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On the log–local principle for the toric boundary

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Abstract

Let X be a smooth projective complex variety and let $D = D_1 + \cdots + D_l$ be a reduced normal crossing divisor on X with each component D_j smooth, irreducible and numerically effective. The log–local principle put forward in van Garrel et al. (*Adv. Math.* 350 (2019) 860–876) conjectures that the genus 0 log Gromov–Witten theory of maximal tangency of (X, D) is equivalent to the genus 0 local Gromov–Witten theory of X twisted by $\bigoplus_{j=1}^{l} \mathcal{O}(-D_j)$. We prove that an extension of the log–local principle holds for X a (not necessarily smooth) $\mathbb{Q}$-factorial projective toric variety, D the toric boundary, and descendant point insertions.

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1 | INTRODUCTION

Let X be a smooth projective complex variety of dimension n and let $D = D_1 + \cdots + D_l$ be an effective reduced normal crossing divisor with each component D_j smooth, irreducible and numerically effective. We can then consider two, a priori very different, geometries associated to the pair (X, D):

- the n-dimensional *log geometry* of the pair (X, D),
- the $(n+l)$-dimensional *local geometry* of the total space $\mathrm{Tot}(\bigoplus_{j=1}^{l} \mathcal{O}_X(-D_j))$.

The genus 0 log Gromov–Witten invariants of (X, D) virtually count rational curves

$$f : \mathbb{P}^1 \to X$$

of a fixed degree $f_*[\mathbb{P}^1] \in \mathrm{H}_2(X, \mathbb{Z})$, with insertions, such as passing through a number of general points, and with prescribed intersections with D. Such an f is said to be *of maximal tangency* if $f(\mathbb{P}^1)$ meets each D_j in only one point of full tangency. On the other hand, the local Gromov–Witten theory of $\mathrm{Tot}(\bigoplus_{j=1}^{l} \mathcal{O}_X(-D_j))$ is a way to study the local contribution of X to the enumerative geometry of a compact $(n+l)$-dimensional variety Y containing X with normal bundle $\bigoplus_{j=1}^{l} \mathcal{O}_X(-D_j)$.

The existence of a relation between the log and the local theory of (X, D) was introduced by the *log–local principle* of [17, Conjecture 1.4].

Conjecture 1. *Let* d *be an effective curve class such that* $\mathsf{d} \cdot D_j > 0$ *for all* $1 \leqslant j \leqslant l$. *After dividing by* $\prod_{j=1}^{l} (-1)^{\mathsf{d}\cdot D_j+1}\, \mathsf{d} \cdot D_j$, *the genus 0 log Gromov–Witten invariants of maximal tangency and class* d *of* (X, D) *equal the genus 0 local Gromov–Witten invariants of class* d *of* $\mathrm{Tot}(\bigoplus_{j=1}^{l} \mathcal{O}_X(-D_j))$ *(with the same insertions).*

Theorem 1.1 [17]. *The log–local principle holds if X is a smooth projective variety and D is smooth and numerically effective.*

There are two natural directions to generalise the log–local principle further. The first is to investigate extensions to correspondences with other invariants. At the level of BPS invariants [12–14, 19, 29], this is proven for the pair of $\mathbb{P}^2$ and smooth cubic in [5, 6] and in higher genus in [9]. In [7, 8], we extend the correspondences to the non-toric and higher genus/refined setting and include open Gromov–Witten invariants, their underlying open BPS counts, as well as quiver Donaldson–Thomas invariants to the set of correspondences. Another direction is the relationship between local and orbifold invariants [3, 30].

The second natural question is to what extent the log–local principle generalises to the case when X and D are not smooth: log Gromov–Witten theory is indeed well-defined for any pair (X, D) which is log smooth, but it is unclear how to define a local geometry in such generality. In the present paper, we consider a situation that goes beyond the smoothness assumptions of Conjecture 1 and where both log and local sides can be defined: we take for X a $\mathbb{Q}$-factorial projective toric variety and for D the toric boundary divisor of X. As X is $\mathbb{Q}$-factorial, it makes sense to require that the components D_j of D are numerically effective. We show in Proposition 2.1 that requiring each D_j to be numerically effective forces X to be a product of fake weighted projective spaces. While such an X is not necessarily smooth, and D is typically not normal crossing, (X, D) can naturally be viewed as a log smooth variety, and so log Gromov-Witten invariants of (X, D) are well-defined. On the other hand, X can be naturally viewed as a smooth Deligne–Mumford stack, and the local geometry $\mathrm{Tot}(\bigoplus_{j=1}^{l} \mathcal{O}_X(-D_j))$ makes sense in the category of orbifolds. The local Gromov–Witten invariants can be defined using orbifold Gromov–Witten theory [2], and it thus makes sense to ask if the genus 0 log invariants of maximal tangency of such a pair (X, D) are related in the sense of Conjecture 1 to the corresponding local invariants. Our main result is Theorem 1.2; we refer to Theorems 3.1–3.4 for precise statements.

Theorem 1.2. *Let X be a $\mathbb{Q}$-factorial projective toric variety and let D be the toric boundary divisor of X. Assume that all the components D_j of D are numerically effective. Then the genus 0 log Gromov–Witten invariants of maximal tangency of (X, D), and the genus 0 local Gromov–Witten invariants of (X, D), both with descendant point insertions, can be computed in closed form for all degrees. As a corollary, the log–local principle holds for the resulting invariants.*

Except for the well-studied case when $X = \mathbb{P}^1$, the log and local Gromov–Witten invariants of (X, D) are non-zero only for one, two or, provided $X = (\mathbb{P}^1)^n$, three point insertions. For $X = (\mathbb{P}^1)^n$ we prove an equality of virtual fundamental classes and refer to well-known techniques to compute the invariants. For the other cases, our proof proceeds by calculating both sides to obtain explicit closed formulae for these invariants for all (X, D) (Theorems 3.2 and 3.3). To compute the log invariants we use the tropical correspondence result [24] and an algorithm of [23] for the tropical multiplicity. The log–local principle of Conjecture 1 then predicts an explicit formula in all degrees for the local invariants, which we verify using local mirror symmetry techniques and a reconstruction result from small to big quantum cohomology.

Relation to [26] and [7, 8]

After this paper was finished, we received the manuscript [26] where the log–local principle is considered for simple normal crossings divisors. The respective strategies have different flavours in the proof and complementary virtues in the outcome: [26] consider the log/local correspondence for X smooth and D_j a hyperplane section, with a beautiful geometric argument reducing the simple normal crossings case to the case of smooth pairs, and with no restrictions on X. The combinatorial pathway we pursued in the toric setting allows on the other hand to relax the hypotheses on the smoothness of X, the normal crossings nature of D, and the very ampleness of D_j, and it lends itself to a wider application to the case when D is not the toric boundary and the refinement to include all-genus invariants. We consider this specifically in the follow-up papers [7, 8], where we prove the log–local principle for log Calabi–Yau surfaces with the components of the anticanonical divisor smooth and numerically effective and suitably reformulate it to, and verify it for, the higher genus theory in these cases. In addition, we extend the correspondences to include open Gromov–Witten invariants, the various underlying BPS counts, and quiver Donaldson–Thomas invariants.

Remark 1. In its most recent version, [26] gives a counter-example in principle to Conjecture 1. It is proven that there is a choice of (unspecified) insertion leading to a counter-example. The geometry however is not log Calabi–Yau and the insertion is not formed of point insertions. It remains open whether the conjecture holds in the more restrictive setting of a log Calabi–Yau variety with only point insertions. The present paper as well as [7, 8] provide evidence for it.

2 | SET-UP

2.1 | Notation

Let X be a $\mathbb{Q}$-factorial projective toric variety of dimension n_X and let $D = D_1 + \cdots + D_{l_D}$ be the toric boundary divisor of X. In the foregoing discussion, we write $r_X := \operatorname{rank} \operatorname{Pic}(X)$ for the rank of the Picard group of X, so that $l_D = n_X + r_X$, and $\chi_X = \chi(X) := \dim_{\mathbb{C}} H(X, \mathbb{C})$ for the dimension of the cohomology of X. The variety X has a natural presentation as a GIT quotient $\mathbb{C}^{n_X + r_X} /\!\!/_t ((\mathbb{C}^\star)^{r_X} \times G_X)$ for G_X a finite abelian group; for every $1 \leqslant j \leqslant l_D$, we write D_j for the divisor corresponding to the $(\mathbb{C}^\star)^{r_X} \times G_X$ reduction to X of the jth coordinate hyperplane in $\mathbb{C}^{n_X + r_X}$. Note in particular that $\sum_{j=1}^{l_D} D_j = -K_X$.

We also fix a further piece of notation, which will turn out to be convenient when dealing with the book-keeping of indices for products of fake weighted projective spaces. Let $m \in \mathbb{N}_0$. If $\mathsf{v} = (v_1, \dots, v_m) \in \mathbb{N}^m$ is a lattice point in the non-negative m-orthant, we write $|\mathsf{v}| = \sum_{i=1}^m v_i$ for its 1-norm; in the following we will consistently use serif fonts for orthant points and italic fonts for their Cartesian coordinates. For R a finitely generated commutative monoid with generators $\alpha_1, \dots, \alpha_m$, $x = \alpha_1^{j_1} \dots \alpha_m^{j_m} \in R$ a reduced word in α_i, and $\mathsf{v} \in \mathbb{N}^m$, we write x^{v} for the product $\prod_i \alpha_i^{j_i v_i} \in R$. We introduce partial orders on the m-orthant by saying that $\mathsf{v} \prec \mathsf{w}$ (respectively, $\mathsf{v} \preceq \mathsf{w}$) if $v_i < w_i$ (respectively, $v_i \leqslant w_i$) for all $i = 1, \dots, m$. Also, we will write $Q^X_{ij} \in \mathbb{Z}$, $i = 1, \dots, r_X$, $j = 1, \dots, n_X + r_X$, for the weight of the ith factor of the $(\mathbb{C}^\star)^{r_X}$ torus action on the jth affine factor of $\mathbb{C}^{n_X + r_X}$.

Definition 1. A numerically effective toric pair (X, D) is a pair given by X a $\mathbb{Q}$-factorial complex projective toric variety with toric boundary divisor $D = D_1 + \dots + D_{l_D}$, such that all the components D_j are numerically effective.

Numerical effectiveness of all the components D_j of the toric bundary divisor imposes strong conditions on X, as the Proposition 2.1 shows.

Definition 2. Let X be a $\mathbb{Q}$-factorial projective toric variety, and let $\mathbb{C}^{n_X + r_X} /\!\!/_t ((\mathbb{C}^\star)^{r_X} \times G_X)$ be its natural GIT description. We say that X is a fake weighted projective space if $\mathbb{C}^{n_X + r_X} /\!\!/_t (\mathbb{C}^\star)^{r_X}$ is a weighted projective space.

