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Fragments of Quasi-Nelson: The Algebraizable Core

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Abstract

This is the second of a series of papers that investigate fragments of quasi-Nelson logic (QNL) from an algebraic logic standpoint. QNL, recently introduced as a common generalization of intuitionistic and Nelson's constructive logic with strong negation, is the axiomatic extension of the substructural logic FL_{ew} (full Lambek calculus with exchange and weakening) by the *Nelson axiom*. The algebraic counterpart of QNL (*quasi-Nelson algebras*) is a class of commutative integral residuated lattices (a.k.a. FL_{ew}-algebras) that includes both Heyting and Nelson algebras and can be characterized algebraically in several alternative ways. The present paper focuses on the algebraic counterpart (a class we dub *quasi-Nelson implication algebras*, QNI-algebras) of the implication–negation fragment of QNL, corresponding to the connectives that witness the algebraizability of QNL. We recall the main known results on QNI-algebras and establish a number of new ones. Among these, we show that QNI-algebras form a congruence-distributive variety (Cor. 3.15) that enjoys equationally definable principal congruences and the strong congruence extension property (Prop. 3.16); we also characterize the subdirectly irreducible QNI-algebras in terms of the underlying poset structure (Thm. 4.23). Most of these results are obtained thanks to twist representations for QNI-algebras, which generalize the known ones for Nelson and quasi-Nelson algebras; we further introduce a Hilbert-style calculus that is algebraizable and has the variety of QNI-algebras as its equivalent algebraic semantics.

Keywords: (Quasi-)Nelson, twist-structure, constructive logic, negation, implication, subreducts

1 Introduction

Quasi-Nelson logic (QNL) may be introduced in a number of equivalent ways, among which the following three are at least worth mentioning: (i) as a non-contractive generalization of intuitionistic logic, (ii) as a non-involutive weakening of Nelson's constructive logic with strong negation [15] and (iii) as the extension of the full Lambek calculus with exchange and weakening (FL_{ew}) by the *Nelson axiom*:

$$((\varphi \Rightarrow (\varphi \Rightarrow \psi)) \wedge (\sim \psi \Rightarrow (\sim \psi \Rightarrow \sim \varphi))) \Rightarrow (\varphi \Rightarrow \psi). \qquad \text{(Nelson)}$$

Accordingly, the algebraic counterpart of QNL (known as *quasi-Nelson algebras* since [22]) may be studied as a common generalization of Heyting and Nelson algebras or as a subclass of (commutative, integral, bounded, distributive, 3-potent) residuated lattices, the latter being the standard algebraic semantics of *FL* [8].

The above perspectives on QNL and quasi-Nelson algebras are discussed at length in the following papers [22, 23, 24], while the more recent papers [17, 18, 20] focus on the issue of characterizing logics/algebras that correspond to some fragments of the language of QNL. Because of the richness of the alternative logico-algebraic signatures in which QNL can be presented and of the weaker interactions among the connectives (in comparison, e.g. to intuitionistic or Nelson logic), the landscape of the fragments of quasi-Nelson logic/algebras appears to be complex. On the other hand,

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the twist representation of quasi-Nelson algebras introduced in [22, 23] (see Section 2) has proved to be a very effective tool in the study of such fragments [17, 18, 21].

In the present paper, which may be viewed as a continuation of [20], we will show that twist constructions can also be successfully employed in the study of the $\{\to, \sim\}$-fragment of quasi-Nelson algebras, corresponding to what may be called the 'algebraizable core' of QNL. In order to justify the interest in this particular fragment, let us introduce and briefly discuss QNL and its language.

One of the most distinctive features of Nelson's constructive logic with strong negation is that it may be equivalently introduced either as an axiomatic strengthening of the full Lambek calculus with exchange and weakening (i.e. a logic in the substructural family) or as a conservative expansion of intuitionistic logic by a new involutive negation. In the former alternative, Nelson's logic is presented in the language of FL_{ew} $\{\wedge, \vee, *, \Rightarrow, 0, 1\}$, consisting of the additive conjunction $(\wedge)$ and disjunction $(\vee)$, the multiplicative or monoid conjunction $(*)$ together with the corresponding residuated implication $(\Rightarrow)$ and the truth constants $(0, 1)$. The negation $(\sim)$ is defined in the standard way as $\sim p := p \Rightarrow 0$. In the latter, one uses the language $\{\wedge, \vee, \to, \sim\}$ consisting of the intuitionistic (additive) conjunction $(\wedge)$, disjunction $(\vee)$ and implication $(\to)$ expanded with a new involutive negation $(\sim)$. The truth constants can be defined by $1 := p \to p$ and $0 := \sim 1$, and a second negation $(\neg)$ can be obtained by letting $\neg p := p \to 0$.

In the literature on Nelson's logic, the residuated implication $\Rightarrow$ is usually called the *strong implication*, while $\to$ is known as the *weak* one; accordingly, the connective $\sim$ is known as the *strong negation*, while $\neg$ is sometimes called the *intuitionistic negation* (but the term 'intuitionistic' may be misleading and is no longer meaningful in the more general setting of QNL). The equivalence between both languages works as follows: within $\{\wedge, \vee, *, \Rightarrow\}$, one can define the weak implication by $p \to q := p \Rightarrow (p \Rightarrow q)$, and within $\{\wedge, \vee, \to, \sim\}$, one may let $p \Rightarrow q := (p \to q) \wedge (\sim q \to \sim p)$ and $p * q := p \wedge q \wedge \sim(p \Rightarrow \sim q)$.

The dual nature of Nelson's logic is shared by QNL as introduced (via its algebraic semantics) in [22], and the above-mentioned equivalences among connectives are still valid. However, since the negation $\sim$ is no longer required to be involutive, other equivalences that hold for Nelson's logic are lost: e.g. the conjunction $(\wedge)$ can no longer be defined as $p \wedge q := \sim(\sim p \vee \sim q)$, and similarly the disjunction is not necessarily given by $p \vee q := \sim(\sim p \wedge \sim q)$; the strong implication $(\Rightarrow)$ is not definable by $p \Rightarrow q := \sim(p * \sim q)$, and so on.[1] These observations suggest that the number of non-equivalent fragments of QNL is much larger than in the Nelson case. In fact, as far as we are aware, even the latter problem (a classification of fragments of Nelson's logic) has never been tackled in a systematic fashion; this may indeed turn out to be a fruitful direction for future research.

The main features that distinguish (quasi-)Nelson logics among the extensions of FL_{ew} arise from the interplay between the implication(s) and the strong negation; this is very compactly expressed by the *Nelson axiom*

$$((p \Rightarrow (p \Rightarrow q)) \wedge (\sim q \Rightarrow (\sim q \Rightarrow \sim p))) \Rightarrow (p \Rightarrow q)$$

whose algebraic alter ego is the identity (Nelson) shown in Definition 2.1. The meaning and implications of the Nelsion axiom/identity are discussed at length in the following papers [22, 23, 24]. In particular, it is shown in [23, Prop. 14] that (Nelson) is equivalent (in the setting of 3-potent residuated lattices) to the following quasi-equational property:

$$\text{if } x \to y = \sim y \to \sim x = 1, \text{ then } x \leq y.$$

[1] For the sake of completeness of information, we may add that the definition $p * q := p \wedge q \wedge \sim(p \Rightarrow \sim q)$ works both for Nelson's logic and QNL, while the more usual $p * q := \sim(p \Rightarrow \sim q)$ is only suitable in the involutive setting.

The latter formulation shows that the property of 'being Nelson' may be expressed using only the weak implication, the negation and the lattice order. These considerations, as well as past research experience, suggest that the $\{\Rightarrow, \sim\}$- and $\{\rightarrow, \sim\}$-fragments lie 'at the core' of (quasi-)Nelson logics; of the two, the former is the more general one, in the sense that the weak implication is definable from the strong, but not the other way round.

Coming to the question of algebraizability, we recall that both Nelson's logic and QNL (as extensions of FL_{ew}) are obviously algebraizable; for QNL presented in the language $\{\wedge, \vee, \rightarrow, \sim\}$ this has been first shown in [11]. The translations that witness the algebraizability (the same for Nelson's logic and QNL) can be defined using different choices of connectives. For the defining equation, one can let $E(\varphi) := \{\varphi \approx \varphi \rightarrow \varphi\}$, or $E(\varphi) := \{\varphi \approx 1\}$, or $E(\varphi) := \{\varphi \approx \varphi \Rightarrow \varphi\}$, or $E(\varphi) := \{\varphi \approx \varphi \leftrightarrow \varphi\}$, or $E(\varphi) := \{\varphi \approx \varphi \Leftrightarrow \varphi\}$, where $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and $\varphi \Leftrightarrow \psi := (\varphi \rightarrow \psi) * (\psi \rightarrow \varphi)$, etc. For the equivalence formulas, one has for instance the following options: $\Delta(\varphi, \psi) := \{\varphi \Leftrightarrow \psi\}$, $\Delta(\varphi, \psi) := \{\varphi \Rightarrow \psi, \psi \Rightarrow \varphi\}$, $\Delta(\varphi, \psi) := \{\varphi \rightarrow \psi, \psi \rightarrow \varphi, \sim\varphi \rightarrow \sim\psi, \sim\psi \rightarrow \sim\varphi\}$, etc. Indeed, one can show that every fragment of QNL containing either $\{\Rightarrow\}$ or $\{\Leftrightarrow\}$ or $\{\rightarrow, \sim\}$ is algebraizable.

Each of the above-mentioned fragments may thus be considered an algebraizable core of QNL. Here we shall focus on the $\{\rightarrow, \sim\}$-fragment, the main reasons motivating this choice being the following. On the one hand, the $\{\rightarrow, \sim\}$-fragment is a minimal one, in the sense that the $\{\sim\}$-fragment is (obviously) not algebraizable, and the $\{\rightarrow\}$-fragment coincides with the well-known implication fragment of intuitionistic logic.[2] On the other hand, despite the seemingly poor language, the $\{\rightarrow, \sim\}$-subreducts of quasi-Nelson algebras (dubbed *QNI-algebras* in [20]) form a rather well-behaved class from a universal algebraic point of view, to which many results from the theory of (quasi-)Nelson algebras may be extended. In particular, we are going to show that a suitable adaptation of the twist construction used to represent quasi-Nelson algebras in [22] yields a representation for QNI-algebras too (regrettably, the purely implicational fragment $\{\Rightarrow\}$ seems to lie beyond the reach of our current techniques). This representation, in turn, is useful both for obtaining further information on QNI-algebras and for a more general exploration of the boundaries of twist-type constructions. We thus both hope and believe that the techniques developed in the present paper may be applied without major difficulties to a number of other fragments of (quasi-)Nelson (and to related classes of algebras), perhaps eventually leading to the systematic classification that is currently lacking.

The paper is organized as follows. In Section 2 we provide the necessary background on the main classes of algebras involved. Section 3 focusses on the $\{\rightarrow, \sim\}$-subreducts of quasi-Nelson algebras, summarizing the known results (in particular, the twist representation) on this class of algebras. Aside from these, the section also establishes a previously unpublished result, namely that QNI-algebras form a variety (Corollary 3.15). Section 4 introduces an alternative twist representation for QNI-algebras. In comparison with the one discussed in Section 3, the new representation has the advantage of establishing a connection between QNI-algebras and a (single-sorted) variety of modal-like algebras (dubbed *nuclear Hilbert semigroups*); we note that our construction is also formally very close to other known representations of non-classical algebras via the so-called nuclei operators (see e.g. [7, 9]). More importantly, the new representation allows us to show that our equational axiomatization of QNI-algebras does indeed characterize the $\{\rightarrow, \sim\}$-subreducts of quasi-Nelson algebras (Corollary 4.20). In Section 4 we also obtain further information on the

[2] The $\{\rightarrow\}$-fragment of intuitionistic logic is well known to be algebraizable, and its algebraic counterpart is the variety of Hilbert algebras. By the way, it is easy to show that the same holds for the $\{\leftrightarrow\}$-fragment of QNL: it is algebraizable and coincides with the intuitionistic one (on the latter, see e.g. [8, p. 118]).

structure of subdirectly irreducible QNI-algebras (Theorem 4.23) and on the lattice of subvarieties of QNI-algebras (Proposition 4.26); for the latter result the twist representation plays again a key role. In Section 5 we introduce a Hilbert-style calculus that axiomatizes the $\{\rightarrow, \sim\}$-fragment of QNL. As anticipated, this calculus is algebraizable and has the variety of QNI-algebras as its equivalent algebraic semantics (Theorems 5.2 and 5.3). Section 6 closes the paper with concluding considerations and suggestions for future research.

2 Quasi-Nelson Algebras as Twist-Algebras

This section contains definitions of classes of algebras that we shall need in the subsequent ones. We begin by introducing the algebraic counterpart of QNL. We assume familiarity with standard results on universal algebra [2] and (residuated) lattices [8].

DEFINITION 2.1.
A *commutative integral bounded residuated lattice* (CIBRL) is an algebra $\mathbf{A} = \langle A; \wedge, \vee, *, \rightarrow, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ such that

(i) $\langle A; *, 1\rangle$ is commutative monoid, (Mon)
(ii) $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded lattice (with order $\leq$), (Lat)
(iii) $a * b \leq c$ iff $a \leq b \Rightarrow c$ for all $a, b, c \in A$. (Res)

On any CIBRL $\mathbf{A}$, the presence of the 0 constant allows us to define a *negation* operation ($\sim$) given by $\sim a := a \Rightarrow 0$ for all $a \in A$. Every Heyting algebra $\mathbf{H}$ can be viewed as a CIBRL on which the $\wedge$ and $*$ operation coincide (hence, the implication $\Rightarrow$ is the residuum of the meet $\wedge$).

DEFINITION 2.2. [22]
A *quasi-Nelson algebra* (QN-algebra) is a CIBRL that satisfies the Nelson identity:

$$(x \Rightarrow (x \Rightarrow y)) \wedge (\sim y \Rightarrow (\sim y \Rightarrow \sim x)) \approx x \Rightarrow y \qquad \text{(Nelson)}$$

A *Nelson algebra* is a quasi-Nelson algebra that satisfies the involutive identity $x \approx \sim\sim x$.

QN-algebras have been introduced only recently but are the subject of a rapidly growing literature [17, 19, 21–23]; Nelson algebras, on the other hand, have been around for more than four decades.[3] Every Heyting algebra satisfies the identity (Nelson) and is therefore an example of a QN-algebra on which the operations $\wedge$ and $*$ coincide (on the other hand, the only Heyting algebras that are also Nelson algebras are the Boolean algebras). The class of quasi-Nelson algebras can thus be viewed as a common generalization of Heyting algebras and Nelson algebras.

We note that the identity $x * y \approx x \wedge y \wedge \sim(x \Rightarrow \sim y)$ is valid on every QN-algebra, suggesting that QN-algebras may be equivalently presented in a language that does not include a primitive symbol for the monoid operation. An alternative language in which (quasi-)Nelson algebras have been traditionally considered is $\{\wedge, \vee, \rightarrow, \sim, 0, 1\}$, which replaces the residuated implication $\Rightarrow$ (in this context known as the *strong implication*) by the *weak implication* $\rightarrow$ and defines $x \Rightarrow y := (x \rightarrow y) \wedge (\sim y \rightarrow \sim x)$. On every QN-algebra $\mathbf{A}$, a second negation $\neg$ can then be defined by the term

[3] In the papers [22, 23], QN-algebras are also called *quasi-Nelson residuated lattices*: the two terms refer to the two presentations (which use either the strong or the weak implication as primitive) of the 'same' class of algebras. In the present paper we shall refrain from employing the term 'quasi-Nelson residuated lattices' or the alternative abstract presentation of [22, 23].

$\neg x := x \to 0$; it is easy to show that $\neg$ only coincides with $\sim$ iff $\mathbf{A}$ is a Heyting algebra. Conversely, the weak implication is definable via the strong one by the term $x \to y := x \Rightarrow (x \Rightarrow y)$. Relying on this equivalence, depending on convenience, we can employ either the strong or the weak implication to express key properties of QN-algebras.

For our present purposes, the main structural result on QN-algebras is the twist representation. We present this result in two slightly different guises, which we will both extend to the class of $\{\to, \sim\}$-subreducts of QN-algebras.

Recall that a *Heyting algebra* is an algebra $\mathbf{H} = \langle H; \wedge, \vee, \to, 0, 1\rangle$ of type $\langle 2, 2, 2, 0, 0\rangle$ such that $\langle H; \wedge, \vee, 0, 1\rangle$ is a bounded lattice (with order $\leq$) and $\to$ is the residuum of $\wedge$, i.e. $a \wedge b \leq c$ iff $a \leq b \to c$, for all $a, b, c \in H$. On each Heyting algebra $\mathbf{H}$, we let $D(\mathbf{H}) := \{a \in H : a \to 0 = 0\}$.

DEFINITION 2.3.
Let $\mathbf{H}_+ = \langle H_+; \wedge_+, \vee_+, \to_+, 0_+, 1_+\rangle$ and $\mathbf{H}_- = \langle H_-; \wedge_-, \vee_-, \to_-, 0_-, 1_-\rangle$ be Heyting algebras (with orders $\leq_+$ and $\leq_-$), let $\nabla \subseteq H_+$ be a filter such that $D(\mathbf{H}_+) \subseteq \nabla$, and let $n\colon L_+ \to L_-$ and $p\colon L_- \to L_+$ be maps satisfying the following conditions[4] :

(i) n is a bounded lattice homomorphism,
(ii) p preserves finite meets and both lattice bounds,
(iii) $n \cdot p = Id_{L_-}$ and $Id_{L_+} \leq_+ p \cdot n$.

The *quasi-Nelson twist-algebra* $Tw(\mathbf{H}_+, \mathbf{H}_-, n, p, \nabla) = \langle A; \wedge, \vee, \sim, \to, 0, 1\rangle$ has universe

$$A := \{\langle a_+, a_-\rangle \in H_+ \times H_- : a_+ \vee_+ p(a_-) \in \nabla,\ a_+ \wedge_+ p(a_-) = 0_+\}$$

and operations given as follows. For all $\langle a_+, a_-\rangle, \langle b_+, b_-\rangle \in H_+ \times H_-$,

$$1 := \langle 1_+, 0_-\rangle,$$
$$0 := \langle 0_+, 1_-\rangle,$$
$$\sim\langle a_+, a_-\rangle := \langle p(a_-), n(a_+)\rangle,$$
$$\langle a_+, a_-\rangle \wedge \langle b_+, b_-\rangle := \langle a_+ \wedge_+ b_+, a_- \vee_- b_-\rangle,$$
$$\langle a_+, a_-\rangle \vee \langle b_+, b_-\rangle := \langle a_+ \vee_+ b_+, a_- \wedge_- b_-\rangle,$$
$$\langle a_+, a_-\rangle \to \langle b_+, b_-\rangle := \langle a_+ \to_+ b_+, n(a_+) \wedge_- b_-\rangle.$$

The twist representation result states that every QN-algebra arises in the way described in Definition 2.3. We proceed to expound the details of this representation.

