho_0}(S_{\text{inst}} - S_0) = \frac{R^2}{\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx d\tau \left[-\frac{v^2}{2} + \frac{\rho^{\gamma-1} - 1}{\gamma - 1} - (\rho - 1)\right], \tag{B.7}$$

The goal is to massage this space-imaginary time integral into a integration on the boundary at infinity. The first step is integration by parts in x

$$-\frac{v^2}{2} = xv\partial_x v - \partial_x\left(\frac{xv^2}{2}\right). \tag{B.8}$$

Using equation of motion (15) for v,

$$-\frac{v^2}{2} = -\partial_\tau (xv) + x\rho^{\gamma-2}\partial_x\rho - \partial_x\left(\frac{xv^2}{2}\right), \tag{B.9}$$

the integral becomes

$$2i\partial_{\rho_0}(S_{\rm inst} - S_0) = \frac{R^2}{\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx d\tau \left[-\partial_\tau(xv) + x\rho^{\gamma-2}\partial_x\rho - \partial_x\left(\frac{xv^2}{2}\right) + \frac{\rho^{\gamma-1}-1}{\gamma-1} - (\rho-1)\right]. \tag{B.10}$$

The terms with density can be absorbed into a total spatial derivative

$$x\rho^{\gamma-2}\partial_x\rho + \frac{\rho^{\gamma-1}-1}{\gamma-1} - (\rho-1) = \partial_x\left(x\frac{\rho^{\gamma-1}-1}{\gamma-1} - u\right), \tag{B.11}$$

where u is the dimensionless displacement field, $\partial_x u = (\rho-1)$. Now, one can apply Stokes' theorem

$$2i\partial_{\rho_0}(S_{\rm inst} - S_0) = \frac{R^2}{\xi}\oint\left[(xv)dx + \left(x\frac{\rho^{\gamma-1}-1}{\gamma-1} - u - \frac{xv^2}{2}\right)d\tau\right]. \tag{B.12}$$

On the boundary at the infinity, $v \approx -\partial_\tau u$ and $(\rho^{\gamma-1}-1) \approx (\gamma-1)(\rho-1) = (\gamma-1)\partial_x u$. The v^2 term decays too fast to give contribution in the integral. The boundary integral is further simplified as

$$2i\partial_{\rho_0}(S_{\rm inst} - S_0) = \frac{R^2}{\xi}\oint\left[(-x\partial_\tau u)dx + (x\partial_x u - u)d\tau\right]. \tag{B.13}$$

By defining complex variable, $z = x + i\tau$, the integral is performed on the complex z-plane

$$2i\partial_{\rho_0}(S_{\rm inst} - S_0) = \frac{R^2}{\xi}\oint\left[\frac{-i}{2}\Big((z+\overline{z})\,\partial_z u - u\Big)dz + c.c.\right]. \tag{B.14}$$

In general, the asymptotic of density at infinity is given by

$$\rho \approx 1 + \frac{\alpha}{2}\left(\frac{1}{z^2} + \frac{1}{\overline{z}^2}\right), \tag{B.15}$$

where α is some constant depending on polytropic index γ. At $\tau = 0$, it becomes the Eq. (45). The corresponding displacement field u at infinity is given by

$$u \approx -\frac{\alpha}{2}\left(\frac{1}{z} + \frac{1}{\overline{z}}\right). \tag{B.16}$$

Substituting asymptotic of u into the integral and performing integration in polar coordinates: $z = r\exp(i\theta)$ and $\overline{z} = r\exp(-i\theta)$, one arrives at Eq. (44):

$$i\partial_{\rho_0}(S_{\rm inst} - S_0) = \frac{R^2}{\xi}\frac{\alpha}{2}\int_0^{2\pi}(1+\cos 2\theta)d\theta = \frac{\pi R^2\alpha}{\xi}. \tag{B.17}$$