Proposition 2.1. *Let X be a $\mathbb{Q}$-factorial projective variety such that every effective divisor on X is numerically effective. Then X is a product of fake weighted projective spaces.*

Proof. By [16, Proposition 5.3], X admits a finite surjective toric morphism $\prod \mathbb{P}^{n_i} \to X$. Let $\Sigma \subset N \otimes \mathbb{R}$ be the fan of $\prod \mathbb{P}^{n_i}$ and $\Sigma' \subset N' \otimes \mathbb{R}$ the fan of X. Then we have an injective morphism of lattices $N \to N'$ of finite index. Identifying N with its image in N', $\Sigma = \Sigma'$. It follows that X is the quotient of $\prod \mathbb{P}^{n_i}$ by N'/N. Hence, X is a product of fake weighted projective spaces. $\square$

By Proposition 2.1, there is $\mathsf{n}_X \in \mathbb{N}^{r_X}$ such that $n_X = |\mathsf{n}_X|$ and X is a product of r_X, $n_i := (\mathsf{n}_X)_i$-dimensional fake weighted projective spaces,

$$X = \prod_{i=1}^{r_X} \mathbb{P}^{G_i}\left(\mathsf{w}_X^{(i)}\right),$$

with $\mathsf{w}_X^{(i)} = ((\mathsf{w}_X)_1^{(i)}, \dots, (\mathsf{w}_X)_{n_i+1}^{(i)}) \in \mathbb{N}^{n_i+1}$, which we may assume not to have any common factors, and

$$\mathbb{P}^{G_i}\left(\mathsf{w}_X^{(i)}\right) := \mathbb{P}\left(\mathsf{w}_X^{(i)}\right) /\!\!/_t G_i,$$

for G_i a finite abelian group. Note that, for fixed i and defining $\varepsilon_i := \sum_{k=1}^{i-1} (n_k + 1)$, we have

$$Q^X_{i, j+\varepsilon_i} = \begin{cases} (\mathsf{w}_X)_j^{(i)} & 1 \leqslant j \leqslant n_i + 1, \\ 0 & \text{else}, \end{cases} \tag{2.1}$$

independent of the G_i. Let $H_i := \mathrm{pr}_i^* c_1(\mathcal{O}_{\mathbb{P}^{G_i}(\mathsf{w}^{(i)})}(1))$ denote the pullback to X of the (orbi-) hyperplane class of the ith factor of X and let $H := H_1 \dots H_{r_X}$. These generate the classical cohomology ring,

$$\mathrm{H}^\bullet(X, \mathbb{C}) = \frac{\mathbb{C}[H_1, \dots, H_{r_X}]}{\left\langle \left\{ H_i^{n_i+1} \right\}_{i=1}^{r_X} \right\rangle}, \tag{2.2}$$

which is independent of the G_i, and we can take a homogeneous linear basis for $H^\bullet(X, \mathbb{C})$ in the form $\{H^{\mathsf{l}}\}_{\mathsf{l}_i \leqslant n_i}$. Note, in particular, that

$$[\mathrm{pt}] = \prod_{i=1}^{r_X} |G_i| \prod_{i,j} (\mathsf{w}_X)_j^{(i)} H^{\mathsf{n}_X}.$$

Indeed, if G_i is trivial, this follows from applying [22, Theorem 1] to each component in the product; and if G_i is non-trivial, then the extra factor comes from the component-wise identification $H^\bullet(\mathbb{P}^{G_i}(\mathsf{w}^{(i)}), \mathbb{C}) = H^\bullet(\mathbb{P}(\mathsf{w}^{(i)}), \mathbb{C})^{G_i}$. We will also write $\mathsf{d} = (d_1, \dots, d_{r_X})$ for the curve class $d_1 H_1 + \dots + d_r H_r$ and

$$\mathsf{d}^{\mathsf{n}_X} := \prod_{i=1}^{r_X} d_i^{n_i}. \tag{2.3}$$

We order the toric divisors D_j of X, $j = 1, \dots, |\mathsf{n}_X| + r_X$, in such a way that

$$Q_{ij}^X = (0, \dots, 0, \overset{i^{\text{th}}}{1}, 0, \dots, 0) \cdot D_j,$$

where the 1 is in the ith position. Finally, we define

$$e_j^X(\mathsf{d}) := \sum_i Q_{ij}^X d_i = \mathsf{d} \cdot D_j, \quad e^X(\mathsf{d}) := \sum_{j=1}^{|\mathsf{n}_X| + r_X} e_j^X(\mathsf{d}) = -\mathsf{d} \cdot K_X. \tag{2.4}$$

2.2 | Log Gromov–Witten invariants

Let (X, D) be a numerically effective toric pair and let d be an effective curve class on X.[†] For the definition of log Gromov–Witten invariants, we endow[‡] X with the divisorial log structure coming from D, and view (X, D) as a log smooth variety. The log structure is used to impose tangency conditions along the components D_j of D: in this paper we consider genus 0 stable maps into X of class d that meet each component D_j in one point of maximal tangency $\mathsf{d} \cdot D_j$. The appropriate moduli space $\overline{\mathrm{M}}_{0,m}^{\log}(X, D, \mathsf{d})$ of genus 0 m-marked maximally tangent stable log maps was constructed (in all generality) in [1, 10, 20]. In this description, we have m marked points that have tangency 0

[†] Note that unlike in Conjecture 1 we do not require that $\mathsf{d} \cdot D_j > 0$ for all $1 \leqslant j \leqslant l_D$.

[‡] We refer to [18] for an introduction to log geometry.

with the boundary (interior marked points), and l_D marked points with maximal tangency with each D_j, respectively. In case $\mathsf{d} \cdot D_j = 0$ for some j, this means that the corresponding maximal tangency marked point is an interior marked point. There is a virtual fundamental class

$$[\overline{\mathrm{M}}^{\log}_{0,m}(X, D, \mathsf{d})]^{\mathrm{vir}} \in \mathrm{H}_{2\mathfrak{vdim}^{(X,D,\mathsf{d})}_{\log}}(\overline{\mathrm{M}}^{\log}_{0,m}(X, D, \mathsf{d})),$$

where

$$\begin{aligned}\mathfrak{vdim}^{(X,D,\mathsf{d})}_{\log} &= -\mathsf{d} \cdot K_X + \dim X - 3 + m - \sum_{j=1}^{l_D} (\mathsf{d} \cdot D_j - 1) \\ &= n_X + m + l_D - 3 = 2n_X + r_X + m - 3.\end{aligned}$$

Evaluating at the marked points p_i yields the evaluation maps

$$\mathrm{ev}_i : \overline{\mathrm{M}}^{\log}_{0,m}(X, D, \mathsf{d}) \longrightarrow X.$$

For L_i the ith tautological line bundle on $\overline{\mathrm{M}}^{\log}_{0,m}(X, D, \mathsf{d})$, whose fiber at $[f : (C, p_1, \dots, p_m) \to X]$ is the cotangent line of C at p_i, there are tautological classes $\psi_i := c_1(L_i)$. We are interested in the calculation of the genus 0 log Gromov–Witten invariants of maximal tangency of (X, D) with one, two or three point insertions and ψ-class insertions at one point, defined as follows:

$$R\mathfrak{p}^X_{\mathsf{d}} := \int_{[\overline{\mathrm{M}}^{\log}_{0,1}(X,D,\mathsf{d})]^{\mathrm{vir}}} \mathrm{ev}_1^*([\mathrm{pt}]) \cup \psi_1^{n_X + r_X - 2}, \tag{2.5}$$

$$R\mathfrak{q}^X_{\mathsf{d}} := \int_{[\overline{\mathrm{M}}^{\log}_{0,2}(X,D,\mathsf{d})]^{\mathrm{vir}}} \mathrm{ev}_1^*([\mathrm{pt}]) \cup \mathrm{ev}_2^*([\mathrm{pt}]) \cup \psi_2^{r_X - 1}. \tag{2.6}$$

The invariant $R\mathfrak{p}^X_{\mathsf{d}}$ (respectively, $R\mathfrak{q}^X_{\mathsf{d}}$) is a virtual count of rational curves in X of degree $\mathsf{d} = (d_1, \dots, d_{r_X})$ that meet each toric divisor D_j in one point of maximal tangency $\mathsf{d} \cdot D_j = \sum_{i=1}^r d_i Q^X_{ij} = e^X_j(\mathsf{d})$ and that pass through one point in the interior with $\psi^{n_X + r_X - 2}$ condition (respectively, two points in the interior, one of which with a $\psi^{r_X - 1}$ condition).

Remark 1. Having a point condition on X cuts down the dimension of the moduli space by n_X. Thus, (2.5) and (2.6) cover all possible invariants with descendant point insertions except for two families of cases. For the first, one distributes the descendant insertions along both points in (2.6). Adapting the log calculations of Section 5 to that case is left as an exercise to the reader, see also Remark 3 for the local side. The second family of cases concerns the invariants of $(\mathbb{P}^1)^n$ with any number of marked points if $n = 1$ and up to three marked points if $n \geqslant 2$. We treat $(\mathbb{P}^1)^n$ separately in Theorem 3.1.

