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# A note on boundary point principles for partial differential inequalities of elliptic type

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## Abstract

In this note we consider boundary point principles for partial differential inequalities of elliptic type. First, we highlight the difference between the conditions required to establish classical strong maximum principles and classical boundary point lemmas for second-order linear elliptic partial differential inequalities. We highlight this difference by introducing a singular set in the domain where the coefficients of the partial differential inequality need not be defined and, in a neighborhood of which, can blow-up. Secondly, as a consequence, we establish a comparison-type boundary point lemma for classical elliptic solutions to quasilinear partial differential inequalities. Thirdly, we consider tangency principles, for  $C^1$  elliptic weak solutions to quasilinear divergence structure partial differential inequalities. We highlight the necessity of certain hypotheses in the aforementioned results via simple examples.

**MSC:** Primary 35R45; 35B51; 35B50; secondary 35J15; 35J62

**Keywords:** Elliptic PDI; Tangency principles; Maximum principles; Boundary point lemmas

## 1 Introduction

In this note we consider boundary point principles for solutions to elliptic partial differential inequalities (PDI). Specifically, we first give a relaxation of Hopf's [4] classical strong maximum principle (CSMP) for classical solutions to linear elliptic PDI, which here, allows the coefficients in the PDI to be unbounded in a neighborhood of a sufficiently regular subset of the spatial domain. The boundary point lemma (BPL) for linear elliptic PDI is obtained as a consequence of this CSMP and complements available results in [5, 13]. Although coefficients in the PDI in the BPL are not necessarily bounded, they are constrained by growth conditions detailed in Sect. 2. As a secondary consideration, we illustrate how to extend BPL for classical solutions to linear elliptic PDI, to comparison-type BPL for elliptic classical solutions of quasilinear PDI, and highlight the importance of specific conditions on this extension via examples. Consequently, we demonstrate that the BPL, as stated in [13, Theorem 2.7.1] is erroneous. Thirdly, we give an extension of a tangency principle for  $C^1$  elliptic weak solutions to divergence structure quasilinear PDI in domains with boundaries that satisfy an interior cone condition, which appeared in [17]. We also highlight that the tangency principle in [13, Theorem 2.7.2], an extension of that in [17], is erroneous.

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We speculate that if detailed proofs for each of the three results in [13, Sect. 2.7] were included, then erroneous statements would not have appeared therein. Consequently, rather than merely provide counterexamples to the aforementioned erroneous theorem statements, we also investigate suitable amendments to the aforementioned results, which are established subject to conditions apparently intended in [13]. Specifically, the amended theorem statements allow for coefficients and functions in the elliptic PDI considered here to allow for local instead of global regularity/bounds, within the limitations of arguments used to prove the results. By providing detailed proofs, we clearly highlight when the mathematical arguments in [13] cease to be valid. Moreover, the proofs provided here illustrate the requirement for supplementary hypotheses to the theorem statements provided in [13], which retain the nonlocal nature of the theorem hypotheses.

We now give a brief account of the historical development of results in this note. The CSMP and BPL for classical solutions to linear elliptic PDI were established by Hopf in [4] for linear elliptic PDI with bounded (uniformly elliptic) coefficients. Although Hopf considered generalizations of the CSMP and BPL to elliptic solutions of nonlinear PDI in [4], more general statements of these results were established by McNabb in [6]. Extensions to the CSMP and BPL for classical solutions to linear elliptic PDI with coefficients that can blow-up or degenerate have been considered by numerous authors, as summarized in [2, 5, 12]. Notably, the consideration of a singular set within a domain for an elliptic PDI where the differential operator is not necessarily defined, and where tangency/maximum principles can be established, date back to Giraud [3]. Additionally, due to the development of a theory for weak solutions to boundary value problems for divergence structure quasilinear elliptic partial differential equations, tangency principles for  $C^1$  elliptic weak solutions to quasilinear PDI were established by Serrin in [17], and extended in [13]. We note that the proof of Serrin relies on an iteration method developed by Moser [7] and a Harnack inequality for quasilinear divergence structure elliptic PDI established by Trudinger [18]. More recently [14–16], tangency principles have been established for  $C^1$  elliptic weak solutions to quasilinear PDI that have conclusions more similar to that of Hopf-type BPL. A broader historical overview of the development of this theory can be found in [10, p.156–158 and p.193–194], [13, p.46] and [1, 8, 11].

The remainder of the note is presented as follows. In Sect. 2, we prove the CSMP and BPL for classical solutions to linear elliptic PDI, and consequently, we establish a comparison-type BPL for elliptic classical solutions to quasilinear PDI. Furthermore, we provide examples that highlight the need for specific conditions given in the statement of the BPL as given here, one of which, is a counterexamples to [13, Theorem 2.7.1]. In Sect. 3, we establish a comparison-type tangency principle for  $C^1$  elliptic weak solutions to quasilinear divergence structure PDI in domains that satisfy an interior cone condition at boundary points. The necessity of several conditions in the BPL statement are highlighted, and furthermore, we demonstrate that [13, Theorems 2.7.2 and 2.7.3], are erroneous. In Sect. 4, we discuss how results in this note can be generalized and placed in a wider context.

## 2 Classical theory

In this section, we establish a CSMP in Theorem 2.3 and BPL in Theorem 2.5 for classical solutions to linear elliptic PDI. The CSMP is noteworthy in that it allows coefficients in the PDI, under constraint, to blow-up in the interior of the domain in the neighborhood of a singular set. After defining the regularity of the singular set and constructing a suitable auxiliary function, the proofs of these results largely follow the description of related

proofs available in [13, Chap. 2]. This allows us to highlight a distinction between the conditions required to establish a CSMP and BPL for classical solutions to linear elliptic PDI. Consequently, we also establish a comparison-type BPL for classical elliptic solutions to quasilinear PDI in Theorem 2.6 using the aforementioned BPL for linear elliptic PDI, refining an analogous statement in [13, Theorem 2.7.1]. We provide a proof using the approach outlined in [13, Sect. 2.7] where it is noteworthy that a full proof is omitted. To conclude the section, we give a simple counterexample to [13, Theorem 2.7.1] and provide a further example to highlight the importance of specific conditions in Theorem 2.6 that are not present in [13, Theorem 2.7.1].

### 2.1 Notation and definitions

For a set  $X \subset \mathbb{R}^n$ , we denote  $\partial X = \bar{X} \setminus \text{int}(X)$ , to be the boundary of  $X$ . In addition, throughout this note,  $\Omega \subset \mathbb{R}^n$  denotes an open connected bounded set (a bounded domain), and we denote the set  $B_R(x_0) \subset \mathbb{R}^n$  to be an open  $n$ -dimensional ball of radius  $R$  (with respect to the Euclidean distance) centered at  $x_0 \in \mathbb{R}^n$ . We also denote the origin in  $\mathbb{R}^n$  by  $O$ . Furthermore, we denote  $R(X)$  to be the set of real-valued functions with domain  $X$ ,  $C(X) \subset R(X)$  to be the set of all continuous functions in  $R(X)$ , and  $C^i(X) \subset C(X)$  to be the set of  $i$ -times continuously differentiable functions in  $C(X)$  for each  $i \in \mathbb{N}$ . Additionally, for  $u \in C^2(\Omega)$  and  $\mathcal{S} \subset \Omega$ , we consider the linear elliptic operator  $L : C^2(\Omega) \rightarrow R(\Omega \setminus \mathcal{S})$  given by

$$L[u] := \sum_{i,j=1}^n a_{ij}u_{x_i x_j} + \sum_{i=1}^n b_i u_{x_i} + cu \quad \text{in } \Omega \setminus \mathcal{S}, \tag{2.1}$$

with  $a_{ij}, b_i, c : \Omega \setminus \mathcal{S} \rightarrow \mathbb{R}$  prescribed functions for  $i, j = 1, \dots, n$ , and such that there exists a nonnegative function  $\Lambda : \Omega \setminus \mathcal{S} \rightarrow \mathbb{R}$  for which,

$$|y|^2 \leq \sum_{i,j=1}^n a_{ij}(x)y_i y_j \leq \Lambda(x)|y|^2 \quad \forall x \in \Omega \setminus \mathcal{S}, y \in \mathbb{R}^n. \tag{2.2}$$

We refer to the set  $\mathcal{S}$  where the linear elliptic operator is not defined for  $u$ , as the *singular set*. Additionally, note that by rescaling the coefficients in the operator in (2.1) by  $\epsilon$ , the left-hand side of (2.2) can be expressed as  $\epsilon|y|^2$ , i.e., with an equivalent frequently used ellipticity condition. Moreover, for  $u \in C^2(\Omega)$  we denote  $Du$  and  $D^2u$  to be the gradient of  $u$  and the Hessian of  $u$  on  $\Omega$ , respectively.