Given a QN-algebra $\mathbf{A} = \langle A; \wedge, \vee, *, \to, 0, 1\rangle$, define the operation $\to$ by the term $x \to y = x \to (x \to y)$. Further define the relation $\equiv$, for all $a, b \in A$, by

$$a \equiv b \quad \text{iff} \quad a \to b = b \to a = 1.$$

The relation $\equiv$ thus obtained is compatible with the operations $\langle \wedge, \vee, *, \to\rangle$, which gives us a quotient $\mathbf{A}_+ = \langle A/{\equiv}; \wedge, *, \vee, \to, 0, 1\rangle$. Moreover, the algebra $\mathbf{A}_+$ is a Heyting algebra (on which the operations $\wedge$ and $*$ coincide). Defining the set $F(A) = \{a \in A : \sim a \leq a\}$, we have that $\nabla_{\mathbf{A}} = F(A)/{\equiv}$ is a lattice filter of $\mathbf{A}_+$ and $D(\mathbf{A}_+) \subseteq \nabla_{\mathbf{A}}$. To obtain a second factor, one considers

[4] In fact, since n and p are monotone maps and $n \cdot p \leq_- Id_{H_-}$ and $Id_{H_+} \leq_+ p \cdot n$, we have that n and p form an adjoint pair from the poset $\langle H_+, \leq_+\rangle$ to the poset $\langle H_-, \leq_-\rangle$. This entails that n preserves arbitrary existing joins and p preserves arbitrary existing meets (cf. Remark 3.8 below).

the set $\sim A = \{\sim a : a \in A\}$ and lets $A_- = \sim A/\equiv$. Then A_- is the universe of a subalgebra of $\mathbf{A}_+$, which we denote by $\mathbf{A}_-$. Lastly, we define maps $n_{\mathbf{A}} \colon A_+ \to A_-$ and $p_{\mathbf{A}} \colon A_- \to A_+$ as follows: $n_{\mathbf{A}}(a/\equiv) = \sim\sim a/\equiv$ and $p_{\mathbf{A}}(\sim a/\equiv) = \sim a/\equiv$. The tuple $\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}} \rangle$ satisfies the required properties for defining a QN twist-algebra $Tw\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}} \rangle$, which allows us to obtain the representation result below.

THEOREM 2.4. [23], Prop. 10

Every quasi-Nelson algebra $\mathbf{A}$ is isomorphic to the quasi-Nelson twist-algebra

$$Tw\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}} \rangle$$

constructed according to Definition 2.3 through the map ι given by $\iota(a) = \langle [a], [\sim a] \rangle$ for all $a \in A$.

Among QN-algebras, the involutive ones (i.e. Nelson algebras) are precisely those algebras $\mathbf{A}$ such that $A = \sim A$. Hence $\mathbf{A}_+ = \mathbf{A}_-$ and $n_{\mathbf{A}}, p_{\mathbf{A}}$ are both the identity map (therefore, a Nelson algebra is determined by just a pair $\langle \mathbf{H}, \nabla \rangle$). Likewise, Heyting algebras correspond to those quasi-Nelson algebras $\mathbf{A}$ such that $\mathbf{A}_+ \cong \mathbf{A}$.

An alternative twist representation for QN-algebras arises from the following observation. By item (iii) of Definition 2.3, the map p is injective; this suggests that the Heyting algebra $\mathbf{H}_-$ may be viewed as a (special) subalgebra of $\mathbf{H}_+$ and, more precisely, as the image of $\mathbf{H}_+$ under a nucleus operator.

Given a Heyting algebra $\mathbf{H} = \langle H; \wedge, \vee, \to, 0, 1 \rangle$, we shall say that a unary operation $\Box \colon H \to H$ is a *nucleus* on $\mathbf{H}$ if the following identities are satisfied:

(i) $\Box(x \wedge y) \approx \Box x \wedge \Box y$
(ii) $x \leq \Box x$
(iii) $\Box x \approx \Box\Box x$
(iv) $\Box 0 \approx 0$.

Nuclei have been extensively studied in the literature on residuated lattices, and those defined on Heyting algebras in particular are the subject, for instance, of [12, 13]. A nucleus can be thought of as a generalization of the double negation operation. Indeed, it is easy to verify that, on every Heyting algebra, letting $\Box x := (x \to 0) \to 0$, one obtains a nucleus. The identity map is also a nucleus on every Heyting algebra. In the present setting, the key observation is the following: given two Heyting algebras $\mathbf{H}_+$ and $\mathbf{H}_-$ related by maps n, p as per Definition 2.3, it is easy to verify that the composition $p \circ n$ is a nucleus on $\mathbf{H}_+$.

The following proposition provides a more concise definition of a nucleus that will be useful to keep in mind when we focus on subreducts of QN-algebras.

PROPOSITION 2.5. [13], Thm. 1.3

Let $\mathbf{H}$ be a Heyting algebra endowed with a unary operation $\Box$ satisfying $\Box 0 = 0$. The following are equivalent:

(i) $\Box$ is a nucleus.
(ii) $a \to \Box b = \Box a \to \Box b$ for all $a, b \in H$.

At this point we are ready to introduce an alternative twist construction for quasi-Nelson algebras.

DEFINITION 2.6.

Let $\mathbf{H} = \langle H, \wedge, \vee, \to, \Box, 0, 1 \rangle$ be a Heyting algebra with a nucleus, and let $\nabla \subseteq H$ be a filter such that $D(\mathbf{H}) \subseteq \nabla$. The *quasi-Nelson twist-algebra* $Tw(\mathbf{H}, \nabla) = \langle A; \wedge, \vee, \sim, \to, 0, 1\rangle$ has universe

$$A := \{\langle a_1, a_2\rangle \in H \times H : a_2 = \Box a_2,\ a_1 \vee a_2 \in \nabla,\ a_1 \wedge a_2 = 0\}$$

and operations given as follows. For all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in H \times H$,

$$1 = \langle 1, 0\rangle$$

$$0 = \langle 0, 1\rangle$$

$$\sim\langle a_1, a_2\rangle = \langle a_2, \Box a_1\rangle$$

$$\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle = \langle a_1 \wedge b_1, \Box(a_2 \vee b_2)\rangle$$

$$\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle = \langle a_1 \vee b_1, a_2 \wedge b_2\rangle$$

$$\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle = \langle a_1 \to b_1, \Box a_1 \wedge b_2\rangle.$$

In the above definition (and throughout the rest of the paper), following standard usage on Nelson algebras, we are overloading the symbols $\wedge, \vee, \to$ so as to refer to operations on $Tw(\mathbf{H}, \nabla)$ as well as on $\mathbf{H}$. A comparison with Definition 2.3 shows that in Definition 2.6 we have replaced the second Heyting algebra $\mathbf{H}_-$ with the direct image $\Box[H]$ of $\mathbf{H}$ under the nucleus. This image is a $\{\wedge, \to, 0, 1\}$-subalgebra of $\mathbf{H}$ but is not necessarily closed under the disjunction, which explains the different definition of the disjunction in the twist-algebra.

The representation result based on the alternative construction is entirely analogous to Theorem 2.4. Given a QN-algebra $\mathbf{A} = \langle A; \wedge, \vee, *, \to, 0, 1\rangle$, one considers the relation $\equiv$, the quotient $\langle A/{\equiv}; \wedge, *, \vee, \to, 0, 1\rangle$ and the filter $\nabla_{\mathbf{A}}$ defined as before. Moreover, having observed that $a \equiv b$ entails $\sim\sim a \equiv \sim\sim b$ for all $a, b \in A$, we enrich the quotient $\langle A/{\equiv}; \wedge, *, \vee, \to, 0, 1\rangle$ with a nucleus defined by $\Box a/{\equiv} := \sim\sim a/{\equiv}$ for each $a \in A$. Letting $H(\mathbf{A}) := \langle A/{\equiv}; \wedge, \vee, \to, \Box, 0, 1\rangle$, we construct the twist-algebra $Tw(H(\mathbf{A}), \nabla_{\mathbf{A}})$ as prescribed by Definition 2.6, obtaining the following result.

THEOREM 2.7.
Every quasi-Nelson algebra $\mathbf{A}$ is isomorphic to the quasi-Nelson twist-algebra $Tw\langle H(\mathbf{A}), \nabla_{\mathbf{A}}\rangle$ constructed according to Definition 2.6 through the map ι given by $\iota(a) = \langle [a], [\sim a]\rangle$ for all $a \in A$.

In the next section we are going to define a twist construction that is more general than the above-defined ones in the sense that it does not require the underlying factor(s) to be Heyting algebra(s), but only to possess a Heyting implication (plus a little additional structure). Such generalization is possible because the component-wise definition of the implication on the twist-algebra (say, in Definition 2.6) only requires the presence of an implication in the first factor, a nucleus and a meet in the second factor; the most important observation being perhaps that we only need to have meets of elements that are in the direct image of the nucleus.

3 QNI-Algebras and Their Twist Representation

We are now going to introduce an abstract class of algebras (QNI-algebras) that corresponds precisely to the $\{\to, \sim\}$-subreducts of quasi-Nelson algebras. This result will be established thanks to a twist representation for these algebras. The basic results contained in the present section were first proved in [20, Sec. 4], to which we refer for further details and proofs.

Let $\mathbf{A} = \langle A; \to, \sim, 0, 1\rangle$ be an algebra of type $\langle 2, 1, 0, 0\rangle$ and let $a, b \in A$. We write $a \preccurlyeq b$ instead of $a \to b = 1$, and $a \equiv b$ instead of $a \to b = b \to a = 1$. We also employ the following new abbreviations: $a \odot b := {\sim}(a \to {\sim} b)$ and

$$q(a,b,c) := (a \to b) \to ((b \to a) \to (({\sim} a \to {\sim} b) \to (({\sim} b \to {\sim} a) \to c))).$$

The idea behind these operations is that the connective $\odot$ acts as a conjunction of sorts (cf. especially items (iv), (xiv) and (xv) in Definition 3.1 and Lemma 3.13), while $q(a,b,c)$ is a generalization of the operation given, on every quasi-Nelson algebra, by $((a \Rightarrow b) \wedge (b \Rightarrow a)) \to c$ (cf. Corollary 3.16).

DEFINITION 3.1. [20], Def. 21

An algebra $\mathbf{A} = \langle A; \to, {\sim}, 0, 1\rangle$ of type $\langle 2,1,0,0\rangle$ is a *quasi-Nelson implication algebra* (QNI-algebra) if the following properties are satisfied, for all $a,b,c,d \in A$:

(i) $1 \to a = a$
(ii) $a \to (b \to a) = a \to a = 0 \to a = 1$
(iii) $a \to (b \to c) = b \to (a \to c) = (a \to b) \to (a \to c)$
(iv) ${\sim} a \to ({\sim} b \to c) = ({\sim} a \odot {\sim} b) \to c$
(v) $q(a,b,a) = q(a,b,b)$
(vi) if ${\sim} a \preccurlyeq {\sim} b$, then ${\sim} a \preccurlyeq {\sim} a \odot {\sim} b$
(vii) $a \odot (b \odot c) \equiv (a \odot b) \odot c$
(viii) $a \odot b \equiv b \odot a$
(ix) if $a \equiv b$ and $c \equiv d$, then $a \to c \equiv b \to d$ and $a \odot c \equiv b \odot d$
(x) ${\sim} a = {\sim}{\sim}{\sim} a$
(xi) ${\sim} 1 = 0$ and ${\sim} 0 = 1$
(xii) $(a \to b) \to ({\sim}{\sim} a \to {\sim}{\sim} b) = 1.$
(xii) $a \preccurlyeq {\sim}{\sim} a$
(xiv) if $a \preccurlyeq b$, then $a \odot c \preccurlyeq b \odot c$ and $c \odot a \preccurlyeq c \odot b$
(xv) $a \odot (a \to b) \equiv a \odot b$
(xvi) $a \odot b \equiv {\sim}{\sim} a \odot {\sim}{\sim} b$
(xvii) ${\sim}(a \to b) \equiv {\sim}({\sim}{\sim} a \to {\sim}{\sim} b)$.

In what follows, we shall denote by QNI the class of QNI-algebras. Notice that the constants $0, 1$ could also be introduced as the following abbreviations: $1 := a \to a$ and $0 := {\sim} 1$. This reflects the observation that, since every quasi-Nelson algebra satisfies $x \to x \approx 1$ and ${\sim} 1 \approx 0$, the class of $\{\to, {\sim}\}$-subreducts of quasi-Nelson algebras coincides with the class of $\{\to, {\sim}, 0, 1\}$-subreducts.

As a sanity check, one can verify that every quasi-Nelson (twist-)algebra satisfies all the properties listed in Definition 3.1. Another natural example of a QNI-algebra is any bounded Hilbert algebra $\langle A; \to, 0, 1\rangle$ (see below for the definition) where one lets ${\sim} a := a \to 0$ for all $a \in A$ (cf. Propositions 3.11 and 4.26.iii).

Regarding the connective $\odot$ it may be helpful to keep in mind that, on a quasi-Nelson twist-algebra $Tw(\mathbf{H}_+, \mathbf{H}_-, n, p, \nabla)$, one has, for all $\langle a_+, b_+\rangle \in H_+$ and all $\langle a_-, b_-\rangle \in H_-$,

$$\langle a_+, a_-\rangle \odot \langle b_+, b_-\rangle = \langle p(n(a_+) \wedge_- n(b_+)),\, n(a_+ \to_+ p(b_-))\rangle.$$

REMARK 3.2.

For ease of reference, the items in Definition 3.1 are precisely those in [20, Def. 21]. However, the ones listed below are redundant and could therefore be omitted:

Item (xiii). Indeed, from (xi) we have $1 = {\sim}{\sim} 1$. Thus, using (i) and (xii), we have $a \to {\sim}{\sim} a = (1 \to a) \to (1 \to {\sim}{\sim} a) = (1 \to a) \to ({\sim}{\sim} 1 \to {\sim}{\sim} a) = 1$.

Item (xvi). By definition of $\odot$ we have $a \odot b = \sim(a \to \sim b)$. Using (xvii) and applying the definition of $\odot$ again, we have $\sim(a \to \sim b) \equiv \sim(\sim\sim a \to \sim\sim\sim b) = \sim\sim a \odot \sim\sim b$.

Also, (xiv) easily implies the second claim in item (ix).

Our next goal is to show that every QNI-algebra embeds into a twist-algebra.

LEMMA 3.3. [20], Lemma 22
Let $\mathbf{A} \in \mathsf{QNI}$ and $a, b, c \in A$.

(i) $a \to 1 = 1$.
(ii) $a \equiv b$ if and only if $a \to c = b \to c$ for all $c \in A$.
(iii) $\sim a \odot \sim b \preccurlyeq \sim a$ and $\sim a \odot \sim b \preccurlyeq \sim b$.
(iv) If $a \preccurlyeq b$ and $b \preccurlyeq c$, then $a \preccurlyeq c$.
(v) $a \equiv 1$ if and only if $a = 1$.
(vi) $\sim a \preccurlyeq \sim b$ if and only if $\sim a \preccurlyeq \sim a \odot \sim b$.
(vii) $a \odot \sim a = 0$.
(viii) If $a \preccurlyeq b$, then $\sim\sim a \preccurlyeq \sim\sim b$.
(ix) $a \odot a \equiv \sim\sim a \odot \sim\sim a \equiv \sim\sim a$.
(ix) The relation $\leq$ defined by $a \leq b$ iff ($a \preccurlyeq b$ and $\sim b \preccurlyeq \sim a$) is a partial order on A, with minimum 0 and maximum 1.

Let $\mathbf{A} = \langle A; \to, \sim, 0, 1\rangle \in \mathsf{QNI}$. Observe that the relation $\preccurlyeq$ is reflexive (Definition 3.1.ii) and transitive (Lemma 3.3.iv). Hence $\equiv$ is an equivalence relation. Letting $A_+ := A/\!\equiv$ and recalling Definition 3.1.ix, we can thus obtain a quotient algebra $\mathbf{A}_+ = \langle A_+; \to_+, 0_+, 1_+\rangle$. We proceed to take a closer look at this structure; we shall need some additional terminology.

Recall that *Hilbert algebras* are algebras $\langle A; \to, 1\rangle$ of type $\langle 2, 0\rangle$ that are precisely the $\{\to, 1\}$-subreducts of Heyting algebras. It is well known that $\langle A; \to, 1\rangle$ is a Hilbert algebra if and only if the following (quasi-)identities are satisfied:

(H1) $x \to (y \to x) \approx 1$
(H2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) \approx 1$
(H3) if $x \to y \approx 1$ and $y \to x \approx 1$, then $x \approx y$.

Every Hilbert algebra has a natural order $\leq$ (not necessarily forming a lattice or even a semilattice) given by $a \leq b$ iff $a \to b = 1$. The top element of $\leq$ is 1. If the natural order also has a minimum element (denoted 0 and sometimes included in the algebraic signature as a constant), then we speak of a *bounded Hilbert algebra*. It is useful to recall that every Hilbert algebra satisfies the commutative identity $x \to (y \to z) \approx y \to (x \to z)$ and that the implication $\to$ is order-reversing in the first argument and order-preserving in the second. These observations can be used to show that every Hilbert algebra satisfies not just (H2) but indeed the stronger (H2'): $x \to (y \to z) \approx (x \to y) \to (x \to z)$.

PROPOSITION 3.4. [20], Prop. 23
For each $\mathbf{A} \in \mathsf{QNI}$, the quotient $\mathbf{A}_+ = \langle A_+, \to_+, 0_+, 1_+\rangle$ is a bounded Hilbert algebra.

Let $\sim A := \{\sim a : a \in A\}$. Observe that, for all $a, b \in A$, if $\sim a \equiv b$, then $b \in \sim A$. Thus $\{b \in A : \sim a \equiv b\} = \{b \in \sim A : \sim a \equiv b\}$ for all $a \in A$. Hence, we can unambiguously let $A_- := \sim A/\!\equiv$, and we have $A_- \subseteq A_+$. We endow A_- with operations as follows. For all $a, b \in A$, let:

$$[\sim a] \wedge_- [\sim b] := [\sim a \odot \sim b] = [\sim(\sim a \to \sim\sim b)]$$
$$0_- := [\sim 1] = [0] = 0_+$$
$$1_- := [\sim 0] = [1] = 1_+.$$

PROPOSITION 3.5. [20], Prop. 24
For each $\mathbf{A} \in \mathsf{QNI}$, the quotient $\mathbf{A}_- = \langle A_-, \wedge_-, 0_-, 1_- \rangle$ is a bounded semilattice.

Although this is not needed for the purpose of the twist representation, one could further define an operation $\to_-$ by $[\sim a] \to_- [\sim b] := [\sim\sim(\sim a \to \sim b)]$, obtaining a bounded Hilbert algebra $\langle A_-, \to_-, 0_-, 1_- \rangle$ and an implicative semilattice $\langle A_-, \wedge_-, \to_-, 0_-, 1_- \rangle$; see Proposition 3.9.

