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Convexity of Sets and Quadratic Functions on the Hyperbolic Space

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Abstract In this paper some concepts of convex analysis on hyperbolic spaces are studied. We first study properties of the intrinsic distance, for instance, we present the spectral decomposition of its Hessian. Next, we study the concept of convex sets and the intrinsic projection onto these sets. We also study the concept of convex functions and present first and second order characterizations of these functions, as well as some optimization concepts related to them. An extensive study of the hyperbolically convex quadratic functions is also presented.

Keywords Hyperbolic space · Convex cone, Convex set · Convex function · Quadratic function

AMS subject classification: 90C30 · 90C26

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1 Introduction

The modeling of various classes of constrained optimization problems via Riemannian manifolds has gained increasing attention in the recent years in both the academic and business community as, in some sense, it outperforms the Euclidean counterpart. Besides the purely theoretical motivations and interests, it aims to obtain practical tools for supporting efficient computational implementations of algorithms for solving such problems. A point that deserves to be highlighted is that endowing the set of constraints of the problem with a suitable Riemannian metric, it allows us to explore its intrinsic algebraic and geometric structures and then significantly reduce the cost of obtaining solutions of the problem in question. For example, a non-convex Euclidean problem can be seen as a convex Riemannian one (as we show in Section 5.2), whose optimization methods for solving it have much less inherent computational complexity, see for instance [24, Example 13.4.1] and [5, 6]. It is worth noting that the concepts of convexity of sets and functions in the Riemannian optimization context is a topic that is of interest in itself, see for example [16, 23, 29, 35].

The hyperbolic space is a non-Euclidean smooth manifold of negative constant sectional curvature, see for example [1, 25]. A concise introductory note on hyperbolic spaces can be found in [3]. Over the recent years several theoretical and practical applications of the hyperbolic space have emerged. Although we are not concerned with practical issues at this time, we emphasize that practical applications appear whenever the natural structure of the data is modeled as an optimization problem on the hyperbolic space. For instance, several problems in machine learning, artificial intelligence, financial networks, as well as procrustes problems and many other practical questions can be modeled in this setting. Papers dealing with these subjects include, for machine learning [20], for artificial intelligence [19], for neural circuits [26], for low-rank approximations of hyperbolic embeddings [13, 28], for procrustes problems [27], for financial networks [14], for complex networks [15, 18], for embeddings of data [32] and the references therein. We also mention that there are many related papers on strain analysis, see for example, [30, 33].

The aim of this paper is to study some concepts related to the convexity of sets and functions on the hyperbolic space in an intrinsic way. Although some of these concepts have already been studied in general Riemannian manifolds, we revisit them in this specific context in order to present some explicit formulas and new properties. To this end, among the various existing models of hyperbolic geometry, we choose the hyperboloid model (also called Lorentz model), see [1, 3]. We first study important properties of the intrinsic distance, for instance, we present the spectral decomposition of its Hessian. Next, we study the concept of convex sets and the intrinsic projection onto these sets. We also study the concept of convex functions and present the first and second order characterizations for these functions. Finally, we present an extensive study of the hyperbolically convex quadratic functions.

The structure of this paper is as follows. In Section 1.1, we recall some notations and basic results. In Section 2, we recall some notations, definitions and basic properties about the geometry of the hyperbolic space used throughout the paper. In Section 2.1 we present some properties of the intrinsic distance from a fixed point. In Section 3 we present a characterization of convex sets in the hyperbolic space and in Section 4 we study properties of the projection onto convex sets. In Section 5 we study the basic properties of convex functions on the hyperbolic space and in Section 5.2 we study hyperbolically convex quadratic functions. In Section 6 we present some concepts of optimization related to hyperbolically convex functions. We conclude this paper by making some final remarks in Section 8.

1.1 Notation and Basics Results

Let \mathbb{R}^m be the m -dimensional Euclidean space. The set of all $m \times n$ matrices with real entries is denoted by $\mathbb{R}^{m \times n}$ and $\mathbb{R}^m \equiv \mathbb{R}^{m \times 1}$. For $M \in \mathbb{R}^{m \times n}$ the matrix $M^\top \in \mathbb{R}^{n \times m}$ denotes the *transpose* of M and $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ stands for the minimum and maximum eigenvalue of the matrix M , respectively. If $x \in \mathbb{R}^m$, then $\text{diag}(x) \in \mathbb{R}^{m \times m}$ denotes a diagonal matrix with (i, i) -th entry equal to x_i , $i = 1, \dots, m$. The matrix I

denotes the $n \times n$ identity matrix. The following characterizations of symmetric positive definite matrices can be found in [12, Theorem 7.2.5, pp. 404].

Lemma 1.1 *Let $M \in \mathbb{R}^{n \times n}$ be symmetric. Then, M is positive definite if and only if $\det M_i > 0$, for $i = 1, \dots, n$, where M_i stands for the principal submatrix M determined by the first i rows and first i columns.*

For a given $M \in \mathbb{R}^{(n+1) \times (n+1)}$, consider the following decomposition:

$$M := \begin{pmatrix} \bar{M} & b \\ b^\top & \theta \end{pmatrix}, \quad \bar{M} \in \mathbb{R}^{n \times n}, \quad b \in \mathbb{R}^{n \times 1}, \quad \theta \in \mathbb{R}. \quad (1)$$

Using the decomposition (1) we have the following characterizations of positive definite matrices and positive semidefinite matrices, for the proof see [11, Propositions 16.1 and 16.2].

Lemma 1.2 *Let $M = M^\top \in \mathbb{R}^{(n+1) \times (n+1)}$. Consider the decomposition of M in the form (1).*

- (i) *M is positive definite if and only if $\theta > 0$ and $\bar{M} - \frac{1}{\theta}bb^\top$ is positive definite;*
- (ii) *If $\theta > 0$, then M is positive semidefinite if and only if $\bar{M} - \frac{1}{\theta}bb^\top$ is positive semidefinite;*
- (iii) *If \bar{M} is positive definite, then M is positive semidefinite if and only if $\theta - b^\top \bar{M}^{-1}b \geq 0$.*

Now, assume that \bar{M} is invertible. In this case the determinant of M is given by the formula of the next lemma, see [12, Section 0.85, pp. 21].

Lemma 1.3 *Let $M \in \mathbb{R}^{(n+1) \times (n+1)}$. Consider the decomposition of M in the form (1). Then, $\det(M) = (\theta - b^\top \bar{M}^{-1}b) \det \bar{M}$.*

Next we present a version of the S-lemma which can be found for example in [22, Theorem 2.2].

Lemma 1.4 *Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric matrices. Assume that $\hat{x}^\top B \hat{x} < 0$ for some $\hat{x} \in \mathbb{R}^{n \times n}$. Then the following two statements are equivalent.*

- (i) *If $x^\top Bx \leq 0$, then $x^\top Ax \geq 0$;*
- (ii) *There exists a $\beta \geq 0$ such that $A + \beta B$ is positive semidefinite.*

We end this section by stating a version of Finsler's lemma, see [10]. A proof of it can be found, for example, in [17, Theorem 2].

Lemma 1.5 *Let $M, N \in \mathbb{R}^{n \times n}$ be two symmetric matrices. If $x^\top N x = 0$ implies $x^\top M x > 0$, then there exists $\lambda \in \mathbb{R}$ such that $M + \lambda N$ is positive definite.*

2 Basics Results About the Hyperbolic Space

In this section we recall some notations, definitions and basic properties about the geometry of the hyperbolic space used throughout the paper. They can be found in many introductory books on Riemannian and Differential Geometry, for example in [1, 25], see also [2].

Let $\langle \cdot, \cdot \rangle$ be the *Lorentzian inner product* of $x := (x_1, \dots, x_n, x_{n+1})^\top$ and $y := (y_1, \dots, y_n, y_{n+1})^\top$ on \mathbb{R}^{n+1} defined as follows

$$\langle x, y \rangle := x_1 y_1 + \dots + x_n y_n - x_{n+1} y_{n+1}. \quad (2)$$

For each $x \in \mathbb{R}^{n+1}$, the *Lorentzian norm (length)* of x is defined to be the complex number

$$\|x\| := \sqrt{\langle x, x \rangle}. \quad (3)$$

Here $\|x\|$ is either positive, zero, or positive imaginary. In order to state the inner product (2) in a convenient form, we take the diagonal matrix J defined by

$$J := \text{diag}(1, \dots, 1, -1) \in \mathbb{R}^{(n+1) \times (n+1)}. \quad (4)$$

By using (4), the Lorentz inner product (2) can be stated equivalently as follows

$$\langle x, y \rangle := x^\top J y, \quad \forall x, y \in \mathbb{R}^{n+1}. \quad (5)$$

Throughout the paper the *n-dimensional hyperbolic space* and its *tangent hyperplane at a point p* are denoted by

$$\mathbb{H}^n := \{p \in \mathbb{R}^{n+1} : \langle p, p \rangle = -1, p^{n+1} > 0\}, \quad T_p \mathbb{H}^n := \{v \in \mathbb{R}^{n+1} : \langle p, v \rangle = 0\}, \quad (6)$$

respectively. It is worth noting that the Lorentzian inner product defined in (2) is not positive definite in the entire space \mathbb{R}^{n+1} . However, one can show that its restriction to the tangent spaces of \mathbb{H}^n is positive definite; see [2, Section 7.6]. Consequently, $\|v\| > 0$ for all $v \in T_p\mathbb{H}^n$ and all $p \in \mathbb{H}^n$ with $v \neq 0$. Therefore, $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ are in fact a positive inner product and the associated norm in $T_p\mathbb{H}^n$, for all $p \in \mathbb{H}^n$. Next we present a basic lemma used in the sequel.

Lemma 2.1 *Let $p, q \in \mathbb{H}^n$. Then, $\langle p, q \rangle \leq -1$ and $\langle p, q \rangle = -1$ if and only if $p = q$.*

Proof Since $p, q \in \mathbb{H}^n$, we have $\langle p, p \rangle = -1$, $p_{n+1} > 0$, $\langle q, q \rangle = -1$ and $q_{n+1} > 0$. Thus, we have $p_{n+1} = \sqrt{u^\top u}$ and $q_{n+1} = \sqrt{v^\top v}$, where $u = (p_1, \dots, p_n, 1)^\top$ and $v = (q_1, \dots, q_n, 1)^\top$. Hence, it follows from (2) that $\langle p, q \rangle = p_1 q_1 + \dots + p_n q_n - \sqrt{u^\top u} \sqrt{v^\top v}$. On the other hand, by taking into account that Cauchy's inequality in the Euclidean space implies that $\sqrt{u^\top u} \sqrt{v^\top v} \geq u^\top v$ and the equality holds if and only if $u = v$, the result follows. \square

Therefore, (2) actually defines a Riemannian metric on \mathbb{H}^n , see [3, pp. 67]. The *Lorentzian projection onto the tangent hyperplane* $T_p\mathbb{H}^n$ is the linear mapping defined by

$$\mathbf{I} + pp^\top \mathbf{J} : \mathbb{R}^{n+1} \rightarrow T_p\mathbb{H}^n, \quad \forall p \in \mathbb{H}^n, \quad (7)$$

where $\mathbf{I} \in \mathbb{R}^{(n+1) \times (n+1)}$ is the identity matrix.

Remark 2.1 The Lorentzian projection (7) is self-adjoint with respect to the Lorentzian inner product (2). Indeed, $\langle (\mathbf{I} + pp^\top \mathbf{J})u, v \rangle = \langle u, (\mathbf{I} + pp^\top \mathbf{J})v \rangle$, for all $u, v \in \mathbb{R}^{n+1}$ and all $p \in \mathbb{H}^n$. Moreover, we also have $(\mathbf{I} + pp^\top \mathbf{J})(\mathbf{I} + pp^\top \mathbf{J}) = \mathbf{I} + pp^\top \mathbf{J}$, for all $p \in \mathbb{H}^n$.

The *intrinsic distance on the hyperbolic space* between two points $p, q \in \mathbb{H}^n$ is defined by

$$d(p, q) := \operatorname{arcosh}(-\langle p, q \rangle). \quad (8)$$

It can be shown that (\mathbb{H}^n, d) is a complete metric space, so that $d(p, q) \geq 0$ for all $p, q \in \mathbb{H}^n$, and $d(p, q) = 0$ if and only if $p = q$. Moreover, (\mathbb{H}^n, d) has the same topology as \mathbb{R}^n . The intersection curve of a plane

though the origin of \mathbb{R}^{n+1} with \mathbb{H}^n is called a *geodesic*. Moreover, each geodesic segment $\gamma: [a, b] \rightarrow \mathbb{H}^n$ is *minimal*, i.e., its arc length is equal to the intrinsic distance $\ell(\gamma) = \operatorname{arcosh}(-\langle \gamma(a), \gamma(b) \rangle)$ between its end points. We say that γ is a *normalized geodesic* if $\|\dot{\gamma}\| = 1$. If $p, q \in \mathbb{H}^n$ and $q \neq p$, then the unique *geodesic segment from p to q* is

$$\gamma_{pq}(t) = \left(\cosh t + \frac{\langle p, q \rangle \sinh t}{\sqrt{\langle p, q \rangle^2 - 1}} \right) p + \frac{\sinh t}{\sqrt{\langle p, q \rangle^2 - 1}} q, \quad \forall t \in [0, d(p, q)].$$

The *exponential mapping* $\exp_p: T_p\mathbb{H}^n \rightarrow \mathbb{H}^n$ is defined by $\exp_p v = \gamma_v(1)$, where γ_v is the geodesic defined by its initial position p , with velocity v at p . Hence, $\exp_p v = p$ for $v = 0$, and

$$\exp_p v := \cosh(\|v\|) p + \sinh(\|v\|) \frac{v}{\|v\|}, \quad \forall v \in T_p\mathbb{H}^n \setminus \{0\}.$$

It is easy to prove that $\gamma_v(1) = \gamma_v(t)$ for any values of t . Therefore, for all $t \in \mathbb{R}$ we have

$$\exp_p tv := \cosh(t\|v\|) p + \sinh(t\|v\|) \frac{v}{\|v\|}, \quad \forall v \in T_p\mathbb{H}^n / \{0\}. \quad (9)$$