2.3 Local Gromov–Witten invariants

Let (X, D) be a numerically effective toric pair as in Definition 1 and write $X^{\mathrm{loc}}_D := \mathrm{Tot}(\bigoplus_i \mathcal{O}_X(-D_i))$ for the target space of the local theory. By Proposition 2.1, we can view X and

$X_D^{\rm loc}$ as the coarse moduli schemes of smooth Deligne–Mumford stacks $\mathcal{X}$ and $\mathcal{X}_D^{\rm loc}$ over $\mathbb{C}$, where

$$\begin{aligned} \mathcal{X} &:= \bigtimes_{i=1}^{r_X} \left[\left(\mathbb{C}^{(n_X)_i} \setminus \{0\} \right) / (\mathbb{C}^\star \times G_i) \right], \\ \mathcal{X}_D^{\rm loc} &:= \bigtimes_{i=1}^{r_X} \left[\left(\left(\mathbb{C}^{(n_X)_i} \setminus \{0\} \right) \times \mathbb{C}^{(n_X)_i+1} \right) / (\mathbb{C}^\star \times G_i) \right]. \end{aligned} \tag{2.7}$$

Even though $X_D^{\rm loc}$ is not proper and may be singular, the locution ‘Gromov–Witten theory of $X_D^{\rm loc}$’ receives a meaning in terms of the orbifold Gromov–Witten theory of $\mathcal{X}$ twisted by $\bigoplus_i \mathcal{O}_{\mathcal{X}}(-D_i)$ [2, 11] and restricted over its non-stacky part, and we refer the reader in particular to [2] for the relevant background on the Gromov–Witten theory of Deligne–Mumford stacks. Let $\overline{\mathrm{M}}_{0,m}(\mathcal{X}, \mathsf{d})$ be the moduli stack of twisted genus 0 m-marked stable maps $[f : C \to \mathcal{X}]$ with $f_*([C]) = \mathsf{d} \in H_2(\mathcal{X}, \mathbb{Q})$, where C is an m-pointed twisted curve[†] [2], and write $\overline{\mathrm{M}}_{0,m}(X, \mathsf{d})$ for the substack of twisted stable maps such that the image of all evaluation maps is contained in the zero-age component of the (rigidified, cyclotomic) inertia stack of $\mathcal{X}$. The stack $\overline{\mathrm{M}}_{0,m}(\mathcal{X}, \mathsf{d})$ can be equipped with a virtual fundamental class [2, Section 4.5], which induces a virtual fundamental class of pure homological degree over the stack $\overline{\mathrm{M}}_{0,m}(X, \mathsf{d})$ of stable maps to the coarse moduli space,

$$\left[\overline{\mathrm{M}}_{0,m}(X, \mathsf{d})\right]^{\rm vir} \in \mathrm{H}_{2\mathfrak{vdim}^{(X,D,\mathsf{d})}}(\overline{\mathrm{M}}_{0,m}(X, \mathsf{d}), \mathbb{Q}),$$

where

$$\mathfrak{vdim}^{(X,D,\mathsf{d})} := -K_X \cdot \mathsf{d} + \dim X + m - 3 = \mathfrak{vdim}_{\log}^{(X,D,\mathsf{d})} + e_X(\mathsf{d}) - l_D.$$

Let now d be such that $\mathsf{d} \cdot D_j > 0$ for all $1 \leqslant j \leqslant l_D$. Then $\mathrm{H}^0(C, f^* \bigoplus_{j=1}^{l_D} \mathcal{O}_X(-D_j)) = 0$ for every twisted stable map $[f : C \to \mathcal{X}]$ with $f_*([C]) = \mathsf{d}$, and so $\mathrm{Ob}_D := R^1\pi_* f^*(\bigoplus_{j=1}^{l_D} \mathcal{O}_X(-D_j))$ is a vector bundle on $\overline{\mathrm{M}}_{0,m}(\mathcal{X}, \mathsf{d})$, which is of rank $\sum_{j=1}^{l_D} (\mathsf{d} \cdot D_j - 1)$ and has fibre $\mathrm{H}^1(C, f^* \bigoplus_{j=1}^{l_D} \mathcal{O}_X(-D_j))$ at a stable map $[f : C \to \mathcal{X}]$. Restricting to the zero-age component defines the virtual fundamental class

$$[\overline{\mathrm{M}}_{0,m}(X_D^{\rm loc}, \mathsf{d})]^{\rm vir} := [\overline{\mathrm{M}}_{0,m}(X, \mathsf{d})]^{\rm vir} \cap c_{\rm top}(\mathrm{Ob}_D) \in \mathrm{H}_{2(\mathfrak{vdim}^{(X,D,\mathsf{d})} + l_D - e_X(\mathsf{d}))}(\overline{\mathrm{M}}_{0,m}(X, \mathsf{d}), \mathbb{Q}), \tag{2.8}$$

and we have

$$\mathrm{vdim}\, \overline{\mathrm{M}}_{0,m}(X_D^{\rm loc}, \mathsf{d}) = \mathfrak{vdim}^{(X,D,\mathsf{d})} - e_X(\mathsf{d}) + l_D = \mathfrak{vdim}_{\log}^{(X,D,\mathsf{d})}.$$

The restriction to the untwisted sector gives well-defined evaluation maps $\mathrm{ev}_i : \overline{\mathrm{M}}_{0,m}(X, \mathsf{d}) \longrightarrow X$, and there are tautological classes $\psi_i := c_1(L_i)$, where the fibre of L_i at a stable map $[f : C \to \mathcal{X}]$ is given by the cotangent line to the coarse moduli space of C at the ith point. The (untwisted)

[†] This means that the coarse moduli space of C is a pre-stable curve in the ordinary sense, with cyclic-quotient stackiness allowed at special points, and satisfying kissing (balancing) conditions for the stacky structures at the nodes. See [2, Section 4] for more details.

local Gromov–Witten invariants of (X,D) are then caps of pullbacks of classes in $H^\bullet(X,\mathbb{C})$ via the evaluation maps against the virtual fundamental class (2.8). In particular, the local counterparts of (2.5) and (2.6) are defined by

$$\mathfrak{p}^X_{\mathsf{d}} := \int_{[\overline{\mathrm{M}}_{0,1}(X^{\mathrm{loc}}_D,\mathsf{d})]^{\mathrm{vir}}} \mathrm{ev}_1^*([\mathrm{pt}]) \cup \psi_1^{n_X+r_X-2}, \tag{2.9}$$

$$\mathfrak{q}^X_{\mathsf{d}} := \int_{[\overline{\mathrm{M}}_{0,2}(X^{\mathrm{loc}}_D,\mathsf{d})]^{\mathrm{vir}}} \mathrm{ev}_1^*([\mathrm{pt}]) \cup \mathrm{ev}_2^*([\mathrm{pt}]) \cup \psi_2^{r_X-1}. \tag{2.10}$$

3 | MAIN RESULTS

We first consider the case of $(\mathbb{P}^1)^n$ and treat the general case thereafter.

Theorem 3.1. *Conjecture 1 holds for $X = (\mathbb{P}^1)^n$ with its toric boundary.*

Proof. For $X = \mathbb{P}^1$, the log–local principle (at the level of the virtual fundamental classes) is a direct consequence of [17] since the toric divisors are disjoint. For $X = (\mathbb{P}^1)^n$ with $n \geqslant 2$, we apply the log product formula [21, 28] on the log side and the product formula [4] on the local side to obtain an equality of virtual fundamental classes. $\square$

Note that computational techniques to compute the invariants of $X = (\mathbb{P}^1)^n$ (with arbitrary numbers of point insertions if $n = 1$) are well-developed. For example, using tropical correspondence results one may show that the maximal tangency three-pointed invariants of $(\mathbb{P}^1)^n$ are $\prod_{j=1}^{2n} \mathsf{d} \cdot D_j = \prod_{i=1}^{n} d_i^2$.

Theorems 3.2 and 3.3 compute the log and local Gromov–Witten invariants defined in Sections 2.2 and 2.3 in all degrees for a numerically effective toric pair (X,D).

Theorem 3.2. *Let (X,D) be a numerically effective toric pair and let d be an effective curve class on X. If there is j such that $\mathsf{d} \cdot D_j = 0$, then $R\mathfrak{p}^X_{\mathsf{d}} = R\mathfrak{q}^X_{\mathsf{d}} = 0$. If $\mathsf{d} \cdot D_j > 0$ for all $1 \leqslant j \leqslant l_D$, then we have*

$$R\mathfrak{p}^X_{\mathsf{d}} = 1, \tag{3.1}$$

$$R\mathfrak{q}^X_{\mathsf{d}} = \prod_{i=1}^{r_X} |G_i| \left(\prod_{i,j} (\mathsf{w}_X)_j^{(i)} \right) \mathsf{d}^{\mathsf{n}_X}. \tag{3.2}$$

We write $\mathring{\prod}_j e_j^X(\mathsf{d})$ to mean the product of $e_j^X(\mathsf{d})$ over $j \in \{1, \dots, |\mathsf{n}_X| + r_X \mid e_j^X(\mathsf{d}) \neq 0\}$.

Theorem 3.3. *Let (X,D) be a numerically effective toric pair and let d be an effective curve class on X. Then*

$$\mathfrak{p}^X_{\mathsf{d}} = \frac{(-1)^{e^X(\mathsf{d}) - n_X - r_X}}{\mathring{\prod}_j e_j^X(\mathsf{d})}, \tag{3.3}$$

$$\begin{aligned}\mathfrak{q}^X_{\mathsf{d}} &= \prod_{i=1}^{r_X} |G_i| \left(\prod_{i,j} (\mathsf{w}_X)^{(i)}_j \right) \mathsf{d}^{\mathsf{n}_X} \mathfrak{p}^X_{\mathsf{d}} \\ &= \prod_{i=1}^{r_X} |G_i| \left(\prod_{i,j} (\mathsf{w}_X)^{(i)}_j \right) \mathsf{d}^{\mathsf{n}_X} \frac{(-1)^{e^X(\mathsf{d}) - n_X - r_X}}{\prod^{\circ}_j e^X_j(\mathsf{d})}. \end{aligned} \tag{3.4}$$

We deduce from these the log–local principle proved in the present paper.

Theorem 3.4. *The log–local principle holds for numerically effective toric pairs (X, D) with descendant point insertions and with no assumptions on $\mathsf{d} \cdot D_j$. That is, for every effective curve class d, the log and local invariants are equal up to the factor*

$$\prod_{j=1}^{l_D} (-1)^{\mathsf{d} \cdot D_j + 1}\, \mathsf{d} \cdot D_j = (-1)^{e^X(\mathsf{d}) - |\mathsf{n}_X| - r_X} \prod_{j=1}^{|\mathsf{n}_X| + r_X} e^X_j(\mathsf{d}).$$

Theorem 3.4 is a direct corollary of the combination of Theorems 3.1–3.3. We will prove Theorem 3.2 using a tropical correspondence principle, and Theorem 3.3 using an equivariant mirror theorem. We review these technical tools in Section 4, and explain how to apply them to the proofs of Theorems 3.2 and 3.3 in Sections 5 and 6, respectively.

4 | COMPUTATIONAL METHODS

4.1 | The log side: Tropical curve counts

Let $X = \prod_{i=1}^{r_X} \mathbb{P}^{G_i}(\mathsf{w}^{(i)})$ as in Section 2.1 and let $\Sigma \subset N_{\mathbb{R}}$ be the fan of $X = X_\Sigma$; here $N \simeq \mathbb{Z}^{|\mathsf{n}_X|}$ and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Define furthermore $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ and let $M := \mathrm{Hom}(N, \mathbb{Z})$ be the dual of N. Denote by $[D_1], \ldots, [D_{|\mathsf{n}_X| + r_X}]$ the rays of Σ corresponding to the irreducible effective toric divisors of X. We use correspondence results with tropical curve counts as developed in [24, 25, 27] (see [18] for an introduction) and state them in the generality needed for our purposes.