We define maps $p_{\mathbf{A}} \colon A_- \to A_+$ and $n_{\mathbf{A}} \colon A_+ \to A_-$ as follows: $p_{\mathbf{A}}$ is the identity map on A_- and $n_{\mathbf{A}}[a] := [\sim\sim a]$ for all $a \in A$.

PROPOSITION 3.6. [20], Prop. 25
Let $\mathbf{A} = \langle A; \to, \sim, 0, 1 \rangle \in \mathsf{QNI}$, with corresponding quotient algebras $\mathbf{A}_+ = \langle A_+; \to_+, 0_+, 1_+ \rangle$, $\mathbf{A}_- = \langle A_-; \wedge_-, 0_-, 1_- \rangle$ and maps $n_{\mathbf{A}} \colon A_+ \to A_-, p_{\mathbf{A}} \colon A_- \to A_+$ defined as above. Then

(i) $n_{\mathbf{A}}$ and $p_{\mathbf{A}}$ are monotone and preserve the bounds.
(ii) $n_{\mathbf{A}} \cdot p_{\mathbf{A}} = Id_{A_-}$ and $Id_{A_+} \leq_+ p_{\mathbf{A}} \cdot n_{\mathbf{A}}$.
(iii) $n_{\mathbf{A}}(a_+) \wedge_- n_{\mathbf{A}}(b_+) = n_{\mathbf{A}}(a_+) \wedge_- n_{\mathbf{A}}(a_+ \to_+ b_+)$, for all $a_+, b_+ \in A_+$.
(iv) $p_{\mathbf{A}}(a_- \wedge_- b_-) \to_+ c_+ = p_{\mathbf{A}}(a_-) \to_+ (p_{\mathbf{A}}(b_-) \to_+ c_+)$, for all $a_-, b_- \in A_-$ and $c_+ \in A_+$.

Propositions 3.4 to 3.7 motivate the following definition.

DEFINITION 3.7.
Let $\mathbf{H}_+ = \langle H_+; \to_+, 0_+, 1_+ \rangle$ be a bounded Hilbert algebra, let $\mathbf{M}_- = \langle M_-; \wedge_-, 0_-, 1_- \rangle$ be a bounded semilattice and let $n \colon H_+ \to M_-$ and $p \colon M_- \to H_+$ be maps satisfying the following properties:

(i) n and p are monotone and preserve the bounds,
(ii) $n \cdot p = Id_{M_-}$ and $Id_{H_+} \leq_+ p \cdot n$.
(iii) $n(a_+) \wedge_- n(b_+) = n(a_+) \wedge_- n(a_+ \to_+ b_+)$, for all $a_+, b_+ \in H_+$.
(iv) $p(a_- \wedge_- b_-) \to_+ c_+ = p(a_-) \to_+ (p(b_-) \to_+ c_+)$, for all $a_-, b_- \in A_-$ and $c_+ \in H_+$.

The algebra $\mathbf{H}_+ \bowtie \mathbf{M}_- = \langle H_+ \times M_-; \to, \sim, 0, 1 \rangle$ is defined as follows. The operations $\to, \sim$ are given as the corresponding ones in Definition 2.3. A *quasi-Nelson implicative twist-algebra* (QNI twist-algebra) $\mathbf{A}$ over $\langle \mathbf{H}_+, \mathbf{M}_-, n, p \rangle$ is a $\{\to, \sim, 0, 1\}$-subalgebra of $\mathbf{H}_+ \bowtie \mathbf{M}_-$ with carrier set A satisfying: $\pi_1[A] = H_+$ and $n(a_+) \wedge_- a_- = 0_-$ for all $\langle a_+, a_- \rangle \in A$.

As observed in [20, p. 18], the set

$$A := \{\langle a_+, a_- \rangle \in H_+ \times M_- : n(a_+) \wedge_- a_- = 0_-\}$$

is closed under the algebraic operations and is therefore the universe of the largest twist-algebra over $\langle \mathbf{H}_+, \mathbf{M}_-, n, p \rangle$.

On every QNI twist-algebra $\mathbf{A}$ over $\langle \mathbf{H}_+, \mathbf{M}_-, n, p \rangle$, we introduce the derived connective $\odot$ as before. Therefore we have $\langle a_+, a_- \rangle \odot \langle b_+, b_- \rangle = \langle p(n(a_+) \wedge_- n(b_+)), n(a_+ \to_+ p(b_-)) \rangle$. We also define the relation $\preccurlyeq$ by

$$\langle a_+, a_- \rangle \preccurlyeq \langle b_+, b_- \rangle \quad \text{iff} \quad \langle a_+, a_- \rangle \to \langle b_+, b_- \rangle = \langle 1_+, 0_- \rangle.$$

It is easy to check that $\langle a_+, a_- \rangle \preccurlyeq \langle b_+, b_- \rangle$ iff $a_+ \leq_+ b_+$. The relation $\equiv$ is defined by

$$\langle a_+, a_- \rangle \equiv \langle b_+, b_- \rangle \quad \text{iff} \quad (\langle a_+, a_- \rangle \preccurlyeq \langle b_+, b_- \rangle \text{ and } \langle b_+, b_- \rangle \preccurlyeq \langle a_+, a_- \rangle).$$

We then have $\langle a_+, a_- \rangle \equiv \langle b_+, b_- \rangle$ iff $a_+ = b_+$. We also have $\sim\langle a_+, a_- \rangle \preccurlyeq \sim\langle b_+, b_- \rangle$ iff $a_- \leq_- b_-$, and therefore $\sim\langle a_+, a_- \rangle \equiv \sim\langle b_+, b_- \rangle$ iff $a_- = b_-$.

REMARK 3.8. [20], Rem. 27

Since n and p are monotone maps and $n \cdot p \leq_- Id_{M_-}$ and $Id_{H_+} \leq_+ p \cdot n$, we have that n and p form an adjoint pair from the poset $\langle H_+, \leq_+ \rangle$ to the poset $\langle M_-, \leq_- \rangle$. This entails that n preserves arbitrary existing joins and p preserves arbitrary existing meets (cf. Definition 2.3 above). Moreover, in our case, for all $a_-, b_- \in M_-$, the meet of $\{p(a_-), p(b_-)\}$ always exists in $\mathbf{H}_+$ and is the element $p(a_- \wedge_- b_-)$.

PROPOSITION 3.9. [20], Prop. 29

Let $\langle \mathbf{H}_+, \mathbf{M}_-, n, p \rangle$ be as per Definition 3.7. For all $a_-, b_- \in M_-$, define the operation $\to_-$ by $a_- \to_- b_- := n(p(a_-) \to_+ p(b_-))$. Then $\mathbf{M}_- = \langle M_-; \wedge_-, \to_-, 0_-, 1_- \rangle$ is a bounded implicative semilattice.[5] Moreover, the map p preserves the implication, i.e. $p(a_- \to_- b_-) = p(a_-) \to_+ p(b_-)$ for all $a_-, b_- \in M_-$.

Proposition 3.9 suggests that in Definition 3.7 we could equivalently have required $\mathbf{M}_-$ to be an implicative semilattice. An interesting consequence of Proposition 3.9 is that $\mathbf{M}_-$ is a *distributive* semilattice in the sense of [10,Sec. II.5], i.e. the lattice of filters of $\mathbf{M}_-$ is distributive. As is well known, the lattice of (implicative) filters of a Hilbert algebra (such as our $\mathbf{H}_+$) is also distributive [5, p. 477].

PROPOSITION 3.10. [20], Prop. 31

Every QNI twist-algebra is a QNI-algebra.

PROPOSITION 3.11.

Let $\langle A; \to, 0, 1 \rangle$ be a bounded Hilbert algebra. Upon defining $\sim a := a \to 0$ for all $a \in A$, the algebra $\langle A; \to, \sim, 0, 1 \rangle$ is a QNI-algebra.

PROOF. The statement could be proved directly, checking that every bounded Hilbert algebra satisfies all items of Definition 3.1. There is, however, a shorter indirect proof. We will reason by contraposition and use the observation that bounded Hilbert algebras (viewed as algebras in the language $\{\to, \sim, 0, 1\}$) are precisely the $\{\to, \sim, 0, 1\}$-subreducts of Heyting algebras. Suppose $\alpha \approx \beta$ is an identity that is not satisfied by all bounded Hilbert algebras. Then, by the above observation, there is some Heyting algebra $\mathbf{A}$ that does not satisfy $\alpha \approx \beta$. Since every Heyting algebra is a quasi-Nelson algebra [23, Prop. 11.iii], we can view $\mathbf{A}$ as a quasi-Nelson twist-algebra [23, Thm. 3] whose $\{\to, \sim, 0, 1\}$-reduct is a QNI twist-algebra. By Proposition 3.10 above, this reduct is a QNI-algebra that witnesses the failure of $\alpha \approx \beta$. This shows that every identity that is satisfied by QNI-algebras is also satisfied by bounded Hilbert algebras. Since both classes are varieties (cf. Corollary 3.15 below), we have that bounded Hilbert algebras are a subvariety of QNI. Hence, the desired result follows. □

Observe that Proposition 3.11 entails that the operations/connectives that are not definable in the $\{\to, \sim, 0, 1\}$-fragment of intuitionistic logic are, a fortiori, not definable in the setting of QNI-algebras and their logic (see Section 5). Thus, in particular, the connectives $\wedge, \vee$ and $*$ of QNL are not definable from $\{\to, \sim, 0, 1\}$; we shall return to the issue of characterizing (other) fragments of QNL in Section 6.

[5] Bounded implicative semilattices are the $\langle \wedge, \to, 0, 1 \rangle$-subreducts of Heyting algebras and correspond to the conjunction-implication–negation fragment of intuitionistic logic. Abstractly, an algebra $\langle M; \wedge, \to, 0, 1 \rangle$ is a bounded implicative semilattice if and only if (i) $\langle M; \wedge, 0, 1 \rangle$ is a bounded semilattice and (ii) $\to$ is the residuum of $\wedge$, i.e. $a \wedge b \leq c$ iff $a \leq b \to c$ for all $a, b, c \in M$.

THEOREM 3.12. [20], Thm. 5
Every $\mathbf{A} \in \mathsf{QNI}$ is isomorphic to a QNI twist-algebra over $\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}} \rangle$ through the map $\iota \colon A \to A_+ \times A_-$ given by $\iota(a) := \langle [a], [\sim a] \rangle$ for all $a \in A$.

Observe that the result of Theorem 3.12 (unlike those of Theorems 2.4 and 2.7) is only an embedding and not an isomorphism. It is, however, quite a useful and informative result, as we proceed to illustrate. First of all, we use it to verify that QNI is a variety.

LEMMA 3.13.
Let $\mathbf{A} \in \mathsf{QNI}$ and $a, b, c \in A$.

(i) $(a \odot b) \to c = \sim\sim a \to (\sim\sim b \to c)$.
(ii) $\sim a \to \sim b \equiv \sim a \to (\sim a \odot \sim b)$.
(iii) $(\sim\sim a \to \sim\sim b) \odot (\sim\sim a \to \sim\sim c) \equiv \sim\sim a \to (b \odot c)$.

PROOF. In the light of Theorem 3.12, we assume that $\mathbf{A}$ is a QNI twist-algebra over some $\langle \mathbf{H}_+, \mathbf{M}_-, n, p \rangle$ and let $a = \langle a_+, a_- \rangle$, $b = \langle b_+, b_- \rangle$, $c = \langle c_+, c_- \rangle$.
(i). Let us compute

$$(\langle a_+, a_- \rangle \odot \langle b_+, b_- \rangle) \to \langle c_+, c_- \rangle = \langle p(n(a_+) \wedge_- n(b_+)) \to_+ c_+, np(n(a_+) \wedge_- n(b_+)) \wedge_- c_- \rangle$$

and

$$\begin{aligned} &\sim\sim\langle a_+, a_- \rangle \to (\sim\sim\langle b_+, b_- \rangle \to \langle c_+, c_- \rangle) \\ &= \langle pn(a_+) \to_+ (pn(b_+) \to_+ c_+), npn(a_+) \wedge_- npn(b_+) \wedge_- c_- \rangle. \end{aligned}$$

Thus $n \cdot p = Id_{M_-}$ (Definition 3.7.ii) immediately implies the equality of the second components; equality of the first follows from Definition 3.7.iv.
(ii). We focus on the first components, which are the only ones that matter. We need to show

$$p(a_-) \to_+ p(b_-) = p(a_-) \to_+ p(np(a_-) \wedge_- np(b_-)).$$

Since p preserves the implication (Proposition 3.9) and $n \cdot p = Id_{M_-}$ (Definition 3.7.ii), we have $p(a_-) \to_+ p(b_-) = p(a_- \to_- b_-)$ and

$$p(a_-) \to_+ p(np(a_-) \wedge_- np(b_-)) = p(a_- \to_- (a_- \wedge_- b_-)).$$

Since every implicative semilattice satisfies $x \to (y \wedge z) \approx (x \to y) \wedge (x \to z)$, we have $a_- \to_- (a_- \wedge_- b_-) = (a_- \to_- a_-) \wedge_- (a_- \to_- b_-) = 1_- \wedge_- (a_- \to_- b_-) = a_- \to_- b_-$. Thus the required result follows.
(iii). As before, we only compute the first components. We need to show that

$$p(n(pn(a_+) \to_+ pn(b_+)) \wedge_- n(pn(a_+) \to_+ pn(c_+))) = pn(a_+) \to_+ p(n(b_+) \wedge_- n(c_+)).$$

Since p preserves the implication (Proposition 3.9) and $n \cdot p = Id_{M_-}$ (Definition 3.7.ii), we have

$$n(pn(a_+) \to_+ pn(b_+)) = np(n(a_+) \to_- n(b_+)) = n(a_+) \to_- n(b_+)$$

and, similarly, $n(pn(a_+) \to_+ pn(c_+)) = n(a_+) \to_- n(c_+)$. Thus the left-hand side of the above equality simplifies to $p((n(a_+) \to_- n(b_+)) \wedge_- (n(a_+) \to_- n(c_+)))$. Similarly, we can simplify the right-hand side as follows:

$$pn(a_+) \to_+ p(n(b_+) \wedge_- n(c_+)) = p(n(a_+) \to_- (n(b_+) \wedge_- n(c_+))).$$

Observe that

$$(n(a_+) \to_- n(b_+)) \wedge_- (n(a_+) \to_- n(c_+)) = n(a_+) \to_- (n(b_+) \wedge_- n(c_+))$$

holds because, as observed above, every implicative semilattice satisfies $x \to (y \wedge z) \approx (x \to y) \wedge (x \to z)$. This immediately implies the required result. □

The following result shows that all the quasi-equational conditions of Definition 3.1 can be replaced by equational ones.

PROPOSITION 3.14.
An algebra $\mathbf{A} = \langle A; \to, \sim, 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ is a QNI-algebra if and only if $\mathbf{A}$ satisfies items (i)–(iii), (v), (vii)–(viii), (x)–(xii), (xv) and (xvii) of Definition 3.1 plus the three conditions stated in Lemma 3.13.

PROOF. It follows from (Definition 3.1 and) Lemma 3.13 that every QNI-algebra satisfies the above conditions. For the converse, observe that, using Definition 3.1.x, it is easy to show that Lemma 3.13.i implies Definition 3.1.iv.

Regarding Definition 3.1.vi, assume $\sim a \preccurlyeq \sim b$. Using Lemma 3.13.ii and Definition 3.1.i, we have $1 = (\sim a \to \sim b) \to (\sim a \to (\sim a \odot \sim b)) = 1 \to (\sim a \to (\sim a \odot \sim b)) = \sim a \to (\sim a \odot \sim b)$. Hence, $\sim a \preccurlyeq \sim a \odot \sim b$, as required.

As observed in Remark 3.2, the second implication in Definition 3.1.ix follows from Definition 3.1.xiv and is therefore redundant. Let us show that the first implication in Definition 3.1.ix is also satisfied. Let then $a, b, c, d \in A$ be such that $a \equiv b$ and $c \equiv d$. Observe that $a \preccurlyeq ((a \to c) \to c)$ holds in general. Indeed, by items (iii) and (ii) of Definition 3.1, we have $a \to ((a \to c) \to c) = (a \to c) \to (a \to c) = 1$. Thus, from the assumption $b \preccurlyeq a$ we obtain $b \preccurlyeq ((a \to c) \to c)$ using the transitivity of $\preccurlyeq$ (see Lemma 3.3.iv: note that the proof of this, which is [20, Lemma 22.iv], only uses items (i)–(iii) of Definition 3.1). Using Definition 3.1.iii, this gives us $1 = b \to ((a \to c) \to c) = (a \to c) \to (b \to c)$. Hence $a \to c \preccurlyeq b \to c$, and a similar reasoning shows that $a \to c \equiv b \to c$. On the other hand, from the assumption $c \preccurlyeq d$, using Definition 3.1.iii and Lemma 3.3.i, we obtain $(b \to c) \to (b \to d) = b \to (c \to d) = b \to 1 = 1$. Thus $b \to c \preccurlyeq b \to d$, and a similar reasoning shows that $b \to d \preccurlyeq b \to c$. Hence, $b \to c \equiv b \to d$. Hence, by the transitivity of $\equiv$, we have $a \to c \equiv b \to d$, as required.

Recall that, as noted in Remark 3.2, items (xiii) and (xvi) of Definition 3.1 are redundant.