We will also use the expression above for denoting the geodesic starting at $p \in \mathbb{H}^n$ with velocity $v \in T_p\mathbb{H}^n$ at p . The *inverse of the exponential mapping* is given by $\log_p q = 0$, for $q = p$, and

$$\log_p q := \frac{\operatorname{arcosh}(-\langle p, q \rangle)}{\sqrt{\langle p, q \rangle^2 - 1}} \left[\mathbf{I} + pp^\top \mathbf{J} \right] q, \quad q \neq p. \quad (10)$$

It follows from (8) and (10) that

$$d(p, q) = \|\log_p q\|, \quad p, q \in \mathbb{H}^n.$$

Let $\Omega \subseteq \mathbb{H}^n$ be an open set and $f: \Omega \rightarrow \mathbb{R}$ a differentiable function. The *gradient on the hyperbolic space* of f is the unique vector field $\Omega \ni p \mapsto \operatorname{grad} f(p) \in T_p\mathcal{M}$ such that $df(p)v = \langle \operatorname{grad} f(p), v \rangle$, see [2, Proposition 7-5, p.162]. Therefore, we have

$$\operatorname{grad} f(p) := \left[\mathbf{I} + pp^\top \mathbf{J} \right] \mathbf{J} \cdot Df(p) = \mathbf{J} \cdot Df(p) + \langle \mathbf{J} \cdot Df(p), p \rangle p, \quad (11)$$

where $Df(p) \in \mathbb{R}^{n+1}$ is the usual gradient of f at $p \in \Omega$. A *vector field* on $\Omega \subseteq \mathbb{H}^n$ is a smooth mapping $X: \Omega \rightarrow \mathbb{R}^{n+1}$ such that $X(p) \in T_p\mathbb{H}^n$. The *covariant derivative* of X at $p \in \Omega$ is map $\nabla X(p): T_p\mathbb{H}^n \rightarrow$

$T_p\mathbb{H}^n$ given by

$$\nabla X(p) := \left[\mathbf{I} + pp^\top \mathbf{J} \right] DX(p),$$

where $DX(p)$ denotes the usual derivative of the vector field X at the point p , see [2, Formula (7.62), p.162]. The *Hessian on the hyperbolic space* of a twice differentiable function $f : \Omega \rightarrow \mathbb{R}$ at a point $p \in \Omega$ is the mapping $\nabla \text{grad} f(p) := \text{Hess} f(p) : T_p\mathbb{H}^n \rightarrow T_p\mathbb{H}^n$ given by

$$\text{Hess} f(p) := \left[\mathbf{I} + pp^\top \mathbf{J} \right] \left[\mathbf{J} \cdot D^2 f(p) + \langle \mathbf{J} \cdot Df(p), p \rangle \mathbf{I} \right], \quad (12)$$

where $D^2 f(p)$ is the usual Hessian (Euclidean Hessian) of the function f at a point p , see [2, Proposition 7.6, p.163]. Let $I \subseteq \mathbb{R}$ be an open interval, $\Omega \subseteq \mathbb{H}^n$ an open set and $\gamma : I \rightarrow \Omega$ a geodesic segment. Since $f : \mathcal{C} \rightarrow \mathbb{R}$ is a differentiable function and $\gamma'(t) \in T_{\gamma(t)}\mathbb{H}^n$ for all $t \in I$, equality (11) implies

$$\frac{d}{dt} f(\gamma(t)) = \langle \text{grad} f(\gamma(t)), \gamma'(t) \rangle = \langle \mathbf{J} \cdot Df(\gamma(t)), \gamma'(t) \rangle, \quad \forall t \in I. \quad (13)$$

Moreover, if the function f is twice differentiable then it holds that

$$\begin{aligned} \frac{d^2}{dt^2} f(\gamma(t)) &= \langle \text{Hess} f(\gamma(t)) \gamma'(t), \gamma'(t) \rangle \\ &= \langle \mathbf{J} \cdot D^2 f(\gamma(t)) \gamma'(t), \gamma'(t) \rangle + \langle \mathbf{J} \cdot Df(\gamma(t)), \gamma(t) \rangle \langle \gamma'(t), \gamma'(t) \rangle, \quad \forall t \in I. \end{aligned} \quad (14)$$

For each $p, q \in \mathbb{H}^n$ the covariant derivative induces the linear isometry relative to the Lorentzian inner product $\langle \cdot, \cdot \rangle$, $P_{pq} : T_p\mathbb{H}^n \rightarrow T_q\mathbb{H}^n$ defined by $P_{pq}v = V(t)$, where V is the unique vector field on the geodesic segment $\gamma : [0, 1] \rightarrow \mathbb{H}^n$ from p to q , i.e., $\gamma(0) = p$ and $\gamma(1) = q$ such that $\nabla V(t)\gamma'(t) = \nabla_{\gamma'(t)} V(t) = 0$ and $V(0) = v$, the so-called *parallel transport* along the geodesic segment γ joining p to q .

The explicitly formula of P_{pq} is given by

$$P_{pq}(v) := v - \frac{\langle v, \log_q p \rangle}{\text{arcosh}^2(-\langle p, q \rangle)} (\log_q p + \log_p q).$$

By using (10), after some algebraic manipulation, the last inequality becomes

$$P_{pq}(v) := \left[\mathbf{I} + \frac{1}{1 - \langle p, q \rangle} (p + q)q^\top \mathbf{J} \right] v.$$

Note that for all geodesic segment $\gamma: [a, b] \rightarrow \mathbb{H}^n$ we have $\gamma'(t) = P_{pq}(\gamma'(a))$, for all $t \in [a, b]$ or equivalently that $\gamma''(t) = 0$, for all $t \in [a, b]$. Next we give two standard notations. We denote the *open* and *closed ball* of radius $\delta > 0$ and center $p \in \mathbb{H}^n$ by $B_\delta(p) := \{q \in \mathbb{H}^n : d(p, q) < \delta\}$ and $\hat{B}_\delta(p) := \{q \in \mathbb{H}^n : d(p, q) \leq \delta\}$, respectively.

Let us recall the *Lorentz group* $G_{\mathcal{L}}$, preserving the norm (3) and metric (5), defined by

$$G_{\mathcal{L}} := \left\{ Q \in \mathbb{R}^{(n+1) \times (n+1)} : Q^\top J Q = J \right\}. \quad (15)$$

Note that $|\det Q| = 1$, for all $Q \in G_{\mathcal{L}}$. Moreover, $Q^{-1}, Q^\top \in G_{\mathcal{L}}$, for all $Q \in G_{\mathcal{L}}$.

Example 2.1 We exhibit two examples of matrices $Q \in G_{\mathcal{L}}$. For the first example, take $u \in \mathbb{R}^{n+1}$ such that $\|u\| > 0$. Thus, $Q = I - (2/\|u\|^2)uu^\top J \in G_{\mathcal{L}}$. For the second example, take $u, w \in \mathbb{R}^{n+1}$ such that $\|u\| = 1$ and $\|w\| = 1$. Therefore, $Q = I + 2wu^\top J - (1/(1 + u^\top Jw))(u + w)(u + w)^\top J \in G_{\mathcal{L}}$.

We end this section by remarking that the Lorentz group (15) preserves geodesics of \mathbb{H}^n .

Remark 2.2 First note that for a given $Q \in G_{\mathcal{L}}$, we have $\|Qv\| = \|v\|$. Moreover, $p \in \mathbb{H}^n$ and $v \in T_p\mathbb{H}^n$ if and only if $Qp \in \mathbb{H}^n$ and $Qv \in T_p\mathbb{H}^n$, i.e., $v \in T_p\mathbb{H}^n \setminus \{0\}$ if and only if $Qv \in T_{Qp}\mathbb{H}^n \setminus \{0\}$. Consequently, it follows from (9) that

$$Q \exp_p tv = \exp_{Qp} tQv, \quad \forall v \in T_p\mathbb{H}^n \setminus \{0\}.$$

Therefore, the Lorentz group preserves the geodesics of \mathbb{H}^n .

2.1 Properties of the Intrinsic Distance on the Hyperbolic Space

In this section, we present some important properties of the intrinsic distance from a fixed point on the hyperbolic space. In particular, we present the spectral decomposition of the Hessian of the intrinsic distance. The *intrinsic distance function on the hyperbolic space from the fixed point* $q \in \mathbb{H}^n$ is the mapping $d_q: \mathbb{H}^n \rightarrow \mathbb{R}$ defined by

$$d_q(p) := \operatorname{arcosh}(-\langle p, q \rangle). \quad (16)$$

The intrinsic distance from q , denoted by d_q , is twice differentiable at $p \in \mathbb{H}^n \setminus \{q\}$. By combining (11) and (16), we can see that the *gradient of the distance from q* at p is given by

$$\text{grad } d_q(p) := \frac{-1}{\sqrt{\langle p, q \rangle^2 - 1}} \left[\mathbf{I} + pp^\top \mathbf{J} \right] q, \quad q \neq p. \quad (17)$$

Moreover, by using (12) and (16), we obtain that the *Hessian of the distance from q* at p is given by

$$\text{Hess } d_q(p) := \frac{\langle p, q \rangle}{\sqrt{\langle p, q \rangle^2 - 1}} \left[\mathbf{I} + pp^\top \mathbf{J} \right] \left[\frac{1}{\langle p, q \rangle^2 - 1} qq^\top \mathbf{J} - \mathbf{I} \right], \quad q \neq p. \quad (18)$$

Before presenting the spectral decomposition of the Hessian of intrinsic distance from a fixed point on the hyperbolic space, we need the following elementary result.

Lemma 2.2 *Let $p, q \in \mathbb{H}^n$ with $q \neq p$. Then $\dim(T_p \mathbb{H}^n \cap T_q \mathbb{H}^n) = n - 1$ and $\langle q + \langle p, q \rangle p, v \rangle = 0$, for all $v \in T_p \mathbb{H}^n \cap T_q \mathbb{H}^n$. As a consequence, taking an orthonormal basis of the subspace $T_p \mathbb{H}^n \cap T_q \mathbb{H}^n$, say $\{v_1, \dots, v_{n-1}\}$ and defining $v_n = (q + \langle p, q \rangle p) / \|q + \langle p, q \rangle p\|$, the set $\{v_1, \dots, v_{n-1}, v_n\}$ is an orthonormal basis of $T_p \mathbb{H}^n$.*

In the next lemma we present a spectral decomposition of the Hessian of the intrinsic distance from a fixed point on the hyperbolic space. The results in this lemma and the next one are closely related to [9], see also its counterparts on the sphere in [4, Lemma 2, Lemma 3].

Lemma 2.3 *Take $q \in \mathbb{H}^n$ and let $\text{Hess } d_q(p) : T_p \mathbb{H}^n \rightarrow T_p \mathbb{H}^n$ be the Hessian of the intrinsic distance from q at the point $p \in \mathbb{H}^n \setminus \{q\}$. Then,*

$$\text{Hess } d_q(p) (q + \langle p, q \rangle p) = 0, \quad \text{Hess } d_q(p) v = \frac{-\langle p, q \rangle}{\sqrt{\langle p, q \rangle^2 - 1}} v, \quad \forall v \in T_p \mathbb{H}^n \cap T_q \mathbb{H}^n. \quad (19)$$

Moreover, $\lambda_1 = 0$ and $\lambda_2 = -\langle p, q \rangle / \sqrt{\langle p, q \rangle^2 - 1} > 0$ are the unique eigenvalues of $\text{Hess } d_q(p)$, with algebraic multiplicity 1 and $n - 1$, respectively. Moreover, the Hessian $\text{Hess } d_q(p)$ is positive semidefinite.

Proof Since $p \neq q$, Lemma 2.1 implies that $\langle p, q \rangle \neq -1$ and from (18) the Hessian is well defined. As $q^\top \mathbf{J} q = -1$, simple calculations give

$$\left[\frac{1}{\langle p, q \rangle^2 - 1} qq^\top \mathbf{J} - \mathbf{I} \right] (q + \langle p, q \rangle p) = -\langle p, q \rangle p.$$

On the other hand, $[I + pp^\top J](-\langle p, q \rangle p) = 0$, which combined with the latter equality and (18), implies the first equality in (19), and we also have that $\lambda_1 = 0$ is an eigenvalue of the Hessian. For proving the second equality in (19), note that the definitions in (6) imply that

$$\langle p, v \rangle = 0, \quad \langle q, v \rangle = 0, \quad \forall v \in T_p \mathbb{H}^n \cap T_q \mathbb{H}^n.$$

Thus, the second inequality in (19) follows from (18) and the last two equalities. In particular, the Hessian is a multiple of the identity in the subspace $T_p \mathbb{H}^n \cap T_q \mathbb{H}^n$. Moreover, due to $\dim T_p \mathbb{H}^n = n$, we conclude, using Lemma 2.2, that the eigenvalues λ_1 and λ_2 have algebraic multiplicity 1 and $n - 1$, respectively, proving the first statement. For proving the second statement, let $\{v_1, \dots, v_{n-1}\}$ be an orthonormal basis of the subspace $T_p \mathbb{H}^n \cap T_q \mathbb{H}^n$. Since $\langle p, q \rangle \neq 1$, we can define $v_n = (q + \langle p, q \rangle p) / \|q + \langle p, q \rangle p\|$. Hence, Lemma 2.2 implies that $\{v_1, \dots, v_{n-1}, v_n\}$ is an orthonormal basis of $T_p \mathbb{H}^n$. Therefore, given $u \in T_p \mathbb{H}^n$, there exist $a_1, \dots, a_{n-1}, a_n \in \mathbb{R}$ such that $u = a_1 v_1 + \dots + a_{n-1} v_{n-1} + a_n v_n$, which, by using the first statement, entails $\langle \text{Hess } d_q(p) u, u \rangle = \lambda_2 (a_1^2 + \dots + a_{n-1}^2)$, completing the proof of the second statement. \square

Take $q \in \mathbb{H}^n$ and define $\rho_q : \mathbb{H}^n \rightarrow \mathbb{R}$ as

$$\rho_q(p) := \frac{1}{2} d_q^2(p). \quad (20)$$

By using the definition of ρ_q in (20) and (12), it is easy to conclude, after some algebra, that

$$\text{Hess } \rho_q(p) = d_q(p) \text{Hess } d_q(p) + [I + pp^\top J] J \cdot Dd_q(p) Dd_q(p)^T, \quad (21)$$

where $Dd_q(p)$ is the usual derivative of d_q at the point p .