Denote by $\overline{\Gamma}$ the topological realisation of a finite connected graph and by Γ the complement of a subset of 1-valent vertices. We require that Γ has no univalent and no bivalent vertices. The set of its vertices, edges, non-compact edges and compact edges is denoted by $\Gamma^{[0]}$, $\Gamma^{[1]}$, $\Gamma^{[1]}_\infty$ and $\Gamma^{[1]}_c$, respectively. Γ comes with a weight function $w : \Gamma^{[1]} \to \mathbb{Z}_{\geqslant 0}$. The non-compact edges come with markings. Weight 0, respectively, positive weight, non-compact edges are *interior*, respectively, *exterior*, markings. There will be one or two interior point markings, which we denote by P_1 and P_2, and $|\mathsf{n}_X| + r_X$ exterior markings corresponding to the toric divisors, which we denote by $[D_1], \ldots, [D_{|\mathsf{n}_X| + r_X}]$ as well.

Definition 3. A *genus 0 degree* d *maximally tangent parametrised marked tropical curve* in X consists of Γ as above and a continuous map $h : \Gamma \to N_{\mathbb{R}}$ satisfying the following.

(i) For $E \in \Gamma^{[1]}$, $h|_E$ is constant if and only if $w(E) = 0$. Otherwise, $h|_E$ is a proper embedding into an affine line with rational slope.

(ii) Let $V \in \Gamma^{[0]}$ with $h(V) \in N_{\mathbb{Q}}$. For edges $E \ni V$, denote by $u_{(V,E)}$ the primitive integral vector at $h(V)$ into the direction $h(E)$ (and set $u_{(V,E)} = 0$ if $w(E) = 0$). The *balancing condition* holds:

$$\sum_{E \ni V} w(E)\, u_{(V,E)} = 0.$$

(iii) For each exterior marking D_j, $h|_{D_j}$ is parallel to the ray $[D_j]$ and $w(D_j) = \mathsf{d} \cdot D_j$.
(iv) The first Betti number $b_1(\Gamma) = 0$.

If (Γ', h') is another such parametrised tropical curve, then an *isomorphism* between the two is given by a homeomorphism $\Phi : \Gamma \to \Gamma'$ respecting the discrete data and such that $h = h' \circ \Phi$. A *genus 0 degree* d *maximally tangent marked tropical curve* then is an isomorphism class of such.

Moreover, we say that an interior marking E satisfies a ψ^k*-condition* if $h(E)$ is a $k+2$-valent vertex.

Denote by $\mathrm{T}(\mathfrak{p})_{\mathsf{d}}^X$ the (moduli) space of genus 0 degree d maximally tangent tropical curves in X with the interior marking equipped with a $\psi^{|\mathsf{n}_X|+r_X-2}$-condition passing through a fixed general point in $\mathbb{R}^{|\mathsf{n}_X|+r_X}$. Denote by $\mathrm{T}(\mathfrak{q})_{\mathsf{d}}^X$ the moduli space of genus 0 degree d maximally tangent tropical curves in X with the two interior markings P_1 and P_2 mapping to two fixed general points in $\mathbb{R}^{|\mathsf{n}_X|+r_X}$ and such that P_2 has a ψ^{r_X-1}-condition. We will see in Theorems 5.1 and 5.4 that each of $\mathrm{T}(\mathfrak{p})_{\mathsf{d}}^X$ and $\mathrm{T}(\mathfrak{q})_{\mathsf{d}}^X$ consist of one element. Since $\mathrm{T}(\mathfrak{p})_{\mathsf{d}}^X$ and $\mathrm{T}(\mathfrak{q})_{\mathsf{d}}^X$ are finite hence, their elements are *rigid* [23, Definition 2.5].

Counts of tropical curves are weighted with appropriate multiplicities. There are a number of ways of defining the multiplicity $\mathrm{Mult}(\Gamma)$ of Γ. The version we use was formulated (for X smooth) in [23, Theorem 1.2]. We state it for our setting. Set $A := \mathbb{Z}[N] \otimes_{\mathbb{Z}} \Lambda^\bullet M$. For $n \in N$ and $\alpha \in \Lambda^\bullet M$, write $z^n \alpha$ for $z^n \otimes \alpha$ and $\iota_n \alpha$ for the contraction of α by n. Recall that if $\alpha \in \Lambda^s M$, then $\iota_n \alpha \in \Lambda^{s-1} M$. For $k \geqslant 1$, define $\ell_k : A^{\otimes k} \to A$ via

$$\ell_k(z^{n_1}\alpha_1 \otimes \cdots \otimes z^{n_k}\alpha_k) := z^{n_1+\cdots+n_k}\iota_{n_1+\cdots+n_k}(\alpha_1 \wedge \cdots \wedge \alpha_k).$$

Let now $h : \Gamma \to N_{\mathbb{R}}$ be in $\mathrm{T}(\mathfrak{p})_{\mathsf{d}}^X$ or $\mathrm{T}(\mathfrak{q})_{\mathsf{d}}^X$ and choose a vertex V_∞ of Γ. Consider the flow on Γ with sink vertex V_∞. To each edge E of Γ, we inductively associate an element $\zeta_E = z^{n_E}\alpha_E \in A$, well-defined up to sign:

- for the exterior markings, set $\zeta_{D_j} = z^{w(D_j)\Delta(j)}$, where $\Delta(j)$ is the primitive generator of $[D_j]$;
- for an interior marking P, set ζ_P to be one of the two generators of $\Lambda^{|\mathsf{n}_X|}M$;
- if $E_1, \dots, E_k$ are the edges flowing into a vertex $V \neq V_\infty$ and E_{out} is the edge flowing out, set $\zeta_{E_{\mathrm{out}}} = \ell_k(\zeta_{E_1} \otimes \cdots \otimes \zeta_{E_k})$.

By [23, Theorem 1.2], $\zeta_\Gamma := \prod_{E \ni V_\infty} \zeta_E \in z^0 \otimes \Lambda^{|\mathsf{n}_X|}M$ and $\mathrm{Mult}(\Gamma)$ is the index of ζ_Γ in $\Lambda^{|\mathsf{n}_X|}M$. It then follows from [24, Theorem 1.1] that $R\mathfrak{p}_{\mathsf{d}}^X$ is the number of Γ in $\mathrm{T}(\mathfrak{p})_{\mathsf{d}}^X$ counted with multiplicity $\mathrm{Mult}(\Gamma)$, and $R\mathfrak{q}_{\mathsf{d}}^X$ is the weighted cardinality of $\{\Gamma \in \mathrm{T}(\mathfrak{q})_{\mathsf{d}}^X\}$, each weighted by $\mathrm{Mult}(\Gamma)$.

Remark 2. Note that a priori [24, Theorem 1.1] is stated for smooth varieties; in the cases of interest to us, however, the curves never meet the deeper toric strata and the arguments of [24] carry through.

4.2 | The local side: Mirror symmetry for toric stacks

The second technical result we will use for the calculation of local Gromov–Witten invariants is Theorem 4.1. Consider a torus $T \simeq \mathbb{C}^\star$ acting on $X_D^{\mathrm{loc}} := \mathrm{Tot}(\bigoplus_i \mathcal{O}_X(-D_i))$ transitively on the fibres and covering the trivial action on the image of the zero section. We will denote by $\lambda := c_1(\mathcal{O}_{\mathbb{P}^\infty}(1))$ the polynomial generator of the T-equivariant cohomology of a point, $\mathrm{H}_T(\mathrm{pt}) = \mathrm{H}(BT) \simeq \mathbb{C}[\lambda]$. The basis elements H^{l} of Section 2.1 for the cohomology of X have canonical T-equivariant lifts, which by a slight abuse of notation we denote with the same symbol, to cohomology classes in X_D^{loc} forming a $\mathbb{C}(\lambda)$ basis of $\mathrm{H}_T(X_D^{\mathrm{loc}})$, where as usual $\mathbb{C}(\lambda)$ is the field of fractions of $\mathrm{H}_T(\mathrm{pt})$. The T-equivariant cohomology $\mathrm{H}_T(X_D^{\mathrm{loc}})$ is furthermore endowed with a non-degenerate, symmetric bilinear form given by the restriction of the T-equivariant Chen–Ruan [15, Section 2.1] pairing on the untwisted component of the inertia stack of $\mathcal{X}_D^{\mathrm{loc}}$,

$$\eta_{\mathsf{lm}} := (H^{\mathsf{l}}, H^{\mathsf{m}})_{X_D^{\mathrm{loc}}} := \int_X \frac{H^{\mathsf{l}} \cup H^{\mathsf{m}}}{\cup_i \mathrm{e}_T(\mathcal{O}_X(-D_i))}, \tag{4.1}$$

where e_T denotes the T-equivariant Euler class.

Let now $\tau \in \mathrm{H}_T(X_D^{\mathrm{loc}})$. The *equivariant big J-function* of X_D^{loc} is the formal power series

$$J_{\mathrm{big}}^{X_D^{\mathrm{loc}}}(\tau, z) := z + \tau + \sum_{\mathsf{d} \in \mathrm{NE}(X)} \sum_{n \in \mathbb{Z}^+} \sum_{\mathsf{l}, \mathsf{m} \le \mathsf{n}_X} \frac{1}{n!} \left\langle \tau, \dots, \tau, \frac{H^{\mathsf{l}}}{z - \psi} \right\rangle_{0, n+1, \mathsf{d}}^{X_D^{\mathrm{loc}}} H^{\mathsf{m}} \eta^{\mathsf{lm}}, \tag{4.2}$$

where we employed the usual correlator notation for Gromov–Witten invariants,

$$\left\langle \tau_1 \psi_1^{k_1}, \dots, \tau_n \psi_n^{k_n} \right\rangle_{0,n,\mathsf{d}}^{X^{\mathrm{loc}}} := \int_{[\overline{\mathrm{M}}_{0,m}(X_D^{\mathrm{loc}}, \mathsf{d})]^{\mathrm{vir}}} \prod_i \mathrm{ev}_i^*(\tau_i) \psi_i^{k_i}, \tag{4.3}$$

and $\eta^{\mathsf{lm}} := (\eta^{-1})_{\mathsf{lm}}$. Restriction to $t = t_0 \mathbf{1}_{H(X)} + \sum_{i=1}^{r_X} t_i H_i$ and use of the Divisor Axiom leads to the *equivariant small J-function* of X_D^{loc},

$$J_{\mathrm{small}}^{X_D^{\mathrm{loc}}}(t, z) := z \mathrm{e}^{\sum t_i \phi_i / z} \left(1 + \sum_{\mathsf{d} \in \mathrm{NE}(X)} \sum_{\mathsf{l}, \mathsf{m} \le \mathsf{n}_X} \mathrm{e}^{\sum t_i d_i} \left\langle \frac{H^{\mathsf{l}}}{z(z - \psi_1)} \right\rangle_{0,1,\mathsf{d}}^{X_D^{\mathrm{loc}}} H^{\mathsf{m}} \eta^{\mathsf{lm}} \right). \tag{4.4}$$

The n-pointed genus 0 Gromov–Witten invariants with one marked descendant insertion (respectively, the one-pointed genus 0 descendant invariants) of X_D^{loc}, and no twisted insertions, can thus be read off from the formal Taylor series expansion of J_{big} (respectively, J_{small}) at $z = \infty$.

