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ON GLOBAL WELL-POSEDNESS OF THE MODIFIED KdV EQUATION IN MODULATION SPACES

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ABSTRACT. We study well-posedness of the complex-valued modified KdV equation (mKdV) on the real line. In particular, we prove local well-posedness of mKdV in modulation spaces $M_s^{2,p}(\mathbb{R})$ for $s \geq \frac{1}{4}$ and $2 \leq p < \infty$. For $s < \frac{1}{4}$, we show that the solution map for mKdV is not locally uniformly continuous in $M_s^{2,p}(\mathbb{R})$. By combining this local well-posedness with our previous work (2018) on an a priori global-in-time bound for mKdV in modulation spaces, we also establish global well-posedness of mKdV in $M_s^{2,p}(\mathbb{R})$ for $s \geq \frac{1}{4}$ and $2 \leq p < \infty$.

1. Introduction.

1.1. Modified KdV equation. We consider the Cauchy problem for the complex-valued modified KdV equation on the real line:

$$\begin{cases} \partial_t u + \partial_x^3 u \pm 6|u|^2 \partial_x u = 0 \\ u|_{t=0} = u_0, \end{cases} \quad (x, t) \in \mathbb{R} \times \mathbb{R}. \quad (1.1)$$

The equation (1.1) is known to be completely integrable and is closely related to the cubic nonlinear Schrödinger equation (NLS):

$$i\partial_t u - \partial_x^2 u \mp 2|u|^2 u = 0. \quad (1.2)$$

See [14, 26, 20, 24]. When the initial data u_0 is real-valued, the corresponding solution u to (1.1) remains real-valued, thus solving the following real-valued mKdV:

$$\partial_t u + \partial_x^3 u \pm 6u^2 \partial_x u = 0. \quad (1.3)$$

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The mKdV enjoys the following scaling symmetry:

$$u(x, t) \mapsto u_\lambda(x, t) = \lambda^{-1}u(\lambda^{-1}x, \lambda^{-3}t), \quad (1.4)$$

which induces the scaling-critical Sobolev regularity $s_{\text{crit}} = -\frac{1}{2}$ in the sense that homogeneous $\dot{H}^{-\frac{1}{2}}$ -norm is invariant under the scaling symmetry (1.4).

The Cauchy problem (1.1) has been studied extensively. In [15], Kato studied (1.1) from a viewpoint of quasilinear hyperbolic equations (in particular, not making use of dispersion) and proved its local well-posedness in $H^s(\mathbb{R})$, $s > \frac{3}{2}$. In [17, 18], Kenig-Ponce-Vega exploited the dispersive nature of the equation and proved local well-posedness of (1.1) in $H^s(\mathbb{R})$, $s \geq \frac{1}{4}$. In [27], Tao gave an alternative proof of the local well-posedness in $H^{\frac{1}{4}}(\mathbb{R})$ by using the Fourier restriction norm method. We also mention recent papers [23, 22] on unconditional uniqueness of solutions to (1.1) in $H^s(\mathbb{R})$, $s > \frac{1}{4}$. Let us now turn our attention to global well-posedness of (1.1). In the real-valued setting, Colliander-Keel-Staffilani-Takaoka-Tao [5] applied the I -method and proved global well-posedness of (1.3) in $H^s(\mathbb{R})$ for $s > \frac{1}{4}$. See Kishimoto [21] for the endpoint global well-posedness in $H^{\frac{1}{4}}(\mathbb{R})$. In a recent paper, Killip-Vişan-Zhang [20] exploited the completely integrable structure of the equation and proved a global-in-time a priori bound on the H^s -norm of solutions to the complex-valued mKdV (1.1) for $-\frac{1}{2} < s < 0$. While it is not written in an explicitly manner,¹ their result is readily extendable to $-\frac{1}{2} < s < 1$ and thus proves global well-posedness of the complex-valued mKdV (1.1) in $H^{\frac{1}{4}}(\mathbb{R})$.²