To complete the proof, let us verify that item Definition 3.1.xiv holds. Observe that, since $a \odot c = \sim(a \to \sim c) = \sim\sim\sim(a \to \sim c) = \sim\sim(a \odot c)$ holds by Definition 3.1.x, Lemma 3.13.iii gives us $(a \odot c) \to (b \odot c) \equiv ((a \odot c) \to \sim\sim b) \odot ((a \odot c) \to \sim\sim c)$. Also observe that Lemma 3.13.i and Lemma 3.3.i entail $((a \odot c) \to \sim\sim c) = \sim\sim a \to (\sim\sim c \to \sim\sim c) = \sim\sim a \to 1 = 1$ (note that the proof of Lemma 3.3.i, which is [20, Lemma 22.i], only uses Definition 3.1.ii). Hence, $(a \odot c) \to (b \odot c) \equiv ((a \odot c) \to \sim\sim b) \odot ((a \odot c) \to \sim\sim c) = ((a \odot c) \to \sim\sim b) \odot 1$. By items (viii) and (i) of Definition 3.1, we have $((a \odot c) \to \sim\sim b) \odot 1 \equiv 1 \odot ((a \odot c) \to \sim\sim b) = \sim(1 \to \sim((a \odot c) \to \sim\sim b)) = \sim\sim((a \odot c) \to \sim\sim b)$. Thus, using the transitivity of $\equiv$, we obtain $(a \odot c) \to (b \odot c) \equiv \sim\sim((a \odot c) \to \sim\sim b)$. By Lemma 3.13.i and Definition 3.1.iii, we have $\sim\sim((a \odot c) \to \sim\sim b) = \sim\sim(\sim\sim a \to (\sim\sim c \to \sim\sim b)) = \sim\sim(\sim\sim c \to (\sim\sim a \to \sim\sim b))$. Hence, $(a \odot c) \to (b \odot c) \equiv \sim\sim(\sim\sim c \to (\sim\sim a \to \sim\sim b))$. Now, assuming $a \preccurlyeq b$, by items (i) and (xii) of Definition 3.1, we have $1 = (a \to b) \to (\sim\sim a \to \sim\sim b) = 1 \to (\sim\sim a \to \sim\sim b) = \sim\sim a \to \sim\sim b$. Therefore, using Lemma 3.3.i, we have $\sim\sim(\sim\sim c \to (\sim\sim a \to \sim\sim b)) = \sim\sim(\sim\sim c \to 1) = \sim\sim 1 = 1$. Thus $(a \odot c) \to (b \odot c) \equiv 1$. By Lemma 3.3.v, this entails $(a \odot c) \to (b \odot c) = 1$, i.e. $a \odot c \preccurlyeq b \odot c$, as required (note that the proof of Lemma 3.3.v, which is [20, Lemma 22.v], only uses items (i) and (ii) of Definition 3.1). Since $a \odot c \equiv c \odot a$ and $b \odot c \equiv c \odot b$ hold by Definition 3.1.viii, the preceding argument (recalling that $\preccurlyeq$ is transitive) also establishes $c \odot a \preccurlyeq c \odot b$, as required. □

COROLLARY 3.15. QNI is a variety.

In the next section we are going to introduce an alternative and in a way more economic representation for QNI-algebras via nuclei; but before that, let us take a look at the congruence lattice of these algebras.

PROPOSITION 3.16. [20], Cor. 34
The term $q(x,y,z)$ is a (commutative, non-regular) ternary deduction term in the sense of [4]. Therefore QNI has equationally definable principal congruences and the strong congruence extension property [4, Thm. 2.12].

PROPOSITION 3.17. [20], Prop. 37
Let $\mathbf{A} \in \mathsf{QNI}$, and let $\mathbf{A}_+$ be the corresponding bounded Hilbert algebra (as given in Theorem 3.12). Let $\theta \in \mathrm{Con}(\mathbf{A})$ and $\eta \in \mathrm{Con}(\mathbf{A}_+)$.

(i) $\theta_+ \in \mathrm{Con}(\mathbf{A}_+)$, where $\theta_+ := \{\langle [a],[b]\rangle \in A_+ \times A_+ : \langle a \to e, b \to e\rangle \in \theta$ for all$e \in A\}$.
(ii) $\eta^{\bowtie} \in \mathrm{Con}(\mathbf{A})$, where $\eta^{\bowtie} := \{\langle a,b\rangle \in A \times A : \langle [a],[b]\rangle, \langle [{\sim} a],[{\sim} b]\rangle \in \eta\}$.
(iii) $(\theta_+)^{\bowtie} = \theta$.
(iv) $(\eta^{\bowtie})_+ = \eta$.

THEOREM 3.18. [20], Thm. 6
For every $\mathbf{A} \in \mathsf{QNI}$, one has $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{A}_+)$ via the mutually inverse maps $(.)^{\bowtie}$ and $(.)_+$.

It is well known that the lattice of congruences of every Hilbert algebra is distributive (see e.g. [5, p. 477]). Thus Theorem 3.18 gives us the following.

COROLLARY 3.19. [20], Cor. 38
QNI is congruence-distributive.

4 Nuclear Representation

In this section we present an alternative twist representation of QNI-algebras via nuclei that is the analogue of the representation of quasi-Nelson algebras stated in Theorem 2.7.

Let $\langle \mathbf{H}_+, \mathbf{M}_-, n, p\rangle$ be given as in Definition 3.7. Define on $\mathbf{H}_+$ the operations $\odot_+$ and $\Box$ as follows: for $a_+, b_+ \in H_+$,

$$a_+ \odot_+ b_+ := p(n(a_+) \wedge_- n(b_+)) \qquad \Box a_+ := a_+ \odot_+ a_+ = pn(a_+).$$

We list without proof a few straightforward properties enjoyed by these operations.

PROPOSITION 4.1.
Let $\langle \mathbf{H}_+, \mathbf{M}_-, n, p\rangle$ be as in Definition 3.7, and let $a_+, b_+ \in H_+$.

(i) $\Box 1_+ = 1_+$ and $\Box_+ 0_+ = 0_+$.
(ii) $\Box(a_+ \to_+ b_+) \leq \Box a_+ \to_+ \Box b_+$.
(iii) $a_+ \leq_+ \Box a_+ = \Box\Box a_+$.
(iv) $a_+ \odot_+ (b_+ \odot_+ c_+) = (a_+ \odot_+ b_+) \odot_+ c_+$.
(v) $a_+ \odot_+ b_+ = b_+ \odot_+ a_+ = a_+ \odot_+ (a_+ \to_+ b_+)$.
(vi) $a_+ \to_+ (a_+ \odot_+ b_+) = a_+ \to_+ b_+$.

(vii) $\Box(a_+ \odot_+ b_+) = \Box a_+ \odot_+ \Box b_+ = a_+ \odot_+ b_+$.
(viii) $(a_+ \odot_+ b_+) \to c_+ = \Box a_+ \to_+ (\Box b_+ \to_+ c_+)$.
(ix) $a_+ \odot_+ 0_+ = 0_+$.
(x) $a_+ \odot_+ 1_+ = \Box a_+$.

The algebra $\langle H_+; \odot_+\rangle$ is thus a commutative semigroup. Letting $H_+^{\Box} := \{\Box a_+ : a_+ \in H_+\}$, the following result is also straightforward.

PROPOSITION 4.2.
$\mathbf{H}_+^{\Box} = \langle H_+^{\Box}; \odot_+, 0_+, 1_+\rangle$ is a bounded semilattice.

DEFINITION 4.3.
Given a bounded Hilbert algebra $\mathbf{H} = \langle H; \to, 0, 1\rangle$, we say that a unary operation $\Box$ is a *nucleus* on $\mathbf{H}$ if, for all $a, b \in H$,

(i) $\Box 0 = 0$,
(ii) $\Box(a \to b) \leq \Box a \to \Box b$.
(iii) $a \leq \Box a = \Box\Box a$.

Hilbert algebras ('positive implication algebras') with nuclei are considered, for instance, in [12, Ch. 13]. If the natural order of $\mathbf{H}$ is a lattice order, then the structure $\mathbf{H} = \langle H; \wedge, \vee \to, \Box, 0, 1\rangle$ is a Heyting algebra with a nucleus.

We proceed to establish and analogue of Proposition 2.5, which suggests an alternative way to define nuclei on bounded Hilbert algebras.

LEMMA 4.4.
Let $\mathbf{H} = \langle H; \to, 0, 1\rangle$ be a bounded Hilbert algebra endowed with a unary operation $\Box$ satisfying $\Box 0 = 0$. The following are equivalent:

(i) $\Box$ is a nucleus.
(ii) $\mathbf{H} \vDash x \to \Box y \approx \Box x \to \Box y$.

PROOF. Assume (i) holds. Since $a \leq \Box a$, we have $\Box a \to \Box b \leq a \to \Box b$ because the Hilbert implication is order-reversing in the first argument. To show that $a \to \Box b \leq \Box a \to \Box b$, we compute:

$$
\begin{aligned}
1 &= (\Box a \to \Box b) \to (\Box a \to \Box b) && \\
&= (\Box a \to (\Box a \to \Box b)) \to (\Box a \to \Box b) && x \to y \approx x \to (x \to y) \\
&= \Box a \to ((\Box a \to \Box b) \to \Box b) && (x \to y) \to (x \to z) \approx x \to (y \to z) \\
&= \Box a \to ((\Box a \to \Box\Box b) \to \Box b) && \Box\Box x \approx \Box x \\
&\leq \Box a \to (\Box(a \to \Box b) \to \Box b) && \Box(x \to y) \leq \Box x \to \Box y \\
&= \Box(a \to \Box b) \to (\Box a \to \Box b) && x \to (y \to z) \approx y \to (x \to z) \\
&\leq (a \to \Box b) \to (\Box a \to \Box b) && x \leq \Box x.
\end{aligned}
$$

Conversely, assume (ii) holds. Observe that, by instantiating $x \to \Box y \approx \Box x \to \Box y$, we can obtain $a \to \Box a = \Box a \to \Box a = 1$ and $1 = \Box a \to \Box a = \Box\Box a \to \Box a$. Hence, for all $a \in H$, we have $a \leq \Box a$ and $\Box\Box a \leq \Box a$. It remains to show that $\Box(a \to b) \leq \Box a \to \Box b$ for all $a, b \in H$. Observe that, from $b \leq \Box b$, we have $a \to b \leq a \to \Box b$ because the Hilbert implication is order-preserving in the second argument. Hence,

$$\begin{aligned}
1 &= (a \to b) \to (a \to \Box b) && \\
&= a \to ((a \to b) \to \Box b) && x \to (y \to z) \approx y \to (x \to z) \\
&= a \to (\Box(a \to b) \to \Box b) && x \to \Box y \approx \Box x \to \Box y \\
&= \Box(a \to b) \to (a \to \Box b) && x \to (y \to z) \approx y \to (x \to z) \\
&= \Box(a \to b) \to (\Box a \to \Box b) && x \to \Box y \approx \Box x \to \Box y.
\end{aligned}$$

□

DEFINITION 4.5.
A *bounded nuclear Hilbert semigroup* (nH-semigroup for short) is an algebra $\mathbf{S} = \langle S; \odot, \to, 0, 1\rangle$ such that

(i) $\langle S; \to, 0, 1\rangle$ is a bounded Hilbert algebra;
(ii) $\langle S; \odot\rangle$ is a commutative semigroup;
(iii) The operation $\Box$ given by $\Box a := a \odot a$ for all $a \in S$ is a nucleus on $\langle S; \to, 0, 1\rangle$.
(iii) For all $a, b \in S$,
(iv) $a \odot b = a \odot (a \to b)$
(v) $\Box a \to (\Box b \to c) = (a \odot b) \to c$
(vi) $a \odot 0 = 0$
(vii) $a \odot 1 = \Box a$.

The above-defined operation $\odot$ can be thought of as a generalization of the one determined, on every Heyting algebra with a nucleus, by the term $\Box(x \wedge y)$ or, equivalently, $\Box x \wedge \Box y$. The following lemma justifies this remark and provides an example of an nH-semigroup, which will be useful later on.

LEMMA 4.6.
Let $\mathbf{M} = \langle M; \wedge, \to, \Box, 0, 1\rangle$ be a bounded implicative semilattice with a nucleus. Upon defining $x \odot y := \Box x \wedge \Box y$, the algebra $\langle M; \odot, \to\ 0, 1\rangle$ is an nH-semigroup.

PROOF. All items of Definition 4.5 are easily verifiable. We show a few examples. Item (i) is clear. It is also clear that, for all $a \in M$, one has $\Box a = a \odot a = \Box a \wedge \Box a$. Regarding item (iv), we have, for all $a, b \in M$, $a \odot b = \Box a \wedge \Box b = \Box(a \wedge b) = \Box(a \wedge (a \to b)) = \Box a \wedge \Box(a \to b) = a \odot (a \to b)$. □

Indeed, any bounded Hilbert algebra can be endowed with an nH-semigroup structure (Proposition 4.8); hence, nH-semigroups can also be viewed as a generalization (rather than a specialization) of bounded Hilbert algebras. To show this, we shall rely on the properties of bounded Hilbert algebras listed in the following lemma. Given a bounded Hilbert algebra $\mathbf{H} = \langle H; \to, 0, 1\rangle$, we abbreviate $\neg x := x \to 0$.

LEMMA 4.7.
Let $\mathbf{H} = \langle H; \to, 0, 1\rangle$ be a bounded Hilbert algebra, and let $a, b, c \in H$.

(i) $a \leq \neg\neg a$.
(ii) $\neg a = \neg\neg\neg a$
(iii) $a \to \neg b = b \to \neg a$.
(iv) $a \to \neg b = \neg\neg a \to \neg b$.

(v) $a \leq b \to \neg(a \to \neg b)$.
(vi) $\neg\neg a \to (\neg\neg b \to c) = \neg(a \to \neg b) \to c$.

PROOF. We do not prove the identities that are well known to hold on all Hilbert algebras (as opposed to bounded ones), such as $x \to x \approx 1$, $x \to 1 \approx 1$ etc. (see e.g. [5, Lemma 1.1]). Also recall that $a \leq b$ if and only if $a \to b = 1$, for all $a, b \in H$.

(i). We have:

$$\begin{aligned} a \to \neg\neg a &= a \to (\neg a \to 0) && \\ &= \neg a \to (a \to 0) && x \to (y \to z) \approx y \to (x \to z) \\ &= \neg a \to \neg a && \\ &= 1 && x \to x \approx 1. \end{aligned}$$

(ii). By item (i), it suffices to prove $\neg\neg\neg a \leq \neg a$. We have:

$$\begin{aligned} \neg a \to \neg\neg\neg a &= \neg a \to (\neg\neg a \to 0) && \\ &= \neg\neg a \to (\neg a \to 0) && x \to (y \to z) \approx y \to (x \to z) \\ &= \neg\neg a \to \neg\neg a && \\ &= 1 && x \to x \approx 1. \end{aligned}$$

(iii). Obviously it suffices to show $a \to \neg b \leq b \to \neg a$. We have:

$$\begin{aligned} (a \to \neg b) \to (b \to \neg a) &= b \to ((a \to \neg b) \to \neg a) && x \to (y \to z) \approx y \to (x \to z) \\ &= b \to ((a \to \neg b) \to (a \to 0)) && \\ &= b \to (a \to (\neg b \to 0)) && \text{by (H2}'\text{)} \\ &= a \to (b \to \neg\neg b) && x \to (y \to z) \approx y \to (x \to z) \\ &= a \to 1 && \text{by item (i)} \\ &= 1 && x \to 1 \approx 1. \end{aligned}$$

(iv). Using items (iii) and (ii), we have $a \to \neg b = b \to \neg a = b \to \neg\neg\neg a = \neg\neg a \to \neg b$.

(v). Since $a \leq b \to a$, we have $(a \to c) \to d \leq ((b \to a) \to c) \to d$ for all $c, d \in H$. Hence,

$$\begin{aligned} 1 &= (a \to \neg b) \to (a \to \neg b) && x \to x \approx 1 \\ &\leq ((b \to a) \to \neg b) \to (a \to \neg b) && \\ &= a \to (((b \to a) \to \neg b) \to \neg b) && x \to (y \to z) \approx y \to (x \to z) \\ &= a \to (((b \to a) \to (b \to \neg b)) \to (b \to 0)) && x \to (x \to 0) \approx \neg x \\ &= a \to ((b \to (a \to \neg b)) \to (b \to 0)) && \text{by (H2}'\text{)} \\ &= a \to (b \to ((a \to \neg b) \to 0)) && \text{by (H2}'\text{)} \\ &= a \to (b \to \neg(a \to \neg b)). && \end{aligned}$$

(vi). By item (iii), $\neg a \leq b \to \neg a = a \to \neg b$. Since the Hilbert implication is order-reversing in the first argument, from the latter we obtain $\neg(a \to \neg b) \leq \neg\neg a$. Similarly, $\neg(a \to \neg b) \leq \neg\neg b$.

From the the latter we have $\neg\neg b \to c \leq \neg(a \to \neg b) \to c$. Using also the observation that the Hilbert implication is order-preserving in the second argument, from $\neg\neg b \to c \leq \neg(a \to \neg b) \to c$ we obtain $\neg\neg a \to (\neg\neg b \to c) \leq \neg\neg a \to (\neg(a \to \neg b) \to c) \leq \neg(a \to \neg b) \to (\neg(a \to \neg b) \to c) = \neg(a \to \neg b) \to c$, where the last equality holds because of the contraction identity $x \to (x \to y) \approx x \to y$. To show that $\neg(a \to \neg b) \to c \leq \neg\neg a \to (\neg\neg b \to c)$, first observe that:

$$\begin{aligned}
\neg\neg a \to (\neg\neg b \to \neg(a \to \neg b)) &= (\neg\neg a \to \neg\neg b) \to (\neg\neg a \to \neg(a \to \neg b)) && \text{by (H2}'\text{)}\\
&= (a \to \neg\neg b) \to (a \to \neg(a \to \neg b)) && \text{by item (iv)}\\
&= a \to (\neg\neg b \to \neg(a \to \neg b)) && \text{by (H2}'\text{)}\\
&= a \to (b \to \neg(a \to \neg b)) && \text{by item (iv)}\\
&= 1 && \text{by item (v).}
\end{aligned}$$

Now, recalling that the identity $x \leq (x \to y) \to y$ holds on every Hilbert algebra, we compute:

$$\begin{aligned}
1 &= \neg\neg a \to (\neg\neg b \to (\neg(a \to \neg b)))\\
&\leq \neg\neg a \to (\neg\neg b \to ((\neg(a \to \neg b) \to c) \to c))\\
&= [\text{by} x \to (y \to z) \approx y \to (x \to z)]\\
&= \neg\neg a \to ((\neg(a \to \neg b) \to c) \to (\neg\neg b \to c))\\
&= [\text{by} x \to (y \to z) \approx y \to (x \to z)]\\
&= (\neg(a \to \neg b) \to c) \to (\neg\neg a \to (\neg\neg b \to c)).
\end{aligned}$$

□

PROPOSITION 4.8.
Let **H** be any algebra having a bounded Hilbert algebra reduct $\langle H; \to, 0, 1\rangle$. Upon defining $x \odot y := \neg(x \to \neg y)$, the algebra $\langle H; \odot, \to 0, 1\rangle$ is an nH-semigroup.

PROOF. Let us verify that items (ii)–(vii) of Definition 4.5 are satisfied.

(ii). Commutativity of $\odot$ follows directly from Lemma 4.7.iii. For associativity, we need to show that $a \odot (b \odot c) = \neg(a \to \neg\neg(b \to \neg c)) = \neg(\neg(a \to \neg b) \to \neg c) = (a \odot b) \odot c$. Thus, it suffices to verify the following equalities:

$$\begin{aligned}
a \to \neg\neg(b \to \neg c) &= \neg(b \to \neg c) \to \neg a && \text{by Lemma 4.7.iii}\\
&= \neg\neg b \to (\neg\neg c \to \neg a) && \text{by Lemma 4.7.vi}\\
&= \neg\neg b \to (a \to \neg\neg\neg c) && \text{by Lemma 4.7.iii}\\
&= \neg\neg b \to (a \to \neg c) && \text{by Lemma 4.7.ii}\\
&= \neg\neg b \to (\neg\neg a \to \neg c) && \text{by Lemma 4.7.iv}\\
&= \neg\neg a \to (\neg\neg b \to \neg c) && x \to (y \to z) \approx y \to (x \to z)\\
&= \neg(a \to \neg b) \to \neg c && \text{by Lemma 4.7.vi.}
\end{aligned}$$

(iii). Observe that $a \to \neg a = a \to (a \to 0) = a \to 0 = \neg a$ for all $a \in H$. Thus, $\Box a = \neg(a \to \neg a) = \neg\neg a$. Using this observation, it is easy to verify that $\Box$ is a nucleus.

