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Finite-size scaling, phase coexistence, and algorithms for the random cluster model on random graphs

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Abstract

For $\Delta \geq 5$ and q large as a function of Δ, we give a detailed picture of the phase transition of the random cluster model on random Δ-regular graphs. In particular, we determine the limiting distribution of the weights of the ordered and disordered phases at criticality and prove exponential decay of correlations and central limit theorems away from criticality. Our techniques are based on using polymer models and the cluster expansion to control deviations from the ordered and disordered ground states. These techniques also yield efficient approximate counting and sampling algorithms for the Potts and random cluster models on random Δ-regular graphs at *all* temperatures when q is large. This includes the critical temperature at which it is known the Glauber and Swendsen-Wang dynamics for the Potts model mix slowly. We further prove new slow-mixing results for Markov chains, most notably that the Swendsen-Wang dynamics mix exponentially slowly throughout an open interval containing the critical temperature. This was previously only known at the critical temperature.

Many of our results apply more generally to Δ-regular graphs satisfying a small-set expansion condition.

1 Introduction

The random cluster model on a graph $G = (V, E)$ with parameters $q, \beta \geq 0$ is the measure μ_G on $\{0,1\}^E$ with

$$\mu_G(A) \coloneqq \frac{q^{c(A)}(e^\beta - 1)^{|A|}}{Z_G(q,\beta)}\,, \qquad Z_G(q,\beta) \coloneqq \sum_{A \subseteq E} q^{c(A)}(e^\beta - 1)^{|A|}\,, \tag{1}$$

where $c(A)$ is the number of connected components of (V, A). Setting $p \coloneqq 1 - e^{-\beta}$ gives a description of μ_G as a tilted bond percolation model with edge probability $p \in [0,1]$:

$$Z_G(q, -\log(1-p)) = \sum_{A \subseteq E} q^{c(A)} \left(\frac{p}{1-p}\right)^{|A|} = \frac{1}{(1-p)^{|E|}} \sum_{A \subseteq E} q^{c(A)} p^{|A|} (1-p)^{|E \setminus A|}\,.$$

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The random cluster model is a generalization of the *q-color ferromagnetic Potts model*, which, for q a positive integer, is the probability distribution on $[q]^V$ defined by

$$\mu_G^{\text{Potts}}(\sigma) := \frac{1}{Z_G^{\text{Potts}}(q,\beta)} \prod_{\{u,v\}\in E} e^{\beta \mathbf{1}_{\sigma_u=\sigma_v}}, \qquad Z_G^{\text{Potts}}(q,\beta) := \sum_{\sigma\in[q]^V} \prod_{\{u,v\}\in E} e^{\beta \mathbf{1}_{\sigma_u=\sigma_v}}.$$

In particular, the case $q = 2$ case is the Ising model. The connection between the Potts and random cluster models is that for integer q,

$$Z_G(q,\beta) = Z_G^{\text{Potts}}(q,\beta),$$

and moreover, there is a natural coupling of the measures μ_G and μ_G^{Potts}. The coupling is as follows. Given an edge set A distributed according to the random cluster measure μ_G, independently assign a uniformly chosen color from $[q]$ to each connected component of (V, A) to obtain a coloring $\sigma \in [q]^V$. The distribution of σ is μ_G^{Potts} [26]. For an introduction to the random cluster model, see [38].

This paper concerns a relatively complete set of results about the probabilistic and algorithmic behavior of the large-q random cluster model on random Δ-regular graphs. In particular, we obtain a detailed description of the phase diagram; establish strong correlation decay and finite-size scaling statements; prove central limit theorems off criticality; obtain efficient approximate counting and sampling algorithms at *all* temperatures $\beta > 0$; and establish slow-mixing of standard Markov chains in a neighborhood of the critical temperature β_c. Many of our results apply more generally to Δ-regular graphs satisfying a small-set expansion condition, see Section 1.1.1. We will shortly give precise statements of these results, but before doing this we briefly give some context and an outline of our methods.

Our techniques are based on polymer models and the cluster expansion, tools developed to investigate the phase diagrams of statistical physics models on lattices [39, 62, 50, 52]. In particular, we adapt to the random graph and expander setting the idea from [52] of analyzing the Potts model phase transition by controlling the ordered and disordered phases of the random cluster model via separate convergent cluster expansions. The key to this approach is obtaining convergent ordered and disordered expansions for parameter regimes that overlap — in particular, the expansions both converge at the critical temperature. These expansions give us strong control on the dominant and sub-dominant contributions to the partition function, and enable us to prove our probabilistic and algorithmic results. While the use of expansion methods to obtain probabilistic results is well-known, algorithmic implications are more recent [43, 46, 53, 29, 16].

The most crucial technical aspect of this paper is thus the development of convergent expansions, and our main innovation here is a polymer model that applies to the ordered phase on expander graphs. This relies on an inductive construction of polymers that circumvents the difficulty created by the non-local weight $q^{c(A)}$ present in the random cluster model. In prior work studying the random cluster model on $\mathbb{Z}^d$ via Pirogov–Sinai theory, this non-local weight was handled by using notions of boundaries arising from the topology of Euclidean space. Our inductive construction defines a notion of 'boundary' that encodes the connected components of A, and hence the computation of $q^{c(A)}$, for typical edge sets A. The success of this encoding, and its usefulness for deriving a convergent expansion, relies crucially on (i) expansion properties of the underlying graph and (ii) the fact that in the ordered phase, typical edge sets A consist of a large fraction of all edges. The second point is one aspect

of the fact that the phase transition of the random cluster model is first order when q is large. This leads to the resulting boundaries being geometrically small, which is important for obtaining a convergent expansion.

The techniques typically used to understand statistical physics models on random graphs are very different from ours: typical methods include the first and second moment methods, the cavity method, and the interpolation method [57, 59, 65, 23, 18, 19]. Using polymer models and the cluster expansion allows us to obtain results that are not, to date, accessible via the aforementioned techniques. The strengths of our approach include: the ability to make statements about every vertex or pair of vertices in a graph (e.g., Theorem 1, part (6)); a precise characterization of phase coexistence (Theorem 2); and control of both the ordered and disordered contributions to the partition function at and away from criticality which leads to strong algorithmic consequences (Theorems 4 and 6). On the other hand, our approach is inherently perturbative in that it requires q to be large, and it does not as readily yield explicit formulae for critical thresholds.

1.1 The phase diagram of the random cluster model on random graphs

On $\mathbb{Z}^d$, meaning on sequences of graphs $G_n \uparrow \mathbb{Z}^d$ in an appropriate sense, a great deal is known about the random cluster model, see [24] and references therein. When $q \geq 1$ these models are known to undergo a phase transition at a critical temperature $\beta_c(q)$ from a *disordered state ($\beta < \beta_c$)* to an *ordered state ($\beta > \beta_c$)*. The nature of this transition depends on the value of q and the dimension d. For the present paper the most relevant results concern when q is large. In this case configurations in the disordered state typically consist of relatively few edges, while in the ordered state typical configurations have relatively few missing edges. Moreover, exactly at β_c typical configurations look like either an ordered or a disordered configuration. In physical parlance, the phase transition is *first-order* [52]. Finer results concerning finite-size scaling are also known [12]. Roughly speaking, these results concern how $|V(G_n)|^{-1} \log Z_{G_n}$ differs from $\lim_{n\to\infty} |V(G_n)|^{-1} \log Z_{G_n}$, i.e., the corrections to the leading order behavior of $\log Z_{G_n}$ as $G_n \uparrow \mathbb{Z}^d$. Below we will identify a first-order phase transition for the random cluster model on random regular graphs (Theorem 1) and determine the finite-size scaling of the (random) log partition function (Theorem 3).

1.1.1 Expansion profiles

To state the class of graphs to which our results apply, we need a refined notion of edge expansion. The *expansion profile* of a Δ-regular graph $G = (V, E)$ is

$$\phi_G(\alpha) \coloneqq \min_{S \subset V, |S| \leq \alpha |V|} \frac{|E(S, S^c)|}{\Delta |S|}, \qquad \alpha \in (0, 1/2], \tag{2}$$

where $E(S, S^c) \subset E$ is the set of edges with one vertex in S, one in S^c. For $\Delta \in \{3, 4, \dots\}$ and $\delta \in (0, 1/2)$ we will be interested in the family $\mathcal{G}_{\Delta,\delta}$ of Δ-regular graphs that satisfy:

1. $\phi_G(1/2) \geq 1/10$,
2. $\phi_G(\delta) \geq 5/9$.

We note that the constants $1/10$ and $5/9$ are somewhat arbitrary; what we use in our proofs is that they are greater than 0 and $1/2$, respectively.

1.1.2 Locally tree-like graphs and local convergence of probability measures

Given a graph G, let $B_T(v)$ denote the depth-T neighborhood of a vertex $v \in V(G)$. A sequence of graphs G_n is *locally tree-like* if for every $T > 0$, with probability tending to one as $n \to \infty$ over the choice of a uniformly random vertex v from G_n, $B_T(v)$ is a tree.

Recall that *random cluster measures on the infinite Δ-regular tree* $\mathbb{T}_\Delta$ can be defined by taking weak limits of measures on finite trees with boundary conditions. Two random cluster measures μ^{free} and μ^{wire} on $\mathbb{T}_\Delta$ are of particular importance: they are respectively obtained by taking weak limits with free boundary conditions and with wired boundary conditions, i.e., all leaves 'wired' into one connected component. See [38, Chapter 10] for more details, including a proof that these weak limits exist and are unique. The measure μ^{free} is particularly simple as it is an independent edge percolation measure.

In what follows we adopt the convention that the index of a graph sequence denotes the number of vertices in the graph: $|V(G_n)| = n$. We further assume n is increasing, but not necessarily through consecutive integers, so as to ensure $n\Delta/2$ is an integer. Limits as $n \to \infty$ are understood in this sense.

For a sequence of Δ-regular graphs G_n we say a corresponding sequence μ_n of probability measures on $\{0,1\}^{E(G_n)}$ *converges locally* to a random cluster measure μ_∞ on the infinite Δ-regular tree $\mathbb{T}_\Delta$, denoted $\mu_n \xrightarrow{loc} \mu_\infty$, if for every $\varepsilon, T > 0$ and n sufficiently large, with probability at least $1 - \varepsilon$ over the choice of a random vertex v from G_n, the distribution of μ_n restricted to $B_T(v)$ is within ε total variation distance of the distribution of μ_∞ restricted to the depth-T neighborhood of the root. See [59, 65] for examples and more details of this notion of convergence.

1.1.3 Main probabilistic results and related literature

The statements of our results require some notation. Fix Δ, δ in the definition of $G_{\Delta,\delta}$ and let $\eta := \min\{1/100, \delta/5\}$. Given a graph $G_n = (V, E)$ on n vertices, let $\Omega_n = \{0,1\}^E$ and let

$$\begin{aligned}\Omega_{\text{dis}} &:= \{A \in \Omega_n : |A| \le \eta|E|\}\\ \Omega_{\text{ord}} &:= \{A \in \Omega_n : |A| \ge (1-\eta)|E|\}\\ \Omega_{\text{err}} &:= \Omega_n \setminus (\Omega_{\text{dis}} \cup \Omega_{\text{ord}})\end{aligned}$$

so that $\Omega_n = \Omega_{\text{dis}} \sqcup \Omega_{\text{ord}} \sqcup \Omega_{\text{err}}$. We write μ_n for the random cluster measure on G_n.

Recall that a sequence of probability measures μ_n on Ω_n has *exponential decay of correlations with rate $\epsilon > 0$* if there exists a $C > 0$ such that

$$|\mu_n(e,f) - \mu_n(e)\mu_n(f)| \le Ce^{-\epsilon \cdot \text{dist}_{G_n}(e,f)}, \qquad \text{for all } e, f \in E(G_n),$$

where $\text{dist}_{G_n}(\cdot,\cdot)$ is the graph distance on G_n and we have used the customary abuse of notation $\mu_n(e) = \mu_n(\{A \subset E : e \in A\})$ and similarly for other marginals.

Finally with $\mathbf{A}$ denoting the random edge subset drawn according to a random cluster measure, given a sequence of random cluster measures μ_n we say $|\mathbf{A}|$ obeys a central limit theorem under μ_n if for each $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{|\mathbf{A}| - \mathbb{E}_{\mu_n}|\mathbf{A}|}{\sqrt{\text{var}_{\mu_n}(|\mathbf{A}|)}} \le t\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx\,.$$

Theorem 1. *Suppose $\Delta \geq 5, \delta > 0$. For $q = q(\Delta, \delta)$ sufficiently large there exists a $\beta_c(q, \Delta)$ so that the following holds for every sequence $G_n \in \mathcal{G}_{\Delta,\delta}$ of locally tree-like graphs.*

1. *The limit $\lim_{n\to\infty} n^{-1} \log Z_{G_n}$ exists and is an analytic function of β on $(0, \infty) \setminus \{\beta_c\}$. For $\beta \leq \beta_c$ the limit equals $f_{\mathrm{dis}} = \log q + \frac{\Delta}{2} \log\left(1 + \frac{e^\beta - 1}{q}\right)$, while for $\beta \geq \beta_c$ the limiting value is given by a function $f_{\mathrm{ord}}(q, \Delta, \beta)$ defined in Section 3.*
2. *For $\beta < \beta_c$, $\limsup_{n\to\infty} n^{-1} \log \mu_n(\Omega_n \setminus \Omega_{\mathrm{dis}}) < 0$.*
3. *For $\beta > \beta_c$, $\limsup_{n\to\infty} n^{-1} \log \mu_n(\Omega_n \setminus \Omega_{\mathrm{ord}}) < 0$.*
4. *For $\beta < \beta_c$, $\mu_n \xrightarrow{loc} \mu^{\mathrm{free}}$ as $n \to \infty$.*
5. *For $\beta > \beta_c$, $\mu_n \xrightarrow{loc} \mu^{\mathrm{wire}}$ as $n \to \infty$.*
6. *For $\beta \neq \beta_c$, μ_n exhibits exponential decay of correlations.*
7. *For $\beta \neq \beta_c$, $|\mathbf{A}|$ obeys a central limit theorem under μ_n.*
8. *For all $\beta \geq 0$, $\mu_n(\Omega_{\mathrm{err}}) = O(e^{-n})$.*

Theorem 1 gives a rather complete description of the phase transition and probabilistic properties of the random cluster model on locally tree-like graphs in $\mathcal{G}_{\Delta,\delta}$. At all temperatures, all but an exponentially small fraction of the measure is on configurations with at most an η- or at least a $(1-\eta)$-fraction of all edges. There is a unique phase transition at β_c, Ω_{dis} has all but an exponentially small fraction of the measure for $\beta < \beta_c$, and Ω_{ord} has all but an exponentially small fraction of the measure for $\beta > \beta_c$. Correlations decay exponentially at $\beta \neq \beta_c$. In fact, as the proof will show, we can make stronger statements about correlation decay conditional on Ω_{dis} or Ω_{ord}; see also Lemma 18 below. As part of our proof we determine the critical point asymptotically in q: $\beta_c(q, \Delta) = (1 + o_q(1)) \frac{2 \log q}{\Delta}$, see Section 3.

For the q-color Potts model, some of these results were known previously, and without the restriction that q is large. Dembo, Montanari, and Sun [23] proved that for *any* sequence G_n of 2Δ-regular locally tree-like graphs the limit $\frac{1}{n} \log Z_{G_n}$ exists for all β and equals the replica-symmetric Bethe formula for the free energy, given implicitly by a variational formula. For random Δ-regular graphs Galanis, Štefankovič, Vigoda, and Yang [32] established a detailed picture of the phase transition, and they determined β_c explicitly:

$$\beta_c(q, \Delta) = \log \frac{q-2}{(q-1)^{1-2/\Delta} - 1} \tag{3}$$

for $q \geq 3$ and $\Delta \geq 3$ integers. They also proved versions of Theorem 1 parts (2), (3), and (7) for the Potts model with slightly different definitions of Ω_{ord} and Ω_{dis}.[1] Taken together with the results of [23] this implies the formula (3) for β_c for integral $q \geq 3$ holds for all sequences of Δ-regular locally tree-like graphs. For the case $q = 2$ of the Ising model, Dembo and Montanari [22] and Montanari, Mossel, and Sly [59] proved local convergence results for locally tree-like graphs. See also [65] for further results on general 2-spin models on locally tree-like graphs. Giardinà, Giberti, van der Hofstad, and Prioriello [35] proved a central limit

[1] The definitions of Ω_{ord} and Ω_{dis} in [32] specify the number of vertices receiving each of the q-colors.

theorem for the magnetization of Ising model in random regular graphs in the uniqueness regime. To the best of our knowledge Theorem 1 gives the first central limit theorem in a supercritical phase of a spin model on a sparse random graph.

It is important to note that β_c in Theorem 1 is *not* the Gibbs uniqueness threshold β_u of the random cluster model on the infinite tree. Instead it is the 'order-disorder threshold' in the terminology of [32]. In particular, $\beta_u < \beta_c$. We also remark that there is another uniqueness threshold $\beta_u^\star > \beta_c$ conjectured by Häggström [42], but this conjecture concerns a class of Gibbs measures that does not include μ^{free}.