On the other hand, it is known that the solution map to (1.1) is not locally uniformly continuous in $H^s(\mathbb{R})$ for $s < \frac{1}{4}$; see [19, 3]. This in particular implies that one can *not* use a contraction argument to construct solutions to (1.1) in this regime. One possible approach to study rough solutions outside $H^{\frac{1}{4}}(\mathbb{R})$ is to use a more robust energy method. In [4], Christ-Holmer-Tataru employed an energy method in the form of the short-time Fourier restriction norm method and proved global existence of solutions to the real-valued mKdV (1.3) in $H^s(\mathbb{R})$ for $-\frac{1}{8} < s < \frac{1}{4}$. Uniqueness of these solutions, however, is unknown at this point.

Another approach is to study the Cauchy problem (1.1) in some other scales of function spaces than the Sobolev spaces $H^s(\mathbb{R})$. In [9], Grünrock studied (1.1) in the Fourier-Lebesgue spaces $\mathcal{FL}^{s,p}(\mathbb{R})$ defined by the norm:

$$\|f\|_{\mathcal{FL}^{s,p}} = \|\langle \xi \rangle^s \widehat{f}(\xi)\|_{L^p},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$, and proved its local well-posedness in $\mathcal{FL}^{s,p}(\mathbb{R})$ for $s \geq \frac{1}{2p}$ and $2 \leq p < 4$. In [10], Grünrock-Vega extended this result to $2 \leq p < \infty$ with the same range of $s \geq \frac{1}{2p}$. Note that the space $\mathcal{FL}^{0,\infty}(\mathbb{R})$ of pseudo-measures is critical in terms of the scaling symmetry (1.4), i.e. the $\mathcal{FL}^{0,\infty}$ -norm remains invariant under (1.4). Hence, by taking $p \rightarrow \infty$, we see that the local well-posedness result in [10] is almost critical. There are two remarks in order; (i) the range of $s \geq \frac{1}{2p}$ in [9, 10] is sharp in the sense that the solution map to (1.1) is not locally uniformly continuous in $\mathcal{FL}^{s,p}(\mathbb{R})$ for $s < \frac{1}{2p}$ and $2 \leq p < \infty$. See Section 5 in [9]. (ii) there seems to be no known global well-posedness of (1.1) in the context of Fourier-Lebesgue spaces, extending local solutions constructed in [9, 10] globally in time.

¹See also Appendix B of [24] for details of a global-in-time a priori bound on the H^s -norm of solutions to the complex-valued mKdV (1.1) for $0 < s < \frac{1}{2}$.

²In a recent preprint [13], by exploiting the completely integrable structure of the equation, Harrop-Griffiths, Killip, and Vişan proved global well-posedness of mKdV (1.1) in $H^s(\mathbb{R})$, $s > -\frac{1}{2}$.

1.2. Main results. Our main goal in this paper is to study the Cauchy problem (1.1) in modulation spaces $M_s^{2,p}(\mathbb{R})$. We first recall the definition of modulation spaces $M_s^{r,p}(\mathbb{R})$ introduced in [6, 7]. Let $\psi \in \mathcal{S}(\mathbb{R})$ such that

$$\text{supp } \psi \subset [-1, 1] \quad \text{and} \quad \sum_{k \in \mathbb{Z}} \psi(\xi - k) \equiv 1.$$

Then, given $s \in \mathbb{R}$, $1 \leq r, p \leq \infty$, the modulation space $M_s^{r,p}(\mathbb{R})$ is defined as the collection of all tempered distributions $f \in \mathcal{S}'(\mathbb{R})$ such that $\|f\|_{M_s^{r,p}} < \infty$, where the $M_s^{r,p}$ -norm is defined by

$$\|f\|_{M_s^{r,p}} = \|\langle n \rangle^s \|\psi_n(D)f\|_{L_x^r(\mathbb{R})}\|_{\ell_n^p(\mathbb{Z})}.$$