(iv). We need to check that $\neg(a \to \neg b) = \neg(a \to \neg(a \to b))$ for all $a, b \in H$. Indeed, this follows from the equality $a \to \neg b = a \to \neg(a \to b)$. The latter holds because, using (H1) and (H2), we have $a \to \neg(a \to b) = a \to ((a \to b) \to 0) = (a \to b) \to (a \to 0) = a \to (b \to 0) = a \to \neg b$.

(v). This is Lemma 4.7 (vi).

(vi). Easy: for all $a \in H$, we have $a \odot 0 = \neg(a \to \neg 0) = \neg(a \to 1) = \neg 1 = 0$.

(vii). Also easy: $a \odot 1 = \neg(a \to \neg 1) = \neg(a \to 0) = \neg\neg a$ for all $a \in H$. $\square$

Note that Proposition 4.8 applies also to (e.g.) Heyting algebras and implicative semilattices but produces an nH-semigroup of a special type (as opposed to the one in Lemma 4.6), namely one where the nucleus coincides with the double negation.

The following lemma will be useful in subsequent proofs.

LEMMA 4.9.
Let $\mathbf{S} = \langle S; \to, \odot, 0, 1\rangle$ be an nH-semigroup and $a, b, c \in S$.

(i) $\Box(a \odot b) = \Box a \odot \Box b = a \odot b$.
(ii) If $a \leq b$, then $\Box a = a \odot b$.
(iii) $\Box a \to (b \odot c) = (\Box a \to \Box b) \odot (\Box a \to \Box c)$.
(iv) $a \to 0 = \Box a \to 0$.

PROOF. (i). The equality $\Box(a \odot b) = \Box a \odot \Box b$ is straightforward: observe that, applying the definition of $\Box$ (and using the associativity and commutativity of $\odot$), we have $\Box(a \odot b) = a \odot b \odot a \odot b = a \odot a \odot b \odot b = \Box a \odot \Box b$. To show that $\Box a \odot \Box b = a \odot b$, recall that every Hilbert algebra satisfies $x \to x \approx 1$. Then, using Definition 4.5.v and Definition 4.3.iii, we have $(\Box a \odot \Box b) \to (a \odot b) = \Box\Box a \to (\Box\Box b \to (a \odot b)) = \Box a \to (\Box b \to (a \odot b)) = (a \odot b) \to (a \odot b) = 1$. Thus $\Box a \odot \Box b \leq a \odot b$. The other inequality follows from the following observation: by Definition 4.3.iii we have $a \odot b \leq \Box(a \odot b) = \Box a \odot \Box b$. Hence $a \odot b = \Box a \odot \Box b$.

(ii). Given $a, b \in S$ such that $a \leq b$ (so $a \to b = 1$), we are going to show that $(a \odot b) \to \Box a = \Box a \to (a \odot b) = 1$. Since every Hilbert algebra satisfies $x \to (y \to x) \approx 1$, by Definition 4.5.v we have $(a \odot b) \to \Box a = \Box a \to (\Box b \to \Box a) = 1$. Also recall that, by Definition 4.5.iv, we have $a \odot b = a \odot (a \to b)$. Thus, using also the assumption Definition 4.5.vii, it is easy to verify that the assumption $a \to b = 1$ entails $\Box a \to (a \odot b) = \Box a \to (a \odot (a \to b)) = \Box a \to (a \odot 1) = \Box a \to \Box a = 1$.

(iii). In the light of Proposition 4.11, it is sufficient to adapt the proof of Lemma 3.13.iii.

(iv). Recall that $\Box 0 = 0$ (Definition 4.3.i). Using this and Lemma 4.4, we have $a \to 0 = a \to \Box 0 = \Box a \to \Box 0 = \Box a \to 0$, as required. $\square$

Given an nH-semigroup $\mathbf{S}$, let $S^{\Box} := \{\Box a : a \in S\} = \{a \in S : a = \Box a\}$.

LEMMA 4.10.
For every nH-semigroup $\mathbf{S}$, the algebra $\mathbf{S}^{\Box} := \langle S^{\Box}; \odot, 0, 1\rangle$ is a bounded semilattice.

PROOF. Definition 4.3.iii implies $1 \leq \Box 1$, therefore $\Box 1 = 1$. Thus $\Box$ preserves the bounds, and we have $0, 1 \in S^{\Box}$. Lemma 4.9.i entails that $S^{\Box}$ is closed under the $\odot$ operation. Thus $\langle S^{\Box}; \odot\rangle$ is a commutative semigroup. For all $a \in S^{\Box}$, items (vi) and (vii) of Definition 4.5 imply $a \odot 0 = 0 \odot a = 0$ and $a \odot 1 = 1 \odot a = a \odot a = \Box a = a$. Moreover, $a \odot a = \Box a = a$. So $\langle S^{\Box}; \odot, 0, 1\rangle$ is a bounded semilattice, as claimed. $\square$

It is also easy to check that, upon defining $a \to^{\Box} b := \Box(a \to b)$, we have that $\langle S^{\Box}; \odot, \to^{\Box}, 0, 1\rangle$ is an implicative semilattice (cf. Proposition 3.9 above). Furthermore, the map $\Box \colon S \to S^{\Box}$ and the identity map $Id_{S^\Box} \colon S^\Box \to S^\Box$ satisfy the following properties (corresponding precisely to those of Definition 3.7).

PROPOSITION 4.11.
Let $\mathbf{S}$ be an nH-semigroup, and let $\mathbf{S}^\Box$ and $\Box \colon S \to S^\Box$ be defined as above. Then:

(i) $\Box$ and $Id_{A^\Box}$ are monotone and preserve the bounds.
(ii) $\Box \cdot Id_{S^\Box} = Id_{S^\Box}$ and $Id_S \leq Id_{S^\Box} \cdot \Box$.
(iii) $\Box a \odot \Box b = \Box a \odot \Box(a \to b)$ for all $a, b \in S$.
(iv) $(a \odot b) \to c = a \to (b \to c)$ for all $a, b \in S^\Box$ and all $c \in S$.

PROOF. (i). The statement clearly holds for the map $Id_{A^\Box}$. Also, we have observed in the proof of Lemma 4.10 that the map $\Box$ preserves the bounds. To check that $\Box$ is monotone, assume $a \leq b$ for some $a, b \in S$, which means $1 = a \to b$. Then, using Definition 4.3.ii, we have $1 = \Box 1 = \Box(a \to b) \leq \Box a \to \Box b$, so $\Box a \leq \Box b$.

(ii). This clearly follows from $\Box$ being a nucleus (Definition 4.3.iii).

(iii). We have observed in the proof of Lemma 4.10 that $\Box a \odot \Box b = \Box(a \odot b)$ for all $a, b \in S$. Using this observation and Definition 4.3.iv, we have $\Box a \odot \Box b = \Box(a \odot b) = \Box(a \odot (a \to b)) = \Box a \odot \Box(a \to b)$, as required.

(iv). This easily follows from Definition 4.5.v together with Definition 4.3.iii. $\Box$

Let $\mathbf{S} = \langle S; \to, \odot, 0, 1\rangle$ be an nH-semigroup. By Proposition 4.11, upon defining $\mathbf{H}_+ := \langle S; \to, \odot, 0, 1\rangle$, $\mathbf{M}_- := \mathbf{S}^\Box$, $n(a) = \Box a$ for all $a \in S$ and $p(a) = a$ for all $a \in S^\Box$, can construct an algebra $\mathbf{H}_+ \bowtie \mathbf{M}_-$ according to Definition 3.7, which we can simply denote by $\mathbf{H}_+ \bowtie \mathbf{H}_+$ since it is completely determined by $\mathbf{H}_+$. We can further consider QNI twist-algebras $\mathbf{A}$ over $\langle \mathbf{H}_+, \mathbf{M}_-, n, p\rangle$ or (more briefly) over $\mathbf{H}_+$. Observe that the elements of A are pairs $\langle a, b\rangle \in S \times S$ satisfying $b = \Box b$ and $\Box a \odot b = 0$. The latter requirement is indeed equivalent to $a \odot b = 0$. We officialize these considerations in the following definition.

DEFINITION 4.12.
Let $\mathbf{S} = \langle S; \odot, \to, 0, 1\rangle$ be an nH-semigroup. The algebra $\mathbf{S} \bowtie \mathbf{S} = \langle S \times S; \to, \sim, 0, 1\rangle$ is defined as follows. For all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in S \times S$,

$$1 := \langle 1, 0\rangle,$$

$$0 := \langle 0, 1\rangle,$$

$$\sim\langle a_1, a_2\rangle := \langle a_2, \Box a_1\rangle,$$

$$\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle := \langle a_1 \to b_1, a_1 \odot b_2\rangle.$$

A *quasi-Nelson implicative twist-algebra* (QNI twist-algebra) $\mathbf{A}$ over $\mathbf{S}$ is a $\{\to, \sim, 0, 1\}$-subalgebra of $\mathbf{S} \bowtie \mathbf{S}$ with carrier set A satisfying: $\pi_1[A] = S$ and, for all $\langle a_1, a_2\rangle \in A$, $\Box a_2 = a_2$ and $a_1 \odot a_2 = 0$.

Keeping the abbreviation $x \odot y := \sim(x \to \sim y)$, we can check that

$$\langle a_1, a_2\rangle \odot \langle b_1, b_2\rangle = \langle a_1 \odot \Box b_1, \Box(a_1 \to b_2)\rangle = \langle a_1 \odot b_1, a_1 \to b_2\rangle.$$

The last equality holds because, on the one hand, by Lemma 4.9.i and Definition 4.3.iii, we have $a_1 \odot \Box b_1 = \Box a_1 \odot \Box\Box b_1 = \Box a_1 \odot \Box b_1 = a_1 \odot b_1$. On the other hand, $a_1 \to b_2 \leq \Box(a_1 \to b_2)$ holds by Definition 4.3.iii, but we also have $\Box(a_1 \to b_2) \leq \Box a_1 \to \Box b_2 = a_1 \to b_2$ by Lemma 4.4 (and the requirement $\Box b_2 = b_2$).

For Definition 4.12 to be sound, we need to check that the set

$$B := \{\langle a_1, a_2\rangle : a_1 \odot a_2 = 0,\ a_2 = \Box a_2\}$$

is the universe of a subalgebra of $\mathbf{S} \bowtie \mathbf{S}$ (and therefore the largest twist-algebra over $\mathbf{S}$). It is clear that $\langle 1, 0\rangle, \langle 0, 1\rangle \in B$. Assuming $\langle a_1, a_2\rangle \in B$, we have $\sim\langle a_1, a_2\rangle = \langle a_2, \Box a_1\rangle \in B$ because $a_2 \odot \Box a_1 = \Box a_2 \odot \Box a_1 = a_2 \odot a_1 = 0$ (Lemma 4.9.i) and $\Box\Box a_1 = \Box a_1$ (Definition 4.3.iii). Regarding the binary operation, assume $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in B$. To see that $\langle a_1 \to b_1, a_1 \odot b_2\rangle \in B$, we observe that $\Box(a_1 \odot b_2) = a_1 \odot b_2$ by Lemma 4.9.i. Furthermore, using the commutativity and associativity of $\odot$ together with items (iv) and (vi) of Definition 4.5, we have $(a_1 \to b_1) \odot (a_1 \odot b_2) = a_1 \odot (a_1 \to b_1) \odot b_2 = a_1 \odot b_1 \odot b_2 = 0 \odot b_2 = 0$, as required.

Before stating the main embedding result (Theorem 4.16 below), we wish to give a direct proof of an interesting observation that could also be derived from Theorem 4.16 together with Theorem 3.18.

LEMMA 4.13. [20], Lemma 36

Let $\mathbf{H} = \langle H; \to, 1\rangle$ be a Hilbert algebra, $a, b \in A$ and $\theta \in \mathrm{Con}(\mathbf{H})$. The following conditions are equivalent:

(i) $\langle a, b\rangle \in \theta$.

(ii) $\langle a \to b, 1\rangle, \langle b \to a, 1\rangle \in \theta$.

PROPOSITION 4.14.

Let $\mathbf{S} = \langle S; \odot, \to, 0, 1\rangle$ be a nH-semigroup and let $\theta \subseteq S \times S$ be an equivalence relation. The following are equivalent:

(i) θ is a congruence of $\mathbf{S}$.

(ii) θ is a congruence of the Hilbert algebra reduct $\langle S; \to, 1\rangle$.

PROOF. Obviously it suffices to show that (ii) implies (i). Let then θ be a congruence of the $\odot$-free reduct of $\mathbf{S}$ such that $\langle a, b\rangle \in \theta$. We claim that, for all $c \in S$, we have $\langle a \odot c, b \odot c\rangle \in \theta$ and (since $\odot$ is commutative) also $\langle c \odot a, c \odot b\rangle \in \theta$. Indeed, from $\langle a, b\rangle \in \theta$ we have $\langle a \to b, b \to b\rangle = \langle a \to b, 1\rangle \in \theta$. Recall that (on every Hilbert algebra) for all $c, d \in S$ such that $c \leq d$, one has that $\langle c, 1\rangle \in \theta$ implies $\langle d, 1\rangle \in \theta$. Indeed, since $c \to d = 1$ and $1 \to d = d$, from $\langle c, 1\rangle \in \theta$ we obtain $\langle c \to d, 1 \to d\rangle = \langle 1, d\rangle \in \theta$. Hence, from $\langle a \to b, 1\rangle \in \theta$ and the inequalities $a \to b \leq \Box(a \to b) \leq \Box a \to \Box b$, which hold by items (iii) and (ii) of Definition 4.3, we have $\langle \Box a \to \Box b, 1\rangle \in \theta$. The same reasoning shows that $\langle \Box c \to (\Box a \to \Box b), 1\rangle \in \theta$ for all $c \in S$. Since $\Box c \to (\Box a \to \Box b) = \Box a \to (\Box c \to \Box b)$, we have $\langle \Box a \to (\Box c \to \Box b), 1\rangle \in \theta$. Also, from $\Box c \to \Box b \leq \Box(\Box c \to \Box b)$, we have $\Box a \to (\Box c \to \Box b) \leq \Box a \to \Box(\Box c \to \Box b)$, which gives us $\langle \Box a \to \Box(\Box c \to \Box b), 1\rangle \in \theta$. By Definition 4.5.vii and Lemma 4.9.iii, we have $\Box(\Box c \to \Box b) = (\Box c \to \Box b) \odot 1 = (\Box c \to \Box b) \odot (\Box c \to \Box c) = \Box c \to (b \odot c)$. Thus, using Definition 4.5.v, we have $\Box a \to \Box(\Box c \to \Box b) = \Box a \to (\Box c \to (b \odot c)) = (a \odot c) \to (b \odot c)$. Hence, $\langle (a \odot c) \to (b \odot c), 1\rangle \in \theta$. Similar reasoning shows that $\langle a, b\rangle \in \theta$ entails $\langle (b \odot c) \to (a \odot c), 1\rangle \in \theta$. By Lemma 4.13, this gives us $\langle a \odot c, b \odot c\rangle \in \theta$, as claimed. This easily entails the desired result.$\Box$

In the converse direction, from every $\mathbf{A} = \langle A; \to, \sim, 0, 1\rangle \in \mathsf{QNI}$, we can retrieve an nH-semigroup as follows. The Hilbert algebra quotient $\langle A/{\equiv}; \to, 0, 1\rangle$ is defined as in the preceding section. On $A/{\equiv}$ we define the operation $\odot$ by $[a] \odot [b] := [a \odot b] = [\sim(a \to \sim b)]$ and we let $S(\mathbf{A}) := \langle A/{\equiv}; \to, \odot, 0, 1\rangle$. Observe that we could let $\Box[a] := [\sim\sim a]$ because $[\sim\sim a] = [a \odot a] = [\sim(a \to \sim a)]$ holds on every QNI-algebra.

PROPOSITION 4.15.
For every $\mathbf{A} = \langle A; \to, \sim, 0, 1\rangle \in \mathsf{QNI}$, the above-defined algebra $S(\mathbf{A}) := \langle A/\equiv; \to, \odot, 0, 1\rangle$ is an nH-semigroup.

PROOF. We verify that all items of Definition 4.5 are satisfied. Item (i) follows from Proposition 3.4. Item (ii) is a consequence of items (vii) and (viii) of Definition 3.1. Regarding item (iii), recall that $\Box[a] = [\sim\sim a]$ for all $a \in A$. It then follows from Definition 3.1.xi that $\Box$ preserves both bounds, and Definition 3.1.xiii implies that $[a] \leq \Box[a]$ for all $a \in A$. That $\Box[a] = \Box\Box[a]$ follows from Definition 3.1.x. Finally, to establish item (ii) of Definition 4.3, we can reason (using twist-algebras) as in [20, Lemma 28]. Item (iv) of Definition 4.5 follows from Definition 3.1.xv, and Definition 4.5.v in Lemma 3.13.i. Item (vi) of Definition 4.5 is easy. Regarding item (vii), observe that $a \odot 1 = 1 \odot a = \sim(1 \to \sim a) = \sim\sim a$ and that, as observed earlier, $\sim\sim a \equiv a \odot a$. $\Box$

We have thus established an embedding result analogue to Theorem 3.12.

THEOREM 4.16.
Every QNI algebra $\mathbf{A}$ is isomorphic to a QNI twist-algebra over $S(\mathbf{A})$ through the map $\iota\colon A \to A/\equiv \times A/\equiv$ given by $\iota(a) := \langle [a], [\Box \sim a]\rangle = \langle [a], [\sim a]\rangle$ for all $a \in A$.

Our next aim is to show that every nH-semigroup is embeddable into a (complete) Heyting algebra with a nucleus. The latter result (taking advantage of Theorems 2.7 and 4.16) will then be used to to show that every QNI-algebra embeds into a quasi-Nelson algebra, thus justifying the claim that QNI-algebras are precisely the $\{\to, \sim, 0, 1\}$-subreducts of quasi-Nelson algebras. We shall need a few lemmas, beginning from the following well-known result on Hilbert algebras.

LEMMA 4.17. [16], Thm. II.4.1
Every bounded Hilbert algebra $\langle H; \to, 0, 1\rangle$ embeds into a complete Heyting algebra (of open sets of a T_0 space).