Lemma 2.4 *Take $q \in \mathbb{H}^n$ and define $\text{Hess } \rho_q(p) : T_p \mathbb{H}^n \rightarrow T_p \mathbb{H}^n$ as the Hessian of ρ_q at the point $p \in \mathbb{H}^n \setminus \{q\}$. Then the following equalities hold:*

$$\text{Hess } \rho_q(p) (q + \langle p, q \rangle p) = q + \langle p, q \rangle p, \quad \text{Hess } \rho_q(p) v = \frac{-\langle p, q \rangle \text{arcosh}(-\langle p, q \rangle)}{\sqrt{\langle p, q \rangle^2 - 1}} v, \quad (22)$$

for all $v \in T_p \mathbb{H}^n \cap T_q \mathbb{H}^n$. As a consequence, $\mu_1 = 1$ and $\mu_2 = \langle p, q \rangle \text{arcosh}(-\langle p, q \rangle) / \sqrt{\langle p, q \rangle^2 - 1}$ are the unique eigenvalues of $\text{Hess } \rho_q(p)$, with algebraic multiplicity 1 and $n - 1$, respectively. Moreover, the Hessian $\text{Hess } \rho_q(p)$ is positive definite.

Proof First note that taking into account that $Dd_q(p) = -(Jq)/\sqrt{\langle p, q \rangle^2 - 1}$, we have

$$J \cdot Dd_q(p) Dd_q(p)^T = \frac{1}{\langle p, q \rangle^2 - 1} qq^T J. \quad (23)$$

Due to $q^T J q = -1$, it follows from the last equality that $J \cdot Dd_q(p) Dd_q(p)^T (q + \langle p, q \rangle p) = q$. On the other hand, $[I + pp^T J]q = q + \langle p, q \rangle p$. Hence, we obtain that

$$[I + pp^T J] J \cdot Dd_q(p) Dd_q(p)^T (q + \langle p, q \rangle p) = q + \langle p, q \rangle p.$$

Therefore, combining the last equality, equation (21) and the first equality in (19), we get that

$$\text{Hess } \rho_q(p)(q + \langle p, q \rangle p) = q + \langle p, q \rangle p,$$

which is the first equality in (22). For proving the second one, note first that the definition of $T_q \mathbb{H}^n$ implies that $q^T J v = 0$ for all $v \in T_p \mathbb{H}^n \cap T_q \mathbb{H}^n$. Then, by using (23), we have

$$[I + pp^T J] Dd_q(p) Dd_q(p)^T v = 0, \quad \forall v \in T_p \mathbb{H}^n \cap T_q \mathbb{H}^n.$$

Hence, equation (21) implies that $\text{Hess } \rho_q(p)v = d_q(p) \text{Hess } d_q(p)v$ for all $v \in T_p \mathbb{H}^n \cap T_q \mathbb{H}^n$. Thus, by using the second equality in (19) and the definition of $d_q(p)$ in (16), the second equality in (22) follows. The remainder of our proof requires similar arguments to those in the proof of Lemma 2.3 (note that in the final part of the proof we must invoke the fact that $\text{arcosh}(-\langle p, q \rangle) > 0$ and $\langle p, q \rangle < 0$, which holds by applying Lemma 2.1 together with $p \neq q$). \square

3 Convex Sets on the Hyperbolic Space

In this section we present some properties of the convex sets of the hyperbolic space. It is worth to remark that the convex sets on the hyperbolic space \mathbb{H}^n are closely related to convex cones belonging to the interior of the Lorentz cone

$$\mathcal{L} := \left\{ x \in \mathbb{R}^{n+1} : x_{n+1} \geq \sqrt{x_1^2 + \cdots + x_n^2} \right\}. \quad (24)$$

Definition 3.1 The set $\mathcal{C} \subseteq \mathbb{H}^n$ is said to be *hyperbolically convex* if for any $p, q \in \mathcal{C}$ the geodesic segment joining p to q is contained in \mathcal{C} .

For each set $\mathcal{A} \subseteq \mathbb{H}^n$, let $\mathcal{H}_{\mathcal{A}}$ be the *cone spanned by \mathcal{A}* , namely,

$$\mathcal{H}_{\mathcal{A}} := \{tp : p \in \mathcal{A}, t \in [0, +\infty)\}. \quad (25)$$

Clearly, $\mathcal{H}_{\mathcal{A}}$ is the smallest cone which contains \mathcal{A} and belongs to the interior of the Lorentz cone \mathcal{L} .

In the next result we relate a hyperbolically convex set with the cone spanned by it.

Proposition 3.1 *The set $\mathcal{C} \subseteq \mathbb{H}^n$ is hyperbolically convex if and only if the cone $\mathcal{H}_{\mathcal{C}}$ is convex.*

Proof Assume that $\mathcal{C} \subseteq \mathbb{H}^n$ is a hyperbolically convex set. Let $x, y \in \mathcal{H}_{\mathcal{C}}$. For proving that $\mathcal{H}_{\mathcal{C}}$ is convex, it suffices to show that

$$z = x + y \in \mathcal{H}_{\mathcal{C}}. \quad (26)$$

The definition of $\mathcal{H}_{\mathcal{C}}$ implies that there exist $p, q \in \mathcal{C}$ and $s, t \in [0, +\infty)$ such that $x = sp$ and $y = tq$.

Hence, due to $z = sp + tq$ with $p, q \in \mathcal{C} \subseteq \mathbb{H}^n$, (26) and $\langle p, q \rangle \leq -1$ imply that $\langle z, z \rangle \leq -(t+s)^2$, which is equivalent to $0 < t+s \leq \sqrt{-\langle z, z \rangle}$. Now we take

$$\gamma_{pq}(t) = \left(\cosh t + \frac{\langle p, q \rangle \sinh t}{\sqrt{\langle p, q \rangle^2 - 1}} \right) p + \frac{\sinh t}{\sqrt{\langle p, q \rangle^2 - 1}} q, \quad \forall t \in [0, d(p, q)]. \quad (27)$$

the normalized segment of geodesic from p to q . To proceed we first need to prove that

$$d(p, az) \leq d(p, q), \quad \gamma_{pq}(d(p, az)) = az, \quad (28)$$

where $a := 1/\sqrt{-\langle z, z \rangle}$. Since the function $[0, +\infty) \ni \tau \mapsto \operatorname{arcosh}(\tau)$ is increasing, it follows from (8) that to prove the inequality in (28) it suffices to show that $-\langle p, az \rangle \leq -\langle p, q \rangle$ or the equivalent inequality

$$0 \leq \langle p, az \rangle - \langle p, q \rangle. \quad (29)$$

Due to $z = sp + tq$ and $\langle p, p \rangle = -1$, direct calculations yield $\langle p, az \rangle - \langle p, q \rangle = a(-s + t\langle p, q \rangle) - \langle p, q \rangle$.

Thus, taking into account that $\langle p, q \rangle \leq -1$ and $s+t \leq \sqrt{-\langle z, z \rangle} = 1/a$, we conclude that

$$\langle p, az \rangle - \langle p, q \rangle \geq a(s\langle p, q \rangle + t\langle p, q \rangle) - \langle p, q \rangle = a(-\langle p, q \rangle)(1/a - s - t) \geq 0,$$

which implies that (29) holds and consequently the inequality in (28) also holds. Our next task is to prove the equality in (28). Thus, by using (27), we have to show that

$$\gamma_{pq}(d(p, az)) = \left(\cosh d(p, az) + \frac{\langle p, q \rangle \sinh d(p, az)}{\sqrt{\langle p, q \rangle^2 - 1}} \right) p + \frac{\sinh d(p, az)}{\sqrt{\langle p, q \rangle^2 - 1}} q = az. \quad (30)$$

It follows from (8) that $\cosh d(p, az) = -a\langle p, z \rangle$, which implies that $\sinh d(p, az) = \sqrt{a^2\langle p, z \rangle^2 - 1}$. Thus, considering that $a^2\langle z, z \rangle = -1$ and $z = sp + tq$ with $p, q \in \mathcal{C} \subseteq \mathbb{H}^n$ and $t \in [0, +\infty)$, we have

$$\sinh d(p, az) = \sqrt{a^2\langle p, z \rangle^2 - 1} = a\sqrt{\langle p, z \rangle^2 + \langle z, z \rangle} = at\sqrt{\langle p, q \rangle^2 - 1} \quad (31)$$

and $\cosh d(p, az) = -a\langle p, z \rangle = as - at\langle p, q \rangle$. Hence, substituting the last equality and (31) into (30) and taking into account that $z = sp + tq$, we obtain that

$$\gamma_{pq}(d(p, az)) = \left(as - at\langle p, q \rangle + \frac{\langle p, q \rangle at\sqrt{\langle p, q \rangle^2 - 1}}{\sqrt{\langle p, q \rangle^2 - 1}} \right) p + \frac{at\sqrt{\langle p, q \rangle^2 - 1}}{\sqrt{\langle p, q \rangle^2 - 1}} q = asp + atq = az.$$

which concludes the proof of the equality in (28). Since \mathcal{C} is a hyperbolically convex set and $d(p, az) \leq d(p, q)$ we obtain that $\gamma_{pq}(d(p, az)) \in \mathcal{C}$, which together with (25) and (28) implies that $z = (1/a)\gamma_{pq}(d(p, az)) \in \mathcal{H}_{\mathcal{C}}$. Thus, $\mathcal{H}_{\mathcal{C}}$ is convex.

Now, assume that the cone $\mathcal{H}_{\mathcal{C}}$ is convex. First note that $\mathcal{C} = \mathcal{H}_{\mathcal{C}} \cap \mathbb{H}^n$. Take $p, q \in \mathcal{C}$ with $q \neq p$. We must prove that the geodesic segment from p to q is contained in \mathcal{C} . As $p, q \in \mathcal{H}_{\mathcal{C}}$ and $\mathcal{H}_{\mathcal{C}} \subseteq \mathcal{L}$, we conclude that $q \neq -p$. Thus, $\langle p, q \rangle < -1$ and $d(p, q) > 0$. Let

$$[0, d(p, q)] \ni t \mapsto \gamma_{pq}(t) = \alpha(t)p + \beta(t)q$$

be the normalized geodesic segment from p to q , where

$$\alpha(t) := \cosh t + \frac{\langle p, q \rangle \sinh t}{\sqrt{\langle p, q \rangle^2 - 1}}, \quad \beta(t) := \frac{\sinh t}{\sqrt{\langle p, q \rangle^2 - 1}}.$$

Since $\gamma_{pq}(t) \in \mathbb{H}^n$, $p, q \in \mathcal{H}_{\mathcal{C}}$ and $\mathcal{H}_{\mathcal{C}}$ is a convex cone, for proving that $\gamma_{pq}(t) \in \mathcal{C}$ for all $t \in [0, d(p, q)]$, it suffices to prove that $\alpha(t) \geq 0$ and $\beta(t) \geq 0$ for all $t \in [0, d(p, q)]$. Due to $\sinh t \geq 0$ for all $t \geq 0$ we conclude that $\beta(t) \geq 0$ for all $t \in [0, d(p, q)]$. We proceed to prove that $\alpha(t) \geq 0$ for all $t \in [0, d(p, q)]$.

For that, we first note that due to hyperbolic tangent being an increasing function, $\cosh d(p, q) = -\langle p, q \rangle$ and $\sinh d(p, q) = \sqrt{\langle p, q \rangle^2 - 1}$, we have

$$\tanh t \leq \tanh d(p, q) = \frac{\sinh d(p, q)}{\cosh d(p, q)} = \frac{\sqrt{\langle p, q \rangle^2 - 1}}{-\langle p, q \rangle}, \quad t \in [0, d(p, q)].$$

Hence, taking into account that $\cosh t \geq 0$ for all $t \in \mathbb{R}$ and $\langle p, q \rangle < -1$, we conclude that

$$\alpha(t) = \cosh t \left(1 + \frac{\langle p, q \rangle}{\sqrt{\langle p, q \rangle^2 - 1}} \tanh t \right) \geq 0, \quad t \in [0, d(p, q)],$$

which completes the proof. \square

Remark 3.1 The hyperbolically convex sets are intersections of the hyperboloid with convex cones which belong to the interior of \mathcal{L} . Indeed, it follows easily from Proposition 3.1, that if $\mathcal{K} \subseteq \text{int}(\mathcal{L})$ is a convex cone, where \mathcal{L} is the Lorentz cone, then $C = \mathcal{K} \cap \mathbb{H}^n$ is a hyperbolically convex set and $\mathcal{K} = \mathcal{K}_{\mathcal{C}}$.

Remark 3.2 Let $\mathcal{C} \subseteq \mathbb{H}^n$ and $Q \in G_{\mathcal{L}}$. First note that $Q\mathcal{C} := \{Qp : p \in \mathbb{H}^n\}$. It follows from Remark 2.2 and Definition 3.1 that \mathcal{C} is hyperbolically convex if and only if $Q\mathcal{C}$ is hyperbolically convex.

4 Intrinsic Projection Onto Hyperbolically Convex Sets

In this section we present some properties of the intrinsic projection onto hyperbolically convex sets on hyperbolic spaces. Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a closed hyperbolically convex set and $p \in \mathbb{H}^n$. Consider the following constrained optimization problem

$$\min_{q \in \mathcal{C}} d(p, q). \quad (32)$$

The minimal value of the function $\mathcal{C} \ni q \mapsto d(p, q)$ is called the *distance of p from \mathcal{C}* and it is denoted by $d_{\mathcal{C}}(p)$, i.e., $d_{\mathcal{C}} : \mathbb{H}^n \rightarrow \mathbb{R}$ is defined by

$$d_{\mathcal{C}}(p) := \min_{q \in \mathcal{C}} d(p, q).$$

Since (\mathbb{H}^n, d) is a complete metric space, we have the following results.

Proposition 4.1 *Let $\mathcal{C} \in \mathbb{H}^n$ be a nonempty subset. Then, $|d_{\mathcal{C}}(p) - d_{\mathcal{C}}(q)| \leq d(p, q)$, for all $p, q \in \mathbb{H}^n$.*

In particular, the function $d_{\mathcal{C}}$ is continuous.

Note that due to \mathcal{C} being a closed set and the distance function continuous, the problem (32) has a solution. The solution of the problem (32) is called *metric projection*, it was first studied in [31]. In the next proposition we explicitly give an important property of the metric projection.