The following theorem provides an explicit hypergeometric presentation of $J_{\mathrm{small}}^{X_D^{\mathrm{loc}}}(t, z)$. Let $\kappa_j := c_1(\mathcal{O}(-D_j))$ be the T-equivariant first Chern class of $\mathcal{O}(D_j)$ and $y_i \in \mathrm{Spec}\mathbb{C}[[t]]$, $i = 1, \dots, r_X$ be variables in a formal disk around the origin. Writing $(x)_n := \Gamma(x+n)/\Gamma(x)$ for the Pochhammer symbol of (x, n) with $n \in \mathbb{Z}$, the *T-equivariant I-functions* of X and X_D^{loc} are defined as the $\mathrm{H}_T(X)$ and $\mathrm{H}_T(X_D^{\mathrm{loc}})$ valued Laurent series

$$I^X(y, z) := z \mathbf{1}_{H(X)} + \prod_i y_i^{H_i/z} \sum_{\mathsf{d} \in \mathrm{NE}(X)} \prod_i y_i^{d_i} z^{\mathsf{d} \cdot K_X} \frac{1}{\prod_j \left(\frac{\kappa_j}{z} + 1 \right)_{\mathsf{d} \cdot D_j}}, \tag{4.5}$$

$$I^{X_D^{\mathrm{loc}}}(y,z) := z\mathbf{1}_{H(X)} + \prod_i y_i^{H_i/z} \sum_{\mathsf{d}\in \mathrm{NE}(X)} \prod_i y_i^{d_i} z^{\mathsf{d}\cdot(K_X+D)-l_D} \frac{\prod_j \kappa_j \left(\frac{\kappa_j}{z}+1\right)_{\mathsf{d}\cdot D_j-1}}{\prod_j \left(\frac{\kappa_j}{z}+1\right)_{\mathsf{d}\cdot D_j}} \tag{4.6}$$

and their *mirror maps* as their formal $\mathcal{O}(z^0)$ coefficient,

$$\begin{aligned} \tilde{t}^i_X(y) &:= \left[z^0 H_i\right] I^X(y,z), \\ \tilde{t}^i_{X_D^{\mathrm{loc}}}(y) &:= \left[z^0 H_i\right] I^{X_D^{\mathrm{loc}}}(y,z). \end{aligned} \tag{4.7}$$

Note that $\mathcal{X}$ and $\mathcal{X}_D^{\mathrm{loc}}$ are smooth toric Deligne–Mumford stacks with coarse moduli schemes X and X_D^{loc} that are projective over their affinisation, and at this level of generality a result of [15] can be applied to provide a Givental-style equivariant mirror statement for them, as follows. In the language of [15], the I-functions (4.5) and (4.6) are the stacky I-functions of [15, Definitions 28 and 29] for $\mathcal{X}$ and $\mathcal{X}_D^{\mathrm{loc}}$, respectively, restricted to insertions in the zero-age sector of their inertia stack. The main result of [15] identifies the small J-function of a semi-projective toric Deligne–Mumford stack to its stacky I-function, up to a change-of-variables given by its $\mathcal{O}(z^0)$ term as in (4.7). In particular, the following statement is a projection to the untwisted sector of $\mathcal{X}$ and $\mathcal{X}_D^{\mathrm{loc}}$ of [15, Theorem 31 and Corollary 32].

Theorem 4.1 [15]. *We have*

$$\begin{aligned} J^X_{\mathrm{small}}(\tilde{t}_X(y),z) &= I^X(y,z), \\ J^{X_D^{\mathrm{loc}}}_{\mathrm{small}}\left(\tilde{t}_X(y)+\tilde{t}_{X_D^{\mathrm{loc}}}(y)-\log y, z\right) &= I^{X_D^{\mathrm{loc}}}(y,z). \end{aligned} \tag{4.8}$$

5 | THE LOG SIDE: PROOF OF THEOREM 3.2

Assume first that there is j such that $\mathsf{d}\cdot D_j = 0$. Given that

$$X = \prod_{i=1}^{r_X} \mathbb{P}^{G_i}\left(\mathsf{w}_X^{(i)}\right)$$

is given its toric boundary, each D_j is of the form (up to reordering of the factors)

$$D^j \times \prod_{i\neq k} \mathbb{P}^{G_i}\left(\mathsf{w}_X^{(i)}\right)$$

for some k and with D^j a prime toric divisor in $\mathbb{P}^{G_k}(\mathsf{w}_X^{(k)})$. As $d=(d_i)_i$, $d\cdot D_j = 0$ implies by ampleness of D^j that $d_k \cdot D^j = 0$ and thus $d_k = 0$. This means that each genus 0 degree d maximally tangent stable log map factors through

$$\mathbb{P}^{G_k}\left(\mathsf{w}_X^{(k)}\right) \times \prod_{i\neq k} \mathbb{P}^{G_i}\left(\mathsf{w}_X^{(i)}\right),$$

and is trivial on the first component. By the log product formula [21, 28], the invariant reduces to the corresponding invariant of $\prod_{i\neq k}\mathbb{P}^{G_i}(\mathsf{w}^{(i)}_X)$. This moduli problem however is in positive virtual dimension and thus $R\mathfrak{p}^X_{\mathsf{d}} = R\mathfrak{q}^X_{\mathsf{d}} = 0$. For the remainder of this section, we therefore assume that $\mathsf{d}\cdot D_j > 0$ for all $1 \leqslant j \leqslant l_D$.

Moving to the general case with one or two point insertions, recall that $X = \prod_{i=1}^{r_X}\mathbb{P}^{G_i}(\mathsf{w}^{(i)})$ is given by the fan $\Sigma \subset N_{\mathbb{R}}$ where $N \simeq \mathbb{Z}^{|\mathsf{n}_X|}$ and $N_{\mathbb{R}} = N\otimes_{\mathbb{Z}}\mathbb{R}$. Then Σ is the product fan of the fans $\Sigma_i \subset (N_i)_{\mathbb{R}}$ of $\mathbb{P}^{G_i}(\mathsf{w}^{(i)})$, where $N_i \simeq \mathbb{Z}^{\mathsf{n}_i}$. Writing $\varepsilon_i := \sum_{k=1}^{i-1}(n_k+1)$, the rays of Σ_i are $[D_{\varepsilon_i+1}],\dots,[D_{\varepsilon_i+n_i+1}]$ and have primitive generators $\Delta(\varepsilon_i+1),\dots,\Delta(\varepsilon_i+n_i+1)$, which satisfy

$$\mathsf{w}^{(i)}_{\varepsilon_i+1}\Delta(\varepsilon_i+1)+\cdots+\mathsf{w}^{(i)}_{\varepsilon_i+n_i+1}\Delta(\varepsilon_i+n_i+1)=0.$$

Write L_i for the sublattice of N_i generated by the $[\Delta(\varepsilon_i+j)]$ and write B_i for the change of basis matrix from a $\mathbb{Z}$-basis of N_i to a $\mathbb{Z}$-basis of L_i. Then

$$|\det B_i| = |N_i/L_i| = |G_i|.$$

Let L be the sublattice of N generated by the L_i and let B be the change of basis matrix from N to L given by the B_i. We have that $|\det B| = \prod_{i=1}^{r_X}|\det B^i| = \prod_{i=1}^{r_X}|G_i|$.

Proposition 5.1. *The set $T(\mathfrak{p})^X_{\mathsf{d}}$ has an unique element Γ of multiplicity 1.*

Proof. Each element Γ of $\mathrm{T}(\mathfrak{p})^X_{\mathsf{d}}$ has $|\mathsf{n}_X|+r_X$ exterior markings (=rays) parallel to the rays $[D_1],\dots,[D_{|\mathsf{n}_X|+r_X}]$ and 1 vertex (=unique interior marking) with valency $|n|+r_X$. Thus, the only possibility is that Γ is the translate of the rays of the fan of X. Write ζ for one of the two generators of $\Lambda^{|\mathsf{n}_X|}M$. Then $\mathrm{Mult}(\Gamma)$ is given by the index of

$$\prod_{j=1}^{|\mathsf{n}_X|+r_X} z^{e^X_j(\mathsf{d})\Delta(j)} = \zeta \in \Lambda^{|\mathsf{n}_X|}M$$

in $\Lambda^{|\mathsf{n}_X|}M$, which equals 1. □

It follows from Proposition 5.1 and the correspondence result of [24] that

$$R\mathfrak{p}^X_{\mathsf{d}} = 1.$$

We calculate the multiplicity of the element of $T(\mathfrak{q})^X_{\mathsf{d}}$ in three steps of increasing generality.

Proposition 5.2. *Assume that X is the fake weighted projective plane $\mathbb{P}^G(\mathsf{w}_1,\mathsf{w}_2,\mathsf{w}_3)$, where we assumed that* $\gcd(\mathsf{w}_1,\mathsf{w}_2,\mathsf{w}_3)=1$. *Then $T(\mathfrak{q})^X_{\mathsf{d}}$ has an unique element of multiplicity* $|G|\,\mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 d^2$.

Proof. From $\mathsf{w}_1\Delta(1)+\mathsf{w}_2\Delta(2)+\mathsf{w}_3\Delta(3)=0$, it follows that $|\Delta(1)\wedge\Delta(2)| = \mathsf{w}_3|\det B|$. Choose the basis $\{\Delta(1),\Delta(2)\}$ of $N_{\mathbb{R}}$. In this basis, choose P_1 to be (1,0) and P_2 to be (0,1). Then the unique genus 0 degree d maximally tangent tropical curve passing through P_1 and P_2 consists of the rays $[D_1],[D_2],[D_3]$, meeting at $0=(0,0)$, and with weights $\mathsf{w}_j d$ on $[D_j]$.