The Heyting algebra and the embedding of Lemma 4.17 are constructed as follows. Given a Hilbert algebra $\langle H; \to, 0, 1\rangle$, one defines an *implicative filter* as a subset $F \subseteq H$ satisfying (i) $1 \in F$ and (ii) $b \in F$ whenever $a, a \to b \in F$, for all $a, b \in H$. An implicative filter F is *irreducible* if $F \neq H$ and $F = F_1 \cap F_2$ (where F_1, F_2 are implicative filters) entails $F = F_1$ or $F = F_2$. The set of all irreducible implicative filters of $\langle H; \to, 0, 1\rangle$, denoted $\mathcal{X}$, is endowed with the following topology. Defining $h(a) := \{F \in \mathcal{X} : a \in F\}$ for all $a \in H$, one considers the topology τ generated by the subbase $\{h(a) : a \in H\}$. On $\mathcal{X}$ one defines a Heyting algebra $\mathbf{A}(\mathcal{X}) = \langle \mathcal{O}(\mathcal{X}); \wedge, \vee, \to, 0, 1\rangle$ as follows. The universe $\mathcal{O}(\mathcal{X})$ is the family of all τ-open subsets of $\mathcal{X}$ and the operations are defined, for all $O_1, O_2 \in \mathcal{O}(\mathcal{X})$, by

$$1 := \mathcal{X},$$

$$0 := \emptyset,$$

$$O_1 \wedge O_2 := O_1 \cap O_2,$$

$$O_1 \vee O_2 := O_1 \cup O_2,$$

$$O_1 \to O_2 := \mathsf{Int}((\mathcal{X} - O_1) \cup O_2),$$

where Int is the interior operator. Then the map $h\colon H \to \mathcal{O}(\mathcal{X})$ is the required embedding, i.e. one has $h(1) = \mathcal{X}$, $h(0) = \emptyset$ and $h(a \to b) = \mathsf{Int}((\mathcal{X} - h(a)) \cup h(b))$ for all $a, b \in H$.

Now, given an nH-semigroup $\mathbf{H} = \langle H; \odot, \to, \Box, 0, 1\rangle$, let $\mathbf{A}(\mathcal{X}) = \langle \mathcal{O}(\mathcal{X}); \wedge, \vee, \to, 0, 1\rangle$ be the complete Heyting algebra defined as above and h the above-defined embedding. Let us expand $\mathbf{A}(\mathcal{X})$

with operations $\Box_{\mathbf{A}}$ and $\odot_{\mathbf{A}}$ defined as follows: for all $O, O', \in \mathbf{A}(\mathcal{X})$,

$$\Box_{\mathbf{A}} O := \bigwedge\{(O \to h(\Box a)) \to h(\Box a) : a \in H\}$$
$$O \odot_{\mathbf{A}} O' := \Box_{\mathbf{A}} O \cap \Box_{\mathbf{A}} O'.$$

Observe that $\mathbf{A}(\mathcal{X})$, being complete, is closed under $\Box_{\mathbf{A}}$ (and therefore also under $\odot_{\mathbf{A}}$). Thus, $\langle \mathcal{O}(\mathcal{X}); \wedge, \vee, \to, \odot_{\mathbf{A}}, \Box_{\mathbf{A}}, 0, 1\rangle$ is an algebra.

THEOREM 4.18.
Let $\mathbf{H} = \langle H; \odot, \to, \Box, 0, 1\rangle$ be an nH-semigroup, and let $\langle \mathcal{O}(\mathcal{X}); \wedge, \vee, \to, \odot_{\mathbf{A}}, \Box_{\mathbf{A}}, 0, 1\rangle$ be the algebra defined as above. Then,

(i) The above-defined map h is an embedding of $\langle H; \odot, \Box\rangle$ into $\langle \mathcal{O}(\mathcal{X}); \odot_{\mathbf{A}}, \Box_{\mathbf{A}}\rangle$.
(ii) $\langle \mathbf{A}(\mathcal{X}), \Box_{\mathbf{A}}\rangle$ is Heyting algebra with a nucleus, and $\langle \mathcal{O}(\mathcal{X}), \odot_{\mathbf{A}}, \to, 0, 1\rangle$ is an nH-semigroup.
(iii) Thus, h is an embedding of $\mathbf{H}$ into $\langle \mathbf{A}(\mathcal{X}), \Box_{\mathbf{A}}\rangle$.

PROOF. (i) We need to show that h preserves the two new operations. Let $a \in H$. Let us show that $\Box_{\mathbf{A}} h(a) = h(\Box a)$. Indeed, on the one hand, for all $b \in H$, we have $h(\Box a) \leq (h(a) \to h(\Box b)) \to h(\Box b)$. Indeed, we have

$$\begin{aligned} \Box a \to ((a \to \Box b) \to \Box b) &= (a \to \Box b) \to (\Box a \to \Box b) && x \to (y \to z) \approx y \to (x \to z) \\ &= (a \to \Box b) \to (a \to \Box b) && \text{by Lemma 4.4} \\ &= 1. \end{aligned}$$

Hence, $h(\Box a \to ((a \to \Box b) \to \Box b) = h(\Box a) \to ((h(a) \to h(\Box b)) \to h(\Box b)) = 1$, i.e. $h(\Box a) \leq (h(a) \to h(\Box b)) \to h(\Box b)$. This means that $h(\Box a) \leq \Box_{\mathbf{A}} h(a)$. On the other hand, since $a \leq \Box a$, we have $(a \to \Box a) \to \Box a = 1 \to \Box a = \Box a$. Thus, $h(\Box a) = (h(a) \to h(\Box a)) \to h(\Box a)$. This entails $h(a) \in \{(h(a) \to h(\Box b)) \to h(\Box b) : b \in H\}$, so $\Box_{\mathbf{A}} h(a) \leq h(\Box a)$.

Now, given $a, b \in H$, let us verify that $h(a \odot b) = h(a) \odot_{\mathbf{A}} h(b)$. If $F \in h(a \odot b)$, then $a \odot b \in F$. By Definition 4.5.v, we have $(a \odot b) \to \Box b = \Box a \to (\Box b \to \Box b) = \Box a \to 1 = 1 \in F$. Thus, by (mp), we have $\Box b \in F$, which gives us $F \in h(\Box b) = \Box_{\mathbf{A}} h(b)$. A similar reasoning shows that $F \in h(\Box a) = \Box_{\mathbf{A}} h(a)$, so $F \in \Box_{\mathbf{A}} h(a) \cap \Box_{\mathbf{A}} h(b) = h(a) \odot_{\mathbf{A}} h(b)$.

Conversely, assume $F \in h(a) \odot_{\mathbf{A}} h(b) = \Box_{\mathbf{A}} h(a) \cap \Box_{\mathbf{A}} h(b) = h(\Box a) \cap h(\Box b)$. Then $\Box a, \Box b \in F$. By Definition 4.5.v, we have $\Box a \to (\Box b \to (a \odot b)) = (a \odot b) \to (a \odot b) = 1 \in F$. Then, applying (mp) twice, from $\Box a, \Box b \in F$ we obtain $a \odot b \in F$. Hence, $F \in h(a \odot b)$, as required.

(ii). It follows from [12, Thm. 13.15] that $\Box_{\mathbf{A}}$ is a nucleus on the Heyting algebra $\mathbf{A}(\mathcal{X})$. Hence, by Lemma 4.6, we have that $\langle \mathcal{O}(\mathcal{X}), \odot_{\mathbf{A}}, \to, 0, 1\rangle$ is an nH-semigroup, as required.

(iii). By the two previous items. □

The preceding theorem allows us to prove the following result.

COROLLARY 4.19.
Every QNI-algebra embeds into a quasi-Nelson algebra.

PROOF. Let $\mathbf{A} \in \mathsf{QNI}$. By Theorem 4.16, we can assume that $\mathbf{A}$ is a subalgebra of a QNI twist-algebra over an nH-semigroup $\mathbf{S}$. By Theorem 4.18, the map h is an embedding of $\mathbf{S}$ into the Heyting algebra with a nucleus $\mathbf{H} = \langle \mathbf{A}(\mathcal{X}), \Box_{\mathbf{A}}\rangle$. Define a map $f \colon A \to H \times H$ by $f(\langle a_1, a_2\rangle) := \langle h(a_1), h(a_2)\rangle$. We claim that f is an embedding of $\mathbf{A}$ into $Tw\langle \mathbf{H}, H\rangle$. It is easy to check that f is injective and preserves the algebraic operations of $\mathbf{A}$. It remains to verify that $h[A] \subseteq Tw\langle \mathbf{H}, H\rangle$. For this, it suffices to observe that, for all $\langle a_1, a_2\rangle \in A$, we have $a_1 \odot_{\mathbf{S}} a_2 = 0_{\mathbf{S}}$ and $\Box_{\mathbf{S}} a_2 = a_2$. Hence, $h(a_1) \odot_{\mathbf{H}} h(a_2) = 0_{\mathbf{H}}$ and $\Box_{\mathbf{H}} h(a_2) = h(a_2)$, as required. □

COROLLARY 4.20.
QNI is the class of $\{\to, \sim, 0, 1\}$-subreducts of quasi-Nelson algebras.

Congruences, subdirectly irreducibles and subvarieties

Taking into account Proposition 4.14, Theorem 4.16 gives us a counterpart of Theorem 3.18:

THEOREM 4.21.
The lattice of congruences of every $\mathbf{A} \in$ QNI is isomorphic to the lattice of congruences of the Hilbert algebra reduct of $H(\mathbf{A})$.

Given an algebra $\mathbf{A}$ with a partial order $\leq$ and maximum 1, we shall say that an element $c \in A$ is the *penultimate element* of A if $c \neq 1$ and, for all $a \in A$ such that $a < 1$, it holds that $a \leq c$. The following observation can be found in [3, p. 69]; the proof presented below is an unpublished result by L. Cabrer and S. Celani (personal communication).

THEOREM 4.22.
A Hilbert algebra $\mathbf{H}$ is subdirectly irreducible if and only if $\mathbf{H}$ has a penultimate element.

PROOF. Recall from [16, p. 26–7] that the lattice of congruences of every Hilbert algebra $\mathbf{H}$ is isomorphic to the lattice of implicative filters of $\mathbf{H}$ and that the implicative filter generated by an element $a \in H$ is $F(a) = \{b \in H : a \leq b\}$.

Now assume $\mathbf{H} = \langle H, \to, 1\rangle$ is a subdirectly irreducible Hilbert algebra. Then $\mathbf{H}$ has a minimal congruence above the identity, to which corresponds an implicative filter F_0 that is minimal among the filters distinct from $\{1\}$. We claim that F_0 has exactly two elements. Indeed, suppose for a contradiction that there were elements $a, b \in F_0$ such that $a \neq b$ and $a, b < 1$. Assume $a \nleq b$. Then $F(a)$ is an implicative filter and $b \notin F(a)$. Hence $F_0 \subseteq F(a)$, by minimality of F_0. But this is impossible, because $b \in F_0$ and $b \notin F(a)$. So F_0 has exactly two elements, say $F_0 = \{c, 1\}$. Moreover, for every $a \neq 1$, we have $F_0 \subseteq F(a)$, so $a \leq c$. Hence, c is the penultimate element.

Conversely, assume $c \in H$ is the penultimate element. We claim that $F(c) = \{c, 1\}$ is minimal among the filters distinct from $\{1\}$. Indeed, let F be a filter such that $F \neq \{1\}$. Then there is an element $a < 1$ such that $a \in F$. We have $a \leq c$, because c is the penultimate element. Hence, $F(c) \subseteq F$, as claimed. By the correspondence between filters and congruences, we conclude that $\mathbf{H}$ has a minimal congruence above the identity, entailing that $\mathbf{H}$ is subdirectly irreducible. □

We have seen in Lemma 3.3.x that, on every $\mathbf{A} \in$ QNI, a partial order $\leq$ can be defined by the prescription $a \leq b$ iff ($a \preccurlyeq b$ and $\sim b \preccurlyeq \sim a$). Thus the above-introduced notion of penultimate element applies to QNI-algebras as well. The result of Cabrer and Celani can be recast in the context of QNI-algebras as follows.

THEOREM 4.23.
A QNI-algebra $\mathbf{A}$ is subdirectly irreducible if and only if $\mathbf{A}$ has a penultimate element.

PROOF. Let us assume that $\mathbf{A} = \langle A; \to, \sim, 0, 1\rangle$ is a QNI twist-algebra over an nH-semigroup $\mathbf{H} = \langle H; \to, \odot, 0, 1\rangle$. By Theorem 4.21, $\mathbf{A}$ is subdirectly irreducible if and only if $\langle H; \to, 1\rangle$ is a subdirectly irreducible Hilbert algebra. Hence, by Theorem 4.22, we have that $\mathbf{A}$ is subdirectly irreducible if and only if $\langle H; \to, 1\rangle$ has a penultimate element. To complete the proof, it suffices to check that $\mathbf{A}$ has a penultimate element precisely when the same holds for $\langle H; \to, 1\rangle$.

Assume $\langle H; \to, 1\rangle$ has a penultimate element c. If $c = 0$, then $\langle H; \to, 0, 1\rangle$ is the two-element Boolean algebra. It is easy to check that there are only two QNI twist-algebras over the two-element Boolean algebra (cf. Remark 4.27), and both have a penultimate element. Let us then assume $0 < c < 1$. We claim that the element $\langle c, 0\rangle$ belongs to A and is the penultimate element of $\mathbf{A}$. To see this, observe first of all that, by the requirement $\pi_1[A] = H$ in Definition 4.12, we have that $c \in H$ entails that there is $d \in H$ such that $\langle c, d\rangle \in A$. Recall also that Definition 4.12 requires $c \odot d = 0$. This entails $d = 0$. Indeed, if $d = 1$, then (by Definition 4.3.iii and Definition 4.5.vii) we would have $0 < c \leq \Box c = c \odot 1 = c \odot d$, contradicting the assumption that $c \odot d = 0$. Since c is the penultimate element of $\langle H; \to, 1\rangle$, we conclude that $d \leq c$. Then $d \to c = 1$ and we can invoke Lemma 4.9.ii to obtain $0 = c \odot d = \Box d$. Since $d \leq \Box d$ (Definition 4.3.iii), we have $d = 0$ as claimed. Now let us verify that $\langle a, b\rangle \leq \langle c, 0\rangle$ for all $\langle a, b\rangle \in A$ with $\langle a, b\rangle \neq \langle 1, 0\rangle$. To this end, observe that $a = 1$ entails $b = 0$. Indeed, reasoning as before, the requirement $a \odot b = 0$ gives us $b \leq \Box b = 1 \odot b = a \odot b = 0$. Hence, if $\langle a, b\rangle \neq \langle 1, 0\rangle$, then $a \neq 1$ and so $a \leq c$. This gives us $\langle a, b\rangle \preccurlyeq \langle c, 0\rangle$. On the other hand we easily obtain ${\sim}\langle c, 0\rangle = \langle 0, \Box c\rangle \preccurlyeq {\sim}\langle a, b\rangle$. Hence $\langle a, b\rangle \leq \langle c, 0\rangle$, as claimed.

The converse is easy. Assume $\mathbf{A}$ has a penultimate element $\langle c, d\rangle$. Then, for every $a \in H$ such that $a \neq 1$ and for every $b \in H$, we have $\langle a, b\rangle \leq \langle c, d\rangle$. Thus, in particular, $\langle a, b\rangle \preccurlyeq \langle c, d\rangle$, which means that $a \leq c$. Hence c is the penultimate element of $\langle H; \to, 1\rangle$. $\Box$

As done in [23, Prop. 11] for quasi-Nelson algebras, it is possible to use the representation of either Theorem 3.12 or Theorem 4.16 to obtain further information on subvarieties of QNI.

LEMMA 4.24.
Let $\mathbf{A} = \langle A; \to, {\sim}, 0, 1\rangle$ be a QNI twist-algebra over an nH-semigroup $\mathbf{H} = \langle H; \to, \odot, 0, 1\rangle$, and let $\alpha \approx \beta$ be an equation in the language of nH-semigroups. The following are equivalent:

(i) $\mathbf{H} \vDash \alpha \approx \beta$.
(ii) $\mathbf{A} \vDash \alpha \to \beta \approx 1$ and $\mathbf{A} \vDash \beta \to \alpha \approx 1$.

PROOF. The result is an easy consequence of the following considerations. Recall that, for all $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in A$, one has $\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle = \langle b_1, b_2\rangle \to \langle a_1, a_2\rangle = 1$ if and only if $a_1 = b_1$. Further observe that the operations $\to, \odot, 0, 1$ are defined on $\mathbf{A}$, for the first component, precisely as in a direct product of the corresponding operations on $\mathbf{H}$. In fact, the preceding reasoning can be extended to include the operation $\Box$ as well, if we considered the operation $\Box$ on QNI-algebras defined as $\Box x := {\sim}{\sim} x$. $\Box$

LEMMA 4.25.
Let $\mathbf{A} = \langle A; \to, {\sim}, 0, 1\rangle \in \mathsf{QNI}$. The following are equivalent:

(i) $\mathbf{A} \vDash (x \to y) \to ({\sim} y \to {\sim} x) \approx 1$.
(ii) $\mathbf{A} \vDash (x \to {\sim} y) \to (y \to {\sim} x) \approx 1$.
(iii) $\mathbf{A} \vDash (x \to 0) \to {\sim} x \approx 1$.
(iv) $\mathbf{A} \vDash {\sim} x \approx x \to 0$.
(v) $\mathbf{A} \vDash {\sim} x \approx x \to {\sim} x$.
(vi) $\mathbf{A} \vDash {\sim}{\sim} x \approx x \odot 1$.
(vii) $\mathbf{A} \vDash x \odot y \approx y \odot x$.
(viii) $\mathbf{A} \vDash {\sim}{\sim} x \odot 1 \approx 1 \odot {\sim}{\sim} x$.

PROOF. Whenever convenient, we will assume that $\mathbf{A}$ is QNI twist-algebra over an nH-semigroup $\mathbf{H} = \langle H; \to, \odot, 0, 1\rangle$. Supposing (i) holds, we have $a \to {\sim} b \preccurlyeq {\sim}{\sim} b \to {\sim} a$ for all $a, b \in A$.

Observe that $\sim\sim b \to \sim a \preccurlyeq b \to \sim a$ because

$$
\begin{aligned}
(\sim\sim b \to \sim a) \to (b \to \sim a) &= b \to ((\sim\sim b \to \sim a) \to \sim a) && \text{by Def.3.1.iii}\\
&= (b \to (\sim\sim b \to \sim a)) \to (b \to \sim a) && \text{by Def.3.1.iii}\\
&= ((b \to \sim\sim b) \to (b \to \sim a)) \to (b \to \sim a) && \text{by Def.3.1.iii}\\
&= (1 \to (b \to \sim a)) \to (b \to \sim a) && \text{by Def.3.1.xiii}\\
&= (b \to \sim a) \to (b \to \sim a) && \text{by Def.3.1.i}\\
&= 1 && \text{by Def.3.1.ii.}
\end{aligned}
$$

Thus, by the transitivity of $\preccurlyeq$ (Lemma 3.3.iv), we have $a \to \sim b \preccurlyeq b \to \sim a$, i.e. $(a \to \sim b) \to (b \to \sim a) = 1$. Hence (ii) holds.