Note that the correlation decay given in part (6) of Theorem 1 is very strong compared to decorrelation statements for random graphs outside the range of tree uniqueness obtained by other methods. The statements in, e.g. [18, 19], assert that the correlation between two randomly chosen vertices in the graph tends to 0 with high probability, while the correlation decay property in part (6) holds for *every* pair of vertices, and moreover the decay is exponential in the distance.

Next we turn our attention more specifically to random Δ-regular graphs, i.e., when G_n is chosen uniformly at random from the set of all Δ-regular graphs on n vertices. Recall we implicitly assume $n\Delta/2$ is an integer. As alluded to above, Theorems 1 and 6 apply to the random Δ-regular graph when $\Delta \geq 5$: in Proposition 37 below we cite results showing that there is a $\delta > 0$ such that realizations of the random graph belong to $\mathcal{G}_{\Delta,\delta}$ with high probability. Here and in what follows, we say a property P_n of graphs on n vertices holds with high probability if $\mathbb{P}[P_n] = 1 - o(1)$ as $n \to \infty$.

Theorem 1 primarily concerned the behavior at $\beta \neq \beta_c$, but it is also interesting to investigate the behavior precisely at β_c. For the Potts model, Galanis, Štefankovič, Vigoda, and Yang [32] showed that with high probability over the choice of G_n, at $\beta = \beta_c$ both $\mu_n(\Omega_{\text{ord}}) \geq n^{-c}$ and $\mu_n(\Omega_{\text{dis}}) \geq n^{-c}$ for some constant $c > 0$, and also that $\mu_n(\Omega \backslash (\Omega_{\text{dis}} \cup \Omega_{\text{ord}}))$ is exponentially small.[2] This is a logarithmic-scale phase coexistence result, and using this, they proved that the Swendsen–Wang dynamics mix slowly at criticality. Our next result gives a complete phase coexistence result for the random cluster model, and hence also the Potts model, on the random Δ-regular graph for $\Delta \geq 5$ and q large by determining precisely the limiting distribution of $\mu_n(\Omega_{\text{dis}})$ and $\mu_n(\Omega_{\text{ord}})$ at criticality.

Theorem 2. *For $\Delta \geq 5$ and $q = q(\Delta)$ large enough, there is a non-constant, positive random variable Q so that for the random cluster model on the random Δ-regular graph at $\beta = \beta_c$:*

1. *The random variable $\mu_n(\Omega_{\text{dis}})$ converges in distribution to $1/(Q+1)$ and $\mu_n(\Omega_{\text{ord}})$ converges in distribution to $Q/(Q+1)$ as $n \to \infty$*
2. *The random cluster measure μ_n conditioned on Ω_{dis} and on Ω_{ord} converges locally to μ^{free} and μ^{wire} respectively.*
3. *$Q/q \to 1$ in probability as $q \to \infty$.*

We will also prove a stronger form of Theorem 2 part (2), showing that this holds for all locally tree-like graphs in $\mathcal{G}_{\Delta,\delta}$ for β in an interval around β_c, see Proposition 33.

Theorem 2 is derived via the following result, in which f_{dis} and f_{ord} are the functions from Theorem 1 part (1). In particular, the result will be used to characterize the distribution

[2] In fact [32] uses slightly different definitions of Ω_{ord} and Ω_{dis}, as was mentioned earlier.

of the random variable Q from Theorem 2. The functions α_k^{dis} and α_k^{ord} in the theorem statement depend on q, Δ, β and are defined in Section 6.

Theorem 3. *Fix $\Delta \geq 5$. Let $Y_1, Y_2, \ldots$ be a sequence of independent Poisson random variables where Y_k has mean $(\Delta-1)^k/(2k)$. For the random Δ-regular graph G_n and $q = q(\Delta)$ large enough,*

1. *For $\beta < \beta_c$, $\log Z_{G_n} - n f_{\mathrm{dis}}$ converges in distribution to W^{dis} given by the almost surely absolutely convergent series*
$$W^{\mathrm{dis}} := \sum_{k \geq 3} \alpha_k^{\mathrm{dis}} Y_k \,.$$

2. *For $\beta > \beta_c$, $\log Z_{G_n} - n f_{\mathrm{ord}}$ converges in distribution to W^{ord} given by the almost surely absolutely convergent series*
$$W^{\mathrm{ord}} := \log q + \sum_{k \geq 3} \alpha_k^{\mathrm{ord}} Y_k \,.$$

3. *For $\beta = \beta_c$,*
$$\frac{Z_{G_n}}{\exp\left(n f_{\mathrm{dis}}(\beta_c)\right)} \longrightarrow \exp(W^{\mathrm{dis}}) + \exp(W^{\mathrm{ord}})$$
in probability as $n \to \infty$.

The random variable Q in Theorem 2 is $\exp(W^{\mathrm{ord}} - W^{\mathrm{dis}})$ with $\beta = \beta_c$. Theorem 3 can be viewed as a kind of finite-size scaling result, as the random variables W^{dis} and W^{ord} capture the deviations in Z_{G_n} from the bulk tree-like behavior. Theorem 2 is a comparison of the size of these corrections for the ordered and disordered contributions to the partition function.

The Poisson random variables Y_k in Theorem 3 correspond to the limiting distribution of the number of cycles of length k in the random regular graph. Similar Poisson random variables arise in analogous results for a class of random constraint satisfaction problems (including random graph coloring) in the replica symmetric regime that have been obtained by combining the small subgraph conditioning method with the second-moment method or rigorous implementations of the cavity method [18, 63, 19]. See also [60, 17, 55] for more on finite-size effects in spin models on random graphs and corrections to the Bethe formula due to short cycles.

Remark 1. *For integer q, the results of Theorems 1 and 2 can be immediately transferred to the Potts model via the Edwards-Sokal coupling. In particular this means that the function $\beta_c(q, \Delta)$ in Theorem 1 must agree with (3) for integer q.*

Remark 2. *The lower bound $\Delta \geq 5$ in Theorems 1 and 2 and Theorem 4 below facilitates some arguments involving small-set expansion. We believe our results could be extended to $\Delta = 3, 4$ with a more delicate analysis.*

While we are able to obtain a much more complete description of the phase diagram of the random cluster and Potts models on random Δ-regular graphs than in previous works, we emphasize that our techniques are inherently perturbative, i.e., rely on taking q large. In particular, our arguments require $q \geq \Delta^{C\Delta}$ for some fixed, but large, C. It would be very interesting to extend these results to all $q > 2$.

1.2 Approximate counting and sampling

One motivation for this paper is to investigate the relationship between phase transitions and computational complexity. The family of graphs most central to the interplay between phase transitions and algorithms are arguably random graphs. Random graphs provide candidate hard instances for several important NP-hard problems like max independent set, q-coloring, and MAX-CUT. Explanations for the computational hardness of these instances are given by structural properties of the relevant statistical physics models (hard-core model, anti-ferromagnetic Ising and Potts models), including *replica symmetry breaking* of the solution space as predicted by the cavity method from statistical physics [51, 1, 58]. On the other hand, another class of models, including the hard-core model on random bipartite graphs and the ferromagnetic Ising and Potts models, do not exhibit replica symmetry breaking: they are replica symmetric over the entire range of parameters [65, 23]. Nonetheless these models still play an important role in computational complexity: they are used as gadgets in hardness reductions for approximate counting and sampling [64, 27, 65, 30, 32].

In this paper we investigate approximate counting and sampling problems in replica symmetric models on random graph through the lens of the q-color ferromagnetic Potts and random cluster models. These models exhibit a first-order 'disorder/order' phase transition (proved for the Potts model in [32], and for the random cluster model in this paper), and this phase transition has been used in the construction of gadgets to show #BIS-hardness of sampling from the Potts model on bounded degree graphs [32]. But are these instances computationally hard themselves? For the case of large q, we establish that the answer is 'no': there are efficient approximate sampling and counting algorithms for the Potts and random cluster models at *all* temperatures. Very few all-temperature algorithms for statistical physics models on random regular graphs are known (examples include the special cases of the Ising and monomer-dimer models for which the problems are tractable on all graphs [48, 49]). To the best of our knowledge this is the first such result for a problem where approximate counting is NP- or #BIS-hard in the worst case (in this case, #BIS-hard). Our counting algorithms are deterministic, and rely crucially on exploiting the first-order phase transition established in Theorem 1.

There are two main computational problems associated to statistical physics models like the random cluster and Potts models: the *counting problem* of computing the partition function Z, and the *sampling problem* of outputting a random configuration distributed as μ. In general these problems are #P-hard even for restricted classes of graphs and parameter settings. As a result, current research is focused on finding efficient *approximate* counting and sampling algorithms. We say $\hat{Z}$ is an *ε-relative approximation to Z* if $e^{-\varepsilon}\hat{Z} \leq Z \leq e^{\varepsilon}\hat{Z}$. A *fully polynomial-time approximation scheme* (FPTAS) is an algorithm that, given a graph G and any $\varepsilon > 0$, outputs an ε-relative approximation to Z_G and runs in time polynomial in $1/\varepsilon$ and $|V(G)|$. A *polynomial-time sampling algorithm* is a randomized algorithm that, given a graph G and any $\varepsilon > 0$, outputs a configuration A with distribution $\hat{\mu}$ in time polynomial in $|V(G)|$ and $1/\varepsilon$ such that $\|\mu_G - \hat{\mu}\|_{TV} \leq \varepsilon$.

There is an extensive literature on approximate counting and sampling from the Potts and random cluster models which we discuss below in Section 1.2.1. First, however, we state our algorithmic results. Recall the class of graphs $\mathcal{G}_{\Delta,\delta}$ from Section 1.1.1.

Theorem 4. *For every $\Delta \geq 5$, $\delta > 0$, and q large enough as a function of Δ, δ, there is an FPTAS and a polynomial-time sampling algorithm for the q-color Potts and random cluster*

models at all inverse temperatures $\beta \geq 0$ over the class $\mathcal{G}_{\Delta,\delta}$.

Corollary 5. *For $\Delta \geq 5$ and $q = q(\Delta)$ large enough, with high probability over the random Δ-regular graph, there is a FPTAS and polynomial-time sampling algorithm for the q-color Potts and random cluster models at all temperatures.*

In particular, there is an algorithm that in polynomial-time decides to accept or reject a random n-vertex, Δ-regular graph G. The acceptance condition is simply that $G \in \mathcal{G}_{\Delta,\delta}$, see Lemma 37 below. For accepted graphs, the algorithm of Theorem 4 can be used.

Our methods also allow us to obtain negative algorithmic results, i.e., to establish exponentially slow mixing of some well-known Markov chains. Precise definitions of the Markov chains appearing in the next theorem, and of mixing times, can be found in Section 4.

The Swendsen–Wang dynamics [66] are non-local dynamics for the Potts model devised to circumvent the problem of phase coexistence by allowing re-coloring of many vertices in a single step of the chain. On the lattice $(\mathbb{Z}/n\mathbb{Z})^d$ (with q sufficiently large) the Swendsen–Wang dynamics are expected to be fast except at criticality, where the mixing time is $\exp(\Omega(n^{d-1}))$ [11, 33]. On the other hand, for the mean-field model (i.e., on the complete graph) the mixing time is exponentially slow in an entire interval around β_c [37, 5, 31, 34]. It has been conjectured that the Swendsen–Wang dynamics for random regular graphs exhibit mean-field behavior, mixing exponentially exponentially slowly for $q > 2$ and β in the entire interval $(\beta_u, \beta_u^\star)$. See [4] for a discussion. The next theorem takes a step towards confirming that the mean-field picture is correct for random regular graphs by proving slow mixing in an interval. Previously slow mixing was only known at criticality, consistent with both lattice and mean-field-type behavior [32, 4].

Theorem 6. *For all $\Delta \geq 5$ and $q = q(\Delta)$ large enough, there is an interval (β_m, β_M) containing β_c so that for any sequence of locally tree-like graphs $G_n \in \mathcal{G}_{\Delta,\delta}$, the mixing times of the random cluster Glauber dynamics and the Swendsen–Wang dynamics[3] are $e^{\Omega(n)}$.*

As remarked above, slow mixing was previously only known at the critical point $\beta = \beta_c$ on random Δ-regular graphs for $q \geq 2\Delta/\log\Delta$ [32]. Our ability to prove Theorem 6 is due to the fact that our methods give us detailed information about the relative probabilities of ordered and disordered configurations in an interval around β_c.

1.2.1 Related algorithmic work

For the special case of the Ising model ($q = 2$), Jerrum and Sinclair [49] gave a fully polynomial-time randomized approximation scheme (FPRAS) for all graphs and all inverse temperatures $\beta \geq 0$. See also [40], which shows that the $q = 2$ random cluster dynamics are rapidly mixing on all graphs. When $q = 1$ the random cluster model is Bernoulli bond percolation, and algorithmic tasks are trivial. The existence of efficient algorithms at all temperatures and on all graphs appears to be a rather special property, though. For $q \notin \{1, 2\}$ most positive algorithmic results to date have been restricted to the high-temperature regime (β small) or, when q is integral, the low-temperature regime (β large).

The most relevant results for the present paper are that, for the case of random Δ-regular graphs, Blanca, Galanis, Goldberg, Štefankovič, Vigoda, and Yang [3] give an efficient

[3]For non-integer q we consider the Chayes-Machta dynamics [15], a generalization of Swendsen–Wang to non-integer q. The results also apply to the Potts Glauber dynamics.

sampling algorithm for the Potts and random cluster models when the temperature is above the uniqueness threshold of the infinite Δ-regular tree, i.e., $\beta < \beta_u(\mathbb{T}_\Delta)$; Blanca and Gheissari then showed $O(n \log n)$ mixing of the random cluster dynamics for $\beta < \beta_u$ [4]. In a more general setting, Bordewich, Greenhill, and Patel showed the Glauber dynamics for the Potts model on graphs of maximum degree at most Δ mix rapidly for $\beta \leq (1 + o_q(1)) \log q/(\Delta - 1)$ and showed slow mixing of the Glauber dynamics on random regular graphs for $\beta \geq (1 + o_q(1)) \log q/(\Delta - 1 - 1/(\Delta - 1))$ [8]. See also [20] for deterministic algorithms in a slightly smaller range of β.

On the hardness side, Goldberg and Jerrum [36] showed that approximating the Potts model partition function Z^{Potts} for $q \geq 3$ on general graphs is #BIS-hard; that is, it is as hard as approximating the number of independent sets in a bipartite graph [25]. Galanis, Štefankovič, Vigoda, and Yang [32] refined these results by showing that for $\beta > \beta_c(q, \Delta)$ (recall (3)), it is #BIS-hard to approximate Z^{Potts} on graphs of maximum degree Δ. The description of the phase diagram of the Potts model on random regular graphs found in [32] and discussed above is a crucial ingredient for this result.

For $q > 2$ the algorithms mentioned above apply only in the high-temperature regime. At very low temperatures efficient sampling and counting algorithms have recently been developed for structured classes of graphs [43, 2, 45] and for expander graphs [46, 28, 14, 29]. By making use of the ideas in [43] in combination with the application of Pirogov-Sinai theory to the random cluster model in [11], Borgs, Chayes, Helmuth, Perkins, and Tetali [9] gave efficient counting and sampling algorithms for the random cluster model on d-dimensional tori $(\mathbb{Z}/n\mathbb{Z})^d$ at *all* temperatures when q is large enough as a function of d. As the tori $(\mathbb{Z}/n\mathbb{Z})^d$ approximate $\mathbb{Z}^d$ as $n \to \infty$, the results of [9] can be interpreted as saying that the phase transition for the q-state random cluster model on $\mathbb{Z}^d$ is not an algorithmic barrier, at least when q is large. Our Theorem 4 has a similar interpretation: informally speaking, it says that the phase transition for the q-state random cluster model on random Δ-regular graphs is not an algorithmic barrier when q is large.

On the other hand, it is shown in [32] that this phase transition does have a link to computational complexity, since it can be used to construct gadgets which show the #BIS-hardness of approximating the Potts partition function for $\beta > \beta_c$. Similarly, while the phase transition in the hard-core model on random Δ-regular bipartite graphs has a direct link to the NP-hardness of approximating the independence polynomial on bounded degree graphs [65, 64, 30, 68], it is still plausible that there are efficient sampling and counting algorithms for the hard-core model on random bipartite graphs at all activities. For activities large enough, efficient algorithms are given in [46, 53, 16], with the last paper obtaining the best known bound of $\lambda = \Omega(\log \Delta/\Delta)$. The authors of [16] observe, however, that $\Omega(\log \Delta/\Delta)$ appears to be a barrier for the type of polymer model argument used in these papers, and so finding efficient algorithms for the hard-core model at and slightly above criticality will likely require new ideas and techniques.