Appendix C. Emptiness Formation and Superconductor with magnetic impurity

The action of the disordered superconductor with broken time reversal invariance, eg. by magnetic impurities, can be represented by the non-linear sigma model [70] as

$$iS[Q] \propto \mathrm{Tr}\left\{-\frac{\eta}{2}[\sigma_z, Q]^2 + 4i\epsilon(\sigma_z Q) + 4i\Delta(i\sigma_y Q)\right\}, \tag{C.1}$$

where $\sigma_{x,y,z}$ are the Pauli matrices in the Nambu space, η is the magnetic impurities concentration, ϵ is the energy and Δ is the superconducting order parameter. The gradient terms is neglected in the action under the assumption that vector potential varies slowly on the scale of superconducting correlation length. The Green function, Q, is the Nambu (and Keldysh) matrix constrained by $Q^2 = 1$. Its retarded component may be parametrized in the Nambu space as

$$Q = \begin{pmatrix} \cosh\theta & \sinh\theta \\ -\sinh\theta & -\cosh\theta \end{pmatrix}, \tag{C.2}$$

where θ is the complex Nambu angle and the density of states (DOS) is given by $\rho(\epsilon, \eta)$ $= \mathrm{Re}[\cosh\theta]$. Performing variation over θ, one obtains the saddle point equation

$$\epsilon = \Delta\coth\theta - i\eta\cosh\theta, \tag{C.3}$$

which describes how DOS, $\mathrm{Re}[\cosh\theta]$, changes with the magnetic impurity strength η.

Back to the hydrodynamics at $n = 0, \gamma = 3$, Riemann invariants, λ and $\overline{\lambda}$, are decoupled. Solution of Eq. (24) with the boundary condition that density is zero within $|x| < R$ at $\tau = 0$ is

$$x - \lambda\tau = R\frac{\lambda}{\sqrt{\lambda^2+1}}, \tag{C.4}$$

where R is the size of emptiness. Changing variables as $\lambda = i\cosh\theta$ this solution can be formulated as

$$x = R\coth\theta + i\tau\cosh\theta. \tag{C.5}$$

One can establish correspondence between Eq. (C.3) and (C.5) by identifying coordinate x as energy ϵ, emptiness size R as superconducting order parameter Δ and imaginary time $-\tau$ as magnetic impurities concentration η. This indicates that DOS of a disordered superconductor is equivalent to the density of the 1D liquid with $n = 0$, forming the emptiness. The observation moment $\tau = 0$ corresponds to the BCS time-reversal invariant case without magnetic impurities, where the gap is given by Δ. Away from this limit the gap is suppressed by the magnetic impurities until one reaches a gapless state at some critical η_c. The shape of the gap on the (ϵ, η) plane is given by the astroid $(\epsilon/\Delta)^{2/3} + (\eta/\eta_c)^{2/3} = 1$ [68].

One may wonder if there are non-linear sigma model representations of the polytropic liquids with $n > 0$, such that their densities coincide with DOS of corresponding superconductors. To this end we parametrize the velocity field as

$$v = i(2n+1)\cosh\theta. \tag{C.6}$$

The equations Eq. (57) and (58) for $n = 1, 2$ become

$$x = R\left(\frac{3}{2}\coth\theta - \frac{1}{2}\coth^3\theta\right) + 3i\tau\cosh\theta, \tag{C.7}$$

$$x = R\left(\frac{15}{8}\coth\theta - \frac{5}{4}\coth^3\theta + \frac{3}{8}\coth^5\theta\right) + 5i\tau\cosh\theta. \tag{C.8}$$

One can construct the corresponding non-linear sigma models, leading to such saddle point equations (with the identification $x \to \epsilon$, $R \to \Delta$ and $\tau \to \eta$).

For $n = 1$,

$$iS[Q] \propto \mathrm{Tr}\left\{-\frac{3}{16}\eta[\sigma_z, Q]^4 + i\epsilon\left(\frac{1}{3}(\sigma_z Q)^3 - 3(\sigma_z Q)\right) + i\Delta\left(\frac{1}{3}(i\sigma_y Q)^3 - 3(i\sigma_y Q)\right)\right\}. \tag{C.9}$$

And for $n = 2$,

$$\begin{aligned} iS[Q] \propto \mathrm{Tr}\Bigg\{&-\frac{5}{16}\eta[\sigma_z, Q]^6 + i\epsilon\left(\frac{3}{10}(\sigma_z Q)^5 - \frac{5}{2}(\sigma_z Q)^3 + 15(\sigma_z Q)\right) \\ &+ i\Delta\left(\frac{3}{10}(i\sigma_y Q)^5 - \frac{5}{2}(i\sigma_y Q)^3 + 15(i\sigma_y Q)\right)\Bigg\}. \end{aligned} \tag{C.10}$$

However, the underlying microscopic models for these non-linear sigma models are yet to be identified.

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