To establish the CSMP in this note, we give the following definition, which will be used to define the structure of the singular set  $\mathcal{S} \subset \Omega$ . We refer to  $\mathcal{S}$  as the singular set since the coefficients  $a_{ij}, b_i$  or  $c$  of  $L$  are allowed, with constraint, to blow-up in neighborhoods of  $\mathcal{S}$ . We note that in [1], alternatively, two-sided ‘hour-glass’ conditions are employed for regularity conditions on singular sets that complement the following definition.

**Definition 2.1** Let  $\Omega \subset \mathbb{R}^n$  be a domain and  $\mathcal{S} \subset \Omega$ . We say that  $\mathcal{S}$  satisfies an *outward ball property* if, given any nonempty relatively closed set  $\mathcal{T} \subset \Omega$  that is a strict subset of  $\Omega$ , there exists  $R > 0$  and  $x_0 \in \Omega \setminus (\mathcal{T} \cup \mathcal{S})$  such that

$$B_R(x_0) \subset \Omega \setminus (\mathcal{T} \cup \mathcal{S}) \quad \text{and} \quad \partial B_R(x_0) \cap \mathcal{T} \neq \emptyset. \tag{2.3}$$

To illustrate some geometric aspects of sets that satisfy an outward ball property, consider the following:

- (i) If  $\mathcal{S}$  consists solely of a finite number of points in  $\Omega$  then  $\mathcal{S}$  satisfies the outward ball property. This follows by considering  $d_{H'} : \mathcal{P}(\mathbb{R}^n) \times \mathcal{P}(\mathbb{R}^n) \rightarrow [0, \infty)$  with  $\mathcal{P}(X)$  denoting the power set of  $X$ , and

$$d_{H'}(X, Y) = \sup_{x \in X} \left( \inf_{y \in Y} |x - y| \right) \quad \forall X, Y \in \mathcal{P}(\mathbb{R}^n),$$

i.e., one component of the Euclidean Hausdorff distance between  $X$  and  $Y$ . Note that if  $|X| = 1$ , then  $d_{H'}$  is the Euclidean Hausdorff distance between the two sets  $X$  and  $Y$ , denoted here by  $d(X, Y)$ . Now, let  $\mathcal{T}$  be as in Definition 2.1. Then, since  $\mathcal{T}$  is nonempty and  $\mathcal{T} \neq \Omega$ , it follows that  $\partial\mathcal{T} \cap \Omega \neq \emptyset$ . If  $d_{H'}(\partial\mathcal{T} \cap \Omega, \mathcal{S}) = 0$ , it follows that  $\mathcal{T} \subseteq \mathcal{S}$ , and we can choose a point  $x_0 \in \Omega \setminus (\mathcal{T} \cup \mathcal{S})$  sufficiently close to  $\mathcal{T}$  such that there exists a ball  $B_R(x_0)$  that satisfies (2.3). Alternatively, if  $d_{H'}(\partial\mathcal{T} \cap \Omega, \mathcal{S}) > 0$ , then since  $\Omega \setminus (\mathcal{T} \cup \mathcal{S})$  is a nonempty open set, we can choose  $x_0 \in \Omega \setminus (\mathcal{T} \cup \mathcal{S})$  so that  $d_{H'}(\{x_0\}, \mathcal{T}) < \frac{1}{2}d_{H'}(\{x_0\}, \mathcal{S} \cup \partial\Omega)$ . Thus, there exists a ball  $B_R(x_0)$  that satisfies (2.3).

- (ii) If  $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$  and

$$\begin{aligned} \mathcal{S} = & \left\{ (x_1, x_2) \in \Omega : (x_1, x_2) = (\phi_1(t), \phi_2(t)) \quad \forall t \in (0, 1) \text{ with} \right. \\ & \phi : (0, 1) \rightarrow \Omega \text{ twice continuously differentiable and injective on} \\ & (0, 1), \phi' > 0 \text{ on } (0, 1), \lim_{t \rightarrow 0} \phi(t) \neq \lim_{t \rightarrow 1} \phi(t) \text{ and} \\ & \left. \lim_{t \rightarrow 0} \phi(t), \lim_{t \rightarrow 1} \phi(t) \neq \phi(s) \quad \forall s \in (0, 1) \right\}, \end{aligned}$$

then  $\mathcal{S}$  satisfies the outward ball property. To see this, let  $\mathcal{T}$  be as in Definition 2.1. If  $d_{H'}(\partial\mathcal{T} \cap \Omega, \mathcal{S}) > 0$ , then a ball that satisfies (2.3) is guaranteed to exist, following the justification in (i). Alternatively, if  $d_{H'}(\partial\mathcal{T} \cap \Omega, \mathcal{S}) = 0$ , then it follows that  $\partial\mathcal{T} \cap \Omega \subseteq \overline{\mathcal{S}}$ . Suppose there exists  $s_0 \in \partial\mathcal{T} \cap \mathcal{S}$ . Then, since  $\mathcal{S}$  is given by a sufficiently smooth curve, for  $s_0$ , there exists a ball  $B_R(s_0) \subset \Omega$  such that

$$\begin{aligned} \mathcal{S} \cap \overline{B_R(s_0)} &= \{(\phi_1(t), \phi_2(t)) : t_1 \leq t \leq t_2\} \\ &=: \mathcal{S}_R \cup \{(\phi_1(t_1), \phi_2(t_1)), (\phi_1(t_2), \phi_2(t_2))\}. \end{aligned}$$

Thus,  $\partial\mathcal{T} \cap B_R(s_0) \subset \mathcal{S}_R$  and hence, via the Jordan Curve Theorem,  $B_R(s_0)$  can be decomposed into the disjoint sets  $\mathcal{S}_R, B_R^1(s_0)$  and  $B_R^2(s_0)$  with  $B_R^1(s_0)$  the connected open set with boundary  $\mathcal{S}_R$  and the arc on  $\partial B_R(s_0)$  connecting  $(\phi_1(t_1), \phi_2(t_1))$  to  $(\phi_1(t_2), \phi_2(t_2))$  in a clockwise direction ( $B_R^2(s_0)$  is defined similarly to  $B_R^1(s_0)$  with clockwise replaced by anticlockwise). Thus,  $\mathcal{T} \cap B_R(s_0)$  is either:  $\mathcal{S}_R \cap \mathcal{T}, \mathcal{S}_R \cup B_R^1(s_0)$  or  $\mathcal{S}_R \cup B_R^2(s_0)$ , and in each case, since  $\mathcal{S}_R$  is defined by a  $C^2$  curve, there exists a ball  $B_{R_1}(x) \subset B_R(s_0) \setminus (\mathcal{T} \cup \mathcal{S})$  that satisfies (2.3). If instead  $d_{H'}(\partial\mathcal{T} \cap \Omega, \mathcal{S}) = 0$  and  $\partial\mathcal{T} \cap \mathcal{S} = \emptyset$  then a similar argument to that in (i) can be used to demonstrate that a ball that satisfies (2.3) exists. It follows analogously from the Jordan–Brouwer Separation Theorem that any set of finitely many disjoint

compact  $(n - 1)$ -dimensional sufficiently smooth  $C^2$  manifolds in a domain  $\Omega \subset \mathbb{R}^n$  also satisfies the outward ball property.

(iii) If  $\Omega = (-1, 1)^2$  and

$$\mathcal{S}' = \{(x_1, x_2) \in \Omega : x_1 = 0 \text{ or } x_2 = 0\},$$

then  $\mathcal{S}'$  does not satisfy the outward ball property. This follows by considering  $\mathcal{T} = \{(0, 0)\}$  and observing that every ball  $B_R(x) \subset \Omega$  such that  $\partial B_R(x) \cap \mathcal{T} \neq \emptyset$ , also satisfies  $B_R(x) \cap \mathcal{S}' \neq \emptyset$ . However, if instead  $\Omega = (-1, 1)^2$  and

$$\mathcal{S} = \{(x_1, x_2) \in [0, 1) \times (-1, 1) : x_1 = 0 \text{ or } x_2 = 0\}$$

then  $\mathcal{S}$  satisfies the outward ball property.

(iv) If  $\mathcal{S}'$  is locally dense on  $B_R(x) \subset \Omega$  then  $\mathcal{S}'$  does not satisfy the outward ball property. This can be observed by choosing  $\mathcal{T}$  to contain any point in  $\mathcal{S}' \cap B_R(x)$ . Consequently, sets that satisfy Definition 2.1 are necessarily measure zero sets with respect to the Lebesgue measure.

(v) If  $\mathcal{S}$  satisfies Definition 2.1, then  $\mathcal{S}$  is 1-porous at each  $s \in \mathcal{S}$  with respect to [20, Definition 2.1]. This follows by considering  $\mathcal{T} = \{s\}$ . However, not all subsets of  $\Omega$  that are 1-porous at every point necessarily satisfy Definition 2.1. For example, consider  $\Omega = (-1, 1)^2$  with

$$\mathcal{S}' = \left\{ (x_1, x_2) \in \Omega : x_1 = \frac{1}{2^n} \text{ for } n \in \mathbb{N} \right\}.$$

Since  $\mathcal{S}'$  consists of a countable set of isolated lines, it follows immediately that  $\mathcal{S}'$  is 1-porous at each  $s \in \mathcal{S}'$ . However, by considering  $\mathcal{T} = (-1, 0) \times (-1, 1) \subset \Omega$ , it follows that  $\mathcal{S}'$  does not satisfy the outward ball property.