1.3 Open problems

For integer q, the explicit formula (3) for $\beta_c(q, \Delta)$ was found in [32]. We conjecture that this formula holds for non-integral q as well:

Conjecture 7. *For all $\Delta \geq 3$, $q > 2$, the critical point of the random cluster model on the*

random Δ-regular graph satisfies

$$\beta_c(q,\Delta) = \log \frac{q-2}{(q-1)^{1-2/\Delta}-1}\,.$$

It would be interesting to see if the methods of [32, 23] could be generalized to the random cluster model to prove this conjecture.

As discussed earlier, our methods rely in an essential way on q being very large. Phenomenologically, one expects the same behavior for all $q > 2$. It would be very interesting to extend our results to this setting. One possible approach, at least for algorithmic results, is to show that a Markov chain started from either $A = \emptyset$ or $A = E$ can be used to obtain approximate samples.

1.4 Organization of the paper

In Section 2 we define ordered and disordered polymer models, and prove estimates showing that their cluster expansions converge in overlapping regions of parameters that cover all inverse temperatures β when q is large as a function of Δ.

The definitions and estimates of Section 2 allows us to study the random cluster model via the polymer models. In Section 3 we exploit this polymer model framework to prove Theorem 1. In Section 5 we show that the random Δ-regular graph belongs to $\mathcal{G}_{\Delta,\delta}$ with high probability and that membership in $\mathcal{G}_{\Delta,\delta}$ can be checked efficiently. In Section 6 we prove Theorems 2 and 3. In Section 4 we prove Theorem 6.

The estimates of Section 2 reduce the algorithmic problems to algorithmic problems concerning the two polymer models. In Section 7 we provide efficient approximate counting and sampling algorithms for the polymer models, proving Theorem 4.

2 Polymer model representations and estimates

Recall the definition of $\mathcal{G}_{\Delta,\delta}$ from Section 1.1.1. In this section we assume the following.

Assumption 1. *We assume $q \geq 1$, $\Delta \geq 5$, $G \in \mathcal{G}_{\Delta,\delta}$, and set parameters*

$$n \coloneqq |V(G)|\,, \quad e^{\beta_0} - 1 \coloneqq q^{1.9/\Delta}, \quad e^{\beta_1} - 1 \coloneqq q^{2.1/\Delta}, \quad \eta \coloneqq \min\{1/100, \delta/5\}. \tag{4}$$

2.1 *A priori* estimates

The intuition for understanding the large-q random cluster model on an expander graph is that at *all* temperatures a typical configuration consists of either very few of the edges or nearly all of the edges. The following lemma will allow us to make this precise. Recall $c(A)$ is the number of connected components induced by an edge set A.

Lemma 8. *Suppose $G \in \mathcal{G}_{\Delta,\delta}$ and $n \geq 360/(\eta\delta)$. For $A \subset E$ such that $\eta|E| \leq |A| \leq (1-\eta)|E|$,*

$$\frac{c(A)}{n} + \frac{|A|}{|E|} \leq 1 - \eta/40\,.$$

Proof. Given A, let n_1 denote the number of vertices in connected components of size 1 and n_2 the number in components of size at least 2 and at most δn. We will prove the lemma by a case analysis based on $n_1 + n_2$. Before doing this, we collect some useful facts.

First, observe that

$$c(A) \le n_1 + n_2/2 + 1/\delta\,. \tag{5}$$

Second, the number of unoccupied edges satisfies

$$|E| - |A| \ge n_1 \frac{\Delta}{2} + n_2 \frac{5}{9}\frac{\Delta}{2} + \min\{n_1 + n_2, n - n_1 - n_2\} \cdot \frac{\Delta}{20} \tag{6}$$

by double counting the unoccupied edges and the definition of $G_{\Delta,\delta}$. Together (5), (6), $|E| = n\Delta/2$, and a little algebra gives

$$\frac{c(A)}{n} + \frac{|A|}{|E|} \le 1 - \frac{1}{18}\frac{n_2}{n} - \frac{\min\{n_1 + n_2, n - n_1 - n_2\}}{10n} + \frac{1}{\delta n}\,. \tag{7}$$

We now perform the case analysis. First, if $\frac{n_1+n_2}{n} \le \eta/2$, then since $|A|/|E| \le 1 - \eta$ and $c(A) \le n_1 + n_2 + (\delta n)^{-1}$, we directly see that

$$\frac{c(A)}{n} + \frac{|A|}{|E|} \le \frac{\eta}{2} + 1 - \eta + \frac{1}{\delta n} \le 1 - \frac{\eta}{2} + \frac{1}{\delta n}\,.$$

Next suppose $\frac{n_1+n_2}{n} > 1 - \eta/2$. If $n_2/n < \eta/2$, then $n_1/n > 1 - \eta$. This implies $(|E| - |A|)/|E| > 1 - \eta$, or $|A| < \eta|E|$ which contradicts the assumption on A. We may therefore assume that $n_2/n \ge \eta/2$ and so (7) gives

$$\frac{c(A)}{n} + \frac{|A|}{|E|} \le 1 - \frac{\eta}{36} + \frac{1}{\delta n}\,.$$

Finally if $\eta/2 \le \frac{n_1+n_2}{n} \le 1 - \eta/2$ then (7) gives

$$\frac{c(A)}{n} + \frac{|A|}{|E|} \le 1 - \frac{\eta}{20} + \frac{1}{\delta n}\,.$$

In each case we've shown that $\frac{c(A)}{n} + \frac{|A|}{|E|} \le 1 - \frac{\eta}{36} + \frac{1}{\delta n}$, and the lemma follows. □

Recall that $\Omega = \{0,1\}^E$, and that $Z = \sum_{A\in\Omega} q^{c(A)}(e^\beta - 1)^{|A|}$. When we write $A \in \Omega$ we call the edges in A *occupied* and the edges not in A *unoccupied*. In light of Lemma 8, we decompose the state space into three pieces. Recall that

$$\begin{aligned}
\Omega_{\text{dis}} &\coloneqq \{A \in \Omega : |A| \le \eta|E|\} \\
\Omega_{\text{ord}} &\coloneqq \{A \in \Omega : |A| \ge (1-\eta)|E|\} \\
\Omega_{\text{err}} &\coloneqq \Omega \setminus (\Omega_{\text{dis}} \cup \Omega_{\text{ord}})\,.
\end{aligned}$$

Let $Z^{\text{ord}}, Z^{\text{dis}}, Z^{\text{err}}$ be the corresponding random cluster model partition functions. Thus

$$Z = Z^{\text{ord}} + Z^{\text{dis}} + Z^{\text{err}}\,. \tag{8}$$

The next lemma shows that Z^{err} represents an exponentially small fraction of the partition function. It also establishes that unless β lies in the interval (β_0, β_1), then up to an exponentially small correction, Z is given by Z^{ord} or Z^{dis}.

Lemma 9. *If q and n are sufficiently large as a function of Δ and δ then the following hold.*

1. *For $\beta \geq 0$, $Z^{\mathrm{err}}/Z \leq e^{-n}$.*
2. *For $\beta \geq \beta_1$, $Z^{\mathrm{dis}}/Z \leq e^{-n}$.*
3. *For $\beta \leq \beta_0$, $Z^{\mathrm{ord}}/Z \leq e^{-n}$.*

Proof. Let $z \coloneqq \max\left\{q, (e^{\beta}-1)^{\Delta/2}\right\}$. Then $Z \geq z^n$ so that for $A \in \Omega$,

$$\frac{q^{c(A)}(e^{\beta}-1)^{|A|}}{Z} \leq z^{2|A|/\Delta + c(A) - n}\,.$$

By Lemma 8, we have for every $A \in \Omega_{\mathrm{err}}$,

$$\frac{2|A|}{\Delta} + c(A) - n \leq -\frac{\eta n}{40}\,.$$

Since $|\Omega_{\mathrm{err}}| \leq |\Omega| = 2^{n\Delta/2}$, it follows that

$$Z^{\mathrm{err}}/Z \leq 2^{n\Delta/2} z^{-\eta n/40} \leq 2^{n\Delta/2} q^{-\eta n/40}\,,$$

which proves part (1) for $q = q(\Delta, \delta)$ large enough.

Next suppose $\beta \geq \beta_1$. To prove part (2), recall that $\eta \leq 1/100$, so using $|\Omega_{\mathrm{dis}}| \leq 2^{n\Delta/2}$,

$$\frac{Z^{\mathrm{dis}}}{Z} \leq (e^{\beta}-1)^{-\Delta n/2} Z^{\mathrm{dis}} \leq (e^{\beta}-1)^{(\eta-1)\Delta n/2} 2^{\Delta n/2} q^n \leq q^{-2.1\cdot.99n/2+n} 2^{\Delta n/2} \leq e^{-n}$$

for $q = q(\Delta)$ large enough.

Lastly, suppose $\beta \leq \beta_0$. Then, using that $A \in \Omega_{\mathrm{ord}}$ implies $c(A)$ is at most ηn,

$$\frac{Z^{\mathrm{ord}}}{Z} \leq q^{-n} Z^{\mathrm{ord}} \leq q^{(\eta-1)n} 2^{\Delta n/2} (e^{\beta_0}-1)^{\Delta n/2} \leq q^{-.99n+.95n} 2^{\Delta n/2} \leq e^{-n}$$

for $q = q(\Delta)$ large enough, which proves part (3). □

Lemma 9 implies the contribution of Z^{err} to Z is negligible at all temperatures, and so it suffices to control Z^{dis}, Z^{ord} or both, depending on the value of β. We will do this by defining two polymer models and proving they have convergent cluster expansions for $\beta \in [0, \beta_1]$ and $\beta \in [\beta_0, \infty)$ respectively. Crucially, since $\beta_0 < \beta_1$, these two intervals overlap.

2.2 Polymer models

Let $\mathcal{P}$ be a collection of (possibly edge-labelled) finite connected subgraphs of some given finite or infinite graph. We refer to the elements of $\mathcal{P}$ as *polymers*. We say that two polymers $\gamma_1, \gamma_2 \in \mathcal{P}$ are *compatible*, denoted $\gamma_1 \sim \gamma_2$, if they are vertex disjoint, and we write $\gamma_1 \nsim \gamma_2$ to denote incompatibility. Let $w\colon \mathcal{P} \to \mathbb{C}$; w is called a *weight function*. The triple $(\mathcal{P}, \sim, w)$ is a special case of a *polymer model* as defined by Kotecký and Preiss [50], generalizing a technique used to study statistical mechanics models on lattices, see, e.g., [39].

Let $\mathcal{P}' \subseteq \mathcal{P}$ be a finite subset of polymers, and let $\Omega(\mathcal{P}')$ denote the family of all sets of pairwise compatible polymers from $\mathcal{P}'$. Then the expression

$$\Xi(\mathcal{P}') \coloneqq \sum_{\Gamma \in \Omega(\mathcal{P}')} \prod_{\gamma \in \Gamma} w(\gamma)$$

is the *partition function* of the polymer model $(\mathcal{P}', \sim, w)$. The cluster expansion is a formal power series for $\log \Xi(\mathcal{P}')$. In order to describe the cluster expansion we require some notation.

Suppose that $\Gamma = (\gamma_1, \ldots, \gamma_t)$ is an ordered tuple of polymers. We define the *incompatibility graph* H_Γ to be the graph on the vertex set $1, \ldots, t$ where $\{i, j\} \in E(H_\Gamma)$ if and only if $i \neq j$ and γ_i is incompatible with γ_j. A *cluster* is an ordered tuple Γ of polymers whose incompatibility graph H_Γ is connected. Given a graph H, the *Ursell function* $\phi(H)$ of H is

$$\phi(H) \coloneqq \frac{1}{|V(H)|!} \sum_{\substack{A \subseteq E(H) \\ \text{spanning, connected}}} (-1)^{|A|} .$$

Let $\mathcal{C}$ be the set of all clusters of polymers from $\mathcal{P}'$. The *cluster expansion* is the formal power series in the weights $w(\gamma)$

$$\log \Xi(\mathcal{P}') = \sum_{\Gamma \in \mathcal{C}} w(\Gamma) , \tag{9}$$

where

$$w(\Gamma) \coloneqq \phi(H_\Gamma) \prod_{\gamma \in \Gamma} w(\gamma) .$$

The convergence of the infinite series on the right-hand side of (9) is not automatic. The following theorem gives a convenient condition for convergence, and a useful consequence.

Let $E(\gamma)$ denote the set of edges in the polymer γ. For a cluster Γ let $\|\Gamma\| \coloneqq \sum_{\gamma \in \Gamma} |E(\gamma)|$ and write $\Gamma \nsim \gamma$ if there exists $\gamma' \in \Gamma$ so that $\gamma \nsim \gamma'$.

Theorem 10 ([50])**.** *Suppose that there exists $r \geq 0$ such that for all polymers $\gamma \in \mathcal{P}$,*

$$\sum_{\gamma' \nsim \gamma} |w(\gamma')| e^{(1+r)|E(\gamma')|} \leq |E(\gamma)| , \tag{10}$$

then the cluster expansion for $\log \Xi(\mathcal{P}')$ converges absolutely for every finite subset $\mathcal{P}' \subseteq \mathcal{P}$. Moreover, for all polymers γ,

$$\sum_{\Gamma \in \mathcal{C},\, \Gamma \nsim \gamma} |w(\Gamma)| \, e^{r\|\Gamma\|} \leq |E(\gamma)| . \tag{11}$$

Our applications of polymers models will involve weights $w(\gamma)$ that are analytic functions of a parameter β. By verifying that (10) holds uniformly for all β in a domain in the complex plane, we will obtain analyticity of $\log \Xi$ in the same domain, as Theorem 10 then implies that the right-hand side of (9) converges uniformly in β in the domain.

Note that when the weights $w(\gamma)$ of a polymer model are all non-negative reals, we can define an associated Gibbs measure ν on $\Omega(\mathcal{P}')$ by

$$\nu(\Gamma) \coloneqq \frac{\prod_{\gamma \in \Gamma} w(\gamma)}{\Xi(\mathcal{P}')} . \tag{12}$$

2.3 Disordered polymer model

In this section we describe a polymer model that captures deviations from the disordered ground state $A_{\text{dis}} = \emptyset$.

Define *disordered polymers* to be connected subgraphs (V', E') of G with $|E'| \le \eta n$. Let $\mathcal{P}_{\text{dis}} = \mathcal{P}_{\text{dis}}(G)$ be the set of disordered polymers in G. Two polymers are compatible if they are vertex disjoint. For a polymer γ, let $|\gamma|$ denote the number of vertices of γ and $|E(\gamma)|$ the number of edges. The weight of the polymer is defined to be

$$w_\gamma^{\text{dis}} := q^{1-|\gamma|}(e^\beta - 1)^{|E(\gamma)|}.$$

The *disordered polymer partition function* is

$$\Xi^{\text{dis}} := \sum_{\Gamma} \prod_{\gamma \in \Gamma} w_\gamma^{\text{dis}},$$

where the sum is over all compatible collections of disordered polymers.

Proposition 11. *If q and n are sufficiently large as a function of Δ and δ, then for all $\beta \in \mathbb{C}$ such that $\left|e^\beta - 1\right| \le e^{\beta_1} - 1$, the disordered polymer model satisfies (10) with $r = \log q/(4\Delta)$.*

Proof. We will show that for $\beta \le \beta_1$ and for every $v \in V(G)$,

$$\sum_{\gamma \ni v} e^{(1+r)|E(\gamma)|} \left|w_\gamma^{\text{dis}}\right| \le \frac{1}{2}. \tag{13}$$

This is sufficient to verify (10) for the disordered polymer model: given a polymer γ', sum (13) over all vertices of γ'. Since $|\gamma'|/2 \le |E(\gamma')|$, we obtain (10). We will prove (13) in three steps.

First we consider polymers with $|E(\gamma)| = 1$ and $|E(\gamma)| = 2$. The contribution to the left-hand side of (13) from such polymers is exactly $e^{1+r}\Delta|e^\beta - 1|q^{-1} + \frac{3}{2}e^{2+2r}\Delta(\Delta-1)|e^\beta - 1|^2 q^{-2}$. Since $\Delta \ge 5$ and $\left|e^\beta - 1\right| \le q^{\frac{2.1}{\Delta}}$, this is at most $1/6$ for $q = q(\Delta)$ large enough.

Next we consider polymers with $2 < |E(\gamma)| < \Delta/2$. Note that $|\gamma| \ge \sqrt{2|E(\gamma)|}$ for any polymer. By [10, Lemma 2.1 (c)] we can bound the number of polymers with k edges containing a fixed vertex v by $(e\Delta)^k$. We bound the contribution of these polymers to the left-hand side of (13) by

$$\sum_{3 \le k < \Delta/2} (e^{2+r}\Delta)^k q^{1-\sqrt{2k}} \left|e^\beta - 1\right|^k \le \sum_{3 \le k < \Delta/2} (e^{2+r}\Delta)^k q^{1-\sqrt{2k}+\frac{2.1}{\Delta}k} \le \sum_{3 \le k < \Delta/2} |e^{2+r}\Delta|^k q^{2.05-\sqrt{2k}}.$$

Since $\sqrt{6} - 2.05 > 1/8$, this is at most $1/6$ for $q = q(\Delta)$ large enough.