Proposition 4.2 *Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a closed hyperbolically convex set and $p \in \mathbb{H}^n$. A point $y^p \in \mathcal{C}$ is a solution of the problem (32) if and only if*

$$\left\langle \left(\mathbf{I} + y^p (y^p)^\top \mathbf{J} \right) p, \left(\mathbf{I} + y^p (y^p)^\top \mathbf{J} \right) q \right\rangle \leq 0, \quad \forall q \in \mathcal{C}. \quad (33)$$

Furthermore, the solution of problem (32) is unique.

Proof First we assume that y^p is a solution of (32). If $p \in \mathcal{C}$ i.e., $p = y^p$, then the inequality trivially holds. Assume that $p \notin \mathcal{C}$, i.e., $p \neq y^p$. Take $q \in \mathcal{C}$ such that $q \neq y^p$ and

$$[0, 1] \ni t \mapsto \exp_{y^p}(t \log_{y^p} q) = \cosh(td(y^p, q))y^p + \frac{\sinh(td(y^p, q))}{d(y^p, q)} \log_{y^p} q \quad (34)$$

be the geodesic segment from y^p to q . Thus, due to \mathcal{C} being a hyperbolically convex set, it follows that $d(p, y^p) \leq d(p, \exp_{y^p}(t \log_{y^p} q))$, for all $t \in [0, 1]$. Hence, using (8) and (34), we conclude that

$$\operatorname{arcosh}(-\langle p, y^p \rangle) \leq \operatorname{arcosh} \left(- \left\langle p, \cosh(td(y^p, q))y^p + \frac{\sinh(td(y^p, q))}{d(y^p, q)} \log_{y^p} q \right\rangle \right),$$

for all $t \in [0, 1]$. Since $1 \leq -\langle p, y^p \rangle$, for all $p \in \mathcal{C}$, and the function $[1, +\infty] \ni s \mapsto \operatorname{arcosh}(s)$ is increasing, we obtain from (34) that

$$\left\langle p, \cosh(td(y^p, q))y^p + \frac{\sinh(td(y^p, q))}{d(y^p, q)} \log_{y^p} q \right\rangle \leq \langle p, y^p \rangle, \quad \forall t \in [0, 1].$$

After some algebra, we conclude from the previous inequality that

$$\frac{\sinh(td(y^p, q))}{td(y^p, q)} \langle p, \log_{y^p} q \rangle \leq \frac{1 - \cosh(td(y^p, q))}{td(y^p, q)} d(y^p, q) \langle p, y^p \rangle, \quad \forall t \in [0, 1].$$

Letting t go to zero in the last inequality we have $\langle p, \log_{y^p} q \rangle \leq 0$, which, in view of (10), yields

$$\frac{\operatorname{arcosh}(-\langle y^p, q \rangle)}{\sqrt{\langle y^p, q \rangle^2 - 1}} \left\langle p, \left(\mathbf{I} + y^p (y^p)^\top \mathbf{J} \right) q \right\rangle \leq 0.$$

Thus, due to $\operatorname{arcosh}(-\langle y^p, q \rangle) > 0$, we have $\langle p, (\mathbf{I} + y^p (y^p)^\top \mathbf{J}) q \rangle \leq 0$. Since

$$\left\langle y^p (y^p)^\top \mathbf{J} p, \left(\mathbf{I} + y^p (y^p)^\top \mathbf{J} \right) q \right\rangle = 0,$$

the desired inequality (33) follows. To establish the converse we assume that y^p satisfies (33). Direct computations show that (33) is equivalent to

$$\langle p, q \rangle + \langle p, y^p \rangle \langle y^p, q \rangle \leq 0, \quad \forall q \in \mathcal{C}. \quad (35)$$

Since $\langle y^p, q \rangle \leq -1$ and $\langle p, y^p \rangle \leq -1$, we have $\langle p, y^p \rangle \langle y^p, q \rangle \geq -\langle p, y^p \rangle$. Thus, (35) implies that

$$1 \leq -\langle p, y^p \rangle \leq -\langle p, q \rangle, \quad \forall q \in \mathcal{C}.$$

Due to the function $[0, +\infty] \ni t \mapsto \operatorname{arcosh}(t)$ being increasing, the last inequality implies that

$$\operatorname{arcosh}(-\langle p, y^p \rangle) \leq \operatorname{arcosh}(-\langle p, q \rangle), \quad \forall q \in \mathcal{C},$$

or equivalently that $d(p, y^p) \leq d(p, q)$, for all $q \in \mathcal{C}$. Therefore, y^p is a solution of (32) and the converse is proved. For the uniqueness, let $p, \hat{p} \in \mathcal{P}_{\mathcal{C}}(p)$. Since $y^p, \hat{y}^p \in \mathcal{C}$ and $\langle y^p, \hat{y}^p \rangle \leq -1$ (see Lemma 2.1), by the first statement, it follows from the equivalence of (33) and (35) that

$$\langle p, \hat{y}^p \rangle \leq -\langle p, \hat{y}^p \rangle \langle y^p, \hat{y}^p \rangle = \langle p, \hat{y}^p \rangle |\langle y^p, \hat{y}^p \rangle|,$$

which implies that $\langle p, \hat{y}^p \rangle \leq \langle p, \hat{y}^p \rangle \langle y^p, \hat{y}^p \rangle^2$. Due to $\langle p, \hat{y}^p \rangle \leq -1$, we obtain that $1 \geq \langle \hat{y}^p, \hat{y}^p \rangle^2$. Hence, taking into account that $\langle \hat{y}^p, \hat{y}^p \rangle \leq -1$, we conclude that $\langle y^p, \hat{y}^p \rangle = -1$. Therefore, from Lemma 2.1 we conclude that $y^p = \hat{y}^p$ and the solution set of the problem (32) is a singleton set, which concludes the proof. \square

It follows from Proposition 4.2 that the *projection mapping* $\mathcal{P}_{\mathcal{C}} : \mathbb{H}^n \rightarrow \mathcal{C}$ given by

$$\mathcal{P}_{\mathcal{C}}(p) := \operatorname{argmin}_{q \in \mathcal{C}} d(p, q) \quad (36)$$

is well defined. Moreover, (36) is equivalent to the following inequality

$$\langle (\mathbf{I} + \mathcal{P}_{\mathcal{C}}(p) \mathcal{P}_{\mathcal{C}}(p)^T \mathbf{J}) p, (\mathbf{I} + \mathcal{P}_{\mathcal{C}}(p) \mathcal{P}_{\mathcal{C}}(p)^T \mathbf{J}) q \rangle \leq 0, \quad \forall q \in \mathcal{C}, \forall p \in \mathbb{H}^n. \quad (37)$$

Considering that Lemma 2.1 implies that for all $p, q \in \mathbb{H}^n$ we have $\langle \mathcal{P}_{\mathcal{C}}(p), q \rangle \leq -1$, we conclude from (10) that (37) can be equivalently stated as follows

$$\langle \log_{\mathcal{P}_{\mathcal{C}}(p)} p, \log_{\mathcal{P}_{\mathcal{C}}(p)} q \rangle \leq 0, \quad \forall q \in \mathcal{C}, \forall p \in \mathbb{H}^n, \quad (38)$$

see [8, Corollary 3.1]. Furthermore, since that the function $[0, +\infty) \ni \tau \mapsto \operatorname{arcosh}(\tau)$ is increasing, it follows from (8) that (36), (37) and (38) are also equivalent to

$$\mathcal{P}_{\mathcal{C}}(p) := \operatorname{argmin}_{q \in \mathcal{C}} (-\langle p, q \rangle). \quad (39)$$

An immediate consequence of (39) is the monotonicity of the projection mapping, stated as follows:

Proposition 4.3 *Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a nonempty closed hyperbolically convex set. Then*

$$\langle \mathcal{P}_{\mathcal{C}}(p) - \mathcal{P}_{\mathcal{C}}(q), p - q \rangle \geq 0, \quad \forall p, q \in \mathcal{C}.$$

Proof Take $p, q \in \mathbb{H}^n$. Since $\mathcal{P}_{\mathcal{C}}(p), \mathcal{P}_{\mathcal{C}}(q) \in \mathcal{C}$, it follows from (39) that $-\langle p, \mathcal{P}_{\mathcal{C}}(p) \rangle \leq -\langle p, \mathcal{P}_{\mathcal{C}}(q) \rangle$ and $-\langle q, \mathcal{P}_{\mathcal{C}}(q) \rangle \leq -\langle q, \mathcal{P}_{\mathcal{C}}(p) \rangle$. Hence, $\langle p, \mathcal{P}_{\mathcal{C}}(p) - \mathcal{P}_{\mathcal{C}}(q) \rangle \geq 0$ and $\langle -q, \mathcal{P}_{\mathcal{C}}(p) - \mathcal{P}_{\mathcal{C}}(q) \rangle \geq 0$. Therefore, summing the last two inequalities the desired inequality follows. \square

Proposition 4.4 *Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a nonempty closed hyperbolically convex set. Then $\mathcal{P}_{\mathcal{C}}$ is continuous.*

Proof Let $\{p^k\} \subseteq \mathbb{H}^n$ be such that $\lim_{k \rightarrow +\infty} p^k = p$. Since Proposition 4.1 implies that $d_{\mathcal{C}}$ is continuous and taking into account that (36) implies

$$d(p^k, \mathcal{P}_{\mathcal{C}}(p^k)) = d_{\mathcal{C}}(p^k), \quad (40)$$

we conclude that $(d(p^k, P_{\mathcal{C}}(p^k)))_{k \in \mathbb{N}}$ is a bounded sequence. Consequently, considering that

$$d(p, P_{\mathcal{C}}(p^k)) \leq d(p, p^k) + d(p^k, P_{\mathcal{C}}(p^k)),$$

we also have that $(P_{\mathcal{C}}(p^k))_{k \in \mathbb{N}}$ is also bounded. Let $q \in \mathcal{C}$ be a cluster point of $(P_{\mathcal{C}}(p^k))_{k \in \mathbb{N}}$ and let $(p^{k_j})_{j \in \mathbb{N}}$ be such that $\lim_{j \rightarrow +\infty} P_{\mathcal{C}}(p^{k_j}) = q$. Hence, using (40) we have $d_{\mathcal{C}}(p^{k_j}) = d(p^{k_j}, P_{\mathcal{C}}(p^{k_j}))$, for all $j \in \mathbb{N}$. Thus, letting j goes to $+\infty$ and using Proposition 4.1 we have $d_{\mathcal{C}}(p) = d(p, q)$, which due to the second part of Proposition 4.2 implies that $q = P_{\mathcal{C}}(p)$. Consequently, $(P_{\mathcal{C}}(p^k))_{k \in \mathbb{N}}$ has only one cluster point, namely, $P_{\mathcal{C}}(p)$. Thus, $\lim_{k \rightarrow +\infty} P_{\mathcal{C}}(p^k) = P_{\mathcal{C}}(p)$ and the proof is concluded. \square

5 Hyperbolically Convex Functions

In this section we study the basic properties of convex functions on the hyperbolic space. In particular, for differentiable convex functions, the first and second order characterizations will be presented.

Definition 5.1 Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a hyperbolically convex set and $I \subseteq \mathbb{R}$ an interval. A function $f : \mathcal{C} \rightarrow \mathbb{R}$ is said to be hyperbolically convex (respectively, strictly hyperbolically convex) if for any geodesic segment $\gamma : I \rightarrow \mathcal{C}$, the composition $f \circ \gamma : I \rightarrow \mathbb{R}$ is convex (respectively, strictly convex) in the usual sense.

In the following remark we state some general properties of hyperbolically convex, which follow directly from Definition 5.1.

Remark 5.1 It follows from Definition 5.1 that $f : \mathcal{C} \rightarrow \mathbb{R}$ is a hyperbolically convex function if and only if the *epigraph* $\text{epi} f := \{(p, \mu) : p \in \mathcal{C}, \mu \in \mathbb{R}, f(p) \leq \mu\}$, is convex in $\mathbb{H}^n \times \mathbb{R}$. Moreover, if $f : \mathcal{C} \rightarrow \mathbb{R}$ is a hyperbolically convex function, then the sub-level sets $\{p \in \mathcal{C} : f(p) \leq a\}$ are hyperbolically convex sets, for all $a \in \mathbb{R}$. Furthermore, if $f, f_1, \dots, f_n : \mathbb{H}^n \rightarrow \mathbb{R}$ are hyperbolically convex in \mathcal{C} , then ζf and $f_1 + \dots + f_n$ are hyperbolically convex in \mathcal{C} , for all $\zeta \geq 0$.

The next proposition follows from Remark 2.2 and Remark 3.2, Definition 3.1 and Definition 5.1.

Proposition 5.1 *Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a hyperbolically convex set, $Q \in G_{\mathcal{C}}$ and $\mathcal{D} := \{Q^{-1}p : p \in \mathcal{C}\}$. The function $f : \mathcal{C} \rightarrow \mathbb{R}$ is hyperbolically convex if and only if $f \circ Q : \mathcal{D} \rightarrow \mathbb{R}$ defined by $f \circ Q(q) := f(Qq)$ is hyperbolically convex.*

5.1 Characterization of Hyperbolically Convex Functions

In this section we present first and second order characterization for hyperbolically convex functions on hyperbolic spaces.

Proposition 5.2 *Let $\mathcal{C} \subseteq \mathbb{H}^n$ be an open hyperbolically convex set and $f : \mathcal{C} \rightarrow \mathbb{R}$ be a differentiable function. The function f is hyperbolically convex if and only if $f(q) \geq f(p) + \langle \text{grad } f(p), \log_p q \rangle$, for all $p, q \in \mathcal{C}$ and $q \neq p$, or equivalently,*

$$f(q) \geq f(p) + \frac{\text{arcosh}(-\langle p, q \rangle)}{\sqrt{1 - \langle p, q \rangle^2}} \left\langle [I + pp^\top]J \cdot Df(p), q \right\rangle, \quad \forall p, q \in \mathcal{C}, q \neq p,$$

where Df is the usual gradient of f .