Later in this section, for  $u \in C^2(\Omega)$ , we consider the quasilinear operator  $Q : C^2(\Omega) \rightarrow R(\Omega)$  given by,

$$Q[u] := \sum_{i,j=1}^n A_{ij}(x, u, Du)u_{x_i x_j} + B(x, u, Du) \quad \text{in } \Omega, \tag{2.4}$$

with  $A_{ij}, B : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  prescribed functions. Specifically, we refer to  $Q$  as elliptic with respect to a specific  $u \in C^2(\Omega)$  if (2.2) holds for  $a_{ij}(x) = A_{ij}(x, u(x), Du(x))$  for all  $x \in \Omega$ .

### 2.2 CSMP and BPL for linear elliptic PDI

Before we establish a CSMP and BPL for classical solutions to  $L[u] \geq 0$  with  $L$  given by (2.1), we give the following lemma that guarantees the existence of a suitable *comparison function*.

**Lemma 2.2** *Let  $R, m > 0$  be constants and set  $k = 2n(\frac{2}{R} + 1) + 3$ . Additionally, suppose that there exists a constant  $\epsilon > 0$  and a continuous nonincreasing function  $\Lambda : (0, \epsilon] \rightarrow (0, \infty)$  such that  $\Lambda \in L^1((0, \epsilon))$ ,*

$$\epsilon \in \left( 0, \min \left\{ 1, \frac{R}{2} \right\} \right) \quad \text{and} \quad \int_0^\epsilon \Lambda(s) ds < \frac{1}{k}, \tag{2.5}$$

and moreover, for  $\Omega = B_R(O) \setminus \bar{B}_{R-\epsilon}(O)$  that

$$|y|^2 \leq \sum_{i,j=1}^n a_{ij}(x)y_i y_j \leq \Lambda(R - |x|)|y|^2 \quad \forall x \in \Omega, y \in \mathbb{R}^n, \tag{2.6}$$

$$|b_i(x)| \leq \Lambda(R - |x|) \quad \forall x \in \Omega, \tag{2.7}$$

$$-c(x) \leq \frac{\Lambda(R - |x|)}{(R - |x|)} \quad \forall x \in \Omega. \tag{2.8}$$

Then, if  $L$  is a linear elliptic operator with coefficients that satisfy (2.6)–(2.8), there exists  $v : \bar{\Omega} \rightarrow [0, m]$ , such that:

- (i)  $v = 0$  on  $\partial B_R(O)$ ,  $v = m$  on  $\partial B_{R-\epsilon}(O)$ , and  $v > 0$  on  $\Omega$ .
- (ii)  $v \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ .
- (iii)  $L[v] > 0$  on  $\Omega$ .
- (iv)  $\partial_\nu v < 0$  on  $\partial B_R(O)$ , where  $\partial_\nu v$  denotes the outward (to  $\Omega$ ) directional derivative of  $v$  normal to  $\partial\Omega$ .

*Proof* Define  $f : [0, \epsilon] \rightarrow [0, \infty)$  to be

$$f(r) = r + k \int_0^r \int_0^s \Lambda(t) dt ds \quad \forall r \in [0, \epsilon]. \tag{2.9}$$

It follows immediately that

$$f \in C^1([0, \epsilon]) \cap C^2((0, \epsilon]), \tag{2.10}$$

with

$$f'(r) = 1 + k \int_0^r \Lambda(t) dt \quad \forall r \in [0, \epsilon], \tag{2.11}$$

$$f''(r) = k\Lambda(r) \quad \forall r \in (0, \epsilon]. \tag{2.12}$$

Now, we define  $\tilde{v} : \bar{\Omega} \rightarrow \mathbb{R}$  to be

$$\tilde{v}(x) = f(R - |x|) \quad \forall x \in \bar{\Omega}. \tag{2.13}$$

It follows from (2.9)–(2.13) that

$$\tilde{v} \in C^1(\bar{\Omega}) \cap C^2(\Omega), \tag{2.14}$$

$$\tilde{v} > 0 \quad \text{on } \Omega, \tag{2.15}$$

and

$$\begin{aligned} L[\tilde{v}](x) &= \frac{(f''(R - |x|)|x| + f'(R - |x|))}{|x|^3} \sum_{i,j=1}^n a_{ij}(x)x_i x_j \\ &\quad - \frac{f'(R - |x|)}{|x|} \sum_{i=1}^n (a_{ii}(x) + b_i(x)x_i) + f(R - |x|)c(x) \end{aligned} \tag{2.16}$$

for all  $x \in \Omega$ . It now follows from substituting (2.9), (2.11), and (2.12) into (2.16), and using (2.5)–(2.8), that

$$\begin{aligned}
 L[\tilde{v}](x) &= \frac{1}{|x|^3} \left( k\Lambda(R - |x|)|x| + \left( 1 + k \int_0^{R-|x|} \Lambda(t) dt \right) \sum_{i,j=1}^n a_{ij}(x)x_i x_j \right. \\
 &\quad \left. - \frac{1}{|x|} \left( 1 + k \int_0^{R-|x|} \Lambda(t) dt \right) \sum_{i=1}^n (a_{ii}(x) + b_i(x)x_i) \right. \\
 &\quad \left. + c(x) \left( (R - |x|) + k \int_0^{R-|x|} \int_0^s \Lambda(t) dt ds \right) \right) \\
 &\geq \Lambda(R - |x|) \left( k - 2n \left( \frac{2}{R} + 1 \right) - 2 \right) \\
 &= \Lambda(R - |x|) \\
 &> 0
 \end{aligned} \tag{2.17}$$

for all  $x \in \Omega$ . Now, define  $v : \bar{\Omega} \rightarrow \mathbb{R}$  to be

$$v(x) = \frac{\tilde{v}(x)m}{f(\epsilon)} \quad \forall x \in \bar{\Omega}. \tag{2.18}$$

Then, via (2.18), (2.13), (2.15), and (2.9),  $v$  satisfies (i). Also, via (2.14)  $v$  satisfies (ii). Additionally, from (2.17) and (2.18),  $v$  satisfies (iii). Moreover, via (2.18), (2.13), (2.11), and (2.9), it follows that

$$\partial_\nu v(x)|_{|x|=R} = \frac{-m}{f(\epsilon)} < 0,$$

and hence  $v$  satisfies (iv), as required. □

We now establish a CSMP for linear elliptic PDI that allows coefficients of  $L$  to blow-up in neighborhoods of interior points of  $\Omega$ . We note that one can recover a standard CSMP for linear elliptic PDI with bounded coefficients of appropriate sign (see for instance [4, 10, 13]) by considering  $\mathcal{S} = \emptyset$  with  $\Lambda$  a sufficiently large constant.

**Theorem 2.3** ((CSMP)) *Let  $\Omega \subset \mathbb{R}^n$  and  $\mathcal{S} \subset \Omega$  satisfy the outward ball property. Suppose that  $u \in C^2(\Omega)$  satisfies the linear elliptic PDI  $L[u] \geq 0$  on  $\Omega \setminus \mathcal{S}$ . In addition, suppose that for each  $B_R(x_0) \subset (\Omega \setminus \mathcal{S})$  for which  $\partial B_R(x_0) \cap \partial\Omega = \emptyset$ , there exists a function  $\Lambda : (0, \frac{R}{2}] \rightarrow (0, \infty)$  that is continuous nonincreasing and such that  $\Lambda \in L^1((0, \frac{R}{2}))$ , and such that the coefficients of  $L$  satisfy*

$$|y|^2 \leq \sum_{i,j=1}^n a_{ij}(x)y_i y_j \leq \Lambda(d(\{x\}, \partial B_R(x_0)))|y|^2 \quad \forall x \in B_R(x_0), y \in \mathbb{R}^n, \tag{2.19}$$

$$|b_i(x)| \leq \Lambda(d(\{x\}, \partial B_R(x_0))) \quad \forall x \in B_R(x_0), \tag{2.20}$$

$$-c(x) \leq \frac{\Lambda(d(\{x\}, \partial B_R(x_0)))}{d(\{x\}, \partial B_R(x_0))} \quad \forall x \in B_R(x_0). \tag{2.21}$$



Additionally, let

$$M_u = \sup_{x \in \Omega} u(x) \tag{2.22}$$

and suppose that either  $M_u = 0$ , or  $M_u > 0$  with  $c$  nonpositive. Then,  $M_u > u(x)$  for all  $x \in \Omega$  or  $u$  is constant on  $\Omega$ .