Here, $\psi_n(D)$ is the Fourier multiplier operator with the multiplier

$$\psi_n(\xi) := \psi(\xi - n).$$

In the following, we only consider $r = 2$. In the case of $r = 2$, we have the following embedding

$$M_s^{2,p}(\mathbb{R}) \supset \mathcal{FL}^{s,p}(\mathbb{R}) \tag{1.5}$$

for $p \geq 2$. The embedding (1.5) is immediate from

$$\|f\|_{\mathcal{FL}^{s,p}} \sim \|\langle n \rangle^s \|\psi_n(\xi)\widehat{f}(\xi)\|_{L_\xi^p(\mathbb{R})}\|_{\ell_n^p(\mathbb{Z})}$$

and the support condition on ψ .

In [24], we extended the work [20] by Killip-Vişan-Zhang on the global-in-time a priori bound for solutions to (1.1) to the modulation space setting and proved the following result.

Proposition 1. *Let $2 \leq p < \infty$ and $0 \leq s < 1 - \frac{1}{p}$.³ Then, there exists $C = C(p) > 0$ such that*

$$\|u(t)\|_{M_s^{2,p}} \leq C(\|u(0)\|_{M_s^{2,p}}) \tag{1.6}$$

for any Schwartz class solution u to the complex-valued mKdV (1.1) and any $t \in \mathbb{R}$.

In [24], we also established the same global-in-time a priori bound for solutions to the cubic NLS (1.2). Combining this with the local well-posedness of (1.2) in $M_s^{2,p}(\mathbb{R})$ for $s \geq 0$ and $2 \leq p < \infty$ by S. Guo [11], we proved global well-posedness of the cubic NLS (1.2) in almost critical modulation spaces⁴ $M_s^{2,p}(\mathbb{R})$ for $s \geq 0$.

On the other hand, there is no known local well-posedness for the modified KdV equation (1.1) in the modulation space $M_s^{2,p}(\mathbb{R})$, which motivated us to prove the following local well-posedness result.

Theorem 1.1. *Let $s \geq \frac{1}{4}$ and $2 \leq p < \infty$. Then, the complex-valued mKdV (1.1) is locally well-posed in $M_s^{2,p}(\mathbb{R})$.*

In [11], S. Guo proved local well-posedness of the cubic NLS (1.2) in the modulation spaces $M_s^{2,p}(\mathbb{R})$ for $s \geq 0$ and $2 \leq p < \infty$. The proof was based on the Fourier restriction norm method adapted to the modulation spaces, where an endpoint version of two-dimensional Fourier restriction estimate played a crucial role.

³The upper bound $1 - \frac{1}{p}$ is not essential and we expect that this restriction can be relaxed by a consideration similar to that in Section 3 of [20].

⁴The modulation spaces are based on the unit cube decomposition of the frequency space and thus there is no scaling for the modulation spaces. We, however, say that $M_0^{2,\infty}(\mathbb{R})$ is a critical space in view of the embedding (1.5) with $s = 0$ and $p = \infty$.

See also [12] for a work on the derivative NLS which employs a similar strategy. In proving Theorem 1.1, we also use the Fourier restriction norm method adapted to the modulation space setting. See (2.3) below. We, however, provide a different approach than [11, 12]. Our argument is based on bilinear estimates; see Lemmas 2.2 and Corollary 1. It is worthwhile to mention that our approach works equally well for the cubic NLS and the derivative NLS, providing an alternative approach to the results in [11, 12].

As a corollary to Proposition 1 and Theorem 1.1, we obtain the following global well-posedness.

Theorem 1.2. *Let $s \geq \frac{1}{4}$ and $2 \leq p < \infty$. Then, there exists a function $C : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is increasing in each argument, such that*

$$\sup_{t \in [-T, T]} \|u(t)\|_{M_s^{2,p}} \leq C(\|u_0\|_{M_s^{2,p}}, T) \quad (1.7)$$

for any $T > 0$ and any Schwartz solution u to (1.1) with $u|_{t=0} = u_0$. In particular, this implies that the complex-valued mKdV (1.1) is globally well-posed in $M_s^{2,p}(\mathbb{R})$.