For larger polymers we need two facts. First, that $\Delta\,|\gamma| \ge 2\,|E(\gamma)| + |E(V(\gamma), V(\gamma)^c)|$. Second, since γ defines a connected subgraph we have $|\gamma| \le |E(\gamma)| + 1 \le 2\eta n \le \delta n$, and so the vertices of γ satisfy the small set expansion condition guaranteed by $G \in \mathcal{G}_{\Delta,\delta}$. Together these facts imply $|\gamma| \ge \frac{9|E(\gamma)|}{2\Delta}$. Using [10, Lemma 2.1 (c)] we bound the contribution of these polymers to the left-hand side of (13) by

$$\sum_{k \ge \Delta/2} (e^{2+r}\Delta)^k q^{1-\frac{9k}{2\Delta}} \left|e^\beta - 1\right|^k \le \sum_{k \ge \Delta/2} (e^{2+r}\Delta)^k q^{1-\frac{9k}{2\Delta}+\frac{2.1k}{\Delta}} = \sum_{k \ge \Delta/2} (e^{2+r}\Delta)^k q^{1-\frac{12k}{5\Delta}},$$

and again this is at most $1/6$ for $q = q(\Delta)$ large enough. □

We remark that it is useful to consider complex β here in order to derive analyticity properties of limiting free energies later (see for example Lemma 28 below).

2.3.1 Disordered polymer measure on edges

In addition to the Gibbs measure for the disordered polymer model given by (12), the disordered polymer model also defines a probability measure $\overline{\nu}_{\mathrm{dis}}$ on $\Omega = \{0,1\}^E$ via projection. To obtain a sample A from $\overline{\nu}_{\mathrm{dis}}$, first sample a configuration of compatible disordered polymers Γ from ν_{dis}, that is, with probability $\prod_{\gamma\in\Gamma} w_\gamma^{\mathrm{dis}}/\Xi^{\mathrm{dis}}$. Second, let

$$A = \bigcup_{\gamma\in\Gamma} E(\gamma)\,.$$

We will show in Section 2.6 that when the disordered cluster expansion converges, i.e., for $\beta \le \beta_1$, the distribution $\overline{\nu}_{\mathrm{dis}}$ is very close to the distribution of the random cluster model measure μ conditioned on Ω_{dis}.

2.4 Ordered polymer expansion

Next we define a polymer model that describes deviations from the ordered ground state $A_{\mathrm{ord}} = E$. We need a more complicated construction compared to the disordered polymer model to handle the non-local cluster weight. The basic idea of the ordered polymer model is that, given an edge configuration A, polymers represent the connected components of the 'boundary' $\mathcal{B}(A)$ of A. We begin by making this precise.

2.4.1 Boundary of occupied edges

The precise notion of boundary is given by the following construction. Given $A \subseteq E$, let $\mathcal{B}_0(A)$ be the set of unoccupied edges $E \setminus A$. To form $\mathcal{B}_{i+1}(A)$ from $\mathcal{B}_i(A)$ we add any edge e incident to a vertex v with at least $5\Delta/9$ incident edges in $\mathcal{B}_i(A)$. This procedure stabilizes and results in a set $\mathcal{B}_\infty(A)$ of edges, of which $\mathcal{B}_0(A)$ are unoccupied and $\mathcal{B}_\infty(A) \setminus \mathcal{B}_0(A)$ are occupied.

Lemma 12. *For any $A \subseteq E$, the algorithm to generate $\mathcal{B}_\infty(A)$ runs in time quadratic in $|\mathcal{B}_0(A)|$. Moreover, $|\mathcal{B}_\infty(A)| \le 10\,|\mathcal{B}_0(A)|$.*

Proof. First, observe that the same set $\mathcal{B}_\infty(A)$ results no matter the order in which edges are added. The first claim therefore will follow from the second as there are at most $10\,|\mathcal{B}_0(A)|$ edges added, and finding the next edge to add (if one exists) takes time at most $10\Delta\,|\mathcal{B}_0(A)|$.

To prove the second claim, observe that each incident edge at a vertex can be 'charged' at most $4/5 = (4\Delta/9)/(5\Delta/9)$ for the new edges added to the boundary. Since each edge in $\mathcal{B}_0(A)$ is incident to 2 vertices it can be charged at most $8/5$ in total, and each subsequent edge can be charged at most $4/5$. Thus

$$|\mathcal{B}_\infty(A)| \le 2|\mathcal{B}_0(A)| \sum_{j=0}^{\infty} (4/5)^j = 10|\mathcal{B}_0(A)|\,. \qquad \square$$

2.4.2 Ordered polymers

We define *ordered polymers* to be connected subgraphs γ of G with an edge labelling $\ell\colon E(\gamma) \to \{\text{occupied}, \text{unoccupied}\}$, subject to (i) $|E_u(\gamma)| \le \eta n$, where $E_u(\gamma)$ denotes the set of unoccupied edges of γ, and (ii) $\mathcal{B}_\infty(E_u(\gamma)) = E(\gamma)$, i.e., the inductive boundary procedure applied to the unoccupied edges of γ stabilizes at γ. Let $\mathcal{P}_{\mathrm{ord}} = \mathcal{P}_{\mathrm{ord}}(G)$ be the set of disordered polymers in G. As usual, two polymers are compatible if they are vertex disjoint.

Let $c'(\gamma)$ denote the number of components of the graph $(V, E \setminus E_u(\gamma))$ with fewer than $n/2$ vertices. We think of these as 'finite components', cf. Lemma 13 below and also Section 3 where we make a similar definition with G replaced by an infinite tree. The weight function for ordered polymers is

$$w_\gamma^{\mathrm{ord}} := q^{c'(\gamma)}(e^\beta - 1)^{-|E_u(\gamma)|}.$$

The *ordered polymer partition function* is

$$\Xi^{\mathrm{ord}} := \sum_{\Gamma} \prod_{\gamma \in \Gamma} w_\gamma^{\mathrm{ord}},$$

where the sum is over compatible collections of ordered polymers.

We end this section with two lemmas that will be used to prove the convergence of the ordered cluster expansion. First, recall the following well-known fact about expander graphs, see e.g. [67, Lemma 2.3].

Lemma 13. *Let $G = (V, E)$ be a graph and let $|E'| \le \eta\,|E|$. Then $(V, E \setminus E')$ contains a connected component of size $\left(1 - \frac{\eta}{2\phi_G(1/2)}\right)|V|$.*

Next we bound the number of unoccupied edges of a polymer in terms of c'.

Lemma 14. *For all ordered polymers γ, $|E_u(\gamma)| \ge \frac{5}{9}\Delta \cdot c'(\gamma)$.*

Proof. Let $S_1, \dots, S_t$ denote the connected components of $(V, E \setminus E_u(\gamma))$. By Lemma 13, since $\phi_G(1/2) \ge 1/10$, we may assume without loss of generality that S_1 contains at least $(1 - 5\eta)\,n \ge (1 - \delta n)$ vertices. Let U be the union of the vertices in $S_2, \dots, S_t$, so $c'(\gamma) = t - 1 \le |U|$. Since any edge leaving U must be unoccupied and $|U| \le \delta n$, the claim follows since $\phi_G(\delta) \ge 5/9$. □

2.4.3 Convergence of Ordered Expansion

Proposition 15. *If $q = q(\Delta)$ is sufficiently large, then for all $\beta \in \mathbb{C}$ such that $\left|e^\beta - 1\right| \ge e^{\beta_0} - 1$ the ordered polymer model satisfies (10) with $r = \log q/(200\Delta)$.*

Proof. We will show that for $\beta \ge \beta_0$ and for every $v \in V(G)$,

$$\sum_{\gamma \ni v} e^{(1+r)|E(\gamma)|} \left|w_\gamma^{\mathrm{ord}}\right| \le \frac{1}{2}\,. \tag{14}$$

As in the proof of Proposition 11, this suffices to verify condition (10).

We index polymers by $k = |E_u(\gamma)|$. By Lemma 12 $|E(\gamma)| \le 10|E_u(\gamma)|$. By [10, Lemma 2.1 (c)] we can bound the number of polymers with $|E_u(\gamma)| = k$ containing a vertex v by $(2e\Delta)^{10k}$,

where the factor of 2 accounts for the choice of occupied/unoccupied for each edge. Then, since $\left|e^{\beta}-1\right| \geq q^{\frac{1.9}{\Delta}}$, by Lemma 14

$$\begin{aligned}\sum_{\gamma \ni v} e^{(1+r)|E(\gamma)|}\left|w_{\gamma}^{\mathrm{ord}}\right| &\leq \sum_{k\geq 1}(2e^{2+r}\Delta)^{10k} q^{\frac{9k}{5\Delta}}\left|e^{\beta}-1\right|^{-k} \\ &\leq \sum_{k\geq 1}(2e^{2+r}\Delta)^{10k} q^{\left(\frac{9}{5\Delta}-\frac{1.9}{\Delta}\right)k} \\ &= \sum_{k\geq 1}(2e^{2+r}\Delta)^{10k} q^{-\frac{k}{10\Delta}},\end{aligned}$$

which is at most $1/2$ for $q = q(\Delta)$ sufficiently large. □

2.4.4 Ordered polymer model measure on edges

Let ν_{ord} be the polymer model measure defined by (12). As for the disordered polymer model we can define a measure $\overline{\nu}_{\mathrm{ord}}$ on $\Omega = \{0,1\}^E$. To obtain a sample A from $\overline{\nu}_{\mathrm{ord}}$ we sample a collection Γ of compatible ordered polymers according to ν_{ord} and then let

$$A = E \setminus \bigcup_{\gamma\in\Gamma} E_u(\gamma)\,.$$

2.5 Consequences of the cluster expansion convergence

This section derives some consequences of Theorem 10. Let $\mathcal{C}_{\mathrm{dis}}$ and $\mathcal{C}_{\mathrm{ord}}$ be the sets of clusters of polymers in $\mathcal{P}_{\mathrm{dis}}$ and $\mathcal{P}_{\mathrm{ord}}$, respectively. For $* \in \{\mathrm{dis},\mathrm{ord}\}$ we write $\mathcal{C}_*^v$ for the set of clusters containing the vertex v. We will always assume that q is large enough that Propositions 11 and 15 apply.

2.5.1 Truncated cluster expansion error bounds

The following lemma will be used extensively in Section 6 and Section 3.

Lemma 16. *For every $v \in G$ and $m \geq 1$,*

$$\sum_{\substack{\Gamma\in\mathcal{C}_*^v \\ \|\Gamma\|\geq m}} |w^*(\Gamma)| \leq q^{-\frac{m}{200\Delta}}\,. \tag{15}$$

Proof. This follows from Propositions 11 and 15 by applying (11) with γ a single edge containing v and $r = \log q/(200\Delta)$. □

An important consequence of Lemma 16 is the following. For $m \geq 1$, define truncated cluster expansions

$$T_m^* \coloneqq \sum_{\substack{\Gamma\in\mathcal{C}_* \\ \|\Gamma\|<m}} w^*(\Gamma), \qquad * \in \{\mathrm{dis},\mathrm{ord}\} \tag{16}$$

By summing (15) over all vertices we have that

$$\sum_{\substack{\Gamma\in\mathcal{C}_*\\ \|\Gamma\|\geq m}} |w^*(\Gamma)| \leq nq^{-\frac{m}{200\Delta}}, \qquad * \in \{\mathrm{dis}, \mathrm{ord}\} \tag{17}$$

and so

$$|T^*_m - \log \Xi_*| \leq nq^{-\frac{m}{200\Delta}}, \qquad * \in \{\mathrm{dis}, \mathrm{ord}\}, \tag{18}$$

which will be used in Section 7.

2.5.2 Probabilistic estimates for measures on edges

We first show that the set of edges contained in a polymer sample is typically small. Recall that for $* \in \{\mathrm{dis}, \mathrm{ord}\}$, ν_* is the Gibbs measure associated to the $*$-polymer model (given by (12)) and $\overline{\nu}_*$ is the measure on $\{0,1\}^E$ induced by ν_* (as defined in Subsections 2.3.1, 2.4.4).

Lemma 17. *Let $* \in \{\mathrm{dis}, \mathrm{ord}\}$ and let $\mathbf{\Gamma}$ be a random configuration of compatible polymers sampled from ν_*. Then for $\alpha > 0$,*

$$\mathbb{P}(\|\mathbf{\Gamma}\| > \alpha n) \leq e^{-2n}$$

for $q = q(\Delta, \alpha)$ sufficiently large.

Proof. Fix $* \in \{\mathrm{dis}, \mathrm{ord}\}$ and let $\mathbf{\Gamma}$ be a random sample from ν_*. Consider the cumulant generating function

$$h_t(\|\mathbf{\Gamma}\|) \coloneqq \log \mathbb{E} e^{t\|\mathbf{\Gamma}\|}.$$

For $t > 0$, we introduce an auxiliary polymer model on $\mathcal{P}_*$ by modifying the polymer weights:

$$\tilde{w}^*_\gamma = w^*_\gamma \cdot e^{t|E(\gamma)|}.$$

Let $\tilde{\Xi}_*$ be the modified partition function. Then $\log \tilde{\Xi}_* - \log \Xi_* = h_t(\|\mathbf{\Gamma}\|)$. Setting $t = 3/\alpha$, the estimates used to prove Propositions 11 and 15 still hold with w^* replaced with $\tilde{w}^*$ if q is sufficiently large, and so, by (17) with $m = 1$ and with the modified weights, we have

$$\log \tilde{\Xi}_* \leq \sum_{\gamma\in\mathcal{C}_*} |\tilde{w}^*(\Gamma)| \leq n.$$

Since $\Xi_* \geq 1$ we then have $h_t(\|\mathbf{\Gamma}\|) \leq n$ also. By Markov's inequality

$$\mathbb{P}(\|\mathbf{\Gamma}\| > \alpha n) \leq e^{-t\alpha n}\mathbb{E}e^{t\|\mathbf{\Gamma}\|} \leq e^{-t\alpha n}e^n \leq e^{-2n}. \qquad \square$$

The next lemma shows that $\overline{\nu}_{\mathrm{dis}}$ and $\overline{\nu}_{\mathrm{ord}}$ exhibit exponential decay of correlations. In the following we let $\mathbf{A}$ denote a random edge subset drawn according to the measure $\overline{\nu}_{\mathrm{dis}}$ or $\overline{\nu}_{\mathrm{ord}}$.

Lemma 18. *1. For $\beta \leq \beta_1$, $\overline{\nu}_{\mathrm{dis}}$ exhibits exponential decay of correlations and $|\mathbf{A}|$ obeys a central limit theorem with respect to $\overline{\nu}_{\mathrm{dis}}$.*

2. *For $\beta \geq \beta_0$, $\overline{\nu}_{\mathrm{ord}}$ exhibits exponential decay of correlations and $|\mathbf{A}|$ obeys a central limit theorem with respect to $\overline{\nu}_{\mathrm{ord}}$.*

Proof. These are standard consequences of the condition (10), so we will only provide a sketch. To prove exponential decay of correlations, we need to show that there exist constants $C, \varepsilon > 0$ so that for all $e, f \in E$,

$$|\overline{\nu}_{\mathrm{dis}}(e,f) - \overline{\nu}_{\mathrm{dis}}(e)\overline{\nu}_{\mathrm{dis}}(f)| \leq Ce^{-\varepsilon \mathrm{dist}(e,f)} . \tag{19}$$

See e.g. [13, Theorem 1.3] for details. Establishing (19) amounts to observing that the correlation between edges e and f, $\overline{\nu}_{\mathrm{dis}}(e,f) - \overline{\nu}_{\mathrm{dis}}(e)\overline{\nu}_{\mathrm{dis}}(f)$, equals a weighted sum over clusters of disordered polymers containing both e and f of the cluster weight. The size of any such cluster is at least $\mathrm{dist}(e,f)$, and a tail bound like (18) shows the total weight of these clusters is exponentially small in $\mathrm{dist}(e,f)$.

Likewise for $\overline{\nu}_{\mathrm{ord}}$, by taking complements and using inclusion-exclusion it is enough to show that

$$\left|\overline{\nu}_{\mathrm{ord}}(\overline{e},\overline{f}) - \overline{\nu}_{\mathrm{ord}}(\overline{e})\overline{\nu}_{\mathrm{ord}}(\overline{f})\right| \leq Ce^{-\varepsilon \mathrm{dist}(e,f)} , \tag{20}$$

where $\overline{\nu}_{\mathrm{ord}}(\overline{e})$ is the probability $e \notin A$ and $\overline{\nu}_{\mathrm{ord}}(\overline{e},\overline{f})$ is the probability that $\{e \notin A\} \wedge \{f \notin A\}$. Again (20) can be expressed as a sum of cluster weights with clusters of size at least $\mathrm{dist}(e,f)$ and so we obtain exponential decay of correlations.