*Proof* Suppose that  $u$  is not constant on  $\Omega$ , and

$$\mathcal{T} = \{x \in \Omega : u(x) = M_u\} \tag{2.23}$$

is not empty. Since  $\mathcal{T}$  is a relatively closed strict subset of  $\Omega$  and  $\mathcal{S}$  satisfies the outward ball property, it follows that there exists a sufficiently small  $B_R(x_0)$  such that  $B_R(x_0) \subset \Omega \setminus (\mathcal{T} \cup \mathcal{S})$ ,  $\partial B_R(x_0) \cap \mathcal{T} = \{y_0\}$  and  $\partial B_R(x_0) \cap \partial \Omega = \emptyset$ . Moreover, it follows from (2.19)–(2.21) and the hypotheses on  $L$  and  $\Lambda$ , that Lemma 2.2 can be applied to a linear elliptic operator  $\tilde{L}$  defined in  $\Omega_0 = B_R(O) \setminus \bar{B}_{R-\epsilon}(O)$ , for sufficiently small  $\epsilon \in (0, R)$ , with coefficients given by

$$\tilde{a}_{ij}(x) = a_{ij}(x + x_0), \quad \tilde{b}_i(x) = b_i(x + x_0), \quad \tilde{c}(x) = \frac{-\Lambda(R - |x|)}{(R - |x|)}, \quad \forall x \in \Omega_0 \tag{2.24}$$

with

$$m = M_u - \left( \sup_{\partial B_{R-\epsilon}(x_0)} u \right) > 0,$$

to guarantee the existence of  $v : \bar{\Omega}_0 \rightarrow [0, m]$  that satisfies the conclusions of Lemma 2.2. Now, define  $w : \bar{\Omega}_0 \rightarrow \mathbb{R}$  to be

$$w(x) = u(x + x_0) + v(x) - M_u \quad \forall x \in \bar{\Omega}_0. \tag{2.25}$$

It follows that  $w \in C^2(\Omega_0) \cap C^1(\bar{\Omega}_0)$  and  $\sup_{\partial \Omega_0} w = w(y_0 - x_0) = 0$ . Additionally, it follows that  $w \leq 0$  on  $\Omega_0$ , for suppose that the converse holds, i.e., that there exists  $x^* \in \Omega_0$  such that  $\sup_{x \in \Omega_0} w(x) = w(x^*) > 0$ . From the hypotheses and Lemma 2.2(iii), we have  $L[u](x^* + x_0) \geq 0$  and  $\tilde{L}[v](x^*) > 0$ , respectively. Thus, via (2.24) and (2.25) we have,

$$\begin{aligned} & \sum_{i,j=1}^n \tilde{a}_{ij}(x^*) w_{x_i x_j}(x^*) + \sum_{i=1}^n \tilde{b}_i(x^*) w_{x_i}(x^*) \\ & > -c(x^* + x_0)u(x^* + x_0) - \tilde{c}(x^*)v(x^*) \\ & \geq \frac{\Lambda(R - |x^*|)}{(R - |x^*|)} (\min\{u(x^* + x_0), 0\} + v(x^*)) \\ & > 0 \end{aligned} \tag{2.26}$$

with the last two inequalities following from (2.21), (2.25), and the hypotheses. However, since there is a local maxima of  $w$  at  $x^*$ , then  $Dw(x^*) = 0$ , and  $D^2w(x^*)$  is negative semidefinite. Consequently, via the Schur Product Theorem, the left-hand side of (2.26) is non-positive, which gives a contradiction, and hence,  $w \leq 0$  on  $\Omega_0$ . Therefore,  $\partial_\nu w(y_0 - x_0) \geq 0$ ,

and hence  $\partial_\nu u(y_0) \geq -\partial_\nu v(y_0 - x_0) > 0$ . However, since there is a local maxima of  $u$  at  $y_0$ , it follows from the regularity of  $u$  that  $Du(y_0) = 0$ , which contradicts  $\partial_\nu u(y_0) > 0$ . Therefore, either  $u$  is constant on  $\Omega$ , or  $u < M_u$  on  $\Omega$ , as required.  $\square$

*Remark 2.4* Note that in Theorem 2.3, the conditions on the coefficients of  $L$  apply on balls that satisfy  $\partial B_R(x_0) \cap \bar{S} \neq \emptyset$  but not on balls that satisfy  $B_R(x_0) \cap \bar{S} \neq \emptyset$ . Thus, although the coefficients of  $L$  can, under constraints (2.19)–(2.21), blow-up as  $x \rightarrow \bar{S}$ , they cannot blow-up (except  $c$  negatively) as  $x \rightarrow x_0$  for  $x_0 \in \Omega \setminus \bar{S}$ . Moreover, observe that the coefficients of  $L$  can blow-up as  $x \rightarrow \partial\Omega$  with conditions (2.19)–(2.21) not required to hold on  $B_R(x_0)$  such that  $\partial B_R(x_0) \cap \partial\Omega \neq \emptyset$ . However, for a BPL to hold for a linear elliptic operator  $L$  on  $\Omega$ , conditions (2.19)–(2.21) are required to hold on balls  $B_R(x_0)$  such that  $\partial B_R(x_0) \cap \partial\Omega \neq \emptyset$ . This is the principal difference in hypothesis between BPL and CSMP for linear elliptic PDI.

A straightforward application of Theorem 2.3 gives an associated BPL for classical solutions to linear elliptic PDI.

**Theorem 2.5** ((BPL)) *Suppose that the hypotheses of Theorem 2.3 hold, with the restriction that ‘for which  $\partial B_R(x_0) \cap \partial\Omega = \emptyset$ ’ is omitted.<sup>1</sup> In addition, suppose that  $u \in C^1(\bar{\Omega})$  and  $\sup_{x \in \Omega} u(x) = u(x_b)$  for some  $x_b \in \partial\Omega$  such that there exists  $B_{R_b}(x'_b) \subset \Omega \setminus S$  that satisfies  $x_b \in \partial B_{R_b}(x'_b)$ . If  $u$  is not constant on  $\Omega$ , then  $\partial_\nu u(x_b) > 0$ .*

*Proof* Since  $u$  satisfies the conditions of Theorem 2.3 and is not constant, it follows that  $u(x) < u(x_b)$  for all  $x \in B_{R_b}(x'_b)$ . A function analogous to  $w$  in (2.25) can now be constructed, from which we can conclude (as in the proof of Theorem 2.3) that  $\partial_\nu u(x_b) > 0$ , as required.  $\square$

### 2.3 Comparison-type BPL for elliptic classical solutions to quasilinear PDI

In this subsection we establish a comparison-type BPL for classical elliptic solutions to quasilinear PDI using the approach described in [13, Chap. 2]. Specifically, via an application of Theorem 2.5, a BPL for classical elliptic solutions to quasilinear PDI can be established. Although the proof is standard, we provide it to inform the discussion that follows.

**Theorem 2.6** ((BPL)) *Suppose that  $u, v : \bar{\Omega} \rightarrow \mathbb{R}$  satisfy  $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$  and the quasilinear PDI  $Q[u] \geq 0$  and  $Q[v] \leq 0$  on  $\Omega$ . Furthermore, suppose that  $Q$  is elliptic with respect to  $u$ , with  $v_{x_i x_j}$  bounded on  $\Omega$  (or instead suppose that  $Q$  is elliptic with respect to  $v$ , with  $u_{x_i x_j}$  bounded on  $\Omega$ ) for  $i, j = 1, \dots, n$ . Suppose that  $u < v$  in  $\Omega$  and  $u = v$  at  $x_b \in \partial\Omega$  for which there exists  $B_{R_b}(x'_b) \subset \Omega$  with  $x_b \in \partial B_{R_b}(x'_b)$ . Suppose that there exists a continuous nonincreasing function  $\Lambda : (0, \frac{R_b}{2}] \rightarrow (0, \infty)$  such that  $\Lambda \in L^1((0, \frac{R_b}{2}))$ ,*

$$\begin{aligned}
 & |A_{ij}(x, z_1, \eta_1) - A_{ij}(x, z_2, \eta_2)| \tag{2.27} \\
 & \leq \Lambda(d(\{x\}, \partial B_{R_b}(x'_b))) \left( \frac{|z_1 - z_2|}{d(\{x\}, \partial B_{R_b}(x'_b))} + \sum_{l=1}^n |\eta_{1l} - \eta_{2l}| \right)
 \end{aligned}$$

<sup>1</sup>i.e., for the BPL, we also impose conditions (2.19)–(2.21) on  $B_R(x_0) \subset \Omega \setminus S$  such that  $\partial B_R(x_0) \cap \partial\Omega \neq \emptyset$ .

for all  $(x, z_1, \eta_1), (x, z_2, \eta_2) \in B_{R_b}(x'_b) \times [-M_z, M_z] \times [-M_\eta, M_\eta]^n$ , and

$$\begin{aligned}
 & B(x, z_1, \eta_1) - B(x, z_2, \eta_2) \\
 & \geq -\Lambda(d(\{x\}, \partial B_{R_b}(x'_b))) \left( \frac{(z_1 - z_2)}{d(\{x\}, \partial B_{R_b}(x'_b))} + \sum_{l=1}^n |\eta_{1l} - \eta_{2l}| \right)
 \end{aligned} \tag{2.28}$$

for all  $(x, z_1, \eta_1), (x, z_2, \eta_2) \in B_{R_b}(x'_b) \times [-M_z, M_z] \times [-M_\eta, M_\eta]^n$  with  $z_1 \geq z_2$ , with

$$M_z = \sup_{x \in B_{R_b}(x'_b)} \{|u(x)|, |v(x)|\} \quad \text{and} \quad M_\eta = \sup_{\substack{x \in B_{R_b}(x'_b) \\ i=1, \dots, n}} \{|u_{x_i}(x)|, |v_{x_i}(x)|\}.$$

Then,  $\partial_v u(x_b) > \partial_v v(x_b)$ .