For $\frac{1}{4} \leq s < 1 - \frac{1}{p}$, Proposition 1 allows us to choose the right-hand side of (1.7) to be independent of $T > 0$. For $s \geq 1 - \frac{1}{p}$, we combine a persistence-of-regularity argument with the global-in-time bound on the $M_{\frac{1}{4}}^{2,p}$ -norms of solutions. See Subsection 3.6.

Remark 1. One can easily adapt the proof of Theorem 1.1 to extend the local well-posedness of (1.1) to $1 \leq p < 2$ (and $s \geq \frac{1}{4}$). Similarly, by establishing persistence of regularity as in [25], we can also prove global well-posedness of (1.1) in $M_s^{2,p}(\mathbb{R})$ for $s \geq \frac{1}{4}$ and $1 \leq p < 2$. See Remark 3.

On the one hand, $\mathcal{F}L^{\frac{1}{4}, \infty}(\mathbb{R})$ scales like $\dot{H}^{-\frac{1}{4}}(\mathbb{R})$ and thus we may say that $M_{\frac{1}{4}}^{2, \infty}(\mathbb{R})$ “scales like” $\dot{H}^{-\frac{1}{4}}(\mathbb{R})$ in view of the embedding (1.5). On the other hand, the $M_s^{2,p}(\mathbb{R})$ -norm is weaker than the $\mathcal{F}L^{s,p}$ -norm for $p > 2$ and the solution map to the mKdV (1.1) fails to be locally uniformly continuous in $M_s^{2,p}(\mathbb{R})$ as soon as $s < \frac{1}{4}$.

Proposition 2. *Suppose that (s, p) satisfies one of the following conditions: (i) $2 \leq p \leq \infty$ and $0 \leq s < \frac{1}{4}$ or (ii) $2 \leq p < \infty$ and $-\frac{1}{p} < s < 0$. Then, the data-to-solution map for (1.1) in the focusing case (with the + sign in (1.1)) : $u_0 \in M_s^{2,p} \mapsto u \in C([-T, T]; M_s^{2,p}(\mathbb{R}))$ is not locally uniformly continuous for any $T > 0$.*

Proposition 2 shows a sharp contrast with the Fourier-Lebesgue case, where local well-posedness was proved via a contraction argument even for some $s < \frac{1}{4}$.

In [19], Kenig-Ponce-Vega proved the failure of local uniform continuity of the solution map for the complex-valued focusing mKdV (1.1) in $H^s(\mathbb{R})$, $-\frac{1}{2} < s < \frac{1}{4}$, by building counterexamples from explicit soliton solutions. See (4.2) below. By making use of breather solutions to the real-valued focusing mKdV (1.1), they also extended this result for the real-valued case. In [3], Christ-Colliander-Tao [3] extended this failure of local uniform continuity below $H^{\frac{1}{4}}(\mathbb{R})$ (for $-\frac{1}{4} < s < \frac{1}{4}$) to the defocusing case by approximating the mKdV dynamics by the cubic NLS dynamics (which was in turn approximated by a dispersionless equation). These (approximate) solutions in [19, 3] depend on a parameter N tending to ∞ and,

as $N \rightarrow \infty$, they start to concentrate at a single point on the frequency side (for $s > 0$). Namely, they are essentially supported on a single unit cube for $N \gg 1$. In this regime, their $M_s^{2,p}$ -norms basically reduce to the H^s -norms, giving rise to the threshold regularity $s = \frac{1}{4}$ even in the modulation space setting. The main difficulty is that calculation required for the modulation space setting is much more involved than that for the Sobolev space setting. Therefore, we only demonstrate the proof for the focusing cases in Section 4. We expect the same result hold for the defocusing case. For the conciseness of the paper, however, we choose not to discuss details for the defocusing case.