Choose 0 to be the sink vertex and let E_1, respectively, E_2, be the edge connecting 0 with P_1, respectively, P_2. Choose moreover $\{e_1, e_2\}$ to be a $\mathbb{Z}$-basis of M with dual basis $\{e_1^*, e_2^*\}$. Then

$$\begin{aligned}\zeta_{E_1} &= \ell_2(\zeta_{D_1} \otimes \zeta_{P_1}) = \ell_2(z^{\mathsf{w}_1 d\Delta(1)} \otimes (e_1^* \wedge e_2^*)) = z^{\mathsf{w}_1 d\Delta(1)} \iota_{\mathsf{w}_1 d\Delta(1)}(e_1^* \wedge e_2^*) \\ &= z^{\mathsf{w}_1 d\Delta(1)}\Big((\iota_{\mathsf{w}_1 d\Delta(1)} e_1^*) \wedge e_2^* - e_1^* \wedge \iota_{\mathsf{w}_1 d\Delta(1)}(e_2^*)\Big) = z^{\mathsf{w}_1 d\Delta(1)}\big(e_1^*(\mathsf{w}_1 d\Delta(1))\, e_2^* - e_2^*(\mathsf{w}_1 d\Delta(1))\, e_1^*\big).\end{aligned}$$

Similarly,

$$\zeta_{E_2} = z^{\mathsf{w}_2 d\Delta(2)}\big(e_1^*(\mathsf{w}_2 d\Delta(2))\, e_2^* - e_2^*(\mathsf{w}_2 d\Delta(2))\, e_1^*\big)$$

and

$$\begin{aligned}\zeta_\Gamma = \zeta_{D_3}\zeta_{E_1}\zeta_{E_2} &= -e_1^*(\mathsf{w}_1 d\Delta(1))\, e_2^*(\mathsf{w}_2 d\Delta(2))\, e_2^* \wedge e_1^* - e_2^*(\mathsf{w}_1 d\Delta(1))\, e_1^*(\mathsf{w}_2 d\Delta(2))\, e_1^* \wedge e_2^* \\ &= \mathsf{w}_1\mathsf{w}_2 d^2 \big(e_1^*(\Delta(1)) e_2^*(\Delta(2)) - e_2^*(\Delta(1)) e_1^*(\Delta(2))\big)\, e_1^* \wedge e_2^* \\ &= \mathsf{w}_1\mathsf{w}_2 d^2\, |\Delta(1) \wedge \Delta(2)|\, e_1^* \wedge e_2^* = \mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 d^2\, |\det B|\, e_1^* \wedge e_2^*,\end{aligned}$$

which is indeed of index $|G|\, \mathsf{w}_1\mathsf{w}_2\mathsf{w}_3 d^2$ in $\Lambda^2 M$. □

Proposition 5.3. *Assume that $r_X = 1$, that is, $X = \mathbb{P}^G(\mathsf{w}_1, \dots, \mathsf{w}_{n+1})$ and that $n \geqslant 3$. Then, for an appropriate choice of marked points P_1 and P_2, the set $T(\mathfrak{q})^X_{\mathsf{d}}$ has a unique element Γ of multiplicity $|G| \prod_{j=1}^{n+1} \mathsf{w}_j\, d^n$.*

Proof. We choose as basis of $N_{\mathbb{R}}$ the basis $\{\Delta(1), \dots, \Delta(n)\}$. We choose our second point (interior marking) P_2 to have coordinate $(a_1, \dots, a_n)$ for $a_i < 0$ and general. We choose our first marked point P_1 to have coordinate $(b, 0, \dots, 0)$ for $b > 0$ large enough so that restricted to the half-space $\{(x_1, \dots, x_n) | x_1 > b\}$, any $h \in \mathrm{T}(\mathfrak{q})^X_{\mathsf{d}}$ is affine linear with image $(b, 0, \dots, 0) + \mathbb{R}_{>0}\Delta(1)$ and weight $e_1(\mathsf{d})$.

For $1 < j \leqslant n$, write $\mathsf{w}_{1j}\Delta(1j) := -\mathsf{w}_1\Delta(1) - \mathsf{w}_j\Delta(j)$ with $\Delta(1j)$ primitive and $\mathsf{w}_{1j} \in \mathbb{N}$. Consider the finite abelian group

$$G^j := (\langle \Delta(1), \Delta(2)\rangle_{\mathbb{R}} \cap N)/\langle \Delta(1), \Delta(2), \Delta(1j)\rangle.$$

Given $\Gamma \in \mathrm{T}(\mathfrak{q})^X_{\mathsf{d}}$, projecting to the plane $\langle \Delta(1), \Delta(2)\rangle_{\mathbb{R}}$ leads to a genus 0 maximally tangent tropical curve in $\mathbb{P}^{G^j}(\mathsf{w}_1, \mathsf{w}_j, \mathsf{w}_{1j})$ passing through two general points. By Proposition 5.2, there is only one such curve (and it has multiplicity $|G^j|\mathsf{w}_1\mathsf{w}_j\mathsf{w}_{1j}d^2$). These curves lift to a unique maximally tangent curve $h : \Gamma \to N_{\mathbb{R}}$.

Choose P_2 to be the sink vertex and consider the associated flow. Since the a_i are chosen to be general, on the set $\{(x_i) | x_i < a_i\}$, h is affine linear with slope parallel to $\Delta(n+1)$. We reorder the $\Delta(j)$ such that following the flow from P_2, the rays that are added to Γ are successively translates of $[D_n], [D_{n-1}], \dots, [D_2]$. Note that all vertices are 3-valent since P_1 and P_2 are in general position. Starting at P_1 and following the flow, we label the compact edges successively $E_1, \dots, E_n$. Choose a $\mathbb{Z}$-basis $e_1, \dots, e_n$ of N. Then

$$\zeta_{E_1} = \ell_2(z^{\mathsf{w}_1 d\Delta(1)} \otimes e_1^* \wedge \cdots \wedge e_n^*) = z^{\mathsf{w}_1 d\Delta(1)} \iota_{\mathsf{w}_1 d\Delta(1)}(e_1^* \wedge \cdots \wedge e_n^*).$$

At the next step,

$$\begin{aligned}\zeta_{E_2} &= z^{\mathsf{w}_1 d\Delta(1)+\mathsf{w}_2 d\Delta(2)} \iota_{\mathsf{w}_1 d\Delta(1)+\mathsf{w}_2 d\Delta(2)} \circ \iota_{\mathsf{w}_1 d\Delta(1)} (e_1^* \wedge \cdots \wedge e_n^*) \\ &= z^{\mathsf{w}_1 d\Delta(1)+\mathsf{w}_2 d\Delta(2)} \iota_{(\mathsf{w}_1 d\Delta(1)+\mathsf{w}_2 d\Delta(2))\wedge \mathsf{w}_1 d\Delta(1)} (e_1^* \wedge \cdots \wedge e_n^*) \\ &= z^{\mathsf{w}_1 d\Delta(1)+\mathsf{w}_2 d\Delta(2)} \iota_{(\mathsf{w}_1 \mathsf{w}_2 d^2 \Delta(1)\wedge\Delta(2))} (e_1^* \wedge \cdots \wedge e_n^*).\end{aligned}$$

Iterating this process, we obtain

$$\zeta_{E_n} = z^{\mathsf{w}_1 d\Delta(1)+\cdots+\mathsf{w}_n d\Delta(n)} \iota_{(\mathsf{w}_1\cdots\mathsf{w}_n d^n \Delta(1)\wedge\cdots\wedge\Delta(n))} (e_1^* \wedge \cdots \wedge e_n^*).$$

Since $\mathsf{w}_1\Delta(1) + \cdots + \mathsf{w}_{n+1}\Delta(n+1) = 0$, $|\Delta(1) \wedge \cdots \wedge \Delta(n)| = \mathsf{w}_{n+1}|\det B|$ and hence

$$\begin{aligned}\zeta_\Gamma &= \iota_{(\mathsf{w}_1\cdots\mathsf{w}_n d^n \Delta(1)\wedge\cdots\wedge\Delta(n))} (e_1^* \wedge \cdots \wedge e_n^*)\, e_1^* \wedge \cdots \wedge e_n^* \\ &= \mathsf{w}_1 \cdots \mathsf{w}_n d^n \, |\Delta(1) \wedge \cdots \wedge \Delta(n)|\, e_1^* \wedge \cdots \wedge e_n^* \\ &= \mathsf{w}_1 \cdots \mathsf{w}_n \mathsf{w}_{n+1} d^n \, |\det B|\, e_1^* \wedge \cdots \wedge e_n^*,\end{aligned}$$

which is indeed of index $|G|\,\mathsf{w}_1 \cdots \mathsf{w}_n\mathsf{w}_{n+1} d^n$ in $\Lambda^n M$. □

Proposition 5.4. *Let $X = \prod_{i=1}^{r_X} \mathbb{P}^{G_i}(\mathsf{w}^{(i)})$ be the product of fake weighted projective spaces. Then the set $T(\mathfrak{q})_{\mathsf{d}}^X$ has a unique element Γ of multiplicity $\prod_{i=1}^{r_X} |G_i|\,(\prod_{i,j}(\mathsf{w}_X)_j^{(i)})\mathsf{d}^{\mathsf{n}_X}$.*

Proof. Label the last r_X divisors $D_{|\mathsf{n}_X|+1}, \dots, D_{|\mathsf{n}_X|+r_X}$ to be coming from distinct components of X. Then $[D_1], \dots, [D_{|\mathsf{n}_X|}] \in \Sigma^{[1]}$ form a $\mathbb{R}$-basis of $N_{\mathbb{R}}$. To calculate $R\mathfrak{q}_{\mathsf{d}}^X$, we choose the marking P_2 with the ψ^{r_X-1} condition to be the origin 0. We choose the marking P_1 a general point that has positive coordinates with respect to the above basis. Then the $r_X + 1$ incoming rays at P_2 are necessarily $D_{|\mathsf{n}_X|+1}, \dots, D_{|\mathsf{n}_X|+r_X}$ with weights $e_j^X(\mathsf{d})$ and a primitive vector in direction $-e_{|\mathsf{n}_X|+1}(\mathsf{d})D_{|\mathsf{n}_X|+1} - \cdots - e_{|\mathsf{n}_X|+r_X}(\mathsf{d})D_{|\mathsf{n}_X|+r_X}$ with appropriate weight.

There is only one way to make a maximally tangent tropical curve Γ passing through P_1 out of it. To see this, for each i, consider the map of fans $\Sigma \to \Sigma_i$ corresponding to the projection to the ith component. In Σ_i the tropical curve becomes straight at 0 and hence we are looking at maximally tangent curves of degree d_i passing through two general points. By Proposition 5.3, there is only one such tropical curve. Moreover, the curve in $N_{\mathbb{R}}$ is uniquely determined by these projections.