*Proof* Let  $w = u - v$  on  $\overline{B_{R_b}(x'_b)}$ . Then, on  $B_{R_b}(x'_b)$ ,

$$\begin{aligned}
 0 & \leq \sum_{i,j=1}^n (A_{ij}(\cdot, u, Du)u_{x_i x_j} - A_{ij}(\cdot, v, Dv)v_{x_i x_j}) + B(\cdot, u, Du) - B(\cdot, v, Dv) \\
 & = \sum_{i,j=1}^n (A_{ij}(\cdot, u, Du)(u_{x_i x_j} - v_{x_i x_j}) + (A_{ij}(\cdot, u, Du) - A_{ij}(\cdot, v, Dv))v_{x_i x_j} \\
 & \quad + (A_{ij}(\cdot, u, Dv) - A_{ij}(\cdot, v, Dv))v_{x_i x_j}) \\
 & \quad + (B(\cdot, u, Du) - B(\cdot, v, Dv)) + (B(\cdot, u, Dv) - B(\cdot, v, Dv)) \\
 & \leq \sum_{i,j=1}^n A_{ij}(\cdot, u, Du)w_{x_i x_j} \\
 & \quad + \Lambda(d(\cdot, \partial B_{R_b}(x'_b))) \sum_{i=1}^n \left( \operatorname{sgn}(w_{x_i}) + \sum_{k,l=1}^n v_{x_k x_l} \operatorname{sgn}(v_{x_k x_l} w_{x_i}) \right) w_{x_i} \\
 & \quad + \left( \frac{\Lambda(d(\cdot, \partial B_{R_b}(x'_b)))}{d(\cdot, \partial B_{R_b}(x'_b))} \sum_{k,l=1}^n v_{x_k x_l} \operatorname{sgn}(v_{x_k x_l} w) \right. \\
 & \quad \left. + \frac{(B(\cdot, u, Dv) - B(\cdot, v, Dv))}{w} \right) w \\
 & =: \sum_{i,j=1}^n \tilde{a} w_{x_i x_j} + \sum_{i=1}^n \tilde{b}_i w_{x_i} + \tilde{c} w,
 \end{aligned}$$

where, for  $i, j = 1, \dots, n$ ,  $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c} : \Omega \rightarrow \mathbb{R}$  are given by,

$$\tilde{a}_{ij} = A_{ij}(\cdot, u, Du), \tag{2.29}$$

$$\tilde{b}_i = \Lambda(d(\cdot, \partial B_{R_b}(x'_b))) \left( \operatorname{sgn}(w_{x_i}) + \sum_{k,l=1}^n v_{x_k x_l} \operatorname{sgn}(v_{x_k x_l} w_{x_i}) \right), \tag{2.30}$$

$$\tilde{c} = \frac{\Lambda(d(\cdot, \partial B_{R_b}(x'_b)))}{d(\cdot, \partial B_{R_b}(x'_b))} \sum_{k,l=1}^n v_{x_k x_l} \operatorname{sgn}(v_{x_k x_l} w) + \frac{B(\cdot, u, Dv) - B(\cdot, v, Dv)}{(u - v)} \tag{2.31}$$

$$\geq -\frac{\Lambda(d(\cdot, \partial B_{R_b}(x'_b)))}{d(\cdot, \partial B_{R_b}(x'_b))} \left( n^2 \sup_{\substack{k,l=1,\dots,n \\ x \in B_{R_b}(x'_b)}} |v_{x_k x_l}(x)| + 1 \right), \tag{2.32}$$

on  $B_{R_b}(x'_b)$ . Thus, it follows that  $\tilde{L}$  is a linear elliptic operator on  $B_{R_b}(x'_b)$ , that satisfies the conditions of Theorem 2.5, provided that we consider  $\Lambda$  in Theorem 2.5 as that in (2.29)–(2.32) after multiplication by a sufficiently large constant. An application of Theorem 2.5 yields  $\partial_\nu w > 0$  at  $x_b$  and hence,

$$\partial_\nu u(x_b) > \partial_\nu v(x_b),$$

as required. □

*Remark 2.7* Note that conditions (2.27) and (2.28) ensure that:  $A_{ij}$  are locally Lipschitz continuous in  $z$  and  $\eta$  on  $\Omega$ ;  $B$  is locally lower Lipschitz in  $z$  and Lipschitz continuous in  $\eta$ ; and the associated Lipschitz and lower Lipschitz constants for  $A_{ij}$  and  $B$  can tend to  $\infty$  as  $x \rightarrow \partial\Omega$  but are constrained by the integrability condition on  $\Lambda$ . Additionally, observe that the conditions in Theorem 2.6 can be readily altered to accommodate  $d(x, \partial\Omega)$  instead of  $d(x, \partial B_{R_b}(x'_b))$ .

We now demonstrate that if the bound on the lower Lipschitz constant for  $B$  in Theorem 2.6 is relaxed to a mere local lower Lipschitz condition, then the conclusion of Theorem 2.6 does not necessarily hold.

*Example 2.8* Suppose that  $\Omega \subset \mathbb{R}^n$  and for  $x_b \in \partial\Omega$  there exists  $B_{R_b}(x'_b) \subset \Omega$  with  $x_b \in \partial B_{R_b}(x'_b)$ . Consider  $u : \bar{\Omega} \rightarrow \mathbb{R}$  given by,

$$u(x) = 0 \quad \forall x \in \bar{\Omega}, \tag{2.33}$$

and  $v : \bar{\Omega} \rightarrow \mathbb{R}$  such that:

$$v \in C^\infty(\bar{\Omega}), \tag{2.34}$$

$$v > 0 \quad \text{in } \Omega, \tag{2.35}$$

$$v(x_b) = 0 \quad \text{and} \quad \partial_\nu v(x_b) = 0. \tag{2.36}$$

Note that (2.34) implies that there exists  $M \geq 0$  such that for  $i, j = 1, \dots, n$ ,

$$|v|, |v_{x_i}|, |v_{x_i x_j}| \leq M \quad \text{for all } x \in \bar{\Omega}. \tag{2.37}$$

Now, for the quasilinear PDI in (2.4), set  $A_{ij} : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  to be

$$A_{ij}(x, z, \eta) = \delta_{ij} \quad \forall (x, z, \eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \tag{2.38}$$

for all  $i, j = 1, \dots, n$  and  $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  to be

$$B(x, z, \eta) = -\frac{z}{v(x)} \sum_{i=1}^n v_{x_i x_i}(x) \quad \forall (x, z, \eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. \tag{2.39}$$

Since the coefficients of  $A_{ij}$  define a Laplacian, it can be seen that  $A_{ij}$  satisfies the conditions of Theorem 2.6 (with, for example,  $\Lambda = 1$ ), and also, that  $Q$  is elliptic with respect to  $u$  with  $v_{x_i x_j}$  bounded on  $\Omega$ . Moreover, observe that  $B$  is independent of  $\eta$ , and

$$B(x, z_1, \eta_1) - B(x, z_2, \eta_2) = -\frac{z_1 - z_2}{v(x)} \sum_{i=1}^n v_{x_i x_i}(x) \geq -M_K(z_1 - z_2), \tag{2.40}$$

for all  $(x, z_1, \eta_1), (x, z_2, \eta_2) \in K$  such that  $z_1 \geq z_2$ , with  $K = \Omega' \times [-M, M] \times [-M, M]^n$  for any  $\Omega'$  that is a compact subset of  $\Omega$ , where via (2.35) and (2.37),

$$M_K = \frac{Mn}{\inf_{x \in \Omega'} \{v(x)\}} > 0. \tag{2.41}$$

Therefore, it follows from (2.40) and (2.41) that  $B(x, z, \eta)$  satisfies the conditions of Theorem 2.6 with the exception of the lower Lipschitz condition, which instead holds only locally on  $\Omega \times \mathbb{R} \times \mathbb{R}^n$ . It follows from (2.33)–(2.36), that  $Q[u] \geq 0$  and  $Q[v] \leq 0$  on  $\Omega$ , for  $Q$  defined by (2.38) and (2.39). Moreover, via (2.33), (2.35), and (2.36)  $u < v$  in  $\Omega$  and  $u(x_b) = v(x_b)$  for  $x_b \in \partial\Omega$ . In conclusion, although  $u, v, A_{ij}$  and  $B$  satisfy all of the conditions of Theorem 2.6 (with the exception of the lower Lipschitz condition on  $B$ , or alternatively (2.28)), via (2.36),

$$\partial_\nu u(x_b) = \partial_\nu v(x_b),$$

which violates the conclusion of Theorem 2.6.