To prove a central limit theorem for $|\mathbf{A}|$ under $\overline{\nu}_{\mathrm{dis}}$ we first center and normalize, letting $Y = (|\mathbf{A}| - \mathbb{E}|\mathbf{A}|)/\sigma$ where $\sigma^2 = \mathrm{var}(|\mathbf{A}|)$. Now by the method of moments (or cumulants) it is enough to show that for each fixed $k \geq 3$, the kth cumulant of Y, $\kappa_k(Y)$, vanishes as $n \to \infty$. Using the cluster expansion we can express

$$\begin{aligned} |\kappa_k(Y)| &= \left| \sum_{\Gamma \in \mathcal{C}_{\mathrm{dis}}} w^{\mathrm{dis}}(\Gamma) \left(\frac{\|\Gamma\|}{\sigma} \right)^k \right| \\ &\leq \frac{1}{\sigma^3} \sum_{\Gamma \in \mathcal{C}_{\mathrm{dis}}} |w^{\mathrm{dis}}(\Gamma)| \|\Gamma\|^k \\ &\leq \frac{\Delta n}{\sigma^3} \sum_{t \geq 1} e^{-kt} t^k = O\left(\frac{n}{\sigma^3} \right) , \end{aligned}$$

where we applied (11) in the last line. A simple conditioning argument (see e.g. [21, Lemma 9]) shows that $\sigma = \Omega(n^{1/2})$, and so for $k \geq 3$, $\kappa_k(Y) \to 0$, as desired. The proof for $\overline{\nu}_{\mathrm{ord}}$ is similar, substituting missing edges for occupied edges. □

Remark 3. *The correlation between edges e and f is a joint cumulant of the indicator random variables that each is in A. The same techniques can be used to show that joint cumulants of the indicators of k edges decay exponentially in the size of the minimum spanning tree connecting the edges in G, see [13] for more details.*

The next lemma states that up to total variation distance ε, the measures induced by $\overline{\nu}_{\mathrm{dis}}$ and $\overline{\nu}_{\mathrm{ord}}$ on the local neighborhood of any vertex v are determined by clusters contained in a (larger) neighborhood of v. To make this precise, for $* \in \{\mathrm{dis}, \mathrm{ord}\}$, $T \in \mathbb{N}$ and a vertex v, define $\overline{\nu}_*^{B_T(v)}$ to be the projection of $\overline{\nu}_*$ to $\{0,1\}^{E(B_T(v))} \subset \{0,1\}^E$. Here $B_T(v)$ is the ball of radius T around v.

Lemma 19. *Suppose $T > 0, \varepsilon > 0$, and $* \in \{\mathrm{dis}, \mathrm{ord}\}$. There is an m large enough as a function of Δ, T, ε so that for any $v \in V$, $\overline{\nu}_*^{B_T(v)}$ is determined up to total variation distance ε by clusters which lie entirely in $B_m(v)$.*

Proof. We prove this for $\overline{\nu}_{\mathrm{dis}}$, the proof for $\overline{\nu}_{\mathrm{ord}}$ is identical. Let $A \subseteq E$ be distributed according to $\overline{\nu}_{\mathrm{dis}}$. We first claim it is enough to give, for each $F \subseteq E(B_T(v))$, a quantity $\kappa(F)$ so that

1. $|\kappa(F) - \overline{\nu}_{\mathrm{dis}}(A \subset F^c)| \leq \varepsilon 2^{-2\Delta^T}$.
2. $\kappa(F)$ depends only on clusters contained in $B_m(v)$ for some $m = m(T, \Delta, \varepsilon)$.

The lemma follows from these two properties by calculating $\overline{\nu}_*^{B_T(v)}(\cdot)$ via inclusion-exclusion, and summing the error bound over all subsets of $E(B_T(v))$.

To find such a $\kappa(F)$, observe there is an exact formula for $\overline{\nu}_{\mathrm{dis}}(A \subset F^c)$ in terms of clusters:

$$\log \overline{\nu}_{\mathrm{dis}}(A \subset F^c) = -\sum_{\Gamma \in \mathcal{C}_{\mathrm{dis}}} w^{\mathrm{dis}}(\Gamma) 1_{\Gamma \cap F \neq \emptyset},$$

where $1_{\Gamma \cap F \neq \emptyset}$ indicates that Γ contains a polymer which contains an edge from F. Since F spans at most $|V(B_T(v))| \leq \Delta^T + 1$ vertices, by Lemma 16 we can truncate the RHS and obtain

$$\left| \log \overline{\nu}_{\mathrm{dis}}(A \subset F^c) + \sum_{\substack{\Gamma \in \mathcal{C}_{\mathrm{dis}} \\ \|\Gamma\| \leq m}} w^{\mathrm{dis}}(\Gamma) 1_{\Gamma \cap F \neq \emptyset} \right| \leq q^{-\frac{m}{200\Delta}} (\Delta^T + 1). \tag{21}$$

Note the quality of (21) is independent of F. The desired quantity is

$$\kappa(F) = \exp\left(-\sum_{\substack{\Gamma \in \mathcal{C}_{\mathrm{dis}} \\ \|\Gamma\| \leq m}} w^{\mathrm{dis}}(\Gamma) 1_{\Gamma \cap F \neq \emptyset} \right).$$

Taking m large enough as a function of Δ, T, ε gives properties (1) and (2) and proves the lemma, as, since $\kappa(F) \in (0,1)$, the accurate multiplicative approximation guaranteed by (21) implies an accurate additive approximation. □

2.6 Polymer model approximation of the partition function

Using the results above we now show that the scaled polymer model partition functions are good approximations to Z^{dis} and Z^{ord}. We also show that the measures $\overline{\nu}_{\mathrm{dis}}$ and $\overline{\nu}_{\mathrm{ord}}$ on edge sets are good approximations to μ_{dis} and μ_{ord}, where μ_{dis} and μ_{ord} are the random cluster measure μ conditioned on Ω_{dis} and Ω_{ord}, respectively.

Lemma 20. *For $q = q(\Delta, \delta)$ sufficiently large and $\beta \leq \beta_1$,*

$$\left| Z^{\mathrm{dis}} - q^n \cdot \Xi^{\mathrm{dis}} \right| \leq e^{-n} Z^{\mathrm{dis}}. \tag{22}$$

Moreover,

$$\|\mu_{\mathrm{dis}} - \overline{\nu}_{\mathrm{dis}}\|_{TV} \leq e^{-n}. \tag{23}$$

Proof. Write $\Xi^{\text{dis}}_{\leq}$ for the contribution to Ξ^{dis} of compatible collections Γ of polymers with $|E(\Gamma)| \leq \eta|E|$ and let $\Xi^{\text{dis}}_{>} = \Xi^{\text{dis}} - \Xi^{\text{dis}}_{\leq}$. By definition we have $Z^{\text{dis}} = q^n \cdot \Xi^{\text{dis}}_{\leq}$. By Lemma 17 we have $\Xi^{\text{dis}}_{>} \leq e^{-2n}\Xi^{\text{dis}} \leq 2e^{-2n}\Xi^{\text{dis}}_{\leq}$. It follows that

$$\left|Z^{\text{dis}} - q^n \cdot \Xi^{\text{dis}}\right| = q^n \Xi^{\text{dis}}_{>} \leq 2q^n e^{-2n} \cdot \Xi^{\text{dis}}_{\leq} = e^{-n} Z^{\text{dis}}\,. \tag{24}$$

The proof of (23) is nearly identical, see, e.g., the proof of [47, Lemma 14]. □

We now turn to Z^{ord}. This requires a preparatory lemma.

Lemma 21. *Let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a collection of compatible polymers, and assume $|E_u(\Gamma)| \leq \eta|E|$. The number of connected components in the graph $G - (E_u(\gamma_1) \cup \ldots \cup E_u(\gamma_k))$ is $c'(\gamma_1) + \ldots + c'(\gamma_k) + 1$.*

Proof. Let $S_1, \ldots, S_t$ denote the vertex sets of the components of $G \setminus E_u(\Gamma)$, and without loss of generality let S_1 be the largest component. As in the proof of Lemma 14, we have that $|S_1| \geq (1-\delta)n$ by Lemma 13 and our assumption on $|E_u(\Gamma)|$. We claim that for for each $i \geq 2$, we have $E(S_i, S_i^c) \subset E_u(\gamma_j)$ for some j. This suffices to prove the lemma.

Suppose, towards a contradiction, that $E(S_2, S_2^c)$ contains edges from more than one of the sets $E_u(\gamma_i)$. Without loss of generality, let the $\ell > 1$ indices for which $E(S_2, S_2^c) \cap E_u(\gamma_i) \neq \emptyset$ be $1, 2, \ldots, \ell$.

Let $T \subset S_2$ denote the set of vertices in S_2 that have fewer than Δ incident edges from each of the sets $E_u(\gamma_1), \ldots, E_u(\gamma_\ell)$. If $T = \emptyset$, pairwise compatibility of polymers implies each vertex in S_2 is incident to Δ edges in exactly one of the sets $E_u(\gamma_i)$, $1 \leq i \leq \ell$. Since S_2 is connected, all edges incident to S_2 must in fact be from the *same* set $E_u(\gamma_i)$, contradicting $\ell > 1$.

To conclude the proof, we show $T = \emptyset$ is the only possibility. Suppose not, i.e., $T \neq \emptyset$. Note that all of the edges in $E(T, T^c)$ belong to $E(\gamma_1) \cup \ldots \cup E(\gamma_\ell)$. Moreover $|T| \leq \delta n$ and so $|E(T, T^c)| \geq 5\Delta|T|/9$ since $\phi_G(\delta) \geq 5/9$. It follows that there is a vertex $u \in T$ incident to $\geq 5\Delta/9$ edges in $E(\gamma_1) \cup \ldots \cup E(\gamma_\ell)$. Without loss of generality, it must be the case that u is incident to $\geq 5\Delta/9$ edges in $E(\gamma_1)$, as u cannot be an endpoint of unoccupied edges in distinct compatible polymers. By the definition of the polymer γ_1, all of the Δ edges incident to u must then belong to $E(\gamma_1)$, contradicting the definition of T. □

Lemma 22. *If $q = q(\Delta, \delta)$ is sufficiently large and $\beta \geq \beta_0$, then*

$$\left|Z^{\text{ord}} - q(e^\beta - 1)^{\frac{\Delta n}{2}} \cdot \Xi^{\text{ord}}\right| \leq e^{-n} Z^{\text{ord}}. \tag{25}$$

Moreover,

$$\|\mu_{\text{ord}} - \overline{\nu}_{\text{ord}}\|_{TV} \leq e^{-n}\,. \tag{26}$$

Proof. Write $\Xi^{\text{ord}}_{\leq}$ for the contribution to Ξ^{ord} of compatible collections Γ of polymers with $|E_u(\Gamma)| \leq \eta|E|$. Then $Z^{\text{ord}} = q(e^\beta - 1)^{\frac{\Delta n}{2}} \Xi^{\text{ord}}_{\leq}$ by Lemma 21. By Lemma 17 we have $\Xi^{\text{ord}}_{>} \leq e^{-2n}\Xi^{\text{ord}} \leq 2e^{-2n}\Xi^{\text{ord}}_{\leq}$. It follows that

$$\left|Z^{\text{ord}} - q(e^\beta - 1)^{\frac{\Delta n}{2}} \Xi^{\text{ord}}\right| = q(e^\beta - 1)^{\frac{\Delta n}{2}} \Xi^{\text{ord}}_{>} \leq 2e^{-2n} q(e^\beta - 1)^{\frac{\Delta n}{2}} \Xi^{\text{ord}}_{\leq} \leq e^{-n} Z^{\text{ord}}.$$

As in the proof of Lemma 20, (26) follows from a similar argument. □

Corollary 23. *If q and n are sufficiently large as a function of Δ and δ, then for all $\beta > 0$*

$$\tilde{Z}(q,\beta) \coloneqq q^n \cdot \Xi^{\mathrm{dis}} + q(e^\beta - 1)^{\Delta n/2} \cdot \Xi^{\mathrm{ord}} \tag{27}$$

is an $e^{-n/2}$-relative approximation to $Z_G(q,\beta)$.

Proof. The result follows by applying Lemmas 9, 20 and 22 in the the ranges $\beta \le \beta_0$, $\beta_0 \le \beta \le \beta_1$ and $\beta \ge \beta_1$. □

3 Phase transitions on trees and random graphs

In this section we prove that the q-state random cluster model on locally tree-like graphs in $\mathcal{G}_{\Delta,\delta}$ has a unique phase transition, and we characterize the critical point. To do this we define ordered and disordered polymer models for the random cluster model on the infinite Δ-regular tree $\mathbb{T}_\Delta$ and its finite depth-L truncations $\mathbb{T}_\Delta^L$, and then relate these models to the random cluster model on locally tree-like Δ-regular graphs.

3.1 The random cluster model and polymer models on finite trees

Fix Δ and let $\mathbb{T}_\Delta$ be the rooted infinite Δ-regular tree with root vertex r. Let $\mathbb{T}_\Delta^L$ denote the finite subtree of $\mathbb{T}_\Delta$ with root r and depth L where each non-leaf vertex has degree Δ.

3.1.1 Free boundary conditions and the disordered polymer model on finite trees

We start with $\mathbb{T}_\Delta^L$. The random cluster model on $\mathbb{T}_\Delta^L$ with free boundary conditions is simply the random cluster model on $\mathbb{T}_\Delta^L$. In particular,

$$Z^{\mathrm{free}}_{\mathbb{T}_\Delta^L}(q,\beta) = \sum_{A \subseteq E(\mathbb{T}_\Delta^L)} q^{c(A)}(e^\beta - 1)^{|A|}\,,$$

where $c(A)$ is the number of connected components of $(V(\mathbb{T}_\Delta^L), A)$. To distinguish these boundary conditions we call the resulting measure on edges the *free random cluster model on* $\mathbb{T}_\Delta^L$. Recall from Section 1.1.2 that μ^{free} is the weak limit of the free random cluster model measures on $\mathbb{T}_\Delta^L$ as $L \to \infty$.

Next we define a disordered polymer model on $\mathbb{T}_\Delta^L$. Define polymers to be connected components of $\mathbb{T}_\Delta^L$. Unlike in Section 2, we do not restrict the size of polymers. The weight of a polymer is again given by $w_\gamma^{\mathrm{dis}} = q^{1-|\gamma|}(e^\beta - 1)^{|E(\gamma)|}$. Call the resulting polymer model partition function $\Xi^{\mathrm{dis}}_{\mathbb{T}_\Delta^L}$.

Lemma 24. *We have the equality*

$$Z^{\mathrm{free}}_{\mathbb{T}_\Delta^L}(q,\beta) = q^{|V(\mathbb{T}_\Delta^L)|} \cdot \Xi^{\mathrm{dis}}_{\mathbb{T}_\Delta^L}\,.$$

Moreover, the induced measure $\overline{\nu}_{\mathrm{dis}}$ on edges is the free random cluster measure on $\mathbb{T}_\Delta^L$.

Proof. This follows from two facts: 1) There is a bijection between subsets $A \subseteq E(\mathbb{T}_\Delta^L)$ and collections of mutually compatible polymers and 2) if $A \subseteq E(\mathbb{T}_\Delta^L)$ corresponds to the set of polymers $\{\gamma_1, \dots, \gamma_k\}$, then $c(A) = |V(\mathbb{T}_\Delta^L)| + \sum_i (1 - |\gamma_i|)$. That is, the bijection of 1) is weight-preserving. □

3.1.2 Wired boundary conditions and the ordered polymer model on finite trees

Informally, the random cluster model on $\mathbb{T}_\Delta^L$ with wired boundary conditions is obtained by declaring that the boundary vertices belong to a single connected component. Formally, we call a subset of $V(\mathbb{T}_\Delta^L)$ *finite* if it contains no boundary vertex. Then,

$$Z^{\text{wire}}_{\mathbb{T}_\Delta^L}(q,\beta) \coloneqq \sum_{A \subseteq E(\mathbb{T}_\Delta^L)} q^{c_w(A)}(e^\beta - 1)^{|A|}\,,$$

where $c_w(A)$ is the number of finite connected components of $(V(\mathbb{T}_\Delta^L), A)$ plus 1; the additional one is to account for the single component containing the boundary vertices. We call the resulting measure on edges the *wired random cluster measure on* $\mathbb{T}_\Delta^L$. Recall from Section 1.1.2 that μ^{wire} is the weak limit of the wired random cluster measures on $\mathbb{T}_\Delta^L$ as $L \to \infty$.

Ordered polymers on $\mathbb{T}_\Delta^L$ are defined as in Section 2, but with no restriction on their size. The weight function is

$$w^{\text{ord}}_\gamma \coloneqq q^{c'(\gamma)}(e^\beta - 1)^{-|E_u(\gamma)|},$$

where $c'(\gamma)$ is the number of finite connected components of the graph $(V(\mathbb{T}_\Delta^L), E(\mathbb{T}_\Delta^L) \setminus E_u(\gamma))$. We have the following analogue of Lemma 21.

Lemma 25. *Let $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ be a collection of compatible ordered polymers on $\mathbb{T}_\Delta^L$. Then*

$$c_w\left(E(\mathbb{T}_\Delta^L) \setminus \bigcup_{i=1}^k E_u(\gamma_i)\right) = 1 + \sum_{i=1}^k c'(\gamma_i)\,. \tag{28}$$

Proof. Let $S_1, \dots, S_t$ denote the vertex sets of the finite components of $E(\mathbb{T}_\Delta^L) \setminus \bigcup_{i=1}^k E_u(\gamma_i)$. In particular, by the definition of c_w, the left hand side of (28) is equal to $t+1$.