Proof By using (13), the usual characterization of scalar convex functions implies that, for all minimal geodesic segment $\gamma : I \rightarrow \mathcal{C}$, the composition $f \circ \gamma : I \rightarrow \mathbb{R}$ is convex if and only if

$$f(\gamma(t_2)) \geq f(\gamma(t_1)) + \langle J \cdot Df(\gamma(t_1)), \gamma'(t_1) \rangle (t_2 - t_1), \quad \forall t_2, t_1 \in I.$$

Note that if $\gamma : [0, 1] \rightarrow \mathcal{C}$ is the geodesic segment from $p = \gamma(0)$ to $q = \gamma(1)$, then it may be represented as $\gamma(t) = \exp_p t \log_p q$. Moreover, $\gamma'(0) = \log_p q$ and $\gamma'(1) = -\log_q p$. Therefore, the first inequality of the proposition is an immediate consequence of the inequality above, Definition 5.1 and equation (10). For concluding the proof, note that equations (11) and (10) together with Remark 2.1 imply the equivalence between the two inequalities of the lemma. \square

Proposition 5.3 *Let $\mathcal{C} \subseteq \mathbb{H}^n$ be an open hyperbolically convex set and $f : \mathcal{C} \rightarrow \mathbb{R}$ a differentiable function. The function f is hyperbolically convex if and only if $\text{grad } f$ satisfies the inequality $\langle \text{grad } f(p), \log_p q \rangle +$*

$\langle \text{grad} f(q), \log_q p \rangle \leq 0$, for all $p, q \in \mathcal{C}$ and $q \neq p$, or equivalently,

$$\langle \mathbf{J} \cdot Df(p) - \mathbf{J} \cdot Df(q), p - q \rangle - (\langle p, q \rangle + 1) [\langle \mathbf{J} \cdot Df(p), p \rangle + \langle \mathbf{J} \cdot Df(q), q \rangle] \geq 0, \quad \forall p, q \in \mathcal{C}, q \neq p,$$

where Df is the usual gradient of f .

Proof Using (13), the usual first order characterization of convex functions implies that, for all minimal geodesic segments $\gamma: I \rightarrow \mathcal{C}$, the composition $f \circ \gamma: I \rightarrow \mathbb{R}$ is convex if and only if

$$[\langle \mathbf{J} \cdot Df(\gamma(t_2)), \gamma'(t_2) \rangle - \langle \mathbf{J} \cdot Df(\gamma(t_1)), \gamma'(t_1) \rangle] (t_2 - t_1) \geq 0, \quad \forall t_2, t_1 \in I.$$

Note that if $\gamma: [0, 1] \rightarrow \mathcal{C}$ is the segment of geodesic from $p = \gamma(0)$ to $q = \gamma(1)$, then it may be represented as $\gamma(t) = \exp_p t \log_p q$. Moreover, $\gamma'(0) = \log_p q$ and $\gamma'(1) = -\log_q p$. Therefore, the first inequality of the proposition follows by combining the previous inequality with Definition 5.1 and (10). For concluding the proof, note that equations (11) and (10) imply the equivalence between the two inequalities of the lemma. \square

Proposition 5.4 *Let $\mathcal{C} \subseteq \mathbb{H}^n$ be an open hyperbolically convex set and $f: \mathcal{C} \rightarrow \mathbb{R}$ be a twice differentiable function. The function f is hyperbolically convex if and only if the Hessian $\text{Hess} f$ on the hyperbolic space satisfies the inequality $\langle \text{Hess} f(p)v, v \rangle \geq 0$, for all $p \in \mathcal{C}$ and all $v \in T_p \mathbb{H}^n$, or equivalently,*

$$\langle \mathbf{J} \cdot D^2 f(p)v, v \rangle + \langle \mathbf{J} \cdot Df(p), p \rangle \langle v, v \rangle \geq 0, \quad \forall p \in \mathcal{C}, \forall v \in T_p \mathbb{H}^n,$$

where $D^2 f(p)$ is the usual Hessian and $Df(p)$ is the usual gradient of f at a point $p \in \mathcal{C}$. If the above inequalities are strict then f is strictly hyperbolically convex.

Proof By using (14), the usual second order characterization of hyperbolically convex functions implies that, for all minimal geodesic segment $\gamma: I \rightarrow \mathcal{C}$, the composition $f \circ \gamma: I \rightarrow \mathbb{R}$ is convex if and only if

$$\langle \mathbf{J} \cdot D^2 f(\gamma(t))\gamma'(t), \gamma'(t) \rangle + \langle \mathbf{J} \cdot Df(\gamma(t)), \gamma(t) \rangle \langle \gamma'(t), \gamma'(t) \rangle \geq 0, \quad \forall t \in I.$$

If the last inequality is strict then $f \circ \gamma$ is strictly convex. Therefore, the result follows by combining the above inequality with Definition 5.1. For concluding the proof, note that equation (12) together with Remark 2.1 imply the equivalence between the two inequalities of the lemma. \square

Example 5.1 Fix $q \in \mathbb{H}^n$. The function $d_q(\cdot) : \mathbb{H}^n \rightarrow \mathbb{R}$ is hyperbolically convex. In general, taking a hyperbolically convex set $\mathcal{C} \subseteq \mathbb{H}^n$, the function $d_q(\cdot) : \mathcal{C} \rightarrow \mathbb{R}$ is hyperbolically convex. Indeed, the hyperbolic convexity of $d_q(\cdot)$ follows by combining Lemma 2.3 with Proposition 5.4.

Example 5.2 Fix $q \in \mathbb{H}^n$. The function $\rho_q : \mathbb{H}^n \rightarrow \mathbb{R}$ defined as $\rho_q(p) := \frac{1}{2}d_q^2(p)$ is strictly hyperbolically convex. In general, taking a hyperbolically convex set $\mathcal{C} \subseteq \mathbb{H}^n$, the function $\rho_q : \mathcal{C} \rightarrow \mathbb{R}$ is strictly hyperbolically convex. Indeed, the result follows by combining Lemma 2.4 with Proposition 5.4.

Example 5.3 Take $\tilde{p} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ and the hyperbolically convex set $\mathcal{C} = \{p \in \mathbb{H}^n : p^1 > 0, \dots, p^n > 0\}$. The function $\psi : \mathcal{C} \rightarrow \mathbb{R}$ defined by $\psi(p) = -\ln(-1 - \langle \tilde{p}, p \rangle)$ is hyperbolically convex. Indeed, considering that $Df(p) = -(1 + \langle \tilde{p}, p \rangle)^{-1}J\tilde{p}$ and $D^2f(p) = (1 + \langle \tilde{p}, p \rangle)^{-2}J\tilde{p}\tilde{p}^\top J$, the hyperbolic convexity of ψ follows by combining Lemma 2.1 and Proposition 5.4.

5.2 Hyperbolically Convex Quadratic Functions

In this section we study the hyperbolic convexity of the quadratic function $f(p) = p^\top A p$, for $A = A^\top \in \mathbb{R}^{(n+1) \times (n+1)}$. We begin with a general characterization.

Corollary 5.1 *Let $A = A^\top \in \mathbb{R}^{(n+1) \times (n+1)}$ and $f : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $f(p) = p^\top A p$. The function f is hyperbolically convex if and only if*

$$v^\top A v + p^\top A p \geq 0, \quad \forall p, v \in \mathbb{R}^{n+1} \quad \text{with} \quad p^\top J p = -1, \quad v^\top J v = 1, \quad p^\top J v = 0.$$

Proof Considering that $Df(p) = 2Ap$, $D^2f(p) = 2A$ and $JJ = I$, we conclude that

$$\langle J \cdot D^2f(p)v, v \rangle + \langle J \cdot Df(p), p \rangle \langle v, v \rangle = 2v^\top A v + 2p^\top A p \langle v, v \rangle.$$

Thus, it follows from Proposition 5.4 that f is hyperbolically convex in \mathbb{H}^n if and only if $v^\top A v + p^\top A p \geq 0$, for all $p \in \mathbb{H}^n$, all $v \in T_p \mathbb{H}^n$ such that $v^\top J v = 1$. Considering that $v \in T_p \mathbb{H}^n$ with $p \in \mathbb{H}^n$ if and only if $v^\top J v = 1$ and $p^\top J v = 0$ with $p \in \mathbb{H}^n$, the result follows. \square

Next, we use the Lorentz group (15) to present some examples of hyperbolically convex quadratic functions. Before that, we need the following result.

Corollary 5.2 *Let $A = A^\top \in \mathbb{R}^{(n+1) \times (n+1)}$ and $Q \in G_{\mathcal{L}}$. Then, $f : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $f(p) = p^\top A p$ is hyperbolically convex if and only if $g : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $g(q) = q^\top (Q^\top A Q) q$ is hyperbolically convex.*

Proof Since $g(q) = f(Qq)$, the equivalence follows from Proposition 5.1. \square

As an application of Corollaries 5.1 and 5.2 in the following we present an example of a hyperbolically convex quadratic function.

Example 5.4 Take a diagonal matrix $D \in \mathbb{R}^{(n+1) \times (n+1)}$ denoted by $D = \text{diag}(d_1, \dots, d_n, d_{n+1})$. Assume that $d_{\min} + d_{n+1} \geq 0$, where $d_{\min} := \min\{d_1, \dots, d_n\}$. Then, for each $Q \in G_{\mathcal{L}}$, the function $g : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $g(p) = p^\top Q^\top D Q p$ is hyperbolically convex. Indeed, take $q, u \in \mathbb{R}^{n+1}$ such that $q^\top J q = -1$, $u^\top J u = 1$ and $p^\top J v = 0$. Thus, we have $q_{n+1}^2 = \sum_{i=1}^n q_i^2 + 1$ and $u_{n+1}^2 = \sum_{i=1}^n u_i^2 - 1$. Hence, since $d_{\min} + d_{n+1} \geq 0$, we obtain that

$$u^\top D u + q^\top D q = \sum_{i=1}^n (d_i + d_{n+1}) u_i^2 + \sum_{i=1}^n (d_i + d_{n+1}) q_i^2 \geq (d_{\min} + d_{n+1}) \left(\sum_{i=1}^n u_i^2 + \sum_{i=1}^n q_i^2 \right) \geq 0.$$

Thus, Corollary 5.1 implies that $f : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $f(p) = p^\top D p$ is hyperbolically convex. Therefore, applying Corollary 5.2 we conclude that g is hyperbolically convex.

To continue with our study of hyperbolic convexity of quadratic functions, we denote the boundary of the Lorentz cone (24) by

$$\partial \mathcal{L} := \left\{ x \in \mathcal{L} : x^\top J x = 0 \right\}.$$

In order to simplify notations, for a given $x \in \mathbb{R}^{n+1}$, we consider the following decomposition:

$$x = (\bar{x}^\top, x_{n+1}) \in \mathbb{R}^{n+1}, \quad \bar{x} := (x_1, \dots, x_n)^\top \in \mathbb{R}^n, \quad x_{n+1} \in \mathbb{R}. \quad (41)$$

Lemma 5.1 *Let $x, y \in \partial \mathcal{L}$. The following three statements are equivalent:*

(i) $x^\top J y \neq 0$;

(ii) $y \neq \alpha x$, for all $\alpha \in \mathbb{R}$;

(iii) $x^\top Jy < 0$.

Proof First, we prove (i) is equivalent to (ii). Assume (i) holds. If there exists $\alpha \in \mathbb{R}$ such that $y = \alpha x$, then $x^\top Jy = x^\top J(\alpha x) = \alpha x^\top Jx = 0$, which contradicts $x^\top Jy \neq 0$. Hence, $y \neq \alpha x$, for all $\alpha \in \mathbb{R}$, and (ii) holds. For the converse, assume (ii) holds. By contradiction, assume that $x^\top Jy = 0$. Since $y, z \in \partial \mathcal{L}$, by using the notation introduced in (41), we have $y^{n+1} = \sqrt{\bar{y}^\top \bar{y}}$ and $z_{n+1} = \sqrt{\bar{z}^\top \bar{z}}$. Thus, due to $y^\top z = 0$, we have $\bar{y}^\top \bar{z} = -y^{n+1} z_{n+1}$, or equivalently

$$\bar{y}^\top \bar{z} = -\sqrt{\bar{y}^\top \bar{y}} \sqrt{\bar{z}^\top \bar{z}}.$$

Hence, Cauchy's inequality implies that there exists a $\alpha \geq 0$ such that $\bar{y} = -\alpha \bar{z}$. Furthermore,

$$y^{n+1} = \sqrt{\bar{y}^\top \bar{y}} = \alpha \sqrt{\bar{z}^\top \bar{z}},$$

which gives $y^{n+1} = \alpha z_{n+1}$. Thus, we conclude that $y = -\alpha Jz$, which implies $y = \alpha x$ and we have a contradiction. Therefore, $x^\top Jy \neq 0$ and (i) holds.

Now, we prove (i) is equivalent to (iii). Assume (i) holds. Since \mathcal{L} is a closed and convex cone, and $x, y \in \partial \mathcal{L} \subseteq \mathcal{L}$, we have $x + y \in \mathcal{L}$. Thus,

$$0 \geq (x + y)^\top J(x + y) = x^\top Jx + 2x^\top Jy + y^\top Jy = 2x^\top Jy.$$

Hence, $x^\top Jy \neq 0$ implies $x^\top Jy < 0$. Therefore, the item (iii) holds. Conversely, (iii) implies (i) is immediate, which concludes the proof. \square

If we make some transformations in Corollary 5.1, we will obtain the following result.

Lemma 5.2 *Let $A = A^\top \in \mathbb{R}^{(n+1) \times (n+1)}$. The following three conditions are equivalent:*

(i) *The function $f : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $f(p) = p^\top Ap$ is hyperbolically convex;*

(ii) *$x^\top Ax + y^\top Ay \geq 0$, for all $x, y \in \mathbb{R}^{n+1}$ with $x, y \in \partial \mathcal{L}$ and $x^\top Jy = -1$;*

(iii) *$z^\top Az + w^\top Aw \geq 0$, for all $z, w \in \mathbb{R}^{n+1}$ with $z, w \in \partial \mathcal{L}$ and $z^\top Jw < 0$.*

Proof First we prove the equivalence between (i) and (ii). For that it is convenient to consider the following invertible transformations

$$p = \frac{1}{\sqrt{2}}(x+y), \quad v = \frac{1}{\sqrt{2}}(x-y) \quad \text{if and only if} \quad x = \frac{1}{\sqrt{2}}(p+v), \quad y = \frac{1}{\sqrt{2}}(p-v), \quad (42)$$

where $x, y, p, v \in \mathbb{R}^{n+1}$. By using the first two equalities in (42), after some calculations, we have

$$\begin{aligned} 2p^\top Jp &= x^\top Jx + 2x^\top Jy + y^\top Jy, \\ 2v^\top Jv &= x^\top Jx - 2x^\top Jy + y^\top Jy, \\ 2p^\top Jv &= x^\top Jx - y^\top Jy. \end{aligned} \quad (43)$$

On the other hand, by using the last two inequalities in (42) we obtain the following three equalities

$$\begin{aligned} 2x^\top Jx &= p^\top Jp + 2p^\top Jv + v^\top Jv, \\ 2y^\top Jy &= p^\top Jp - 2p^\top Jv + v^\top Jv, \\ 2x^\top Jy &= p^\top Jp - v^\top Jv. \end{aligned} \quad (44)$$

Moreover, the equalities in (42) also imply that

$$v^\top Av + p^\top Ap = x^\top Ax + y^\top Ay. \quad (45)$$

First we prove (i) implies (ii). Take $x, y \in \partial\mathcal{L}$ and $x^\top Jy = -1$, and consider the transformation (42). Thus, by using (43), we conclude that $p^\top Jp = -1$, $v^\top Jv = 1$ and $p^\top Jv = 0$. Hence, item (i) together with Corollary 5.1 implies that $v^\top Av + p^\top Ap \geq 0$. Therefore, by using (45), we conclude that $x^\top Ax + y^\top Ay \geq 0$ and item (ii) holds.