*Remark 2.9* We note that  $u, v, A_{ij}$  and  $B$  in Example 2.8 satisfy all of the conditions of [13, Theorem 2.7.1], but violate the conclusion. This occurs since an unconstrained local lower Lipschitz constant is supposed on  $B$  with respect to  $z$  in [13, Theorem 2.7.1], which is an error. It is noteworthy that essentially the same error can be found in the statement of a BPL for classical solutions to linear parabolic PDI given in [10, p.174, Theorem 7], as illustrated in [9]. We also highlight that in both of these instances, a direct proof of the associated BPL is not given, but instead, only the main ideas of the proofs are described.

*Remark 2.10* If  $\partial_\nu v(x_b) > 0$  in Example 2.8, by considering  $K = B_R(x'_b) \times [-M, M] \times [-M, M]^n$  with  $0 < R < R_b$ , it follows from (2.34) that as  $R \rightarrow R_b$ ,

$$M_K \geq \frac{2Mn}{v_{\nu\nu}(x_b)(R_b - R)^2 + O((R - R_b)^3)} \geq \frac{Mn}{v_{\nu\nu}(x_b)d(\partial B_R(x'_b), \partial B_{R_b}(x'_b))^2}. \tag{2.42}$$

Thus, we observe that  $B$  in (2.40) satisfies the conditions of Theorem 2.6 with the exception of  $\Lambda \in L^1((0, \frac{R_b}{2}))$  in (2.28). This follows from letting  $R \rightarrow R_b$  in (2.42), which implies that  $\Lambda$  necessarily satisfies

$$\Lambda(d) \geq \frac{Mn}{v_{\nu\nu}(x_b)d} \quad \text{as } d \rightarrow 0^+.$$

Now, we highlight the necessity of the bound on  $v_{x_i x_j}$  (or  $u_{x_i x_j}$ ) in Theorem 2.6. Note that this condition is not present in [13, Theorem 2.7.1].

*Example 2.11* Let  $\Omega = (0, 1) \subset \mathbb{R}$  and  $u, v : \bar{\Omega} \rightarrow \mathbb{R}$  be given by

$$u(x) = x^{1+\alpha}, v(x) = 2x^{1+\alpha} \quad \forall x \in \bar{\Omega}, \tag{2.43}$$

with constant  $\alpha \in (0, 1)$ . It follows that  $u, v \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ ,  $v > u$  in  $\Omega$ , and for  $x_b = 0 \in \partial\Omega$ , we have  $u(x_b) = v(x_b) = 0$ . Now, consider the quasilinear operator  $Q$  with  $A : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  given by

$$A(x, z, \eta) = 1 + \frac{2}{x^{1+\alpha}} \left( \frac{3}{2} x^{1+\alpha} - z \right) \quad \forall (x, z, \eta) \in \Omega \times \mathbb{R} \times \mathbb{R} \tag{2.44}$$

with  $B = 0$  on  $\Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ . Since

$$Q[u] = A(\cdot, u, Du)u_{xx} = 2u_{xx} \geq 0, \quad Q[v] = A(\cdot, v, Dv)v_{xx} \leq 0, \tag{2.45}$$

on  $\Omega$ , it follows that  $Q$  is elliptic with respect to  $u$ , and that  $Q$  satisfies the conditions (2.27) and (2.28) in Theorem 2.6 with  $\Lambda : (0, 1] \rightarrow (0, \infty)$  given by

$$\Lambda(d) = \frac{2}{d^\alpha} \quad \forall d \in \left( 0, \frac{1}{2} \right].$$

Since  $\Lambda$  is continuous nonincreasing and  $\Lambda \in L^1((0, \frac{1}{2}))$ , it follows from (2.43)–(2.45) that  $u$  and  $v$  satisfy all of the conditions of Theorem 2.6 with the exception of  $v_{xx}$  being bounded on  $\Omega$ . However, via (2.43),

$$u_v(x_b) = v_v(x_b) = 0,$$

which violates the conclusion of Theorem 2.6.

### 3 Weak theory

In this section, we establish a comparison-type tangency principle, for  $C^1$  weak elliptic solutions to divergence structure PDI that is a correction of that stated in [13, Theorem 2.7.2]. The proof largely follows that of [13, Theorem 2.7.2] with additional details included to highlight the additional hypotheses, and missing details in the proof of [13, Theorem 2.7.2]. We also provide simple counterexamples to [13, Theorems 2.7.2 and 2.7.3].

#### 3.1 Notation and definitions

The quasilinear divergence structure PDI we consider are given by:

$$\operatorname{div}(A(\cdot, u, Du)) + B(\cdot, u, Du) \geq 0 \quad \text{on } \Omega, \tag{3.1}$$

$$\operatorname{div}(A(\cdot, v, Dv)) + B(\cdot, v, Dv) \leq 0 \quad \text{on } \Omega, \tag{3.2}$$

with  $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ . Specifically, we consider  $C^1$  weak solutions to (3.1) (and analogously (3.2)) that satisfy:  $u \in C^1(\bar{\Omega})$ ,  $A(\cdot, u, Du), B(\cdot, u, Du) \in L^1_{\text{loc}}(\Omega)$  and

$$\int_{\Omega} A(x, u(x), Du(x)) \cdot D\psi(x) \, dx \leq \int_{\Omega} B(x, u(x), Du(x))\psi(x) \, dx \tag{3.3}$$

for any test function  $\psi \in C^1(\bar{\Omega})$  such that  $\psi \geq 0$  on  $\Omega$  and  $\psi$  has compact support in  $\Omega$ . Moreover, we say that  $u$  (and analogously  $v$ ) is an elliptic solution to (3.1) if  $a_{ij}(x) = (A_i)_{\eta_j}(x, u(x), Du(x))$  satisfies the left inequality in (2.2) for all  $x \in \Omega$ . Furthermore, in this section we consider  $\Omega$  with boundary  $\partial\Omega$  that satisfies an *interior cone condition*, i.e., at each point  $x_b \in \partial\Omega$  there exists a cone of finite height in  $\Omega$  with apex  $x_b$ . We denote the interior of such a cone by  $\Omega_b$ .

Additionally, for a function  $u \in C(\bar{\Omega})$  we say that  $u$  has a finite-order zero at  $x_b \in \bar{\Omega}$  if  $u(x_b) = 0$  and there exists  $x_1 \in \mathbb{R}^n$  and  $m \in \mathbb{N}$  such that  $x_b + hx_1 \in \bar{\Omega}$  for all  $h \in (0, 1)$ , and

$$\lim_{h \rightarrow 0^+} \frac{u(x_b + hx_1)}{h^m} \neq 0. \tag{3.4}$$

Conversely we say that  $u$  has an infinite order 0 at  $x_b$  if  $u(x_b) = 0$  and there does not exist  $m \in \mathbb{N}$  such that (3.4) is satisfied.

### 3.2 A comparison-type tangency principle for weak elliptic solutions to quasilinear divergence structure PDI

**Theorem 3.1** ((Tangency Principle)) *Let  $x_b \in \partial\Omega$  satisfy the interior cone condition, and  $u, v : \bar{\Omega} \rightarrow \mathbb{R}$  be such that:  $u, v \in C^1(\bar{\Omega})$ ;  $u, v$  satisfy (3.1) and (3.2), respectively;  $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is continuous and continuously differentiable with respect to  $z$  and  $\eta$ ;  $A_z$  is uniformly bounded and  $A_\eta$  is uniformly continuous on  $\Omega_b \times [u(x_b) - M_z, u(x_b) + M_z] \times [-M_\eta, M_\eta]^n$  for some constants  $M_z, M_\eta > 0$ ;  $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies*

$$B(x, z_1, \eta_1) - B(x, z_2, \eta_2) \geq -b_z(z_1 - z_2) - b_\eta \sum_{l=1}^n |\eta_{1l} - \eta_{2l}| \tag{3.5}$$

for all  $(x, z_1, \eta_1), (x, z_2, \eta_2) \in \Omega_b \times [u(x_b) - M_z, u(x_b) + M_z] \times [-M_\eta, M_\eta]^n$  with  $z_1 \geq z_2$  for some constants  $b_z, b_\eta \geq 0$ ;  $u$  is an elliptic solution of (3.1) with respect to  $A$  in  $\Omega_b$ ;  $u < v$  in  $\Omega_b$ ; and  $u(x_b) = v(x_b)$ . Then, the zero of  $v - u$  at  $x_b$  is of finite order.