Choose P_2 to be the sink vertex. Then the multiplicity of Γ is calculated as in Proposition 5.3 to be $\prod_{i=1}^{r_X} |G_i|\,(\prod_{i,j}(\mathsf{w}_X)_j^{(i)})\mathsf{d}^{\mathsf{n}_X}$. □

The correspondence principle of [24] then entails that

$$R\mathfrak{q}_{\mathsf{d}}^X = \prod_{i=1}^{r_X} |G_i| \left(\prod_{i,j} (\mathsf{w}_X)_j^{(i)} \right) \mathsf{d}^{\mathsf{n}_X},$$

concluding the calculations of the logarithmic invariants of Theorem 3.2.

6 | THE LOCAL SIDE: PROOF OF THEOREM 3.3

6.1 | The Poincaré pairing

As in Section 4.2, we consider the scalar $T \simeq \mathbb{C}^\star$ action on X_D^{loc} that covers the trivial action on the base X, and denote $\lambda = c_1(\mathcal{O}_{B\mathbb{C}^\star}(1))$ for the corresponding equivariant parameter. Note that for any $\mathsf{l} \preceq \mathsf{n}_X$, the Gram matrix η_{lm} for the restriction to the untwisted sector of the T-equivariant Chen–Ruan pairing (4.1) of X_D^{loc} satisfies

$$\eta_{\mathsf{l}\mathsf{n}_X} = \int_{[X]} \frac{H^{\mathsf{l}+\mathsf{n}_X}}{\mathrm{e}_T(N_{X/X_D^{\mathrm{loc}}})} = \int_{[X]} \frac{H^{\mathsf{l}+\mathsf{n}_X}}{(\mathrm{e}_T(N_{X/X_D^{\mathrm{loc}}}))^{[0]}} = \int_{[X]} \frac{H^{\mathsf{l}+\mathsf{n}_X}}{\prod_{i=1}^{|\mathsf{n}_X|+r_X} \lambda} = \begin{cases} \dfrac{\prod_i |G_i| \prod_{i,j} (\mathsf{w}_X)_j^{(i)}}{\lambda^{|\mathsf{n}_X|+r_X}} & \mathsf{l} = 0, \\ 0 & \text{else}, \end{cases} \tag{6.1}$$

for degree reasons. Also, $\eta_{\mathsf{lm}} = 0$ if $|\mathsf{l}| + |\mathsf{m}| > |\mathsf{n}_X|$ for the same reason: this means that η_{lm} is upper anti-triangular, and $\eta^{\mathsf{lm}} := (\eta^{-1})_{\mathsf{lm}}$ is lower anti-triangular with anti-diagonal elements $\eta^{\mathsf{l},\mathsf{n}_X-\mathsf{l}} = 1/\eta_{\mathsf{l},\mathsf{n}_X-\mathsf{l}}$.

6.2 | One pointed descendants

In the following, let $y = y_1 \dots y_{r_X}$ and $Q = \mathrm{e}^{t_1+\dots+t_{r_X}}$. From (4.4), we have

$$J_{\mathrm{small}}^{X_D^{\mathrm{loc}}}(t,z) := z \prod_{i=1}^{r_X} \mathrm{e}^{t_i H_i/z} \left[1 + \sum_{\mathsf{d},a,\mathsf{l},\mathsf{m}} Q^{\mathsf{d}} z^{-a-2} \left\langle H^{\mathsf{l}} \psi^a \right\rangle_{0,1,\mathsf{d}}^{X_D^{\mathrm{loc}}} \eta^{\mathsf{lm}} H^{\mathsf{m}} \right] =: \sum_{\mathsf{m}} (J_{\mathrm{small}}^{X_D^{\mathrm{loc}}})^{[\mathsf{m}]} H^{\mathsf{m}}. \tag{6.2}$$

Using (6.1), we get that the component of the small, twisted J-function along the identity class is

$$\begin{aligned} (J_{\mathrm{sm}}^{X_D^{\mathrm{loc}}})^{[0]} &:= z \left[1 + \sum_{\mathsf{d},a,\mathsf{l}} Q^{\mathsf{d}} z^{-a-2} \left\langle H^{\mathsf{l}} \psi^a \right\rangle_{0,1,\mathsf{d}}^{X_D^{\mathrm{loc}}} \eta^{\mathsf{l}0} \right], \\ &= z \left[1 + \frac{\lambda^{|\mathsf{n}_X|+r_X}}{\prod_i |G_i| \prod_{i,j} (\mathsf{w}_X)_j^{(i)}} \sum_{\mathsf{d},a} Q^{\mathsf{d}} z^{-a-2} \left\langle H^{\mathsf{n}_X} \psi^a \right\rangle_{0,1,\mathsf{d}}^{X_D^{\mathrm{loc}}} \right], \\ &= z \left[1 + \lambda^{|\mathsf{n}_X|+r_X} \sum_{\mathsf{d},a} Q^{\mathsf{d}} z^{-a-2} \langle [\mathrm{pt}] \psi^a \rangle_{0,1,\mathsf{d}}^{X_D^{\mathrm{loc}}} \right]. \end{aligned} \tag{6.3}$$

Therefore, our first set of invariants (2.9) can be computed from (6.3) as

$$\mathfrak{p}_{\mathsf{d}}^X := \left\langle [\mathrm{pt}] \psi^{|\mathsf{n}_X|+r_X-2} \right\rangle_{0,1,\mathsf{d}} = \frac{1}{\lambda^{|\mathsf{n}_X|+r_X}} \left[z^{-|\mathsf{n}_X|-r_X} \mathrm{e}^{t\cdot \mathsf{d}} \right] (J_{\mathrm{small}}^{X_D^{\mathrm{loc}}})^{[0]}. \tag{6.4}$$

To compute the right-hand side, we use Theorem 4.1. For quantities $a(j)$ depending on $e_j^X(\mathsf{d})$, the notation $\overset{\circ}{\prod}_j a(j)$ refers to the product of $a(j)$ over $j \in \{1, \dots, |\mathsf{n}_X| + r_X \mid e_j^X(\mathsf{d}) \neq 0\}$. From (4.5) and

(4.6), the I-functions of X and X_D^{loc} are

$$I^X(y,z) := z \sum_{\mathsf{d}} \prod_{i=1}^{r_X} y_i^{H_i/z+d_i} \mathring{\prod_j} \frac{1}{\prod_{m_j=1}^{e^X(\mathsf{d})}\left(m_j z + \sum_i Q_{ij}^X H_i\right)} =: \sum_{\mathsf{m}} (I^{X_{\mathrm{loc}}})^{[\mathsf{m}]} H^{\mathsf{m}}, \tag{6.5}$$

$$I^{X_D^{\mathrm{loc}}}(y,z) := z \sum_{\mathsf{d}} \prod_{i=1}^{r_X} y_i^{H_i/z+d_i} \mathring{\prod_j} \frac{\prod_{m_j=0}^{e_j^X(\mathsf{d})-1}\left(\lambda - m_j z - \sum_i Q_{ij}^X H_i\right)}{\prod_{m_j=1}^{e^X(\mathsf{d})}\left(m_j z + \sum_i Q_{ij}^X H_i\right)} =: \sum_{\mathsf{m}} (I^{X_{\mathrm{loc}}})^{[\mathsf{m}]} H^{\mathsf{m}}. \tag{6.6}$$

Lemma 6.1. *The mirror maps of X and X_D^{loc} are trivial,*

$$\tilde{t}_i^X(y) = \tilde{t}_i^{X_D^{\mathrm{loc}}}(y) = \log y_i. \tag{6.7}$$

Proof. This is a straightforward calculation from (6.5) and (6.6). Keeping track of the powers of z in the general summands entails that $I^X(y,z) = z + \sum_i \log y_i H_i + \mathcal{O}(1/z) = I^{X_D^{\mathrm{loc}}}(y,z)$, from which the claim follows. □

By the previous lemma and (6.4), to compute $\mathfrak{p}_{\mathsf{d}}^X$ we just need to evaluate the component of the I-function of X_D^{loc} along the identity, divide by $\lambda^{|\mathsf{n}_X|+r_X}$, and isolate the coefficient of $\mathcal{O}(z^{-|\mathsf{n}_X|-r_X})$. We have

$$\begin{aligned}
(I^{X_D^{\mathrm{loc}}})^{[0]} &= z \sum_{\mathsf{d}} y^{\mathsf{d}} \frac{\mathring{\prod_j} \prod_{m_j=0}^{e_j^X(\mathsf{d})-1} (\lambda - m_j z)}{\mathring{\prod_j} \prod_{m_j=1}^{e_j^X(\mathsf{d})} (m_j z)} \\
&= z \sum_{\mathsf{d}} y^{\mathsf{d}} \frac{\mathring{\prod_j} \prod_{m_j=0}^{e_j^X(\mathsf{d})-1} (\lambda - m_j z)}{z^{e^X(\mathsf{d})} \mathring{\prod_j} (e_j^X(\mathsf{d}))!}.
\end{aligned} \tag{6.8}$$

The numerator in the general summand of (6.8) is divisible by $\lambda^{|\mathsf{n}_X|+r_X}$ (corresponding to setting all $m_j = 0$ in the product):

$$\mathring{\prod_j} \prod_{m_j=0}^{e_j^X(\mathsf{d})-1} (\lambda - m_j z) = \lambda^{|\mathsf{n}_X|+r_X} \mathring{\prod_j} \prod_{m_j=1}^{e_j^X(\mathsf{d})-1} (\lambda - m_j z), \tag{6.9}$$

hence dividing by $\lambda^{|\mathsf{n}_X|+r_X}$ we get

$$\mathring{\prod_j} \prod_{m_j=1}^{e_j^X(\mathsf{d})-1} (\lambda - m_j z) = (-z)^{e^X(\mathsf{d})-|\mathsf{n}_X|-r_X} \left(\mathring{\prod_j} (e_j^X(\mathsf{d})-1)! + \mathcal{O}(1/z) \right). \tag{6.10}$$

In particular, this implies that

$$\mathfrak{p}_{\mathsf{d}}^X = \left\langle [\mathrm{pt}]\psi^{|\mathsf{n}_X|+r_X-2} \right\rangle_{0,1,\mathsf{d}} = \frac{1}{\lambda^{|\mathsf{n}_X|+r_X}} \left[z^{-|\mathsf{n}_X|-r_X} y^{\mathsf{d}} \right] (I^{X_{\mathrm{loc}}})^{[0]} = \frac{(-1)^{e^X(\mathsf{d})-|\mathsf{n}_X|-r_X}}{\prod^{\circ} e_j^X(\mathsf{d})}, \tag{6.11}$$

proving the first part of Theorem 3.4.