Remark 2. In a recent preprint [2], the authors independently proved local well-posedness of (1.1) analogous to Theorem 1.1 for $s \geq \frac{1}{4}$ and $2 \leq p \leq \infty$. While the result in [2] only refers to local well-posedness, it contains the $p = \infty$ case. In view of the embedding $M_s^{2,\infty}(\mathbb{R}) \subset M_{\frac{1}{4}}^{2,p}(\mathbb{R})$ for

$$(s - \frac{1}{4})p > 1, \tag{1.8}$$

a combination of the a priori bound (1.7) in Theorem 1.2 (with $s > \frac{1}{4}$ and $p < \infty$ satisfying (1.8)) and a persistence-of-regularity argument as in Subsection 3.6 seems to yield global well-posedness of (1.1) for $s > \frac{1}{4}$ and $p = \infty$. On the other hand, the global well-posedness issue at the endpoint case: $s = \frac{1}{4}$ and $p = \infty$ remains open.

2. Preliminaries. We write $A \lesssim B$ to denote an estimate of the form $A \leq CB$. Similarly, we write $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$ and use $A \ll B$ when we have $A \leq cB$ for small $c > 0$.

Given dyadic $N \geq 1$, we denote by \mathbf{P}_N the Littlewood-Paley projector onto the (spatial) frequencies $\{|\xi| \sim N\}$. We use the following convention; any summation over capitalized variables such as N_1, N_2, \dots , are presumed to be over dyadic numbers of the form $2^k, k \in \mathbb{N} \cup \{0\}$.

For $n \in \mathbb{Z}$, let

$$\widehat{\Pi_n f}(\xi) = \psi_n(\xi)\widehat{f}(\xi) = \psi(\xi - n)\widehat{f}(\xi). \tag{2.1}$$

By Bernstein’s inequality, we have

$$\begin{aligned} \|\mathbf{P}_N f\|_{L_x^p} &\lesssim N^{\frac{1}{q} - \frac{1}{p}} \|f\|_{L_x^q}, \\ \|\Pi_n f\|_{L_x^p} &\lesssim \|f\|_{L_x^q} \end{aligned} \tag{2.2}$$

for any $1 \leq q \leq p \leq \infty$.

In the seminal work [1], Bourgain introduced the $X^{s,b}$ -space defined by the norm:

$$\|u\|_{X^{s,b}} := \|\langle \xi \rangle^s \langle \tau - \xi^3 \rangle^b \widehat{u}(\xi, \tau)\|_{L_{\tau,\xi}^2}.$$

In this paper, we use the following version of the $X^{s,b}$ -space adapted to the modulation spaces $M_s^{2,p}(\mathbb{R})$:

$$\begin{aligned} \|u\|_{X_p^{s,b}} &:= \left(\sum_{n \in \mathbb{Z}} \langle n \rangle^{sp} \|\langle \tau - \xi^3 \rangle^b \widehat{u}(\xi, \tau)\|_{L_{\tau,\xi}^2(\mathbb{R} \times [n, n+1])}^p \right)^{\frac{1}{p}} \\ &\sim \|\|\Pi_n u\|_{X^{s,b}}\|_{\ell_n^p}. \end{aligned} \tag{2.3}$$

When $p = 2$, the space $X_p^{s,b}$ reduces to the usual $X^{s,b}$ -space. When $b > \frac{1}{2}$, the following embedding holds:

$$X_p^{s,b} \subset C(\mathbb{R}; M_s^{2,p}(\mathbb{R})). \tag{2.4}$$

Let $p \geq q \geq 1$. Since $\ell_n^q(\mathbb{Z}) \subset \ell_n^p(\mathbb{Z})$, we have

$$\|u\|_{X_p^{s,b}} \leq \|u\|_{X_q^{s,b}}. \tag{2.5}$$

On the other hand, from Hölder’s inequality, we have

$$\|\mathbf{P}_N u\|_{X_q^{s,b}} \lesssim N^{\frac{1}{q}-\frac{1}{p}} \|\mathbf{P}_N u\|_{X_p^{s,b}}. \tag{2.6}$$

Given a time interval $I \subset \mathbb{R}$, we also define the local-in-time version $X_p^{s,b}(I)$ of the $X_p^{s,b}$ -space as the collection of functions u such that

$$\|u\|_{X_p^{s,b}(I)} := \inf \{ \|v\|_{X_p^{s,b}} : v|_I = u \}$$

is finite.