Now, assuming (ii), by items (xi) and (i) of Definition 3.1, we have $(a \to \sim 1) \to (1 \to \sim a) = (a \to 0) \to \sim a = 1$. Hence we have (iii).

To show that (iii) implies (iv), observe that $\sim x \to (x \to 0) \approx 1$ is satisfied by every QNI (twist-) algebra $\mathbf{A}$. Indeed, for $\langle a_1, a_2\rangle \in A$, we have $\sim\langle a_1, a_2\rangle = \langle a_2, \Box a_1\rangle$ and, using Definition 4.5.vii, $\langle a_1, a_2\rangle \to \langle 0, 1\rangle = \langle a_1 \to 0, \Box a_1 \odot 1\rangle = \langle a_1 \to 0, \Box\Box a_1\rangle = \langle a_1 \to 0, \Box a_1\rangle$. Thus the second components are equal, and it suffices to check that $a_2 \leq a_1 \to 0$. From the requirement $a_1 \odot a_2 = 0$ of Definition 4.12, we have $(a_1 \odot a_2) \to 0 = 0 \to 0 = 1$. Moreover, using Definition 4.5.v, we have $1 = (a_1 \odot a_2) \to 0 = \Box a_1 \to (\Box a_2 \to 0)$. Recall that $\Box a_2 = a_2$ (by Definition 4.12) and $a_1 \leq \Box a_1$. Since the Hilbert implication is order-reversing in the first argument, we thus have $1 = \Box a_1 \to (\Box a_2 \to 0) = \Box a_1 \to (a_2 \to 0) \leq a_1 \to (a_2 \to 0) = a_2 \to (a_1 \to 0)$. Hence $a_2 \leq a_1 \to 0$, as claimed. This indeed means that a QNI-algebra $\mathbf{A}$ satisfies $(x \to 0) \to \sim x \approx 1$ if and only if $\mathbf{A}$ satisfies $\sim x \approx x \to 0$, so (iii) and (iv) are equivalent.

We claim that (iv) and (v) are equivalent because, actually, every QNI (twist-)algebra satisfies $x \to \sim x \approx x \to 0$. To verify this, let $\langle a_1, a_2\rangle \in A$. Let us preliminary observe that $a_1 \to 0 = a_1 \to a_2$. Indeed, the inequality $a_1 \to 0 \leq a_1 \to a_2$ holds because $\to$ is order-preserving in the second argument. As to the other inequality, since $a_1 \to a_2 \leq \Box(a_1 \to a_2)$, it will suffice to show that $\Box(a_1 \to a_2) \to (a_1 \to 0) = 1$. We have:

$$
\begin{aligned}
\Box(a_1 \to a_2) \to (a_1 \to 0) &= \Box(a_1 \to a_2) \to (\Box a_1 \to 0) && \text{by Lemma 4.9.iv}\\
&= ((a_1 \to a_2) \odot a_1) \to 0 && \text{by Def. 4.5.v}\\
&= (a_1 \odot a_2) \to 0 && \text{by Def. 4.5.iv}\\
&= 0 \to 0 && a_1 \odot a_2 = 0\\
&= 1 && x \to x \approx 1.
\end{aligned}
$$

Thus (since we are in a Hilbert algebra) we have $\Box(a_1 \to a_2) \leq a_1 \to 0$, which gives us $a_1 \to 0 = a_1 \to a_2$. Now, as shown earlier in the proof, $\langle a_1, a_2\rangle \to \langle 0, 1\rangle = \langle a_1 \to 0, \Box a_1\rangle$. On the other hand, we have $\langle a_1, a_2\rangle \to \sim\langle a_1, a_2\rangle = \langle a_1 \to a_2, a_1 \odot \Box a_1\rangle = \langle a_1 \to a_2, \Box a_1\rangle = \langle a_1 \to 0, \Box a_1\rangle$. Regarding the second components, the last equality holds because, by Lemma 4.9.ii and the inequality $a_1 \leq \Box a_1$, we have $\Box a_1 = a_1 \odot \Box a_1$.

It is easy to check that (iv), which we have seen to be equivalent to (v), implies (vi): just observe that $x \odot 1 = \sim(x \to \sim 1)$, and that every QNI-algebra satisfies $\sim(x \to \sim 1) \approx \sim(x \to 0)$.

Let us show that (vi) implies (vii). Let $\langle a_1, a_2\rangle \in A$. Observe that $\Box(a_1 \to 0) = a_1 \to 0$. Indeed, the inequality $a_1 \to 0 \leq \Box(a_1 \to 0)$ holds because of Definition 4.3.iii. As to the converse, observe that $a_1 \leq \Box a_1$ entails $\Box a_1 \to 0 \leq a_1 \to 0$. Then, using items (ii) and (i) of Definition 4.3, we easily obtain $\Box(a_1 \to 0) \leq \Box a_1 \to \Box 0 = \Box a_1 \to 0 \leq a_1 \to 0$. Hence, assuming (v) holds, we have $\langle \Box a_1, a_2 \rangle = {\sim}{\sim}\langle a_1, a_2 \rangle = \langle a_1, a_2 \rangle \odot \langle 1, 0 \rangle = \langle \Box a_1 \odot \Box 1, \Box(a_1 \to 0) \rangle = \langle \Box\Box a_1, a_1 \to 0 \rangle = \langle \Box a_1, a_1 \to 0 \rangle$. Thus (v) entails, like (iii) and (iv), that every element of $\mathbf{A}$ is of the form $\langle a_1, a_1 \to 0 \rangle$. Thus, to verify that (vi) holds, it suffices to check that $\langle a_1, a_1 \to 0 \rangle \odot \langle b_1, b_1 \to 0 \rangle = \langle b_1, b_1 \to 0 \rangle \odot \langle a_1, a_1 \to 0 \rangle$ for all $a_1, b_1 \in H$. This follows easily from the commutativity of the Hilbert implication: we have $\langle a_1, a_1 \to 0 \rangle \odot \langle b_1, b_1 \to 0 \rangle = \langle \Box a_1 \odot \Box b_1, \Box(a_1 \to (b_1 \to 0)) \rangle = \langle \Box a_1 \odot \Box b_1, \Box(b_1 \to (a_1 \to 0)) \rangle = \langle b_1, b_1 \to 0 \rangle \odot \langle a_1, a_1 \to 0 \rangle$.

It is clear that (vii) entails (viii). To conclude the proof, assume (viii) holds, and let us show that (i) must hold as well. On a twist-algebra, this means that, for all $\langle a_1, a_2 \rangle \in A$, we have ${\sim}{\sim}\langle a_1, a_2 \rangle \odot \langle 1, 0 \rangle = \langle \Box\Box a_1 \odot \Box 1, \Box(a_1 \to 0) \rangle = \langle \Box a_1, \Box(a_1 \to 0) \rangle = \langle \Box a_1, a_2 \rangle = \langle \Box a_1, \Box a_2 \rangle = \langle \Box 1 \odot \Box\Box a, \Box(1 \to b) \rangle = \langle 1, 0 \rangle \odot {\sim}{\sim}\langle a, a_2 \rangle$. Thus, in particular, $\Box(a_1 \to 0) = a_2$, which means that every element of $\mathbf{A}$ has the form $\langle a_1, \Box(a_1 \to 0) \rangle$ for some $a_1 \in H$. Observe that, in fact, $\Box(a_1 \to 0) = a_1 \to 0$, which brings us back to the condition considered in item (iv). Indeed, the inequality $a_1 \to 0 \leq \Box(a_1 \to 0)$ holds because of Definition 4.3.iii. As to the converse, observe that $a_1 \leq \Box a_1$ entails $\Box a_1 \to 0 \leq a_1 \to 0$. Then, using items (ii) and (i) of Definition 4.3, we easily obtain $\Box(a_1 \to 0) \leq \Box a_1 \to \Box 0 = \Box a_1 \to 0 \leq a_1 \to 0$. Hence every element of $\mathbf{A}$ has the form $\langle a_1, a_1 \to 0 \rangle$ for some $a_1 \in H$. It is then easy to verify that (i) holds. Indeed, for all $a_1, b_1 \in H$, we have $(\langle a_1, a_1 \to 0 \rangle \to \langle b_1, b_1 \to 0 \rangle) \to ({\sim}\langle b_1, b_1 \to 0 \rangle \to {\sim}\langle a_1, a_1 \to 0 \rangle) = \langle (a_1 \to b_1) \to ((b_1 \to 0) \to (a_1 \to 0)), \Box(a_1 \to b_1) \odot \Box(b_1 \to 0) \odot \Box a_1 \rangle$. Regarding the second component of this expression, we use the observation that $\Box$ distributes over $\odot$ to obtain $\Box(a_1 \to b_1) \odot \Box(b_1 \to 0) \odot \Box a_1 = \Box((a_1 \to b_1) \odot (b_1 \to 0) \odot a_1)$. Using the commutativity of $\odot$ and items (iv) and (vi) of Definition 4.5, we have $\Box((a_1 \to b_1) \odot (b_1 \to 0) \odot a_1) = \Box(a_1 \odot b_1 \odot (b_1 \to 0)) = \Box(a_1 \odot b_1 \odot 0) = \Box 0 = 0$. As to the first component, using the commutativity of the Hilbert implication together with (H2') and (H1), we have $(a_1 \to b_1) \to ((b_1 \to 0) \to (a_1 \to 0)) = (b_1 \to 0) \to ((a_1 \to b_1) \to (a_1 \to 0)) = (b_1 \to 0) \to (a_1 \to (b_1 \to 0)) = 1$. Hence $(\langle a_1, a_1 \to 0 \rangle \to \langle b_1, b_1 \to 0 \rangle) \to ({\sim}\langle b_1, b_1 \to 0 \rangle \to {\sim}\langle a_1, a_1 \to 0 \rangle) = \langle 1, 0 \rangle$, as was required to prove. $\Box$

PROPOSITION 4.26.
Let $\mathbf{A} = \langle A; \to, \sim, 0, 1 \rangle$ be a QNI twist-algebra over an nH-semigroup $\mathbf{H} = \langle H; \to, \odot, 0, 1 \rangle$.

(i) $\mathbf{A} \vDash {\sim}{\sim}x \to x \approx 1$ iff $\mathbf{A} \vDash {\sim}{\sim}x \approx x$ iff $\mathbf{A} \vDash x \approx 1 \odot x$ (1 is the neutral element for $\odot$ on the left) iff $\mathbf{A} \vDash (x \odot x) \to x \approx 1$ iff $\mathbf{H} \vDash \Box x \to x \approx 1$ iff $\mathbf{H} \vDash \Box x \approx x$ iff the natural order of the Hilbert algebra reduct of $\mathbf{H}$ forms a bounded meet-semilattice with $\odot$ as meet.

(ii) $\mathbf{A} \vDash ({\sim}{\sim}x \to {\sim}{\sim}y) \to {\sim}{\sim}(x \to y) \approx 1$ iff $\mathbf{A} \vDash {\sim}{\sim}x \to {\sim}{\sim}y \approx {\sim}{\sim}(x \to y)$ iff $\mathbf{H} \vDash (\Box x \to \Box y) \to \Box(x \to y) \approx 1$ iff $\mathbf{H} \vDash \Box x \to \Box y \approx \Box(x \to y)$.

(iii) $\mathbf{A}$ satisfies any of the identities of Lemma 4.25 (e.g. the operation $\odot$ is commutative) iff $h \colon \mathbf{H} \cong \mathbf{A}$, where $h(a) := \langle a, a \to 0 \rangle$ for all $a \in H$, iff $\mathbf{A}$ is a bounded Hilbert algebra.

(iv) $\mathbf{A} \vDash x \approx x \odot x$ (the operation $\odot$ is idempotent) iff $\mathbf{A} \vDash x \approx x \odot 1$ (1 is the neutral element for $\odot$ on the right) iff $\mathbf{A} \vDash ({\sim}x \to {\sim}y) \to (y \to x) \approx 1$ iff $\mathbf{A} \vDash {\sim}x \to {\sim}y \approx y \to x$ iff $\mathbf{A}$ is a Boolean algebra (on which $\odot$ is the meet) iff $\mathbf{H}$ is a Boolean algebra and $h \colon \mathbf{H} \cong \mathbf{A}$, where $h(a) := \langle a, a \to 0 \rangle$ for all $a \in H$.

(v) $\mathbf{A} \vDash ((x \to y) \to x) \to x \approx 1$ iff $\mathbf{H}$ is a Boolean algebra.

PROOF. (i). The equivalence between ${\sim}{\sim}x \approx x$ and $x \approx 1 \odot x$ is straightforward: just observe that, by Definition 3.1.i, we have $1 \odot x \approx {\sim}(1 \to {\sim}x) \approx {\sim}{\sim}x$. Now, letting $\langle a_1, a_2 \rangle \in A$, let us compute ${\sim}{\sim}\langle a_1, a_2 \rangle = \langle \Box a_1, \Box a_2 \rangle = \langle \Box a_1, a_2 \rangle$ and $\langle a_1, a_2 \rangle \odot \langle a_1, a_2 \rangle = {\sim}(\langle a_1, a_2 \rangle \to$

$\sim\langle a_1, a_2\rangle) = \sim\langle a_1 \to a_2, \Box a_1 \odot \Box a_1\rangle = \sim\langle a_1 \to a_2, a_1 \odot a_1 \odot a_1 \odot a_1\rangle = \sim\langle a_1 \to a_2, \Box\Box a_1\rangle = \sim\langle a_1 \to a_2, \Box a_1\rangle = \langle \Box a_1, \Box(a_1 \to a_2)\rangle$. Hence $\sim\sim\langle a_1, a_2\rangle \equiv \langle a_1, a_2\rangle \odot \langle a_1, a_2\rangle$, which implies that $\sim\sim\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle = \langle 1, 0\rangle$ iff $\langle a_1, a_2\rangle \odot \langle a_1, a_2\rangle \to \langle b_1, b_2\rangle = \langle 1, 0\rangle$ for all $\langle b_1, b_2\rangle \in A$. This gives us the equivalence between $\mathbf{A} \vDash \sim\sim x \to x \approx 1$ and $\mathbf{A} \vDash (x \odot x) \to x \approx 1$. The equivalence between $\mathbf{A} \vDash \sim\sim x \to x \approx 1$ and $\mathbf{A} \vDash \sim\sim x \approx x$ is easy: let us compute $\sim\sim\langle a_1, a_2\rangle \to \langle a_1, a_2\rangle = \langle \Box a_1 \to a_1, \Box\Box a_1 \odot a_2\rangle = \langle \Box a_1 \to a_1, \Box a_1 \odot a_2\rangle$. Thus $\sim\sim\langle a_1, a_2\rangle \to \langle a_1, a_2\rangle = \langle 1, 0\rangle$ entails $\Box a_1 \leq a_1$, which (recalling Definition 4.3.iii) gives us $\Box a_1 = a_1$. Hence $\sim\sim\langle a_1, a_2\rangle = \langle a, a_2\rangle$. That $\mathbf{H} \vDash \Box x \approx x$ entails any of the above conditions is easily proved. To conclude the proof, observe that, on the one hand, if $\odot$ is a semilattice operation on H, then $a_1 = a_1 \odot a_1 = \Box a_1$ for all $a_1 \in H$. On the other hand, if $\mathbf{H} \vDash \Box x \approx x$, then $\odot$ is a semilattice operation and one has $a_1 \to b_1 = 1$ iff $a_1 \odot b_1 = a_1$ for all $a_1, b_1 \in H$. To see that $\odot$ is a semilattice operation, it suffices to observe that (using $\Box a_1 = a_1$) we have $a_1 \odot a_1 = \Box a_1 = a_1$. Observe next that, assuming $a_1 \to b_1 = 1$, by Definition 4.5.iv we have $a_1 \odot b_1 = a_1 \odot (a_1 \to b_1) = a_1 \odot 1 = \Box a_1 = a_1$. Conversely, if $a_1 \odot b_1 = a_1$, then Definition 4.5.v gives us $a_1 \to b_1 = (a_1 \odot b_1) \to b_1 = \Box a_1 \to (\Box b_1 \to b_1) = a_1 \to (b_1 \to b_1) = 1$, as required.

(ii). Assume $\mathbf{A} \vDash (\sim\sim x \to \sim\sim y) \to \sim\sim(x \to y) \approx 1$. Let $a_1, b_1, a_2, b_2 \in H$. Using Definition 4.3.iii, we have $\sim\sim\langle a_1, a_2\rangle \to \sim\sim\langle b_1, b_2\rangle = \langle \Box a_1, \Box a_2\rangle \to \langle \Box b_1, \Box b_2\rangle = \langle \Box a_1 \to \Box b_1, \Box\Box a_1 \odot b_2\rangle = \langle \Box a_1 \to \Box b_1, \Box a_1 \odot b_2\rangle$. On the other hand, we have $\sim\sim(\langle a_1, a_2\rangle \to \langle b_1, b_2\rangle) = \langle \Box(a_1 \to b_1), \Box(\Box a_1 \odot b_2)\rangle$. Recall from the proof of Lemma 4.10 that $\Box$ distributes over $\odot$, i.e. $\Box(\Box a_1 \odot b_2) = \Box\Box a \odot \Box b_2$. Also recall that $b_2 = \Box b_2$ holds because $\langle b_1, b_2\rangle \in A$. Using also Definition 4.3.iii, we have $\langle \Box(a_1 \to b_1), \Box(\Box a_1 \odot b_2)\rangle = \langle \Box(a_1 \to b_1), \Box\Box a_1 \odot \Box b_2\rangle = \langle \Box(a_1 \to b_1), \Box a_1 \odot b_2\rangle$. It is thus clear that $\mathbf{A} \vDash (\sim\sim x \to \sim\sim y) \to \sim\sim(x \to y) \approx 1$ is equivalent to requiring $\Box a_1 \to \Box b_1 \leq \Box(a_1 \to b_1)$ for all $a_1, b_1 \in H$. Since the inequality $\Box(a_1 \to b_1) \leq \Box a_1 \to \Box b_1$ holds on all nH-semigroups, we also have that $\mathbf{A} \vDash (\sim\sim x \to \sim\sim y) \to \sim\sim(x \to y) \approx 1$ is equivalent to $\mathbf{A} \vDash \sim\sim x \to \sim\sim y \approx \sim\sim(x \to y)$.

(iii). Recall that every bounded Hilbert algebra $\mathbf{H}$ is a QNI-algebra if we let $\sim x := x \to 0$. (Proposition 3.11). Now assume any of the conditions of Lemma 4.25 holds. As we have seen, each of them is equivalent to requiring that every element of $\mathbf{A}$ be of the form $\langle a_1, a_1 \to 0\rangle$ for some $a_1 \in H$. Observe that, since $\sim\langle a_1, a_1 \to 0\rangle = \langle a_1 \to 0, \Box a_1\rangle$, this entails $\Box a_1 = (a_1 \to 0) \to 0$ for all $a \in H$. The preceding considerations show that the map h given by $h(a_1) := \langle a_1, a_1 \to 0\rangle$ for all $a_1 \in H$ is bijective. It is clear that h is a homomorphism in the first component, and this is all that matters, since the second component of each pair in $\mathbf{A}$, as we have seen, is determined by the first. Hence $h\colon \mathbf{H} \cong \mathbf{A}$ (which implies that $\mathbf{A}$ is itself a bounded Hilbert algebra).