If $S \subseteq V(\mathbb{T}_\Delta^L)$ is finite, then $|E(S, S^c)| \geq (\Delta - 2)|S|$. It then follows, as in the proof of Lemma 21, that each of the sets $S_1, \dots, S_t$ is incident to edges from precisely one of the polymers $\gamma_1, \dots, \gamma_k$. The result follows. □

With this we prove that the polymer model partition function equals the wired random cluster partition function after scaling.

Lemma 26. *We have the equality*

$$Z^{\text{wire}}_{\mathbb{T}_\Delta^L}(q,\beta) = q(e^\beta - 1)^{|E(\mathbb{T}_\Delta^L)|}\Xi^{\text{ord}}_{\mathbb{T}_\Delta^L}\,.$$

Moreover, the induced measure $\overline{\nu}_{ord}$ on edges is the wired random cluster measure on $\mathbb{T}_\Delta^L$.

Proof. This follows from Lemma 25, which implies there is a weight-preserving bijection between sets $A \subseteq E(\mathbb{T}_\Delta^L)$ and collections of mutually compatible ordered polymers. □

3.2 Infinite trees and limiting free energies

To motivate the definitions that follow, we begin by rewriting the cluster expansions for Ξ^{ord} and Ξ^{dis} for a given finite Δ-regular graph G on n vertices. For a cluster Γ, let $u(\Gamma)$ be the

number of distinct vertices contained in Γ. Write $\mathcal{C}^v_{\mathrm{dis}}(G)$ for the set of disordered clusters containing v, and similarly for ord. Then

$$\log \Xi^{\mathrm{dis}} = \sum_{v \in V(G)} \sum_{\Gamma \in \mathcal{C}^v_{\mathrm{dis}}(G)} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma), \tag{29}$$

$$\log \Xi^{\mathrm{ord}} = \sum_{v \in V(G)} \sum_{\Gamma \in \mathcal{C}^v_{\mathrm{ord}}(G)} \frac{1}{u(\Gamma)} w^{\mathrm{ord}}(\Gamma)\,. \tag{30}$$

Using this as a model, we consider ordered and disordered polymer models on the infinite Δ-regular tree $\mathbb{T}_\Delta$ rooted at r. Here we define polymers and weights exactly as for $\mathbb{T}^L_\Delta$ above in Section 3.1, but with the additional condition that the polymers be finite. In particular, for an ordered polymer γ the weight function is

$$w^{\mathrm{ord}}_\gamma \coloneqq q^{c'(\gamma)}(e^\beta - 1)^{-|E_u(\gamma)|},$$

where $c'(\gamma)$ is the number of finite connected components of the graph $(V(\mathbb{T}_\Delta), E(\mathbb{T}_\Delta) \setminus E_u(\gamma))$; here we mean finite in the usual sense of finite cardinality.

Lemma 27. *For $q = q(\Delta)$ sufficiently large the following hold with $r = \log q/(200\Delta)$:*

1. *For $\beta \in \mathbb{C}$ such that $\left|e^\beta - 1\right| \le e^{\beta_1} - 1$, the disordered polymer model on $\mathbb{T}_\Delta$ satisfies* (10)*.*
2. *For $\beta \in \mathbb{C}$ such that $\left|e^\beta - 1\right| \ge e^{\beta_0} - 1$, the ordered polymer model on $\mathbb{T}_\Delta$ satisfies* (10)*.*

Proof. We can mimic the proofs of Propositions 11 and 15 once we note that the tree $\mathbb{T}_\Delta$ satisfies the following optimal expansion condition: for a finite set $S \subset V(\mathbb{T}_\Delta)$, $|E(S, S^c)| \ge (\Delta - 2)|S|$. □

Let $\mathcal{C}^r_{\mathrm{ord}}$ and $\mathcal{C}^r_{\mathrm{dis}}$ be the respective sets of clusters on $\mathbb{T}_\Delta$ containing the root r. Define

$$\overline{f}_{\mathrm{ord}}(\beta, q) \coloneqq \sum_{\Gamma \in \mathcal{C}^r_{\mathrm{ord}}} \frac{1}{u(\Gamma)} w^{\mathrm{ord}}(\Gamma), \qquad \overline{f}_{\mathrm{dis}}(\beta, q) \coloneqq \sum_{\Gamma \in \mathcal{C}^r_{\mathrm{dis}}} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma)\,,$$

By Lemma 27 and (11) of Theorem 10, for q large and $\beta \in [\beta_0, \beta_1]$ these series converge and are functions of Δ, β, q. Further define

$$f_{\mathrm{ord}}(\beta, q) \coloneqq \frac{\Delta}{2} \log(e^\beta - 1) + \overline{f}_{\mathrm{ord}}, \qquad f_{\mathrm{dis}}(\beta, q) \coloneqq \log q + \overline{f}_{\mathrm{dis}}. \tag{31}$$

Proposition 28. *For any sequence of locally tree-like graphs $G_n \in \mathcal{G}_{\Delta,\delta}$ and $q = q(\Delta, \delta)$ large enough, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{dis}}_{G_n} = f_{\mathrm{dis}}, \qquad \left|e^\beta - 1\right| \le e^{\beta_1} - 1 \tag{32}$$

$$\lim_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{ord}}_{G_n} = f_{\mathrm{ord}}, \qquad \left|e^\beta - 1\right| \ge e^{\beta_0} - 1. \tag{33}$$

Moreover these limits are uniform on the given regions of β.

Proof. We give the proof of the first statement, as the proof of the second statement is the same up to changes in notation. Fix $\varepsilon > 0$ and let $m = \log(2/\varepsilon)$. Let V_m be the set of vertices of G_n whose depth-m neighborhood is not a tree. Recall that we write $\mathcal{C}^v_{\mathrm{dis}}(G_n)$ for the set of clusters on G_n that contain a vertex v. Then using Lemma 20,

$$\begin{aligned}
\left|\log Z^{\mathrm{dis}}_{G_n} - n f_{\mathrm{dis}}\right| &\le \left|\log(q^n \Xi^{\mathrm{dis}}_{G_n}) - n f_{\mathrm{dis}}\right| + 2e^{-n} \\
&= \left|\sum_{v\in V}\left(\sum_{\Gamma\in\mathcal{C}^v_{\mathrm{dis}}(G_n)} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma) - \sum_{\Gamma\in\mathcal{C}^r_{\mathrm{dis}}} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma)\right)\right| + 2e^{-n} \\
&\le \sum_{v\in V}\left|\sum_{\Gamma\in\mathcal{C}^v_{\mathrm{dis}}(G_n)} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma) - \sum_{\Gamma\in\mathcal{C}^r_{\mathrm{dis}}} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma)\right| + 2e^{-n} \\
&\le \varepsilon n + \sum_{v\in V}\left|\sum_{\substack{\Gamma\in\mathcal{C}^v_{\mathrm{dis}}(G_n)\\ \|\Gamma\|\le m}} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma) - \sum_{\substack{\Gamma\in\mathcal{C}^r_{\mathrm{dis}}\\ \|\Gamma\|\le m}} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma)\right| + 2e^{-n} \\
&\le \varepsilon n + 2|V_m| + 2e^{-n}\,.
\end{aligned}$$

The penultimate inequality follows from Lemma 16 applied to each series inside the absolute values. The final inequality follows from applying Lemma 16 to $v \in V_m$ to bound each series by 1 in absolute value and noting that if $v \notin V_m$, the two series inside the absolute values are identical. Since G_n is locally tree-like, $|V_m|/n \to 0$ and so $\limsup |\frac{1}{n}\log Z^{\mathrm{dis}}_{G_n} - f_{\mathrm{dis}}| \le \varepsilon$. Taking $\varepsilon \to 0$ proves the statement.

The same proof shows that $\lim_{n\to\infty} \frac{1}{n}\log Z^{\mathrm{ord}}_{G_n} = f_{\mathrm{ord}}$ for $\left|e^\beta - 1\right| \ge e^{\beta_0} - 1$. □

3.3 Determining the critical point

We will define the critical point $\beta_c(q,\Delta)$ implicitly in terms of the functions f_{ord} and f_{dis}. It will be convenient to first obtain the formula for f_{dis} given in Theorem 1.

Lemma 29. *For $\beta \in \mathbb{C}$ such that $\left|e^\beta - 1\right| \le e^{\beta_1} - 1$*

$$f_{\mathrm{dis}}(\beta, q) = \log q + \frac{\Delta}{2}\log\left(1 + \frac{e^\beta - 1}{q}\right). \tag{34}$$

Proof. This proof uses the following generalization of Section 3.1.1. Given a finite subtree T of $\mathbb{T}_\Delta$, define the disordered polymer model on T just as we did for $\mathbb{T}^L_\Delta$ and let $\mathcal{C}_{\mathrm{dis}}(T)$ denote the collection of clusters of disordered polymers in T. For a cluster Γ let $\mathcal{G}(\Gamma)$ denote the graph union of all polymers in Γ. As in Lemma 24, the polymer model partition function on T is a scaling of the random cluster model partition function.

Note that the random cluster measure on a finite tree with free boundary conditions has a very simple description: it is independent edge percolation with the probability of retaining each edge being $\frac{e^\beta - 1}{e^\beta - 1 + q}$ [38, Chapter 10]. This independence implies that any joint cumulant involving indicators of at least two edges vanishes, i.e., for all trees T with at least two edges,

$$\sum_{\substack{\Gamma\in\mathcal{C}_{\mathrm{dis}}(T):\\ \mathcal{G}(\Gamma)=T}} w^{\mathrm{dis}}(\Gamma) = 0\,, \tag{35}$$

since the left-hand side is the joint cumulant of the edges of T in the random cluster model on T.

To conclude, note that we have

$$\overline{f}_{\mathrm{dis}} = \sum_{\Gamma \in \mathcal{C}_r^{\mathrm{dis}}} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma) = \sum_T \frac{1}{|V(T)|} \sum_{\substack{\Gamma \in \mathcal{C}_{\mathrm{dis}}(T): \\ \mathcal{G}(\Gamma)=T}} w(\Gamma)$$

where the first sum on the right hand side is over all finite subtrees of $\mathbb{T}_\Delta$ containing the root. By Lemma 27 part (1), these sums are absolutely convergent. By (35), only trees consisting of a single edge contribute to the sum over T, and there are Δ of these. Each contributes $\log\left(1 + \frac{e^\beta - 1}{q}\right)$, and this gives the result. □

Proposition 30. *For all $\Delta \geq 5$ and $q = q(\Delta, \delta)$ large enough, there is a unique $\beta_c(q, \Delta) \in (\beta_0, \beta_1)$ such that $f_{\mathrm{ord}}(\beta) = f_{\mathrm{dis}}(\beta)$. Moreover, $f_{\mathrm{ord}} < f_{\mathrm{dis}}$ for $\beta \in [\beta_0, \beta_c)$ and $f_{\mathrm{ord}} > f_{\mathrm{dis}}$ for $\beta \in (\beta_c, \beta_1]$.*

Proof of Proposition 30. Our proof of this proposition follows the strategy of [52].

We begin with a computation. Let $\beta \in [\beta_0, \beta_1]$ so that both the ordered and disordered expansions converge. Then by Proposition 28 and Lemma 29,

$$\begin{aligned}\frac{d}{d\beta}(f_{\mathrm{ord}} - f_{\mathrm{dis}}) &= \frac{d}{d\beta} \lim_{n\to\infty} \frac{1}{n} \log Z^{\mathrm{ord}}_{G_n} - \frac{\Delta}{2} \cdot \frac{e^\beta}{q + e^\beta - 1} \\ &= \lim_{n\to\infty} \frac{1}{n} \frac{d}{d\beta} \log Z^{\mathrm{ord}}_{G_n} - \frac{\Delta}{2} \cdot \frac{e^\beta}{q + e^\beta - 1}.\end{aligned}$$

The interchange of the derivative and limit is valid since f_{ord} is a uniform limit of analytic functions by Proposition 28. To bound the first term we note that

$$\frac{1}{n} \cdot \frac{e^\beta - 1}{e^\beta} \frac{d}{d\beta} \log Z^{\mathrm{ord}}_{G_n}$$

is the expected number of edges in a random cluster configuration conditioned on Ω_{ord} and is therefore at least $(1 - \eta)n\Delta/2$. It follows that

$$\frac{d}{d\beta}(f_{\mathrm{ord}} - f_{\mathrm{dis}}) \geq e^\beta \frac{\Delta}{2} \left[\frac{1 - \eta}{e^\beta - 1} - \frac{1}{q + e^\beta - 1}\right] > 0,$$

since $\eta \leq 1/100$ and $\beta \in [\beta_0, \beta_1]$.

Next, note that

$$f_{\mathrm{dis}} - f_{\mathrm{ord}} = \frac{\Delta}{2} \log\left(\frac{q^{2/\Delta}}{e^\beta - 1} + q^{2/\Delta - 1}\right) - \overline{f}_{\mathrm{ord}}. \tag{36}$$

By Lemma 27 part (2) and Lemma 16 (with $\mathbb{T}_\Delta$ in place of G) we have $|\overline{f}_{\mathrm{ord}}| \leq q^{\frac{1}{200\Delta}}$. It follows that for q sufficiently large, if $\beta = \beta_0$ then $f_{\mathrm{dis}} > f_{\mathrm{ord}}$ and if $\beta = \beta_1$ then $f_{\mathrm{ord}} > f_{\mathrm{dis}}$. Since $f_{\mathrm{ord}} - f_{\mathrm{dis}}$ is a continuous and strictly increasing function of β on $[\beta_0, \beta_1]$, we obtain that there is a unique $\beta_c \in (\beta_0, \beta_1)$ at which $f_{\mathrm{ord}} = f_{\mathrm{dis}}$. □

Corollary 31. *For all $\Delta \geq 5$ and $q = q(\Delta)$ large enough, $\beta_c(q, \Delta)$ is given by*

$$\beta_c(q, \Delta) = (1 + o_q(1)) \frac{2 \log q}{\Delta}. \tag{37}$$

Proof. By Proposition 30, the claim follows by equating f_{ord} and f_{dis} and solving for β, see (36). □

Proposition 30 implies there is a unique transition point on locally tree-like sequences of finite graphs satisfying our expansion hypotheses. The next proposition shows that the transition is first order.

Proposition 32. *For any sequence of locally tree-like graphs G_n from $\mathcal{G}_{\Delta,\delta}$, if $q = q(\Delta, \delta)$ is large enough then*

1. *For $\beta < \beta_c$, $\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(\Omega \setminus \Omega_{\text{dis}}) < 0$.*
2. *For $\beta > \beta_c$, $\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(\Omega \setminus \Omega_{\text{ord}}) < 0$.*

Proof. The lemma follows by combining Proposition 28 with the estimates of Lemma 9. If $\beta \in (\beta_0, \beta_c)$, then by Proposition 28 and Lemma 9 part (1),

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(\Omega \setminus \Omega_{\text{dis}}) \leq \max\{-1, f_{\text{ord}} - f_{\text{dis}}\} < 0$$

where the last inequality follows from by Proposition 30. Similarly if $\beta \in (\beta_c, \beta_1)$, then the quantity in part (2) is at most $\max\{-1, f_{\text{dis}} - f_{\text{ord}}\} < 0$.

If $\beta \leq \beta_0$, then by Lemma 9, $\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(\Omega \setminus \Omega_{\text{dis}}) \leq -1$. Similarly if $\beta \geq \beta_1$, then $\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(\Omega \setminus \Omega_{\text{ord}}) \leq -1$. □

3.4 Local convergence and proof of Theorem 1

Recall from Sections 2.3.1 and 2.4.4 that the disordered and ordered polymer measures on a graph G_n induce measures $\overline{\nu}^n_{\text{dis}}$ and $\overline{\nu}^n_{\text{ord}}$ on edges.

Proposition 33. *Let G_n be a sequence of locally tree-like graphs from $\mathcal{G}_{\Delta,\delta}$, $\Delta \geq 5$. Then for q large,*

1. *If $\beta \leq \beta_1$, $\overline{\nu}^n_{\text{dis}} \xrightarrow{loc} \mu^{\text{free}}$.*
2. *If $\beta \geq \beta_0$, $\overline{\nu}^n_{\text{ord}} \xrightarrow{loc} \mu^{\text{wire}}$.*

Proof. We begin with the first statement. To ease notation let us denote $\overline{\nu}^n_{\text{dis}}$ by $\overline{\nu}_{\text{dis}}$. Recall that for $T > 0$ and $v \in G_n$, $B_T(v)$ denotes the depth-T neighborhood of v. Recall also that $\overline{\nu}^{B_T(v)}_{\text{dis}}$ denotes the projection of $\overline{\nu}_{\text{dis}}$ to $\{0,1\}^{E(B_T(v))}$.