Next we prove that (ii) implies (i). Assume that the item (ii) holds, and take $p, v \in \mathbb{R}^{n+1}$ with $p^\top Jp = -1$, $v^\top Jv = 1$ and $p^\top Jv = 0$, and consider (42). Hence, by using (44) we have x (or $-x$) $\in \partial\mathcal{L}$, y (or $-y$) $\in \partial\mathcal{L}$ and $x^\top Jy = -1$, and item (ii) implies that $x^\top Ax + y^\top Ay \geq 0$. Thus, (45) implies that $v^\top Av + p^\top Ap \geq 0$, which implies that item (i) holds.

We proceed to prove the equivalence between (ii) and (iii). Assume that item (ii) holds and take $z, w \in \partial\mathcal{L}$ and $z^\top Jw < 0$. Since $z^\top Jw < 0$ we define

$$x = \frac{z}{\sqrt{-z^\top Jw}}, \quad y = \frac{w}{\sqrt{-z^\top Jw}}. \quad (46)$$

Thus, considering that $z, w \in \partial\mathcal{L}$ and $z^\top Jw < 0$, some calculations show that $x, y \in \partial\mathcal{L}$ and $x^\top Jy = -1$.

Therefore, using (46) together item (ii), we conclude that

$$z^\top Az + w^\top Aw = -z^\top Jw (x^\top Ax + y^\top Ay) \geq 0,$$

and item (iii) holds. Finally, (iii) implies (ii) is immediate, which concludes the proof. \square

In the following theorem we present a characterization for hyperbolically convex quadratic functions in term of the matrix defining it. In particular, we show that the study of hyperbolically convex quadratic functions reduces to the study of their behavior on the boundary of the Lorentz cone.

Theorem 5.1 *Let $A = A^\top \in \mathbb{R}^{(n+1) \times (n+1)}$. The following four conditions are equivalent:*

- (i) *The quadratic function $f : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $f(p) := p^\top Ap$ is hyperbolically convex;*
- (ii) *The matrix A is $\partial\mathcal{L}$ -copositive, i.e., $x^\top Ax \geq 0$ for all $x \in \partial\mathcal{L}$;*
- (iii) *There exists $\alpha \in \mathbb{R}$ such that $A + \alpha J$ is positive semidefinite;*
- (iv) *The function f is bounded from below, i.e., there exist an $\alpha \in \mathbb{R}$ such that $f(p) \geq \alpha$, for all $p \in \mathbb{H}^n$.*

Proof We first prove that (i) implies (ii). Assume that f is hyperbolically convex. Let $x \in \partial\mathcal{L}$ such that $x \neq 0$. Take $y \in \partial\mathcal{L}$ not parallel to x , i. e., such that $y \neq \alpha x$, for all $\alpha \in \mathbb{R}$. Define the sequence $(y^k)_{k \in \mathbb{N}}$, where $y^k := (1/k)y$, for all $k \in \mathbb{N}$. Since $y \in \partial\mathcal{L}$ and $y \neq \alpha x$ for all $\alpha \in \mathbb{R}$, we also have $y^k \in \partial\mathcal{L}$ and $y^k \neq \alpha x$, for all $\alpha \in \mathbb{R}$ and all $k \in \mathbb{N}$. Thus, by using Lemma 5.1, we conclude that $x^\top Jy^k < 0$, for all $k \in \mathbb{N}$. Hence, considering that $x, y^k \in \partial\mathcal{L}$ and $x^\top Jy^k < 0$, for all $k \in \mathbb{N}$, and f is hyperbolically convex, by using Lemma 5.2, we obtain that $x^\top Ax + (y^k)^\top Ay^k \geq 0$, for all $k \in \mathbb{N}$. Therefore, owing to $y^k := (1/k)y$, for all $k \in \mathbb{N}$, we have

$$x^\top Ax + \frac{1}{k^2} y^\top Ay \geq 0, \quad k \in \mathbb{N}.$$

Therefore, taking the limit in the latter inequality, we obtain that $x^\top Ax \geq 0$. In conclusion, A is $\partial\mathcal{L}$ -copositive.

Next we prove that (ii) implies (iii). Assume that A is $\partial\mathcal{L}$ -copositive. Thus, for all $x \in \mathbb{R}^{n+1}$ with $x^\top Jx = 0$ we have $x^\top Ax \geq 0$. Consequently, for all $x \in \mathbb{R}^{n+1}$ with $x^\top Jx = 0$ and $x \neq 0$ we have $x^\top (A + (1/k)I)x > 0$, for all $k \in \mathbb{N}$. Hence, by Lemma 1.5, there exists $\alpha_k \in \mathbb{R}$ such that $A + (1/k)I + \alpha_k J$ is positive definite, for all $k \in \mathbb{N}$. We claim that the sequence $(\alpha_k)_{k \in \mathbb{N}}$ is bounded. Indeed, assume by absurd that $(\alpha_k)_{k \in \mathbb{N}}$ is unbounded. Since $A + (1/k)I + \alpha_k J$ is positive definite, for each $x \in \mathbb{R}^{n+1}$ with $x \neq 0$ we conclude that

$$\frac{1}{\alpha_k} x^\top Ax + \frac{1}{k\alpha_k} x^\top x + x^\top Jx > 0,$$

for all $k \geq \bar{k}$ and some $\bar{k} \in \mathbb{N}$. Hence, by taking the limit in the last inequality as k goes to infinity, we conclude that $x^\top Jx \geq 0$ for all $x \in \mathbb{R}^{n+1}$, which is absurd. Therefore, the claim is proved. Since the sequence $(\alpha_k)_{k \in \mathbb{N}}$ is bounded, we can take a subsequence $(\alpha_{k_j})_{j \in \mathbb{N}}$ and $\alpha \in \mathbb{R}$ such that $\lim_{j \rightarrow \infty} \alpha_{k_j} = \alpha$.

On the other hand, considering that $A + (1/k)I + \alpha_k J$ is positive definite for all $k \in \mathbb{N}$, we have

$$x^\top \left(A + \frac{1}{k_j} I + \alpha_{k_j} J \right) x > 0,$$

for all $x \in \mathbb{R}^{n+1}$ such that $x \neq 0$. Therefore, by taking the limit in the last inequality as k goes to infinity, we have $x^\top (A + \alpha J)x \geq 0$, for all $x \in \mathbb{R}^{n+1}$, which implies that $A + \alpha J$ is positive semidefinite.

Next we prove that (iii) implies (i). Assume that there is an $\alpha \in \mathbb{R}$ such that $A + \alpha J$ is positive semidefinite. Due to $A + \alpha J$ being positive semidefinite and $v^\top Av + p^\top Ap = v^\top Av + \alpha + p^\top Ap - \alpha$, some calculations show

$$v^\top Av + p^\top Ap = v^\top (A + \alpha J)v + p^\top (A + \alpha J)p \geq 0, \quad (47)$$

for all $p, v \in \mathbb{R}^{n+1}$ with $p^\top Jp = -1$, $v^\top Jv = 1$ and $p^\top Jv = 0$. Therefore, by using (47) and Corollary 5.1, we conclude that f is hyperbolically convex.

Next we prove that (iii) implies (iv). Assume that there exists $\alpha \in \mathbb{R}$ such that $A + \alpha J$ is positive semidefinite. Hence, $p^\top A p + \alpha p^\top J p = p^\top (A + \alpha J) p \geq 0$, for all $p \in \mathbb{R}^n$. Since $p \in \mathbb{H}^n$ implies $p^\top J p = -1$, we conclude that $p^\top A p \geq \alpha$, for all $p \in \mathbb{H}^n$. Therefore, f is bounded from below on \mathbb{H}^n .

Finally, to conclude the proof, we prove that (iv) imply (iii). Assume that f is bounded from below on \mathbb{H}^n . Thus, there exists an $\alpha \in \mathbb{R}$ such that $f(p) \geq \alpha$, for all $p \in \mathbb{H}^n$, or equivalently,

$$p^\top (A + \alpha J) p \geq 0, \quad \forall p \in \mathbb{H}^n. \quad (48)$$

In order to apply Lemma 1.4, we will first prove the statement: if $x^\top J x \leq 0$, then $x^\top (A + \alpha J) x \geq 0$. Let $x \in \mathbb{R}^n$ such that $x^\top J x \leq 0$. Take a sequence $(x_k)_{k \in \mathbb{N}} \in \mathbb{R}^n$ such that $\lim_{k \rightarrow +\infty} x_k = x$ and $x_k^\top J x_k < 0$. Define

$$y^k = \frac{x_k}{\sqrt{-x_k^\top J x_k}}, \quad k \in \mathbb{N}. \quad (49)$$

Since $x_k^\top J x_k < 0$, by using (49), we conclude that $y^{k \top} J y^k = -1$. Thus, $y^k \in \mathbb{H}^n$ and (48) implies that $(y^k)^\top (A + \alpha J) y^k \geq 0$. Using, again (49) we obtain that $x_k^\top (A + \alpha J) x_k \geq 0$, for all $k \in \mathbb{N}$. Hence, by letting $k \rightarrow \infty$, we conclude that $x^\top (A + \alpha J) x \geq 0$, which proves the statement. Therefore, after applying Lemma 1.4, we conclude that there exists a $\beta \in \mathbb{R}$ such that $A + (\alpha + \beta) J$ is positive definite. Hence, by Theorem 5.1, the function f is hyperbolically convex. \square

Example 5.5 Let $A \in \mathbb{R}^{(n+1) \times (n+1)}$ be a positive semidefinite matrix and $\alpha \in \mathbb{R}$. Since $A = (A + \alpha J) - \alpha J$ is a positive semidefinite matrix, by applying Theorem 5.1, we conclude that the function $f_\alpha : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $f_\alpha(p) := p^\top (A + \alpha J) p$ is hyperbolically convex.

For simplifying the statement and proof of the next results it is convenient to introduce the following notation. For a given $A \in \mathbb{R}^{(n+1) \times (n+1)}$, consider the following decomposition:

$$A := \begin{pmatrix} \bar{A} & a \\ a^\top & \sigma \end{pmatrix}, \quad \bar{A} \in \mathbb{R}^{n \times n}, \quad a \in \mathbb{R}^{n \times 1}, \quad \sigma \in \mathbb{R} \quad (50)$$

and denote by $\bar{I} \in \mathbb{R}^{n \times n}$ is the identity matrix. For any $\alpha \in \mathbb{R}$, the decomposition (50) yields

$$A + \alpha J = \begin{pmatrix} \bar{A} + \alpha \bar{I} & a \\ a^\top & \sigma - \alpha \end{pmatrix}. \quad (51)$$

Proposition 5.5 Let $A \in \mathbb{R}^{(n+1) \times (n+1)}$ be a symmetric matrix and $\alpha \in \mathbb{R}$. Consider the decomposition

(50) and the following statements:

(i) The quadratic function $f : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $f(p) := p^\top A p$ is hyperbolically convex;

(ii) The matrix $A + \alpha J$ is positive definite;

(iii) The number $\alpha \in (-\lambda_{\min}(\bar{A}), \sigma)$ and $\det(A + \alpha J) > 0$;

(iv) The number $\alpha \in (-\lambda_{\min}(\bar{A}), \sigma)$ and $\sigma - \alpha - a^\top (\bar{A} + \alpha \bar{I})^{-1} a > 0$.

Then (ii), (iii) and (iv) are equivalent and any of them implies item (i).

Proof We first prove the equivalence between (ii) and (iii). Assume that (ii) holds. Hence, by applying Lemmas 1.1 and 1.2, and taking into account (51), we have $\det(A + \alpha J) > 0$, the matrix $\bar{A} + \sigma \bar{I}$ is positive definite and $\sigma - \alpha > 0$. Since $\bar{A} + \sigma \bar{I}$ is positive definite, we obtain that $\alpha > -\lambda_{\min}(\bar{A})$. Therefore, we conclude that $\alpha \in (-\lambda_{\min}(\bar{A}), \sigma)$ and $\det(A + \alpha J) > 0$. Hence, (iii) holds. Reciprocally, assume that (iii) holds. Thus, we have $\alpha > -\lambda_{\min}(\bar{A})$, $\sigma > \alpha$ and $\det(A + \alpha J) > 0$. Hence, the matrix $\bar{A} + \sigma \bar{I}$ is positive definite, $\sigma - \alpha > 0$ and $\det(A + \alpha J) > 0$. Therefore, by using Lemmas 1.1 and 1.2 and the decomposition (50), we conclude that $A + \alpha J$ is positive definite and (ii) holds.

Next, we prove the equivalence between (iii) and (iv). First note that, for any $\alpha \in (-\lambda_{\min}(\bar{A}), \sigma)$, we obtain that $\lambda_{\min}(\bar{A}) + \alpha > 0$. Thus, the matrix $\bar{A} + \alpha \bar{I}$ is positive definite, which implies that $\det(\bar{A} + \alpha \bar{I}) > 0$. In particular, the matrix $\bar{A} + \alpha \bar{I}$ is invertible. Thus, applying Lemma 1.3, we have

$$\det(A + \alpha J) = \left(\sigma - \alpha - a^\top (\bar{A} + \alpha \bar{I})^{-1} a \right) \det(\bar{A} + \alpha \bar{I}). \quad (52)$$

Since under the assumption $\alpha \in (-\lambda_{\min}(\bar{A}), \sigma)$ we have $\det(\bar{A} + \alpha \bar{I}) > 0$, it follows from (52) that $\det(A + \alpha J) > 0$ is equivalent to $\sigma - \alpha - a^\top (\bar{A} + \alpha \bar{I})^{-1} a > 0$. Therefore, (iii) is equivalent to (iv), and the proof of the first statement of the proposition is concluded.