6.2.1 | Two pointed descendants

Let us now turn to the computation of $\mathfrak{q}_{\mathsf{d}}^X$. We start with the following observation: from (6.6), we have

$$I^{X_D^{\mathrm{loc}}}(y,z) := z + \sum_{\mathsf{l} \preceq \mathsf{n}_X} \frac{1}{z^{|\mathsf{l}|-1}} \left[\prod_{i=1}^{r_X} \frac{\log^{l_i} y_i \, H_i^{l_i}}{l_i!} + \mathcal{O}\left(\frac{1}{z}\right) \right]. \tag{6.12}$$

This follows immediately from the fact that

$$\frac{\prod_{j=1}^{|\mathsf{n}_X|+r_X} \prod_{m_j=0}^{e_j^X(\mathsf{d})-1} \left(\lambda - m_j z - \sum_i Q_{ij}^X H_i \right)}{\prod_{j=1}^{|\mathsf{n}_X|+r_X} \prod_{m_j=1}^{e_j^X(\mathsf{d})} \left(m_j z + \sum_i Q_{ij}^X H_i \right)} = \mathcal{O}\left(z^{-\sum_j \theta(e_j^X(\mathsf{d}))} \right), \tag{6.13}$$

where $\theta(x) = 0$ (respectively, $\theta(x) = 1$) for $x = 0$ (respectively, $x > 0$).

From this, we deduce the following lemma. For $t \in \mathrm{H}_T(X_D^{\mathrm{loc}})$, let $\hat{\star}_t$ denote the big quantum cohomology product,

$$H^{\mathsf{l}} \hat{\star}_t H^{\mathsf{m}} := \sum_{\mathsf{d} \in \mathrm{NE}(X)} \sum_{n \in \mathbb{N}} \sum_{\mathsf{i},\mathsf{k} \preceq n} \left\langle H^{\mathsf{l}}, H^{\mathsf{m}}, H^{\mathsf{i}}, t, \dots, t \right\rangle_{0,3+n,\mathsf{d}}^{X_D^{\mathrm{loc}}} \eta^{\mathsf{ik}} H^{\mathsf{k}} \tag{6.14}$$

and $\star_y$ its restriction to small quantum cohomology at $t = \sum_i \log y_i H_i$,

$$H^{\mathsf{l}} \star_y H^{\mathsf{m}} := \sum_{\mathsf{k}} c_{\mathsf{lm}}^{\mathsf{k}}(y) H^{\mathsf{k}} := \sum_{\mathsf{d} \in \mathrm{NE}(X)} \sum_{\mathsf{i},\mathsf{k}} \left\langle H^{\mathsf{l}}, H^{\mathsf{m}}, H^{\mathsf{i}} \right\rangle_{0,3,\mathsf{d}}^{X_D^{\mathrm{loc}}} y^{\mathsf{d}} \eta^{\mathsf{ik}} H^{\mathsf{k}}. \tag{6.15}$$

Write $t = \sum_{\mathsf{l} \preceq \mathsf{n}_X} t_{\mathsf{l}} H^{\mathsf{l}}$. In the following, we denote $\nabla_{H^{\mathsf{l}}} := \partial_{t^{\mathsf{l}}}$ and, for any function $f : \mathrm{H}_T(X_D^{\mathrm{loc}}) \to \mathbb{C}(\lambda)$, $f|_{\mathrm{sqc}}$ indicates its restriction to small quantum cohomology, $t \to \sum_{i=1}^{r_X} t_i H_i$.

Lemma 6.2. *For* $\mathsf{l} \prec \mathsf{n}_X$, *we have*

$$(\star_y)_{i=1}^{r_X} H_i^{\star_y l_i} = \cup_{i=1}^{r_X} H_i^{\cup l_i} =: H^{\mathsf{l}}. \tag{6.16}$$

Moreover,

$$z \nabla_{H^{\mathsf{l}}} J_{\mathrm{big}}^{X_D^{\mathrm{loc}}}(t,z)|_{\mathrm{sqc}} = \prod_i \left(z y_i \partial_{y_i} \right)^{l_i} I^{X_D^{\mathrm{loc}}}(y,z). \tag{6.17}$$

Remark 2. This proposition is a variation of the well-known statement that for $\mathbb{P}^n$ the small quantum product is the same as the cup product for all degrees up to and excluding n.

Proof. Recall that the components of $J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t,z)$ are a set of flat coordinates for the Dubrovin connection in big quantum cohomology,

$$z\nabla_{H^{\mathsf{l}}}\nabla_{H^{\mathsf{m}}}J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t,z) = \nabla_{H^{\mathsf{l}}\hat{\star}_t H^{\mathsf{m}}}J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t,z). \tag{6.18}$$

Write now $(\mathcal{I})_{[k]} := [z^{-k}]\mathcal{I}$ for any Laurent series $\mathcal{I}\in\mathbb{C}((z))$ and suppose $|\mathsf{l}| = |\mathsf{m}| = 1$. We have that

$$\nabla_{H^{\mathsf{l}}}\nabla_{H^{\mathsf{m}}}\left(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t)\right)_{[s]}\Big|_{\mathrm{sqc}} = \nabla_{H^{\mathsf{l}}\star_y H^{\mathsf{m}}}(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t))_{[s-1]} = \sum_{\mathsf{k}} c^{\mathsf{k}}_{\mathsf{lm}}(y)\nabla_{\mathsf{k}}\left(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t)\right)_{[s-1]}\Big|_{\mathrm{sqc}}. \tag{6.19}$$

Now,

$$c^{\mathsf{k}}_{\mathsf{lm}}(y) = (y_l\partial_{y_l})(y_m\partial_{y_m})(J^{X_D^{\mathrm{loc}}}_{\mathrm{sm}}(y))^{[\mathsf{k}]}_{[1]} = \delta^{\mathsf{k}}_{\mathsf{l+m}} = c^{\mathsf{k}}_{\mathsf{lm}}(0) \tag{6.20}$$

from (6.12) and the fact that $(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}})^{[\mathsf{k}]}$ is the $[\mathsf{k}]$-component of the gradient of the genus 0 Gromov–Witten potential. Then

$$\begin{aligned}(y_l\partial_{y_l})(y_m\partial_{y_m})I^{X_D^{\mathrm{loc}}}_{[s]}(y) &= \nabla_{H^{\mathsf{l}}}\nabla_{H^{\mathsf{m}}}\left(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t)\right)_{[s]}\Big|_{\mathrm{sqc}} = \sum_{\mathsf{k}} c^{\mathsf{k}}_{\mathsf{lm}}(0)\nabla_{\mathsf{k}}\left(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t)\right)_{[s-1]}\Big|_{\mathrm{sqc}}\\ &= \nabla_{H^{\mathsf{l+m}}}\left(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t)\right)_{[s-1]}\Big|_{\mathrm{sqc}}.\end{aligned} \tag{6.21}$$

Now, for $|\mathsf{m}| = 1$ and by induction on $1 \leqslant |\mathsf{l}| < |\mathsf{n}_X|$ we have, from (6.12), that

$$c^{\mathsf{k}}_{\mathsf{lm}}(y) = \nabla_{H^{\mathsf{l}}}\nabla_{H^{\mathsf{m}}}\left(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t)\right)^{[\mathsf{k}]}_{[1]}\Bigg|_{\mathrm{sqc}} = \left(\prod_i y_{l_i}\partial_{y_{l_i}}\right)(y_m\partial_{y_m})\left(I^{X_D^{\mathrm{loc}}}(y)\right)^{[\mathsf{k}]}_{|\mathsf{l}|} = \delta^{\mathsf{k}}_{\mathsf{l+m}} = c^{\mathsf{k}}_{\mathsf{lm}}(0), \tag{6.22}$$

and for $s \geqslant |\mathsf{l}|$,

$$\begin{aligned}\left(\prod_i y_{l_i}\partial_{y_{l_i}}\right)(y_m\partial_{y_m})\left(I^{X_D^{\mathrm{loc}}}(y)\right)_{[s]} &= \nabla_{H^{\mathsf{l}}}\nabla_{H^{\mathsf{m}}}\left(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t)\right)_{[s-|\mathsf{l}|+1]}|_{\mathrm{sqc}}\\ &= \sum_{\mathsf{k}} c^{\mathsf{k}}_{\mathsf{lm}}(0)\nabla_{\mathsf{k}}\left(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t)\right)_{[s-|\mathsf{l}|]}|_{\mathrm{sqc}}\\ &= \nabla_{H^{\mathsf{l+m}}}\left(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t)\right)_{[s-|\mathsf{l}|]}|_{\mathrm{sqc}}.\end{aligned} \tag{6.23}$$

□

Corollary 6.3. *We have*

$$\mathfrak{q}^X_{\mathrm{d}} = \prod_i |G_i| \left(\prod_{i,j} (\mathsf{w}_X)^{(i)}_j \right) \mathrm{d}^{\mathsf{n}_X} \mathfrak{p}^X_{\mathrm{d}}. \tag{6.24}$$

Proof. From the previous lemma we have, in particular,

$$\nabla_{H^{\mathsf{n}_X}} \left(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t) \right)_{[s]} \Big|_{\mathrm{sqc}} = \left(\prod_i \left(y_i \partial_{y_i} \right)^{n_i} \right) \left(I^{X_D^{\mathrm{loc}}}(y) \right)_{[s+|\mathsf{n}_X|-1]}. \tag{6.25}$$

From (2.10) and (6.25), we have

$$\begin{aligned} \mathfrak{q}^X_{\mathrm{d}} &= [y^{\mathrm{d}}] \eta_{\mathsf{n}_X 0} \nabla_{H^{\mathsf{n}_X}} \left(J^{X_D^{\mathrm{loc}}}_{\mathrm{big}}(t) \right)^{[0]}_{[r_X+1]} \Big|_{\mathrm{sqc}} \\ &= \frac{\prod_i |G_i| \prod_{i,j} (\mathsf{w}_X)^{(i)}_j}{\lambda^{|\mathsf{n}_X|+r_X}} [y^{\mathrm{d}}] \prod_i \left(y_i \partial_{y_i} \right)^{n_i} \left(I^{X_D^{\mathrm{loc}}}(y) \right)^{[0]}_{[|\mathsf{n}_X|+r_X]} \\ &= \prod_i |G_i| \prod_{i,j} (\mathsf{w}_X)^{(i)}_j \prod_i d_i^{n_i} \mathfrak{p}^X_{\mathrm{d}}, \end{aligned} \tag{6.26}$$

concluding the proof. □

Remark 3. The statement of Lemma 6.2 also immediately reconstructs explicitly two-point descendant invariants where the powers of ψ-classes are distributed among the two marked points by standard structure results about $g = 0$ Gromov–Witten theory (namely the symplecticity of the S-matrix, which is a consequence of WDVV and the string equation: this is [23, Lemma 17]). Their agreement with the corresponding log invariants is an easy exercise left to the reader.

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