For $L > 0$, let $\nu_{\text{dis},L}$ denote the Gibbs measure associated to the disordered polymer model on $\mathbb{T}^L_\Delta$ as defined in Section 3.1.1. By Lemma 24 the induced measure $\overline{\nu}_{\text{dis},L}$ on $\{0,1\}^{E(\mathbb{T}^L_\Delta)}$ is the free random cluster measure on $\mathbb{T}^L_\Delta$.

We let r denote the root of the tree $\mathbb{T}^L_\Delta$. We will show that for $T > 0$ and $\varepsilon > 0$, for all L sufficiently large the distribution $\overline{\nu}^{B_T(v)}_{\text{dis}}$ of a randomly chosen $v \in G_n$ is within distance

2ε of $\overline{\nu}_{\mathrm{dis},L}^{B_T(r)}$ in total variation distance. This suffices to prove part (1) since μ^{free} is the weak limit of $\overline{\nu}_{\mathrm{dis},L}$ as $L \to \infty$.

We will apply Lemma 19. Given $\varepsilon > 0$, let $m = m(\Delta, T, \varepsilon)$ large enough as required by the lemma. Since G_n is locally tree-like, with high probability over the choice of v, the depth-m neighborhood of v will be a tree, so we can condition on this. Lemma 19 tells us that up to total variation distance ε, $\overline{\nu}_{\mathrm{dis}}^{B_T(v)}$ is determined by clusters contained in $B_m(v)$.

By Lemma 27, the cluster expansion of the disordered polymer model on $\mathbb{T}_\Delta^L$ converges for all L. For $L \geq m$, we may apply the proof of Lemma 19 to show that up to total variation distance ε, $\overline{\nu}_{\mathrm{dis},L}^{B_T(r)}$ is determined by clusters contained in $B_m(r)$.

Since $B_m(v)$ and $B_m(r)$ are identical, we have

$$\|\overline{\nu}_{\mathrm{dis}}^{B_T(v)} - \overline{\nu}_{\mathrm{dis},L}^{B_T(r)}\|_{TV} \leq 2\varepsilon$$

as required.

The proof of the second claim is identical, using Lemma 26 in place of Lemma 24. □

Proof of Theorem 1. Claim (1) follows from Proposition 28. The limit $\lim_{n\to\infty} \frac{1}{n} \log Z_{G_n}$ is analytic for $\beta \in (0,\infty) \setminus \{\beta_c\}$ since f_{dis} is analytic on $(0, \beta_1]$ and f_{ord}, as a uniform limit of analytic functions, is analytic on $[\beta_0, \infty)$. The formula for f_{dis} when $\beta < \beta_c$ follows from Lemma 29. Claims (2) and (3) follow immediately from Proposition 32. Claim (8) follows from Lemma 9.

To emphasize the dependence on n, write μ_{dis}^n and μ_{ord}^n denote the distributions of μ_n conditioned on Ω_{dis} and Ω_{ord}, respectively. To conclude, we will prove the following strengthening of Claims (4), (5), (6), and (7):

(i) For $\beta \leq \beta_1$, $\mu_{\mathrm{dis}}^n \xrightarrow{loc} \mu^{\mathrm{free}}$ as $n \to \infty$.

(ii) For $\beta \geq \beta_0$, $\mu_{\mathrm{ord}}^n \xrightarrow{loc} \mu^{\mathrm{wire}}$ as $n \to \infty$.

(iii) For $\beta \leq \beta_1$, μ_{dis}^n exhibits exponential decay of correlations and $|\mathbf{A}|$ obeys a central limit theorem with respect to μ_{dis}^n.

(iv) For $\beta \geq \beta_0$, μ_{ord}^n exhibits exponential decay of correlations and $|\mathbf{A}|$ obeys a central limit theorem with respect to μ_{ord}^n.

Given Lemmas 20 and 22 it is enough to prove (i)–(iv) for $\overline{\nu}_{\mathrm{dis}}^n$ and $\overline{\nu}_{\mathrm{ord}}^n$ in place of μ_{dis}^n and μ_{ord}^n. Claims (i) and (ii) then follow from Proposition 33. Claims (iii) and (iv) follow from Lemma 18 combined with the observation that since the diameter of an expander graph is $O(\log n)$, the total variation distance error e^{-n} from Lemmas 20 and 22 can be absorbed in the constant in the exponential decay bound. □

4 Slow mixing of Markov chains

In this section we prove Theorem 6. We will give the proof for *Chayes-Machta (CM) dynamics* [15] and then indicate how to adapt the proof for the (much simpler) case of random cluster and Potts Glauber dynamics.

We begin by recalling the definition of the *Chayes-Machta (CM) dynamics* [15], a generalization of Swendsen-Wang dynamics for the Potts model to the setting of the random cluster model. Given a random cluster configuration $A \in \Omega = \{0,1\}^E$, one step of the CM dynamics is defined as follows:

1. declare each component of A to be 'active' independently with probability $1/q$, and declare all vertices in active components to be active;
2. delete all edges in A that connect two active vertices;
3. add each edge in E that connects two active vertices independently with probability $p = 1 - e^{-\beta}$.

We use $P_{\mathrm{CM}}(\cdot,\cdot)$ to denote the transition matrix of the CM dynamics, and μ_A^t the t-step distribution of the chain started at configuration A. The *mixing time* of the CM dynamics is:

$$\tau_{\mathrm{mix}} = \inf\{t : \max_{A \subset E} \|\mu - \mu_A^t\|_{TV} \leq 1/4\,.$$

Our general strategy follows one previously used at $\beta = \beta_c$, e.g., [11, 32]. Our ability to extend slow mixing to an interval around β_c stems from our ability to control the contribution of subdominant phases off criticality (Lemma 35 below).

We begin with a lemma that says CM dynamics are unlikely to transition from an ordered configuration to a disordered configuration.

Lemma 34. *For $q = q(\Delta)$ sufficiently large and $\beta \in (\beta_0, \beta_1)$, $P_{\mathrm{CM}}(A, \Omega_{\mathrm{dis}}) < e^{-n\Delta/40}$ for all $A \in \Omega_{\mathrm{ord}}$.*

Proof. Let $U \subseteq V$ denote the set of vertices declared active at Step 1 in the definition of CM dynamics and let A' denote the random edge configuration resulting from Steps 1,2 and 3. Let $m = |E \cap \binom{U}{2}|$, that is, the number of edges of G joining two active vertices. Note that the number of edges removed from the configuration in Step 2 is at most m and so if $m < |E|/2$, then $|A'| > |A| - |E|/2 \geq (1/2 - \eta)|E|$. Therefore $A' \notin \Omega_{\mathrm{dis}}$ and so we may assume that $m \geq |E|/2$.

Letting X denote the number edges added at Step 3 we have $|A'| > |A| - m + X$. If $X \geq pm/2$, it follows that

$$|A'| \geq (1-\eta)|E| - (1-p/2)m \geq (p/4 - \eta)|E| > \eta|E|$$

Since $p = 1 - e^{-\beta} = 1 - o_q(1)$ for $\beta \in (\beta_0, \beta_1)$, we have $p > 8\eta$ for q large. As a result, $A' \notin \Omega_{\mathrm{dis}}$. The result follows by noting that $\mathbb{P}(X < pm/2) \leq e^{-pm/8}$ by Chernoff's bound. □

The next lemma says that near β_c, it is exponentially more likely to see a disordered or ordered configuration than a configuration in Ω_{err}.

Lemma 35. *If $q = q(\Delta, \delta)$ is sufficiently large, $|\beta - \beta_c| \leq \frac{1}{20\Delta}$, and n is sufficiently large, then*

$$\mu_n(\Omega_{\mathrm{dis}}) \geq e^{-n/20} \quad \textit{and} \quad \mu_n(\Omega_{\mathrm{ord}}) \geq e^{-n/20}\,.$$

Proof. If $\beta \in (\beta_0, \beta_1)$, then by Proposition 28 and Lemma 9 part (1),

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_n(\Omega_{\mathrm{dis}}) \geq \min\{0, f_{\mathrm{dis}} - f_{\mathrm{ord}}\} \ .$$

By the argument of Proposition 30 we have

$$\left| \frac{d}{d\beta}(f_{\mathrm{ord}} - f_{\mathrm{dis}}) \right| < \frac{e^\beta}{e^\beta - 1} \cdot \frac{\Delta}{2} < \Delta \, .$$

Since $f_{\mathrm{dis}} - f_{\mathrm{ord}} = 0$ at $\beta = \beta_c$, it follows that $f_{\mathrm{dis}} - f_{\mathrm{ord}} > -1/20$ for $\beta \in (\beta_0, \beta_c + \frac{1}{20\Delta})$. The bound on $\mu_n(\Omega_{\mathrm{dis}})$ follows. The same argument shows that $\lim_{n\to\infty} \frac{1}{n} \log \mu_n(\Omega_{\mathrm{ord}}) > -1/20$ for $\beta \in (\beta_c - \frac{1}{20\Delta}, \beta_1)$. □

Proof of Theorem 6 for CM dynamics. We will establish slow mixing of CM dynamics by bounding the *conductance* of CM dynamics defined as

$$\Phi_{CM} = \min_{\emptyset \subset S \subset \Omega} \Phi_{CM}(S) \quad \text{where} \quad \Phi_{CM}(S) = \frac{\sum_{A\in S} \mu(A) P_{\mathrm{CM}}(A, S^c)}{\mu(S)\mu(S^c)} \, .$$

Note that P_{CM} and μ depend on the given graph $G = (V, E)$, and in particular, on $n = |V(G)|$. We leave this implicit. By a standard argument (see [61]), it suffices to show that $\Phi_{CM} \leq e^{-\Omega(n)}$ for $\beta \in (\beta_m, \beta_M)$. This is straightforward from the lemmas above:

$$\begin{aligned}
\Phi_{CM} \leq \Phi_{CM}(\Omega_{\mathrm{dis}}) &= \frac{\sum_{A\in\Omega_{\mathrm{dis}}} \mu(A) P_{\mathrm{CM}}(A, \Omega_{\mathrm{dis}}^c)}{\mu(\Omega_{\mathrm{dis}})\mu(\Omega_{\mathrm{dis}}^c)} \\
&\leq e^{n/10} \left(\sum_{A\in\Omega_{\mathrm{dis}}} \mu(A) P_{\mathrm{CM}}(A, \Omega_{\mathrm{ord}}) + \sum_{A\in\Omega_{\mathrm{dis}}} \mu(A) P_{\mathrm{CM}}(A, \Omega_{\mathrm{err}}) \right) \\
&\leq e^{n/10} \left(\mu(\Omega_{\mathrm{dis}}) e^{-n\Delta/40} + \sum_{A\in\Omega_{\mathrm{err}}} \mu(A) P_{\mathrm{CM}}(A, \Omega_{\mathrm{dis}}) \right) \\
&\leq e^{n/10} \left(e^{-n\Delta/40} + e^{-n} \right) \\
&\leq 2e^{-n/40} \, .
\end{aligned}$$

For the second inequality we used Lemma 35. For the third inequality we applied Lemma 34 and reversibility, and for the fourth inequality we used Lemma 9. □

We conclude this section by noting that the above proof adapts easily to the to the cases of random cluster and Potts model Glauber dynamics. First we recall their definitions. Given a random cluster configuration $A \in \Omega = \{0, 1\}^E$, one step of the *random cluster Glauber dynamics* transitions to a new configuration A' as follows:

1. select an edge $e \in E$ uniformly at random;
2. set $A' = A \cup \{e\}$ with probability $\frac{\mu_G(A\cup\{e\})}{\mu_G(A\cup\{e\}) + \mu_G(A\setminus\{e\})}$
3. otherwise set $A' = A \setminus \{e\}$.

Given a Potts configuration $\sigma \in [q]^V$, one step of the *Potts model Glauber dynamics* transitions to a new configuration σ' as follows:

1. select a vertex $v \in V$ uniformly at random;
2. set $\sigma'(v) = k$ with probability $\mu_G^{\text{Potts}}(\tau(v) = k \mid \tau(w) = \sigma(w)\ \forall w \neq v)$ and set $\sigma'(u) = \sigma(u)$ for all $u \neq v$.

We note that by considering the monochromatic edges in each Potts model configuration, the above dynamics naturally induces dynamics on the random cluster model. The proof of Theorem 6 for the Glauber dynamics for the random cluster model and Potts model are similar to the proof for CM dynamics but simpler, as the associated dynamics cannot transition directly from Ω_{dis} to Ω_{ord}. We omit the details.

5 Application to random Δ-regular graphs

In this section we prove that for $\Delta \geq 5$, there is some $\delta > 0$ so that the random Δ-regular graph belongs to $\mathcal{G}_{\Delta,\delta}$ with high probability. We use the following result on the expansion profile of the random regular graph which is a combination of [7, Theorem 1] and [44, Theorem 4.16].

Theorem 36. *Let $\Delta \geq 3$ and let G be a Δ-regular graph on n vertices chosen uniformly at random. Let $0 < x < 1$ be such that*

$$2^{4/\Delta} < (1-x)^{1-x}(1+x)^{1+x}\,, \tag{38}$$

then with high probability $\phi_G(1/2) \geq (1-x)/2$. Moreover, for every $\varepsilon > 0$, there exists $\delta > 0$ such that with high probability $\phi_G(\delta) \geq (\Delta - 2 - \varepsilon)/\Delta$.

Proposition 37. *For every $\Delta \geq 5$, there exists $\delta > 0$ so that a uniformly chosen Δ-regular graph on n vertices is in $\mathcal{G}_{\Delta,\delta}$ with probability $1 - o(1)$ as $n \to \infty$. Moreover, there is a polynomial-time algorithm that accepts/rejects graphs that (i) only accepts G if $G \in \mathcal{G}_{\Delta,\delta}$ and (ii) it accepts with probability $1 - o(1)$ for a randomly chosen Δ-regular graph.*

Proof. Let G be a uniformly chosen Δ-regular graph on n vertices. By substituting $x = 1/10$ into (38), Theorem 36 shows that $\phi_G(1/2) \geq 1/10$ with high probability. Moreover, taking $\varepsilon = 2/9$, the second half of Theorem 36 shows that there exists $\delta > 0$ such that $\phi_G(\delta) \geq 5/9$ with high probability. This proves the first claim. We remark that one can extract explicit sufficient conditions on δ from the proof of [44, Theorem 4.16].

Note that it also holds that for some $\tilde{\varepsilon} > 0$, $\phi_G(1/2) \geq 1/10 + \tilde{\varepsilon}$ and $\phi_G(\delta) \geq 5/9 + \tilde{\varepsilon}$ with high probability. Then using the approximation algorithm from [41], we can approximate $\phi_G(1/2)$ and $\phi_G(\delta)$ and with high probability get a certificate that $\phi_G(1/2) \geq 1/10$ and $\phi_G(\delta) \geq 5/9$. □

6 Finite Size Scaling

In this section G_n is always a random Δ-regular graph on n vertices. Our objective is to determine the limiting distribution of $\log Z^{\text{ord}}_{G_n} - n f_{\text{ord}}$ and $\log Z^{\text{dis}}_{G_n} - n f_{\text{dis}}$ as $n \to \infty$. This

will prove Theorems 2 and 3. To ease notation, we let Z^{dis} and Z^{ord} denote $Z^{\mathrm{dis}}_{G_n}$ and $Z^{\mathrm{ord}}_{G_n}$, respectively.

To state the key proposition we need to introduce a class of graphs that will capture the way in which a Δ-regular graph locally deviates from being a tree. Assume $\Delta \geq 3$. Let $\mathbb{T}_{\Delta-2,\Delta}$ denote the rooted infinite tree whose root has $\Delta - 2$ children and for which every other vertex is degree Δ. For $k \geq 3$ the *Δ-regular tree rooted at C_k* is the graph $\mathbb{T}^{C_k}_{\Delta}$ obtained by attaching to each vertex of a k-cycle C_k a copy of $\mathbb{T}_{\Delta-2,\Delta}$, and rooting the resulting graph at a distinguished vertex r in C_k.

We define disordered and ordered polymers on $\mathbb{T}^{C_k}_{\Delta}$ exactly as we did for $\mathbb{T}_{\Delta}$ in Section 3.2. For $q = q(\Delta)$ large enough the cluster expansions for these polymer models converge provided $\beta \leq \beta_1$ and $\beta \geq \beta_0$, respectively. This can be established by repeating the proof of Lemma 27. For $* \in \{\mathrm{dis}, \mathrm{ord}\}$ let $\mathcal{C}^r_*(\mathbb{T}^{C_k}_{\Delta})$ denote the set of $*$-clusters that contain the root r of $\mathbb{T}^{C_k}_{\Delta}$. To help distinguish notation, in this section we write $\mathcal{C}^r_*(\mathbb{T}_{\Delta})$ for the sets of $*$-clusters on the rooted Δ-regular tree that contain the root. We then let

$$\alpha^{\mathrm{dis}}_k \coloneqq \sum_{\Gamma \in \mathcal{C}^r_{\mathrm{dis}}(\mathbb{T}^{C_k}_{\Delta})} w^{\mathrm{dis}}(\Gamma) - \sum_{\Gamma \in \mathcal{C}^r_{\mathrm{dis}}(\mathbb{T}_{\Delta})} w^{\mathrm{dis}}(\Gamma)$$
$$\alpha^{\mathrm{ord}}_k \coloneqq \sum_{\Gamma \in \mathcal{C}^r_{\mathrm{ord}}(\mathbb{T}^{C_k}_{\Delta})} w^{\mathrm{ord}}(\Gamma) - \sum_{\Gamma \in \mathcal{C}^r_{\mathrm{ord}}(\mathbb{T}_{\Delta})} w^{\mathrm{ord}}(\Gamma)\,.$$

Note that α^{dis}_k is well-defined as the difference of two absolutely convergent power series when $\beta \leq \beta_1$, and similarly for α^{ord}_k when $\beta \geq \beta_0$.