By using the implication (iii) \implies (i) in Theorem 5.1, the last statement of the proposition follows from the first one.

Consider the decomposition (50) of a symmetric matrix $A \in \mathbb{R}^{(n+1) \times (n+1)}$. Then, it follows from Proposition 5.5 that we can decide if $f(p) := p^\top A p$ is hyperbolically convex by solving the following optimization problem:

$$\inf \left\{ \sigma - \alpha - a^\top (\bar{A} + \alpha \bar{I})^{-1} a : \alpha \in (-\lambda_{\min}(\bar{A}), \sigma) \right\}.$$

By using decompositions (41) and (50), Corollary 5.1 can be stated equivalently in the following form.

Corollary 5.3 *Let $A = A^\top \in \mathbb{R}^{(n+1) \times (n+1)}$ and $f : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $f(p) = p^\top A p$. The function f is hyperbolically convex if and only if*

$$\bar{v}^\top \bar{A} \bar{v} + \bar{p}^\top \bar{A} \bar{p} + 2v_{n+1} \bar{v}^\top a + 2p^{n+1} \bar{p}^\top a + \sigma v_{n+1}^2 + \sigma (p^{n+1})^2 \geq 0, \quad (53)$$

for all $p, v \in \mathbb{R}^{n+1}$ with

$$\bar{p}^\top \bar{p} - (p^{n+1})^2 = -1 \quad \bar{v}^\top \bar{v} - v_{n+1}^2 = 1, \quad \bar{v}^\top \bar{p} - v_{n+1} p^{n+1} = 0. \quad (54)$$

Example 5.6 Let $A := \begin{pmatrix} a_i^j \end{pmatrix} \in \mathbb{R}^{(n+1) \times (n+1)}$ and $f : \mathbb{H}^n \rightarrow \mathbb{R}$ be defined by $f(p) = p^\top A p$. If f is hyperbolically convex, then

$$\frac{1}{n} \sum_{i,j=1}^n a_i^j + \sigma \geq 0. \quad (55)$$

Indeed, take $\bar{v} = (1/\sqrt{n}, \dots, 1/\sqrt{n}) \in \mathbb{R}^n$, $v_{n+1} = 0$ and $\bar{p} = 0 \in \mathbb{R}^n$, $p^{n+1} = 1$. Thus, (53) becomes

$$\bar{v}^\top \bar{A} \bar{v} + \bar{p}^\top \bar{A} \bar{p} + 2v_{n+1} \bar{v}^\top a + 2p^{n+1} \bar{p}^\top a + \sigma v_{n+1}^2 + \sigma (p^{n+1})^2 = \frac{1}{n} \sum_{i,j=1}^n a_i^j + \sigma.$$

Since $p = (\bar{p}, p^{n+1}) \in \mathbb{R}^{n+1}$, $v = (\bar{v}, v_{n+1}) \in \mathbb{R}^{n+1}$ satisfy (54) and considering that f is hyperbolically convex, the inequality (55) follows from applying Corollary 5.3.

Theorem 5.2 *Let $A \in \mathbb{R}^{(n+1) \times (n+1)}$ and $f : \mathbb{H}^n \rightarrow \mathbb{R}$ be defined by $f(p) = p^\top A p$. Then, considering decompositions (41) and (50), the following statements hold:*

- (i) *If f is hyperbolically convex, then $\lambda_{\min}(\bar{A}) \geq -\sigma$;*
- (ii) *If $\sigma \geq -\lambda_{\min}(\bar{A})$ and $a = 0$, then f is hyperbolically convex;*
- (iii) *If $\sigma + \lambda_{\min}(\bar{A}) > 2\sqrt{a^\top a}$, then f is hyperbolically convex.*

Proof To prove (i), assume that f is hyperbolically convex and take any $p, v \in \mathbb{R}^{n+1}$ with $\bar{p} = 0, p^{n+1} = 1, v^{n+1} = 0, \bar{v}^\top \bar{v} = 1$. Then, conditions (54) in Corollary 5.3 are satisfied. Hence, it follows from (53) that $\bar{v}^\top \bar{A} \bar{v} \geq -\sigma$, for all $\bar{v} \in \mathbb{R}^n$ with $\bar{v}^\top \bar{v} = 1$. Therefore, the result follows.

We proceed to prove (ii). Assume that $\lambda_{\min}(\bar{A}) \geq -\sigma$ and $a = 0$. In this case, we have $\lambda_{\min}(\bar{A} + \sigma \bar{I}) \geq 0$, where $\bar{I} \in \mathbb{R}^{n \times n}$ is the identity matrix. Hence, both matrices $\bar{A} + \sigma \bar{I}$ and

$$A + \sigma J = \begin{pmatrix} \bar{A} + \sigma \bar{I} & 0 \\ 0 & 0 \end{pmatrix}$$

are positive semidefinite. Thus, by applying Proposition 5.5 with $\alpha = \sigma$, the proof of item (ii) follows.

To prove item (iii), first we introduce the auxiliary quadratic polynomial $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$g(t) = (\sigma - t) (\lambda_{\min}(\bar{A}) + t) - a^\top a.$$

The roots of the quadratic polynomial g are given by

$$\begin{aligned} \mu &= \frac{1}{2} \left(\sigma - \lambda_{\min}(\bar{A}) - \sqrt{[\sigma + \lambda_{\min}(\bar{A})]^2 - 4a^\top a} \right), \\ \eta &= \frac{1}{2} \left(\sigma - \lambda_{\min}(\bar{A}) + \sqrt{[\sigma + \lambda_{\min}(\bar{A})]^2 - 4a^\top a} \right). \end{aligned}$$

Since $\sigma + \lambda_{\min}(\bar{A}) > 2\sqrt{a^\top a}$ we have $\mu < \eta$. Thus, take $\beta \in (\mu, \eta)$. Hence, $g(\beta) > 0$. Since

$$\beta > \mu \geq \frac{1}{2} \left(\sigma - \lambda_{\min}(\bar{A}) + \sqrt{[\sigma + \lambda_{\min}(\bar{A})]^2 - 4a^\top a} \right) = -\lambda_{\min}(\bar{A}),$$

we obtain $\lambda_{\min}(\bar{A}) + \beta > 0$. Thus, we conclude that $\bar{A} + \beta \bar{I}$ is positive definite. It follows that

$$\begin{aligned} \sigma - \beta - a^\top (\bar{A} + \beta \bar{I})^{-1} a &\geq \sigma - \beta - \lambda_{\max}(\bar{A} + \beta \bar{I})^{-1} a^\top a \\ &= \sigma - \beta - \frac{1}{\lambda_{\min}(\bar{A}) + \beta} a^\top a \\ &= \frac{g(\beta)}{\lambda_{\min}(\bar{A}) + \beta} > 0. \end{aligned}$$

Therefore, by using Lemma 1.3, it follows from the positive definiteness of the matrix $\bar{A} + \beta \bar{I}$ that

$$\det(A + \beta J) = \left(\sigma - \beta - a^\top (\bar{A} + \beta \bar{I})^{-1} a \right) \det(\bar{A} + \beta \bar{I}) > 0. \quad (56)$$

Since $\bar{A} + \beta\bar{I}$ is positive definite, combining Lemma 1.4 with (56), we conclude that $A + \beta J$ is positive definite. Therefore, Proposition 5.5 implies that f is hyperbolically convex, and the proof of item (iii) is concluded. \square

Corollary 5.4 *Let $A \in \mathbb{R}^{(n+1) \times (n+1)}$ and $f : \mathbb{H}^n \rightarrow \mathbb{R}$ be defined by $f(p) = p^\top A p$. Consider the decomposition (50) and assume that $a = 0$. Then, f is hyperbolically convex if and only if $\lambda_{\min}(\bar{A}) \geq -\sigma$.*

Proof The proof is an immediate consequence of items (i) and (ii) of Theorem 5.2. \square

In the next proposition we present a characterization for the case $a \neq 0$ in (50), which completes the result of Corollary 5.4.

Proposition 5.6 *Let $A \in \mathbb{R}^{(n+1) \times (n+1)}$ be a symmetric matrix and the decomposition (50). If $a \neq 0$, then the following statements are equivalent:*

- (i) *The quadratic function $f : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $f(p) := p^\top A p$ is hyperbolically convex;*
- (ii) *There exists $\alpha \in \mathbb{R}$ such that the matrix $A + \alpha J$ is positive semidefinite;*
- (iii) *There exists $\alpha \in \mathbb{R}$ such that $\sigma > \alpha$ and the matrix $\bar{A} + \alpha\bar{I} - \frac{1}{\sigma - \alpha} a a^\top$ is positive semidefinite;*

Proof First we prove that (ii) and (iii) are equivalent. Since the matrix $A + \alpha J$ is positive semidefinite, by using the decompositions (51) and [12, Corollary 7.15, p. 398], we conclude that

$$(\bar{a}_{ii} + \alpha)(\sigma - \alpha) \geq a_i^2, \quad i = 1, \dots, n, \quad (57)$$

where \bar{a}_{ii} is the ii -entry of the matrix \bar{A} and a_i is the i -entry of the vector a . Considering that $a \neq 0$, and all elements in the diagonal of a positive semidefinite matrix are nonnegative, it follows from (57) that $\sigma > \alpha$. Thus, applying item (iii) of Lemma 1.2 we conclude that items (ii) and (iii) are equivalent. The equivalence of (i) and (ii) follows from the equivalence of (iii) and (i) in Theorem 5.1. \square

Proposition 5.7 *Let $A \in \mathbb{R}^{(n+1) \times (n+1)}$ be a symmetric matrix and the decomposition (50). Consider the following statements:*

- (i) The quadratic function $f : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $f(p) := p^\top A p$ is hyperbolically convex;
- (ii) There exists $\alpha \in \mathbb{R}$ such that $\bar{A} + \alpha \bar{I}$ is positive definite and $A + \alpha J$ is positive semidefinite;
- (iii) There exists $\alpha \in \mathbb{R}$ such that $\bar{A} + \alpha \bar{I}$ is positive definite and $\sigma - \alpha - a^\top (\bar{A} + \alpha \bar{I})^{-1} a \geq 0$.

Then, items (ii) and (iii) are equivalent and any of them implies (i).

Proof It follows from item (iii) of Lemma 1.2 that items (ii) and (iii) are equivalent. The equivalence of (i) and (ii) follows from the equivalence of (iii) and (i) in Theorem 5.1. \square

Proposition 5.8 Let $A \in \mathbb{R}^{(n+1) \times (n+1)}$ be a symmetric matrix and the decomposition (50). Consider the following statements:

- (i) The quadratic function $f : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by $f(p) := p^\top A p$ is hyperbolically convex;
- (ii) There exists $\alpha \in \mathbb{R}$ such that the matrix $A + \alpha J$ is positive definite;
- (iii) There exists $\alpha \in \mathbb{R}$ such that $\sigma > \alpha$ and the matrix $\bar{A} + \alpha \bar{I} - \frac{1}{\sigma - \alpha} a a^\top$ is positive definite.

Then, items (ii) and (iii) are equivalent and any of them implies item (i).

Proof The equivalence between items (ii) and (iii) follows by direct application of item (i) of Lemma 1.2. By using the implication (iii) \implies (i) in Theorem 5.1, the last statement of the proposition follows from the first one. \square

6 Optimization Concepts on the Hyperbolic Space

In this section we present some concepts of optimization related to hyperbolically convex function. In order to do that, consider a differentiable function $f : \mathbb{H}^n \rightarrow \mathbb{R}$ and the following unconstrained optimization problem

$$\text{Minimize}_{p \in \mathbb{H}^n} f(p). \quad (58)$$

It follows from (11) that a necessary optimality condition for the unconstrained problem (58) is:

$$\left[\mathbf{I} + p p^\top J \right] J \cdot Df(p) = 0. \quad (59)$$

Remark 6.1 If f is a hyperbolically convex function, then Proposition 5.2 implies that all points satisfying (59) are global solutions of problem (58), i.e., (59) is also a sufficient optimality condition.

Let $\mathcal{C} \subset \mathbb{H}^n$ be a hyperbolically convex set and consider the constrained optimization problem:

$$\text{Minimize}_{p \in \mathcal{C}} f(p). \quad (60)$$

In the following proposition we state the necessary optimality condition for the problem (60).

Proposition 6.1 *It the point $\bar{p} \in \mathcal{C}$ is a solution of problem (60), then*

$$\left\langle [\mathbf{I} + \bar{p}\bar{p}^\top \mathbf{J}] \mathbf{J} \cdot Df(\bar{p}), p \right\rangle \geq 0, \quad \forall p \in \mathcal{C},$$

where $Df(\bar{p}) \in \mathbb{R}^{n+1}$ is the usual gradient of f at \bar{p} .

Proof Take $p \in \mathcal{C}$ and let $\bar{p} \in \mathcal{C}$ be a solution to (60). Let $[0, 1] \ni t \mapsto \gamma_{\bar{p}p}(t) = \exp_{\bar{p}}(t \log_{\bar{p}} p)$, be the geodesic from \bar{p} to p . Since \mathcal{C} is hyperbolically convex and $p, \bar{p} \in \mathcal{C}$, we conclude that $\gamma_{\bar{p}p}(t) \in \mathcal{C}$ for all $t \in [0, 1]$. Hence, as $\bar{p} \in \mathcal{C}$ is a solution to the problem in (60), we have $(f(\gamma_{\bar{p}p}(t)) - f(\bar{p}))/t \geq 0$, for all $t \in [0, 1]$. Thus, taking the limit when t tends to zero, we obtain, by using (13) and $\gamma'_{\bar{p}p}(0) = \log_{\bar{p}} p$, that $\langle \text{grad} f(\bar{p}), \log_{\bar{p}} p \rangle \geq 0$. Therefore, the result follows from (10) and (11), by taking into account that $\text{arcosh}(-\langle \bar{p}, p \rangle) \geq 0$. \square

Proposition 6.2 *Let f be a hyperbolically convex function in \mathcal{C} . The point $\bar{p} \in \mathcal{C}$ is a solution of the problem in (60) if and only if*

$$\left\langle [\mathbf{I} + \bar{p}\bar{p}^\top \mathbf{J}] \mathbf{J} \cdot Df(\bar{p}), p \right\rangle \geq 0, \quad \forall p \in \mathcal{C},$$

where $Df(\bar{p}) \in \mathbb{R}^{n+1}$ is the usual gradient of f at \bar{p} .