To conclude the proof, assume $\mathbf{A}$ is a bounded Hilbert algebra, and let us verify that $\mathbf{A}$ satisfies $(x \to 0) \to \sim x \approx 1$. Let $\langle a_1, a_2\rangle \in A$, and observe that $(\langle a_1, a_2\rangle \to \langle 0, 1\rangle) \to \sim\sim(\langle a_1, a_2\rangle \to \sim\langle a_1, a_2\rangle) = \langle 1, 0\rangle$. To see this, by Lemma 3.3.v, it is sufficient to check equality of the first components, i.e. that $(a_1 \to 0) \to \Box(a_1 \to a_2) = 1$. The latter holds true because $a_1 \to 0 \leq a_1 \to a_2 \leq \Box(a_1 \to a_2)$. Further observe that, Lemma 3.3.ix, we have $\langle a_1, a_2\rangle \odot \langle a_1, a_2\rangle = \sim(\langle a_1, a_2\rangle \to \sim\langle a_1, a_2\rangle) \equiv \sim\sim\langle a_1, a_2\rangle$. Since $\mathbf{A}$ is a Hilbert algebra, by (H3) we have that the relation $\equiv$ coincides with the equality on A. Hence we have $\sim(\langle a_1, a_2\rangle \to \sim\langle a_1, a_2\rangle) = \sim\sim\langle a_1, a_2\rangle$, which implies $\sim\sim(\langle a_1, a_2\rangle \to \sim\langle a_1, a_2\rangle) = \sim\sim\sim\langle a_1, a_2\rangle = \sim\langle a_1, a_2\rangle$. Thus we have $(\langle a_1, a_2\rangle \to \langle 0, 1\rangle) \to \sim\sim(\langle a_1, a_2\rangle \to \sim\langle a_1, a_2\rangle) = (\langle a_1, a_2\rangle \to \langle 0, 1\rangle) \to \sim\langle a_1, a_2\rangle = \langle 1, 0\rangle$. This implies that $a_1 \to 0 \leq a_2$ for all $\langle a_1, a_2\rangle \in A$. As observed in the proof of Lemma 4.25, the latter is in turn equivalent to $a_1 \to 0 = a_2$ for all $\langle a_1, a_2\rangle \in A$. Hence $\langle a_1, a_2\rangle \to \langle 0, 1\rangle = \sim\langle a_1, a_2\rangle$, which means that $\mathbf{A}$ satisfies $(x \to 0) \approx \sim x$, as required.

(iv). For $\langle a_1, a_2\rangle \in A$, let us compute $\langle a_1, a_2\rangle \odot \langle 1, 0\rangle = \langle \Box(a_1 \odot 1), \Box(a_1 \to 0)\rangle = \langle \Box\Box a_1, \Box(a_1 \to 0)\rangle = \langle \Box a_1, a_1 \to 0\rangle$. The last passage is justified by (Definition 4.3.iii and) the equality $\Box(a_1 \to 0) = a_1 \to 0$, which was shown in the proof of Lemma 4.25. Also observe that $\langle a_1, a_2\rangle \odot \langle a_1, a_2\rangle = \langle a_1 \odot a_1, a_1 \to a_2\rangle = \langle \Box a_1, a_1 \to 0\rangle$. The last equality is justified by Definition 4.5.iii (for the first component) and, for the second component, by the equality $a_1 \to a_2 = a_1 \to 0$, which we have seen to hold in the proof of Lemma 4.25. Thus, every QNI (twist-)algebra satisfies $x \odot x \approx x \odot 1$, which gives us the equivalence of the first two conditions in the statement. Our computations also show that $x \odot 1 \approx 1$ amounts to the requirement $\langle \Box a_1, a_1 \to 0\rangle = \langle a_1, a_2\rangle$ for all $\langle a_1, a_2\rangle \in A$. This implies that $\mathbf{H} \vDash \Box x \approx x$ (i.e. item (i) above holds) and that every element of $\mathbf{A}$ is of the form $\langle a_1, a_1 \to 0\rangle$ for some $a_1 \in H$ (i.e. all the conditions of Lemma 4.25 hold as well). In particular, $\mathbf{A} \vDash (\sim x \to \sim y) \to (y \to x) \approx 1$ because, by involutivity, $\mathbf{A} \vDash y \to x \approx \sim\sim y \to \sim\sim x$, so the result follows easily from Lemma 4.25.i.

Further observe that $\mathbf{A} \vDash (\sim x \to \sim y) \to (y \to x) \approx 1$ entails items (i) and (iii) above. Indeed, using items (x) and (i) of Definition 3.1, we can instantiate the identity $(\sim x \to \sim y) \to (y \to x) \approx 1$ as follows: $1 = (\sim a \to \sim\sim\sim a) \to (\sim\sim a \to a) = (\sim a \to \sim a) \to (\sim\sim a \to a) = 1 \to (\sim\sim a \to a) = \sim\sim a \to a$ for all $a \in A$. Thus (i) holds, and $\sim\sim a = a$ for all $a \in A$. Hence, for all $a, b \in A$, we have $(a \to b) \to (\sim b \to \sim a) = (\sim\sim a \to \sim\sim b) \to (\sim b \to \sim a) = 1$, which is item (iii) above. It is well known that the identities (H1), (H2) and $(\sim x \to \sim y) \to (y \to x) \approx 1$ constitute a presentation of Boolean algebras in the language $\{\to, \sim\}$. Hence $\mathbf{A}$ is a Boolean algebra, which entails that $\mathbf{A} \vDash ((x \to y) \to x) \to x \approx 1$, i.e. item (v) holds as well. Then $\mathbf{H}$ is a Boolean algebra (on which, by item (i) above, the $\Box$ is the identity map).

Conversely, assume $\mathbf{H}$ is a Boolean algebra and $h\colon \mathbf{H} \cong \mathbf{A}$ via the map given by $h(a_1) := \langle a_1, a_1 \to 0\rangle$ for all $a \in H$. Then $\mathbf{A}$ is a Boolean algebra (in which $\odot$ is the meet), so all the other statements in (iv) are satisfied.

(v). It is easy to show that $\mathbf{A} \vDash ((x \to y) \to x) \to x \approx 1$ if and only if $\mathbf{H} \vDash ((x \to y) \to x) \to x \approx 1$. The latter condition implies that the $\{\to, 1\}$-reduct of $\mathbf{H}$ is a Tarski algebra; Tarski algebras are precisely the $\{\to, 1\}$-subreducts of Boolean algebras and can be axiomatized (relative to Hilbert algebras) precisely by the identity $((x \to y) \to x) \to x \approx 1$. In a bounded Tarski algebra all Boolean operations are definable ($x \to 0$ defines the negation, $(x \to (y \to 0)) \to 0$ defines the conjunction, etc.). Hence $\mathbf{H}$ is a Boolean algebra (in which $\odot$ is the meet), as claimed. Conversely, if the latter holds, then it is obvious that $\mathbf{A} \vDash ((x \to y) \to x) \to x \approx 1$. $\Box$

REMARK 4.27.
Regarding Proposition 4.26, observe that item (iv) implies all the other items, and that (vii) implies (vi); moreover, items (iii) and (vii) jointly imply (iv). We also note that not only the identity in item (iv) but also the one in item (v) define a finitely generated subvariety of QNI. Let us call $\mathbb{I}_3$ the variety axiomatized by the identity in item (v). Theorem 4.21 tells us that $\mathbb{I}_3$ is generated by those QNI-algebras $\mathbf{A}$ such that the corresponding nH-semigroup $\mathbf{H}$ has a subdirectly irreducible Boolean algebra reduct. Since the only subdirectly irreducible Boolean algebra is the two-element one, we have that H consists of two elements. It is easy to check that, on a two-element universe, the only possible nucleus is the identity map; items (vi) and (vii) of Definition 4.5 further imply that $\odot$ must be the lattice meet operation associated to the natural order of the algebra. Thus the two-element nH-semigroup is unique up to isomorphism; let us denote it by $\mathbf{H}_2$. Over $\mathbf{H}_2 \bowtie \mathbf{H}_2$ only two (both involutive) QNI twist-algebras can be built: a two-element one (call it $\mathbf{A}_2$) with universe $\{\langle 0,1\rangle, \langle 1,0\rangle\}$ and a three-element one $\mathbf{A}_3$ with universe $\{\langle 0,1\rangle, \langle 1,0\rangle, \langle 0,0\rangle\}$. The algebra is $\mathbf{A}_2$ isomorphic to the two-element Boolean algebra (where $\to$ is the Boolean implication, $\sim$ is the complement operation and $\odot$ is the meet), and it is obviously a subalgebra of $\mathbf{A}_3$. Thus $\mathbb{I}_3$ is generated by $\mathbf{A}_3$ alone, which is the reduct of the (unique, up to isomorphism) three-element Nelson algebra.

This algebra, in turn, is known to be isomorphic to the three-element MV (or Wajsberg) algebra; observe, however, that $\mathbf{A}_3$ cannot be viewed as a MV/Wajsberg algebra, because the implication $\rightarrow$ is not the Łukasiewicz implication (which coincides with the strong Nelson implication $\Rightarrow$). Indeed, having $\Rightarrow$ in the language, one would be able to define $x \rightarrow y := x \Rightarrow (x \Rightarrow y)$, but from $\rightarrow$ one cannot define $\Rightarrow$ unless the meet operation is also present.

Characterizations similar to those in Proposition 4.26 can be obtained by considering other identities. In general, it is easy to show that, if t is a term in the language of QNI-algebras that only involves the implication, then $\mathbf{A} \vDash t \approx 1$ if and only if $\mathbf{H} \vDash t \approx 1$. Thus one can verify, e.g. that $\mathbf{A}$ satisfies $((x \rightarrow y) \rightarrow z) \rightarrow (((y \rightarrow x) \rightarrow z) \rightarrow z) \approx 1$ if and only if $\mathbf{H}$ satisfies $((x \rightarrow y) \rightarrow z) \rightarrow (((y \rightarrow x) \rightarrow z) \rightarrow z) \approx 1$, which is the well-known prelinearity identity often considered in the context of fuzzy logics.

5 The Logic of QNI

In this section we introduce a Hilbert-style calculus $\vdash_{\mathsf{QNI}}$ that is algebraizable (in the sense of Blok and Pigozzi [1]) and has QNI as its equivalent algebraic semantics. The main results of this section have been established in [14], to which we refer for further details and proofs.

Fm will denote the set formulas built over a denumerable set *Var* of propositional variables using the propositional connectives $\rightarrow$ and $\sim$ (we do not include truth constants in the language, but these may be defined by letting $1 := \varphi \rightarrow \varphi$ and $0 := \sim 1$). In keeping with the notation of the previous sections, given formulas $\varphi, \psi, \gamma \in Fm$, we abbreviate $\varphi \odot \psi := \sim(\varphi \rightarrow \sim \psi)$ and

$$q(\varphi, \psi, \gamma) := (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow ((\sim \varphi \rightarrow \sim \psi) \rightarrow ((\sim \psi \rightarrow \sim \varphi) \rightarrow \gamma))).$$

The calculus $\vdash_{\mathsf{QNI}}$ is defined in a standard way by the following axiom schemata:

$$\varphi \rightarrow (\psi \rightarrow \varphi) \tag{A1}$$

$$(\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \gamma)) \tag{A2}$$

$$q(\varphi, \psi, \psi) \rightarrow q(\varphi, \psi, \varphi) \tag{A3}$$

$$q(\varphi, \psi, \varphi) \rightarrow q(\varphi, \psi, \psi) \tag{A4}$$

$$(\varphi \odot (\psi \odot \gamma)) \rightarrow ((\varphi \odot \psi) \odot \gamma) \tag{A5}$$

$$((\varphi \odot \psi) \odot \gamma) \rightarrow (\varphi \odot (\psi \odot \gamma)) \tag{A6}$$

$$\sim\sim\sim\varphi \rightarrow \sim\varphi \tag{A7}$$

$$(\varphi \rightarrow \psi) \rightarrow (\sim\sim\varphi \rightarrow \sim\sim\psi) \tag{A8}$$

$$\varphi \rightarrow \sim\sim\varphi \tag{A9}$$

$$(\varphi \odot (\varphi \rightarrow \psi)) \rightarrow (\varphi \odot \psi) \tag{A10}$$

$$\sim\sim\varphi \rightarrow (\sim\psi \rightarrow \sim(\varphi \rightarrow \psi)) \tag{A11}$$

$$\sim(\varphi \rightarrow \psi) \rightarrow \sim\psi \tag{A12}$$

$$\sim(\varphi \rightarrow \psi) \rightarrow \sim\sim\varphi \tag{A13}$$

$$\sim(\varphi \rightarrow \varphi) \rightarrow \psi \tag{A14}$$

$$\sim\sim(\sim\varphi \rightarrow \sim\psi) \rightarrow (\sim\varphi \rightarrow \sim\psi). \tag{A15}$$

The only rule of $\vdash_{\mathsf{QNI}}$ is *modus ponens* (mp): from φ and $\varphi \to \psi$, derive ψ.

As expected, it is not difficult to check that all the axioms of $\vdash_{\mathsf{QNI}}$ are sound w.r.t. Nelson logic (provided one interprets $\to$ as the weak implication and $\sim$ as the strong negation; see e.g. [25]) and w.r.t. intuitionistic logic. From the presence of axioms (A1), (A2) and the observation that (mp) is the only rule of the logic, we also obtain that $\vdash_{\mathsf{QNI}}$ enjoys the deduction–detachment theorem in its classical form.

THEOREM 5.1 (Deduction–detachment theorem). For all $\Gamma \cup \{\varphi, \psi\} \subseteq Fm$, we have $\Gamma, \varphi \vdash_{\mathsf{QNI}} \psi$ iff $\Gamma \vdash_{\mathsf{QNI}} \varphi \to \psi$.

THEOREM 5.2. [14], Prop. 1

$\vdash_{\mathsf{QNI}}$ is finitely and regularly algebraizable [6, Def. 3.49] with equivalence formulas $\Delta(\varphi, \psi) := \{\varphi \to \psi, \psi \to \varphi, \sim\varphi \to \sim\psi, \sim\psi \to \sim\varphi\}$ and defining equation $E(\varphi) := \{\varphi \approx \varphi \to \varphi\}$.

One can obtain an axiomatization of the class of $\vdash_{\mathsf{QNI}}$-algebras in the standard way [1, Thm. 2.17]. The above results entail that the term $x \to x$ defines an algebraic constant on every $\vdash_{\mathsf{QNI}}$-algebra; thus, letting $1 := x \to x$ and $0 := \sim 1$, we can view $\vdash_{\mathsf{QNI}}$-algebras as a quasivariety of algebras in the same language as QNI. In fact, it is not difficult to show that the class $\vdash_{\mathsf{QNI}}$-algebras (modulo the above-mentioned language formalities) coincides with the variety QNI.

THEOREM 5.3. [14], Props. 3 and 4

QNI is the equivalent algebraic semantics of $\vdash_{\mathsf{QNI}}$.

Taking into account the well-known bridge theorems regarding algebraizable logics, the latter result could be used to obtain alternative proofs of some of the statements established in Section 3 (see for instance Proposition 3.16).

6 Conclusions

In these final remarks we would like reflect on what we have learnt so far about the problem of characterizing fragments of quasi-Nelson algebras/logic. As mentioned in the Introduction (and illustrated throughout the paper), the twist representation has been a most valuable tool in this endeavour, and indeed it is fair to say that the limits of our current knowledge about fragments of quasi-Nelson coincide with the current limitations of the twist construction. The vagueness of the notion of 'twist construction' makes it hard to formulate in general necessary/sufficient conditions for a twist representation of some sort to be possible. However, the fragments that have been successfully characterized (namely $\{\wedge, \vee, \sim\}$ in [18], $\{\wedge, \vee, \sim, \neg\}$ in [17] and $\{\to, \sim\}$ in [20] and in the present paper) provide some insight. In all three cases, the presence of the negation ($\sim$) appears to be an essential ingredient in the definition of the embeddings of (e.g.) Theorems 3.12 and 4.16; one then invokes the Nelson identity (in one of its equivalent formulations) to ensure that the maps are actually injective. The lattice structure (in particular, the structure of the prime spectrum) played a prominent role in the study of the $\{\wedge, \vee, \sim\}$-fragment, but it turns out to be inessential if the language is sufficiently expressive, as in the case of $\{\wedge, \vee, \sim, \neg\}$ and $\{\to, \sim\}$.

A further lesson to be learnt from the above-mentioned case studies is that, even if the non-involutive setting seems to require a more complex representation of an algebra $\mathbf{A}$ via two factors $(\mathbf{A}_+, \mathbf{A}_-)$, only the positive factor seems essential, for $\mathbf{A}_-$ may be recovered as a special subalgebra of $\mathbf{A}_+$ (this, indeed, appears to be a characteristic consequence of the Nelson identity: see, by contrast, the representation of *semi-Kleene lattices* [18], in which both $\mathbf{A}_+$ and $\mathbf{A}_-$ are essential). This suggests that the operations that require a richer structure on the negative factor than on the positive one (for instance the monoid operation, as defined component-wise on quasi-Nelson twist-algebras) may turn out to be the hardest to account for. Another difficulty is in general represented by the operations that are not compatible with the relation $\equiv$ defined in Section 2, such as the strong implication ($\to$); this may of course be overcome if the operation is definable from the compatible ones that are present in the language (as is the case of $\to$ in the presence of $\sim$).

Regarding the above problems, the treatment of QNI-algebras developed in the present paper suggests a potentially successful strategy. Indeed, even if an algebra $\mathbf{A}$ does not possess a certain operation (say, the meet $\wedge$) that would be required to define the corresponding one on the quotient (say, the meet on $\mathbf{A}_{-}$), it is sometimes possible to find a term (in this case letting $x \odot y := \sim(x \to \sim y)$ was enough) that is not a meet on $\mathbf{A}$ but acts as a meet on the quotient $\mathbf{A}_{-}$. This observation led us to the discovery of the 'right' counterpart of QNI-algebras, i.e. nH-semigroups, a class of algebras that (we note) could not be simply obtained from Heyting algebras with nuclei by eliding certain algebraic operations.

It is clear that the preceding considerations are far from constituting a general explanation of the regularities and irregularities in the landscape of fragments of quasi-Nelson. This notwithstanding, we speculate that a number of other interesting fragments may lie within the reach of (suitable extensions of) the techniques developed in the present paper, and we wish our efforts to be instrumental in reaching a higher point of observation, from which the pattern of such (ir)regularities may start to emerge.

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