Proposition 38. *Let $(Y_1, Y_2, \dots)$ be a sequence of independent Poisson random variables where Y_k has mean $(\Delta-1)^k/(2k)$.*

1. *For $\beta \leq \beta_1$, $W^{\mathrm{dis}}_n \coloneqq \log Z^{\mathrm{dis}} - n f_{\mathrm{dis}}$ converges in distribution to W^{dis} given by the almost surely absolutely convergent series*
$$W^{\mathrm{dis}} \coloneqq \sum_{k \geq 3} \alpha^{\mathrm{dis}}_k Y_k\,.$$

2. *For $\beta \geq \beta_0$, $W^{\mathrm{ord}}_n \coloneqq \log Z^{\mathrm{ord}} - n f_{\mathrm{ord}}$ converges in distribution to W^{ord} given by the almost surely absolutely convergent series*
$$W^{\mathrm{ord}} \coloneqq \log q + \sum_{k \geq 3} \alpha^{\mathrm{ord}}_k Y_k\,.$$

3. *For $\beta = \beta_c$, $W_n \coloneqq \log Z^{\mathrm{ord}} - \log Z^{\mathrm{dis}}$ converges in distribution to W given by the almost surely absolutely convergent series*
$$W \coloneqq \log q + \sum_{k \geq 3} (\alpha^{\mathrm{ord}}_k - \alpha^{\mathrm{dis}}_k) Y_k\,.$$
Moreover, letting $Q \coloneqq e^W$, we have $Q/q \to 1$ in probability as $q \to \infty$.

To prove Proposition 38 we will need several results about the distribution of short cycles in random Δ-regular graphs.

Lemma 39. *For $k \geq 3$, let X_k denote the number of cycles of length k in the random Δ-regular graph on n vertices. Then*

1. *For $3 \leq k \leq \frac{\log n}{5 \log \Delta}$, $\mathbb{E}X_k = (1 + O(k^2/n))\frac{(\Delta-1)^k}{2k}$ [56].*
2. *For any fixed T, the joint distribution of $X_3, \ldots, X_T$ converges to that of independent Poisson random variables of means $\frac{(\Delta-1)^k}{2k}$, $k = 3, \ldots, T$ [69, 6].*
3. *For every fixed $T > 0$, with high probability over the choice of G, the depth-t neighborhood of every vertex contains at most one cycle of length at most T for $t = \frac{\log n}{5 \log \Delta}$ [54, Lemma 2.1].*

With this we can prove Proposition 38.

Proof. Throughout this proof we assume the high probability event that $G_n \in \mathcal{G}_{\Delta,\delta}$, with δ chosen as in Section 5 occurs. We begin by establishing the claimed limits, deferring the claims about absolute convergence almost surely to the end. Towards claim (1), let X_k denote the number of k-cycles in G_n, and define

$$\tilde{W}_n^{\mathrm{dis}}(T) \coloneqq \sum_{k=3}^{T} \alpha_k^{\mathrm{dis}} X_k \,.$$

We will show that for any $\varepsilon > 0$, there is T large enough so that for all $t \geq T$ we have

$$\limsup_{n\to\infty} \mathbb{P}\left[\left|\tilde{W}_n^{\mathrm{dis}}(t) - W_n^{\mathrm{dis}}\right| \geq \varepsilon\right] \leq \varepsilon \,.$$

By Lemma 39, part (2), the joint distribution of $X_3, \ldots, X_t$ converges to that of $Y_3, \ldots, Y_t$, and so this will prove that $W_n^{\mathrm{dis}} = \log Z^{\mathrm{dis}} - n f_{\mathrm{dis}}$ converges to $W^{\mathrm{dis}} = \sum_{k\geq 3} \alpha_k^{\mathrm{dis}} Y_k$ in distribution.

Fix $\varepsilon > 0$. We begin with the formula from the proof of Proposition 28 for W_n^{dis}. Writing G in place of G_n,

$$W_n^{\mathrm{dis}} = \sum_{v\in V} \left(\sum_{\Gamma \in \mathcal{C}^v_{\mathrm{dis}}(G)} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma) - \sum_{\Gamma \in \mathcal{C}^r_{\mathrm{dis}}(\mathbb{T}_\Delta)} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma) \right).$$

Let $m = 200\Delta \log(4n/\varepsilon)/\log q$. We can apply Lemma 16 to obtain

$$\left| W_n^{\mathrm{dis}} - \sum_{v\in V} \left(\sum_{\substack{\Gamma \in \mathcal{C}^v_{\mathrm{dis}}(G) \\ \|\Gamma\| \leq m}} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma) - \sum_{\substack{\Gamma \in \mathcal{C}^r_{\mathrm{dis}}(\mathbb{T}_\Delta) \\ \|\Gamma\| \leq m}} \frac{1}{u(\Gamma)} w^{\mathrm{dis}}(\Gamma) \right) \right| \leq \varepsilon/2 \,. \tag{39}$$

Note that the terms inside the parentheses cancel exactly unless there is a cycle, necessarily of length at most m, in the m-neighborhood of v. To measure the error in $\tilde{W}_n^{\mathrm{dis}}(T) - W_n^{\mathrm{dis}}$ due to these cycles we will reformulate (39) in a way that takes cancellations into account.

A cluster Γ appears in only one of the two sums in (39) for only two possible reasons: because the cluster contains a cycle in G, or because the cycle prevents a cluster on the tree

from occurring in G. For a cycle of length k these possibilities only occur for clusters of size at least k because smaller clusters in G match the tree clusters exactly. To account for the fact that a single cycle will appear in the neighborhood of many vertices in the sum above, we instead sum over the cycles in G and remove the factor $1/u(\Gamma)$. Formally, for each cycle C let $v(C)$ be a distinguished vertex on the cycle, and let $\mathrm{Cyc}_m(G)$ be the set of all cycles of length at most m of G. Then the sum over v in V in (39) can be rewritten as

$$W_n^{\mathrm{dis}}(m) \coloneqq \sum_{C \in \mathrm{Cyc}_m(G)} \left(\sum_{\substack{\Gamma \in \mathcal{C}_{\mathrm{dis}}^{v(C)}(G) \\ |C| \le \|\Gamma\| \le m}} w^{\mathrm{dis}}(\Gamma) - \sum_{\substack{\Gamma \in \mathcal{C}_{\mathrm{dis}}^{r}(\mathbb{T}_\Delta) \\ |C| \le \|\Gamma\| \le m}} w^{\mathrm{dis}}(\Gamma) \right).$$

If G satisfies conclusion (3) of Lemma 39 then there is at most one cycle in the depth-m neighborhood of each vertex, and hence

$$W_n^{\mathrm{dis}}(m) = \sum_{C \in \mathrm{Cyc}_m(G)} \left(\sum_{\substack{\Gamma \in \mathcal{C}_{\mathrm{dis}}^{r}(\mathbb{T}_\Delta^C) \\ |C| \le \|\Gamma\| \le m}} w^{\mathrm{dis}}(\Gamma) - \sum_{\substack{\Gamma \in \mathcal{C}_{\mathrm{dis}}^{r}(\mathbb{T}_\Delta) \\ |C| \le \|\Gamma\| \le m}} w^{\mathrm{dis}}(\Gamma) \right),$$

where $\mathbb{T}_\Delta^C$ is the Δ-regular tree rooted at the cycle C. We have used here that any cluster containing a polymer that is not contained in the m-neighborhood of r has size larger than m, so there is no need to truncate $\mathbb{T}_\Delta^C$ to a finite depth. Moreover, (39) can be rewritten as

$$\left| W_n^{\mathrm{dis}} - W_n^{\mathrm{dis}}(m) \right| \le \varepsilon/2. \tag{40}$$

By Lemma 16 we have that

$$\left| \sum_{\substack{\Gamma \in \mathcal{C}_{\mathrm{dis}}^{r}(\mathbb{T}_\Delta^C) \\ |C| \le \|\Gamma\| \le m}} w^{\mathrm{dis}}(\Gamma) - \sum_{\substack{\Gamma \in \mathcal{C}_{\mathrm{dis}}^{r}(\mathbb{T}_\Delta) \\ |C| \le \|\Gamma\| \le m}} w^{\mathrm{dis}}(\Gamma) \right| \le 2q^{-|C|/200\Delta}$$

Using Lemma 39 part (1) this means the expected contribution to the error from cycles of length at least T is at most

$$\mathbb{E} \left| \tilde{W}_n^{\mathrm{dis}}(T) - W_n^{\mathrm{dis}}(m) \right| \le \sum_{t \ge T} 2 \frac{(\Delta-1)^t}{2t} q^{-t/200\Delta}.$$

Then if $q \ge \Delta^{400\Delta}$, the expected contribution is at most Δ^{-T}. If we take $T = |\log_\Delta(\varepsilon^2/4)|$ then by Markov's inequality

$$\mathbb{P}\left[\left| \tilde{W}_n^{\mathrm{dis}}(t) - W_n^{\mathrm{dis}}(m) \right| \ge \frac{\varepsilon}{2} \right] \le \frac{\varepsilon}{2}.$$

for all $t \ge T$. Combining this with (40) we obtain

$$\mathbb{P}\left[\left| \tilde{W}_n^{\mathrm{dis}}(t) - W_n^{\mathrm{dis}} \right| \ge \varepsilon \right] \le \varepsilon,$$

for all $t \geq T$ as desired.

Part (2) of Proposition 38 for W_n^{ord} can be proven in the same way, and part (3) follows by combining the first two parts since the cycle counts are coupled identically. Next we show that $Q/q \to 1$ in probability as $q \to \infty$. It suffices to prove that

$$\mathbb{E}\left[\left|\sum_{k\geq 3}(\alpha_k^{\mathrm{ord}} - \alpha_k^{\mathrm{dis}})Y_k\right|\right] = o_q(1)$$

as $q \to \infty$. We can bound this by

$$\begin{aligned}
\mathbb{E}\left[\left|\sum_{k\geq 3}(\alpha_k^{\mathrm{ord}} - \alpha_k^{\mathrm{dis}})Y_k\right|\right] &\leq \sum_{k\geq 3}\mathbb{E}[Y_k]\left|\alpha_k^{\mathrm{ord}} - \alpha_k^{\mathrm{dis}}\right| \\
&\leq \sum_{k\geq 3}\frac{(\Delta-1)^k}{2k}\cdot 4q^{-\frac{k}{200\Delta}} \\
&\leq \sum_{k\geq 3}\exp\left[k\left(\log(\Delta-1) - \frac{\log q}{200\Delta}\right)\right] \\
&= o_q(1)
\end{aligned}$$

for $q = q(\Delta)$ sufficiently large, i.e., $q \geq \Delta^{400\Delta}$.

To conclude, observe that this last calculation (and exactly analogous computations for parts (1) and (2)) verifies the conditions of Kolmogorov's two-series theorem, implying the claimed almost sure absolute convergence. □

Proof of Theorem 2. Claim (1) follows by combining claims (1) and (2) of Proposition 38, and claim (3) is part of Proposition 38 claim (3).

Claim (2) follows from the stronger statements in the proof of Theorem 1 that for $\beta \leq \beta_1$, $\mu_{\mathrm{dis}}^n \xrightarrow{loc} \mu^{\mathrm{free}}$, and for $\beta \geq \beta_0$, $\mu_{\mathrm{ord}}^n \xrightarrow{loc} \mu^{\mathrm{wire}}$. That proof also implies that the conditional measures both exhibit exponential decay of correlations (and central limit theorems) at β_c. □

Proof of Theorem 3. The first two parts of this proposition are special cases of Proposition 38. The third part follows from the first two and Lemma 9 as in the proof of Theorem 2 above. □

7 Algorithms

The polymer models and estimates in Section 2 yield efficient approximate counting and sampling algorithms by adapting the polymer model algorithms from [43, 46, 9] to our current setup. In particular, if we assume $\varepsilon > e^{-n/2}/2$, then by Lemmas 20 and 22 and Corollary 23 it suffices to find an FPTAS for Ξ^{dis} when $\beta \leq \beta_1$ and for Ξ^{ord} when $\beta \geq \beta_0$, as well as polynomial-time sampling algorithms for $\overline{\nu}_{\mathrm{dis}}$ and $\overline{\nu}_{\mathrm{ord}}$.

7.1 Approximate counting

The approximate counting algorithms from [43, 46, 9] based on truncating the cluster expansion have two main requirements: 1) condition (18), or a similar statement giving an exponentially small error bound, holds, and 2) one can list all polymers of size at most m and compute their weight functions in time $\exp(O(m + \log n))$.

Lemma 40. *There is an algorithm that lists all ordered polymers of size at most m with running time $\exp(O(m + \log n))$.*

Proof. We can enumerate all connected edge sets of size at most m in time $n(e\Delta)^m$, and hence can create a list L of all labelled connected edge sets of size at most m in time $m \exp(O(m))$.

By Lemma 12 it takes time $\mathrm{poly}(K)$ to create the ordered polymer corresponding to set of K unoccupied edges. For each element (γ, ℓ) of L we apply the ordered polymer construction to the subset of edges of γ labelled 'unoccupied'. If this construction returns (γ, ℓ) we retain the element, otherwise we discard it. In time $\mathrm{poly}(m) \exp(O(m)) = \exp(O(m))$ we obtain a complete list of all ordered polymers. □

Proof of Theorem 4, FPTAS. Suppose $\varepsilon > e^{-n/2}$. Note that a polynomial-time algorithm to compute T_m^{ord} (as defined in (16)) with $m = O(\log n/\varepsilon)$ yields a polynomial-time algorithm for an ϵ-approximation to Ξ^{ord} by (18) whenever $\beta > \beta_0$, as in this case the cluster expansion converges. The same statement holds true for $\beta < \beta_1$ for approximation Ξ^{dis} by T_m^{dis}. In turn, Corollary 23 implies that this gives an FPTAS for Z. The existence of a polynomial-time algorithm to compute T_m^{ord} and T_m^{dis} can be seen as follows.

By [9, Lemma 2.2], an algorithm for computing T_m^{ord} (resp. T_m^{dis}) exists provided there are polynomial-time algorithms to

1. list all polymers of given size $m \le O(\log n)$,
2. compute the weights of all polymers of size $m \le O(\log n)$.

Listing and computing the weights of disordered polymers of size m in time exponential in m is elementary: all connected subgraphs on k edges can be listed in time $n(e\Delta)^k$, and the weights are given by an explicit formula.

Listing ordered polymers of a given size m in time exponential in m can be done by Lemma 40. To compute the weights in time polynomial in n requires computing the number $c'(\gamma)$ of connected components induced by a polymer. Since it takes time $|C(x)|$ to determine the connected component $C(x)$ of a vertex x, this can be done in time n for each polymer.

This completes the proof when $\varepsilon > e^{-n/2}$. When $\varepsilon \le e^{-n/2}$, one can obtain an FPTAS by brute-force enumeration, as the total number of configurations is $2^{\Delta n/2}$. □

7.2 Approximate sampling

Since an efficient sampling algorithm for the Potts model when q is a positive integer follows from an efficient algorithm for the random cluster model by the Edwards–Sokal coupling, we describe our efficient sampling algorithm only for the random cluster model.

Proof of Theorem 4, Sampling. We consider only $\varepsilon > e^{-n/2}$, as smaller ϵ can be handled by brute force.

When the disordered (respectively, ordered) cluster expansion converges we obtain an efficient approximate sampling algorithm for the measure $\bar{\nu}_{\text{dis}}$ (respectively, $\bar{\nu}_{\text{ord}}$) induced by the disordered polymer model by [43, Theorem 10]; note that we have verified the conditions of this theorem in the previous section. By Lemma 20, we thus obtain an efficient approximate sampling algorithm for μ_{dis}, the random cluster model conditional on the event that the configuration lies in Ω_{dis}, when $\beta \leq \beta_1$. Similarly we obtain efficient approximate sampling algorithms for μ_{ord} when $\beta \geq \beta_0$. By the approximate counting part of Theorem 4, which we have already proved, we can efficiently approximate the relative probabilities of Ω_{ord} and Ω_{dis}. We thus obtain an efficient approximate sampling algorithm for the q-random cluster model by Corollary 23. □

7.3 Application to random Δ-regular graphs

Corollary 5 follows directly from Theorem 4 and Proposition 37 in Section 5.

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