Proof If the point $\bar{p} \in \mathcal{C}$ is a solution of (60), then the inequality follows from Proposition 6.1. Conversely, take $p, \bar{p} \in \mathcal{C}$, $p \neq \bar{p}$ and assume that $\langle [\mathbf{I} + \bar{p}\bar{p}^\top \mathbf{J}] \mathbf{J} \cdot Df(\bar{p}), p \rangle \geq 0$. As f is hyperbolically convex in \mathcal{C} , we conclude from Proposition 5.2 that

$$f(p) \geq f(\bar{p}) + \frac{\text{arcosh}(-\langle \bar{p}, p \rangle)}{\sqrt{1 - \langle \bar{p}, p \rangle^2}} \left\langle [\mathbf{I} + \bar{p}\bar{p}^\top \mathbf{J}] \mathbf{J} \cdot Df(\bar{p}), p \right\rangle, \quad \forall p \in \mathcal{C}, p \neq \bar{p}.$$

Since $\langle [\mathbf{I} + \bar{p}\bar{p}^\top \mathbf{J}] \mathbf{J} \cdot Df(\bar{p}), p \rangle \geq 0$ and $\operatorname{arcosh}(-\langle \bar{p}, p \rangle) \geq 0$, the latter inequality implies that $f(p) \geq f(\bar{p})$, for all $p \in \mathcal{C}$. Therefore, \bar{p} is a global solution of the problem in (60). \square

Next we present an equivalent form for Proposition 4.2, whose proof follows by combining Proposition 5.4 with Lemma 2.4, Proposition 6.2 and (17),

Corollary 6.1 *Let $\mathcal{C} \subseteq \mathbb{H}^n$ be a closed hyperbolically convex set and $\bar{p} \in \mathbb{H}^n$. Consider the function $\mathbb{H}^n \ni p \mapsto \rho_{\bar{p}}(p) := \frac{1}{2}d_{\bar{p}}^2(p)$ defined in (20). Then, $\mathcal{P}_{\mathcal{C}}(\bar{p}) = \arg \min_{p \in \mathcal{C}} \rho_{\bar{p}}(p)$ if and only if*

$$\left\langle \left(\mathbf{I} + \mathcal{P}_{\mathcal{C}}(\bar{p}) \mathcal{P}_{\mathcal{C}}(\bar{p})^\top \mathbf{J} \right) \bar{p}, p \right\rangle \leq 0, \quad \forall p \in \mathcal{C}.$$

For $f : \mathbb{H}^n \rightarrow \mathbb{R}$ and $g_i : \mathbb{H}^n \rightarrow \mathbb{R}^m$, $i = 1, \dots, m$ differentiable hyperbolically convex functions and

$$\mathcal{C} = \{p \in \mathbb{R}^n : g_i(p) \leq 0, i = 1, \dots, m\}$$

a hyperbolically convex set, see Remark 5.1. Consider the following particular instance of the hyperbolically convex optimization problem (60):

$$\text{Minimize}_{x \in \mathcal{C}} f(x). \tag{61}$$

Proposition 6.3 *Suppose that $\bar{p} \in \mathcal{C}$ and there exists $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{R}_+^m$ such that*

$$\left[\mathbf{I} + \bar{p}\bar{p}^\top \mathbf{J} \right] \left[\mathbf{J} \cdot Df(\bar{p}) + \sum_{i=1}^m \mu_i \mathbf{J} \cdot Dg_i(\bar{p}) \right] = 0, \quad \sum_{i=1}^m \mu_i g_i(\bar{p}) = 0, \tag{62}$$

where $Df(\bar{p}) \in \mathbb{R}^{n+1}$ is the usual gradient of f at \bar{p} . Then \bar{p} is a solution of the problem (61).

Proof Since $f, g_i : \mathbb{H}^n \rightarrow \mathbb{R}$ are hyperbolically convex functions and $\mu_i \geq 0$, for $i = 1, \dots, m$, it follows that $h : \mathbb{H}^n \rightarrow \mathbb{R}$ defined by

$$h(p) = f(p) + \sum_{i=1}^m \mu_i g_i(p)$$

is hyperbolically convex. Moreover, $f(p) \geq h(p)$, for all $p \in \mathcal{C}$. Now, from the first equality in (62) we obtain that $[\mathbf{I} + p p^\top \mathbf{J}] \mathbf{J} \cdot Dh(\bar{p}) = 0$. Thus, since h is hyperbolically convex, we can apply Proposition 6.2 with $f = h$ and $\mathcal{C} = \mathbb{R}^{n+1}$ to conclude that \bar{p} is a minimizer of h in \mathbb{R}^{n+1} . Hence, from $f(p) \geq h(p)$,

for all $p \in \mathcal{C}$ and the second equality in (62), we have $f(p) \geq h(p) \geq h(\bar{p}) = f(\bar{p})$, for all $p \in \mathcal{C}$. Since $\bar{p} \in \mathcal{C}$, the last inequality implies that it is a solution of (61). \square

Remark 6.2 If \bar{p} is a solution of the problem (61), then under mild conditions on the problem (61) the point \bar{p} satisfies (62), for details see [29, Section 9]; see also [34, Theorem 4.4].

7 Convexity on Others Model of Hyperbolic Geometry

There are several models of hyperbolic geometry, the four commonly used ones are *the Klein model*, *the Poincaré disk model*, *the Poincaré half-plane model* and *the Lorentz or hyperboloid model*, see for example [1,3,25]. Among them, we have chosen the hyperboloid model, because it has several similarities with the Euclidean sphere. It is worth noting that we can choose any of the aforementioned models. Let us recall the general concept of an isometry.

Definition 7.1 Let $(\mathcal{N}, \langle \cdot, \cdot \rangle)$ and $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ be Riemannian manifolds. A mapping $\Phi : \mathcal{N} \rightarrow \mathcal{M}$ is called an isometry, if Φ is continuously differentiable, and for all $q \in \mathcal{N}$ and $u, v \in T_q \mathcal{N}$, we have $\langle u, v \rangle = \langle d\Phi_q u, d\Phi_q v \rangle$, where $d\Phi_q : T_q \mathcal{N} \rightarrow T_{\Phi(q)} \mathcal{M}$ is the differential of Φ at $q \in \mathcal{N}$.

The next result is an important property of isometries, its prove is in [21, Proposition 5.6.1, p. 196].

Proposition 7.1 Let \mathcal{N}, \mathcal{M} be Riemannian manifold and $\Phi : \mathcal{N} \rightarrow \mathcal{M}$ an isometry. If γ is a geodesic in \mathcal{N} , then $\Phi \circ \gamma$ is a geodesic in \mathcal{M} . Moreover, Φ preserve the Riemannian distance.

Straight combination of Definition 7.1 with Proposition 7.1 give us the following result.

Theorem 7.1 Let \mathcal{N}, \mathcal{M} be Riemannian manifolds and the function $\Phi : \mathcal{N} \rightarrow \mathcal{M}$ be an isometry. The function $g : \mathcal{M} \rightarrow \mathbb{R}$ is convex if and only if $f : \mathcal{N} \rightarrow \mathbb{R}$ defined by $f(p) = (g \circ \Phi)(p)$ is convex.

It is well known that the Klein model, the Poincaré disk model, the Poincaré half-plane model and the hyperboloid model are isometric to each other, see isometries between them in [1, Chapter A]. Therefore, it follows from Theorem 7.1 that the concepts of convexity and consequently the results studied in the previous sections have via isometries their counterparts in any model isometric to the hyperboloid model.

8 Final Remarks

This paper is inspired by the papers [4, 7], where we studied some intrinsic properties of the spherically convex functions and spherically quadratic functions, respectively. Despite some of our ideas being similar with the ideas in the aforementioned papers, the convex functions in hyperbolic spaces turned out to have a completely different structure. For example there is no constant globally convex function on the whole sphere, but there are many such functions on the hyperbolic space, as shown in Section 5.2. A related remark is that the class of convex functions on constant curvature manifolds is widening with the decrease of the sign of the curvature. We also expect that the class of convex functions on a proper convex subset of the hyperbolic space to be much more wider than the convex functions on the corresponding proper convex subset of the sphere. This property has already been established for corresponding intersections of the sphere and hyperbolic space with the positive orthant [7], but for other cones it needs to be investigated. Although several applications of optimization in hyperbolic spaces have emerged, a comprehensive study from this point of view of these spaces is still lacking. The results of this paper are the first step in this direction. We foresee significant progress in this topic in the nearby future.

Finally, let us present some basics formulas similar to the ones in Section 2 for an n -dimensional hyperbolic space with constant negative curvature $-K < 0$. For that, let us rescale the Lorentzian inner product (2) as follows: Let $K > 0$ and $\langle \cdot, \cdot \rangle_K$ be the K -Lorentzian inner product of $x := (x_1, \dots, x_n, x_{n+1})^\top$ and $y := (y_1, \dots, y_n, y_{n+1})^\top$ on \mathbb{R}^{n+1} defined by

$$\langle x, y \rangle_K := Kx_1y_1 + \dots + Kx_ny_n - Kx_{n+1}y_{n+1}. \quad (63)$$

Note that $\langle x, y \rangle_K = K\langle x, y \rangle$. For each $x \in \mathbb{R}^{n+1}$, the K -Lorentzian norm (length) of x is the complex number

$$\|x\|_K := \sqrt{\langle x, x \rangle_K}.$$

In order to state the inner product (63) in a convenient form, we take the diagonal matrix J_K defined by

$$J_K := \text{diag}(K, \dots, K, -K) \in \mathbb{R}^{(n+1) \times (n+1)}. \quad (64)$$

By using (64), the Lorentz inner product (63) can be written equivalently as follows

$$\langle x, y \rangle_K := x^\top \mathbf{J}_K y, \quad \forall x, y \in \mathbb{R}^{n+1}.$$

By considering the K -Lorentzian inner product. $\langle \cdot, \cdot \rangle_K$ defined in (63), we define the n -dimensional K -hyperbolic space as follows

$$\mathbb{H}_K^n := \{p \in \mathbb{R}^{n+1} : \langle p, p \rangle_K = -1, p^{n+1} > 0\}, \quad K > 0.$$

It is worth noting that the n -dimensional K -hyperbolic space \mathbb{H}_K^n can also be written as follows

$$\mathbb{H}_K^n := \left\{ p \in \mathbb{R}^{n+1} : \langle p, p \rangle_K = -\frac{1}{K}, p^{n+1} > 0 \right\}, \quad K > 0. \quad (65)$$

We know that \mathbb{H}_K^n has sectional curvature $-K$. It follows from (65) that \mathbb{H}_1^n is the n -dimensional hyperbolic space \mathbb{H}^n . The tangent plane of \mathbb{H}_K^n at a point $p \in \mathbb{H}_K^n$ is given by

$$T_p \mathbb{H}_K^n := \{v \in \mathbb{R}^{n+1} : \langle p, v \rangle_K = 0\}.$$

The *intrinsic distance* on the K -hyperbolic space \mathbb{H}_K^n between two points $p, q \in \mathbb{H}_K^n$ is given by

$$d^K(p, q) := \frac{1}{\sqrt{K}} \operatorname{arcosh}(-\langle p, q \rangle_K).$$

If $p, q \in \mathbb{H}^n$ and $q \neq p$, then the unique *geodesic segment from p to q* is given by

$$\gamma_{pq}^K(t) = \left(\cosh(\sqrt{K}t) + \frac{\langle p, q \rangle_K \sinh(\sqrt{K}t)}{\sqrt{\langle p, q \rangle_K^2 - 1}} \right) p + \frac{\sinh(\sqrt{K}t)}{\sqrt{\langle p, q \rangle_K^2 - 1}} q, \quad \forall t \in [0, d^K(p, q)].$$

The *exponential mapping* $\exp_p^K : T_p \mathbb{H}_K^n \rightarrow \mathbb{H}_K^n$ at a points $p \in \mathbb{H}_K^n$ is given by

$$\exp_p^K v := \cosh(\|v\|_K) p + \sinh(\|v\|_K) \frac{v}{\|v\|_K}, \quad \forall v \in T_p \mathbb{H}_K^n \setminus \{0\}.$$

If γ_v^K is the geodesic defined by its initial position p , with velocity v at p , then $\gamma_v^K(t) = \exp_p^K tv$. The *inverse of the exponential mapping* is given by $\log_p^K q = 0$, for $q = p$, and

$$\log_p^K q := \frac{1}{\sqrt{K}} \operatorname{arcosh}(-\langle p, q \rangle_K) \frac{1}{\sqrt{\langle p, q \rangle_K^2 - 1}} \left[\mathbf{I} + pp^\top \mathbf{J}_K \right] q, \quad q \neq p. \quad (66)$$

It follows from (8) and (10) that $d_K(p, q) = \|\log_q^K p\|_K$, for all $p, q \in \mathbb{H}_K^n$. The explicitly formula of parallel transport P_{pq}^K is given by

$$P_{pq}^K(v) := v - \frac{\langle v, \log_q^K p \rangle_K}{\operatorname{arcosh}^2(-\langle p, q \rangle_K)} (\log_q^K p + \log_p^K q) = \left[\mathbf{I} + \frac{1}{1 - \langle p, q \rangle_K} (p + q) q^\top J_K \right] v.$$

By rescaling the Lorentzian inner product (2) to (63), we can obtain similar results to the previous sections. It follows from a rescaled version of Theorem 5.1 that if a quadratic function is K_0 -hyperbolic convex for a $K_0 > 0$, then it is K -hyperbolic convex with respect to all $K > 0$ (where K -hyperbolic convexity in \mathbb{H}_K^n can be defined similarly to hyperbolic convexity in \mathbb{H}^n). However, the only K -hyperbolically convex quadratic functions that remain convex when $K \rightarrow 0$ are the ones with A positive semidefinite. It is interesting to study a similar question for more general functions.

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