*Proof* For a contradiction, assume that  $w = v - u$  has a zero of infinite order at  $x_b \in \partial\Omega$ . Via regularity on  $w$ , it follows that  $Dw(x_b) = 0$ . Moreover, for each  $\epsilon \in (0, \min\{M_z, M_\eta/2\})$ , there exists a cone of finite height in  $\Omega$  with apex  $x_b$ , without loss of generality denoted by  $\Omega_b$ , such that  $(w(x), Dw(x)) \in (0, \epsilon] \times [-\epsilon, \epsilon]^n$  for all  $x \in \Omega_b$ , there exists a constant

$$a_z = \sup_{\substack{\Omega_b \times [-\epsilon, \epsilon] \times [-\epsilon, \epsilon]^n \\ i=1, \dots, n}} |(A_i)_z| \in [0, \infty) \tag{3.6}$$

and

$$|(A_i)_{\eta_j}(x, u(x), \eta^{(1)}) - (A_i)_{\eta_j}(x, u(x), \eta^{(2)})| < \frac{1}{2M^2} \tag{3.7}$$

for all  $(x, \eta^{(1)}), (x, \eta^{(2)}) \in \Omega_b \times [-2\epsilon, 2\epsilon]^n$  and  $i, j = 1, \dots, n$ . From (3.1) and (3.2), we have

$$\begin{aligned} & \operatorname{div}(A(x, v, Dv) - A(x, u, Du)) + (B(x, v, Dv) - B(x, u, Du)) \\ &= \operatorname{div}(\tilde{A}(x, w, Dw)) + \tilde{B}(x, w, Dw) \leq 0 \end{aligned} \tag{3.8}$$

on  $\Omega_b$ . The function  $\tilde{A} : \Omega_b \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  arises from repeated application of the mean-value theorem in (3.8), e.g.,

$$\tilde{A}_i(x, z, \eta) = (A_i)_z(x, \tilde{z}^{(i)}(x), Dv(x))z + \sum_{j=1}^n (A_i)_{\eta_j}(x, u(x), \tilde{\eta}^{(i)}(x))\eta_j \tag{3.9}$$

for all  $(x, z, \eta) \in \Omega_b \times \mathbb{R} \times \mathbb{R}^n$ ,  $\tilde{z}^{(i)} : \Omega_b \rightarrow [0, \epsilon]$  and  $\tilde{\eta}^{(i)} : \Omega_b \rightarrow [-2\epsilon, 2\epsilon]^n$  for  $i = 1, \dots, n$ . Similarly, via (3.8), we define  $\tilde{B} : \Omega_b \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  as

$$\tilde{B}(x, z, \eta) = \left( \frac{B(x, v(x), Dv(x)) - B(x, u(x), Dv(x))}{v(x) - u(x)} \right) z + \sum_{i=1}^n \check{B}_i(x)\eta_{n+1-i} \tag{3.10}$$

for all  $(x, z, \eta) \in \Omega_b \times \mathbb{R} \times \mathbb{R}^n$ , with  $\check{B}_i : \Omega_b \rightarrow \mathbb{R}$  given by

$$\check{B}_i(x) = \frac{B(x, u(x), \eta^{(i)}(x)) - B(x, u(x), \eta^{(i+1)}(x))}{Dv_{n+1-i}(x) - Du_{n+1-i}(x)} \quad \forall x \in R_i$$

and with  $\check{B}_i = 0$  on  $\Omega_b \setminus R_i$  for  $i = 1, \dots, n$ , such that

$$\mathcal{R}_i = \{x \in \Omega_b : Du_{n+1-i}(x) \neq Dv_{n+1-i}(x)\}$$

for  $i = 1, \dots, n$ , and  $\eta^{(i)} : \Omega_b \rightarrow \mathbb{R}^n$  given by

$$\eta_j^{(i)}(x) = \begin{cases} Dv_j(x), & j \leq n + 1 - i, \\ Du_j(x), & n + 2 - i \leq j, \end{cases}$$

for  $i = 1, \dots, n + 1$ . Since  $Q$  is elliptic with respect to  $u$ , it follows from (3.6)–(3.9) that

$$\begin{aligned} & \eta^T \cdot \tilde{A}(x, z, \eta) \\ &= \sum_{i=1}^n (A_i)_z(x, \tilde{z}^{(i)}(x), Dv(x))z\eta_i + \sum_{i,j=1}^n (A_i)_{\eta_j}(x, u(x), Du(x))\eta_i\eta_j \\ & \quad + \sum_{i,j=1}^n ((A_i)_{\eta_j}(x, u(x), \tilde{\eta}_j^{(i)}(x)) - (A_i)_{\eta_j}(x, u(x), Du(x)))\eta_i\eta_j \\ & \geq -na_z z|\eta| + |\eta|^2 - \frac{1}{2}|\eta|^2 \\ & \geq \frac{1}{4}|\eta|^2 - (na_z)^2 z^2 \end{aligned} \tag{3.11}$$

for  $(x, z, \eta) \in \Omega_b \times [0, \infty) \times \mathbb{R}^n$ . Additionally, via the regularity hypotheses on  $A$  and  $B$  it follows that there exists a constant  $a_\eta \geq 0$  such that

$$|\tilde{A}(x, z, \eta)| \leq a_\eta |\eta| + \sqrt{na_z} z \tag{3.12}$$

for all  $(x, z, \eta) \in \Omega_b \times [0, \infty) \times \mathbb{R}^n$ , and

$$\tilde{B}(x, z, \eta) \geq -nb_\eta |\eta| - b_z z \tag{3.13}$$



for all  $(x, z, \eta) \in \Omega_b \times [0, \infty) \times \mathbb{R}^n$ . It follows from (3.8)–(3.13) that on any  $B_r(x) \subset \Omega_b$ ,  $w$ ,  $\tilde{A}$  and  $\tilde{B}$  satisfy the conditions of Trudinger’s weak Harnack inequality [18, Theorem 1.2] with constants required in the hypotheses and conclusion, independent of the ball, i.e., there exists a constant  $C$  independent of  $B_{2r}(x) \in \Omega_b$  such that

$$\frac{1}{r^n} \int_{B_{2r}(x)} w \, dx \leq C \min_{B_r(x)} w \quad \forall B_{2r}(x) \subset \Omega_b. \tag{3.14}$$

Note that [18, Theorem 1.2] remains true if  $u > 0$  in  $\Omega$ , inequalities (1.2) hold on  $\Omega \times [0, \infty) \times E^n$ , and the second and third inequalities in (1.2) for  $\alpha = 2$  are replaced by  $p \cdot A(x, u, p) \geq a_5|p|^2 - a_2u^2$  and  $B(x, u, p) \geq -b_1|p| - b_2u$  for constants  $a_2, b_1, b_2 \geq 0$  and  $a_5 > 0$ .

Now, since  $x_b$  is the apex of the cone  $\Omega_b \subset \Omega$ , it follows that there exists a sequence of balls  $\{B_{r_k}(y_k)\}_{k \in \mathbb{N}_0}$ : that have boundaries that tangentially intersect  $\partial\Omega_b$ ; such that  $B_{r_k/3}(y_k) \subset B_{2r_{k+1}/3}(y_{k+1})$  for all  $k \in \mathbb{N}_0$ ; for which  $y_k \rightarrow x_b$  as  $k \rightarrow \infty$ ;  $r_{k+1} < r_k$  for  $k \in \mathbb{N}_0$ ; and by denoting  $\theta$  to be the half-angular opening of the cone, we can set

$$\frac{r_{k+1}}{r_k} = \frac{|y_{k+1} - x_b|}{|y_k - x_b|} = \frac{1 + (\frac{1}{3}) \sin(\theta)}{1 + (\frac{2}{3}) \sin(\theta)} = \kappa \in (0, 1) \tag{3.15}$$

for all  $k \in \mathbb{N}_0$ . It follows immediately that

$$\min_{B_{r_k/3}(y_k)} w \leq \frac{3^n}{\omega_n r_k^n} \int_{B_{r_k/3}(y_k)} w \, dx \leq \frac{3^n}{\omega_n r_{k+1}^n} \int_{B_{2r_{k+1}/3}(y_{k+1})} w \, dx \quad \forall k \in \mathbb{N}_0, \tag{3.16}$$

with  $\omega_n$  denoting the volume of a Euclidean unit ball in  $\mathbb{R}^n$ . By combining (3.14) and (3.16), we have

$$\min_{B_{r_k/3}(y_k)} w \geq L^k \min_{B_{r_0/3}(y_0)} w \quad \forall k \in \mathbb{N}, \tag{3.17}$$

with

$$L = \frac{\omega_n}{C3^n}.$$

Now, via our initial assumption,  $w$  has a zero of infinite order at  $x_b$  and hence via (3.15), for each  $m \in \mathbb{N}$  there exists a positive constant  $c$  independent of  $k$  such that

$$w(y_k) \leq c|y_k - x_b|^m = c|y_0 - x_b|^m \kappa^{mk} \quad \forall k \in \mathbb{N}_0. \tag{3.18}$$

Now, via (3.17) and (3.18), it follows that there exists a positive constant  $c$  independent of  $k$  such that

$$L^k \leq c\kappa^{mk} \quad \forall k \in \mathbb{N}_0. \tag{3.19}$$

Letting  $k \rightarrow \infty$  in (3.19) implies that

$$\kappa^m \geq L. \tag{3.20}$$

However, via (3.15)  $\kappa^m \rightarrow 0$  as  $m \rightarrow \infty$  and hence for all sufficiently large  $m$ , it follows that (3.20) yields a contradiction. Therefore, the zero of  $w$  at  $x_b$  is of finite order, as required.  $\square$

*Remark 3.2* Observe that via the bounds in (3.11)–(3.13), we have ensured that the constant  $C$  in (3.14) exists independently of the choice of ball in  $\Omega_b$ . Alternatively, using the conditions of [13, Theorem 2.7.2], although bounds analogous to (3.11)–(3.13) hold on any ball in  $\Omega_b$ , the same constant  $C$  is not necessarily valid for every ball, i.e.,  $C$  is potentially dependent on  $k$ . Consequently, in the proof of [13, Theorem 2.7.2], although Theorem [18, Theorem 1.2] can be applied to any ball in  $\Omega_b$ , as in (3.14) and (3.17), the constant  $c$  that arises, as in (3.19), is not necessarily independent of  $k$ , which is the source of the error in the proof.

*Example 3.3* Suppose that  $\Omega \subset \mathbb{R}^n$  and for  $x_b \in \partial\Omega$  satisfies an interior cone condition. Consider  $u : \bar{\Omega} \rightarrow \mathbb{R}$  and  $v : \bar{\Omega} \rightarrow \mathbb{R}$  as given in Example 2.8 such that additionally,  $v$  has a zero of infinite order at  $x_b$ , i.e.,

$$\partial_v^{(m)} v(x_b) = 0 \quad \forall m \in \mathbb{N}. \tag{3.21}$$

For the quasilinear partial differential inequalities in (3.1) and (3.2) set  $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  to be

$$A(x, z, \eta) = \eta \quad \forall (x, z, \eta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \tag{3.22}$$

with  $B : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  as in (2.39). It follows that  $A$  satisfies the conditions of Theorem 3.1, and also that  $u$  and  $v$  are elliptic solutions of (3.1) and (3.2), respectively. Via (2.40) and (2.41), observe that  $B$  is independent of  $\eta$ , and satisfies the conditions of Theorem 3.1 with the exception of the lower Lipschitz condition in (3.5), which instead holds locally on  $\Omega \times \mathbb{R} \times \mathbb{R}^n$ . Moreover, via (2.33) and (2.35), it follows that  $u < v$  in  $\Omega$ . In conclusion, although  $\Omega, u, v, A$  and  $B$  satisfy all of the conditions of Theorem 3.1 (with the exception of the lower Lipschitz condition on  $B$ ), via (3.21),

$$\partial_v^{(m)} u(x_b) = \partial_v^{(m)} v(x_b) \quad \forall m \in \mathbb{N},$$

which violates the conclusion of Theorem 3.1. We also note here that the conditions on  $A_\eta$  and  $A_z$  in Theorem 3.1 cannot be relaxed to those in [13, Theorem 2.7.2], which can be observed via similarly constructed counterexamples.

*Remark 3.4* We note that the erroneous tangency principle stated in [13, Theorem 2.7.2] was intended to be a relaxation of that in [17] to allow for weaker constraints on the nonlinearities  $A$  and  $B$  in (3.1) and (3.2) as  $x \rightarrow \partial\Omega$ . However, the constraint on  $B$ , appears to arise from the very same condition on  $B$  in the erroneous BPL stated in [13, Theorem 2.7.1].

To conclude the section, we note that in [13, Theorem 2.7.3], a strong maximum principle and tangency principle is stated with the regularity conditions on  $u$  and  $v$  in [13, Theorem 2.7.2] relaxed to  $u, v \in C(\bar{\Omega})$  but so that  $u$  and  $v$  also possess strong derivatives

in  $L^2_{loc}(\Omega)$ . To compensate for these relaxed regularity conditions on  $u$  and  $v$ , stricter regularity conditions are imposed on  $A$  and  $B$  that we now demonstrate, are insufficient to establish the conclusion. This establishes that all three theorems in [13, Sect. 2.7] are erroneous.

*Example 3.5* For  $\epsilon \in (0, 1)$  consider  $\Omega = B_1(0) \setminus \overline{B_{1-\epsilon}(0)}$  with  $u, A = A(\eta)$  and  $B = B(x, z)$  as in Example 3.3. Here, consider  $v$  given by

$$v(x) = \begin{cases} e^{-1/(1-|x|^2)}, & x \in B_1(0) \setminus B_{1-\epsilon}(0), \\ 0, & x \in \partial B_1(0). \end{cases} \tag{3.23}$$

Observe that  $u, v \in C^\infty(\bar{\Omega})$  and that the zero of  $v - u$  on  $\partial B_1(0)$  is of infinite order. Additionally, note that  $A$  is locally bounded on  $\mathbb{R}^n$  and  $B$  is locally bounded and locally lower Lipschitz on  $\Omega \times \mathbb{R}$ . Furthermore, for  $i = 1, \dots, n$ , we have

$$v_{x_i x_i}(x) = \frac{(4x_i^2 - 8x_i^2(1 - |x|^2) - 2(1 - |x|^2)^2)v(x)}{(1 - |x|^2)^4} \quad \forall x \in \Omega. \tag{3.24}$$

Via (3.23) and (3.24), for sufficiently small  $\epsilon > 0$ , it follows that

$$\sum_{i=1}^n v_{x_i x_i} > 0 \quad \text{on } \Omega.$$

For such  $\epsilon > 0$ , it follows that  $B(x, z)$ , as given by (2.39), is nonincreasing in  $z$  on  $\Omega$ . Therefore, although  $\Omega, u, v, A$ , and  $B$  satisfy the conditions of [13, Theorem 2.7.3], the conclusion that the zero of  $v - u$  on  $\partial B_1(0)$  is of finite order is violated.

### 4 Discussion

In Theorem 2.3, the outward ball condition on  $\mathcal{S}$  in Definition 2.1 can be generalized to an outward  $C^{1,Dim}$  condition, provided that the conditions on the coefficients of  $L$  are appropriately constrained. This can be achieved with more restrictive conditions in the statement of Theorem 2.3, by replacing the function in Lemma 2.2 with a suitable alternative (for instance, the regularized distance functions constructed in [5, Sects. 1 and 2]).

It is also noteworthy that although the singular sets for elliptic PDI considered in Sect. 2 satisfy a 1-porous condition (see [20, Definition 2.1]) the review articles [19, 20] do not indicate that a link has been established between porous sets and singular sets for elliptic PDI. Although it is not immediately clear if strengthening the link between porosity and singular sets for elliptic PDI will be a fruitful pursuit, it is potentially worthy of further consideration.

In relation to Theorem 2.6, a fully nonlinear version can be established without substantial additional technicality (see, for example [4, 13]). Moreover, the condition bounding  $v_{x_i x_j}$  can be relaxed provided that the right-hand side of (2.29)–(2.32) can be expressed (for instance, by further constraining the growth of  $\Lambda(d)$  as  $d \rightarrow 0$ ) so that Theorem 2.5 can be applied.

With regard to Theorem 3.1, we note that allowable blow-up in  $A$  and  $B$  as  $x \rightarrow x_b$  can be accommodated by using the more general integrability conditions on coefficients in Theorem [18, Theorem 1.2], i.e., by using Theorem [18, Theorem 5.1]. Also, complementary

results are contained in [14–16] where BPL for quasilinear elliptic PDI are established under more regular domain and PDI constraints, but that guarantee the existence of nonzero (first) outward directional derivatives. It is also pertinent to note that in [16] the author highlights two further distinct incorrect statements of BPL from those highlighted here and in [9].

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#### Abbreviations

The following abbreviations are used throughout: BPL—boundary point lemma(s); CSMP—classical strong maximum principle(s); and PDI—partial differential inequality(ies).

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#### Ethics approval and consent to participate

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The author has no competing interests to declare that are relevant to the content of this article.

#### Authors' contributions

The main idea of this paper was proposed by JM. JM prepared the manuscript initially and performed all the steps of the proofs in this research. JM read and approved the final manuscript.

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