The following linear estimates follow from the characterization (2.3) and the corresponding linear estimates for the standard $X^{s,b}$ -spaces. See [8] for the proof.

Lemma 2.1. (i) (Homogeneous linear estimate). *Given $1 \leq p < \infty$ and $s, b \in \mathbb{R}$, we have*

$$\|e^{-t\partial_x^3} f\|_{X_p^{s,b}([0,T])} \lesssim \|f\|_{M_s^{2,p}}$$

for any $0 < T \leq 1$.

(ii) (Nonhomogeneous linear estimate). *Let $s \in \mathbb{R}$, $1 \leq p < \infty$, and $-\frac{1}{2} < b' \leq 0 \leq b \leq 1 + b'$. Then, we have*

$$\left\| \int_0^t e^{-(t-t')\partial_x^3} F(t') dt' \right\|_{X_p^{s,b}([0,T])} \lesssim T^{1+b'-b} \|F\|_{X_p^{s,b'}([0,T])}$$

for any $0 < T \leq 1$.

In the following, we list various estimates in proving the crucial trilinear estimate (Proposition 3). The following inequality will be convenient in dealing with the resonant case in Section 3. From Hölder’s and Young’s inequalities, we have

$$\sum_{\substack{m,n \in \mathbb{Z} \\ m \neq n}} \frac{a_m b_n}{|m-n|\langle n \rangle^\varepsilon} \leq C_\varepsilon \|a_n\|_{\ell^p(\mathbb{Z})} \|b_n\|_{\ell^{p'}(\mathbb{Z})} \tag{2.7}$$

for any $\varepsilon > 0$, where p' denotes the Hölder conjugate of p .

Next, we recall a bilinear estimate from [9]. Given $\theta > 0$, let $I^\theta = (-\partial_x^2)^{\frac{\theta}{2}}$ denote the Riesz potential of order $-\theta$. We also define I_-^θ by

$$\mathcal{F}_x(I_-^\theta(f, g))(\xi) := \int_{\xi=\xi_1+\xi_2} |\xi_1 - \xi_2|^\theta \widehat{f}(\xi_1) \widehat{g}(\xi_2) d\xi_1.$$

Then, we have the following bilinear estimate. See Lemma 3.1 and Corollary 3.2 in [10].

Lemma 2.2. *Let $I^{\frac{1}{2}}$ and $I_-^{\frac{1}{2}}$ be as above (with $\theta = \frac{1}{2}$). Then, we have⁵*

$$\|I^{\frac{1}{2}} I_-^{\frac{1}{2}}(u, v)\|_{L_{x,t}^2(\mathbb{R}^2)} \lesssim \|u\|_{X^{0, \frac{1}{2}+}} \|v\|_{X^{0, \frac{1}{2}+}}.$$

The following two estimates are immediate corollary of Lemma 2.2.

⁵We use $a+$ (and $a-$) to denote $a + \varepsilon$ (and $a - \varepsilon$, respectively) for arbitrarily small $\varepsilon \ll 1$, where an implicit constant is allowed to depend on $\varepsilon > 0$ (and it usually diverges as $\varepsilon \rightarrow 0$).

Corollary 1. (i) *Let $N_1, N_2 \geq 1$ be dyadic such that $N_1 \gg N_2$. Then, we have*

$$\|\mathbf{P}_{N_1} u \mathbf{P}_{N_2} v\|_{L^2_{x,t}(\mathbb{R}^2)} \lesssim \frac{1}{N_1} \|\mathbf{P}_{N_1} u\|_{X^{0, \frac{1}{2}+}} \|\mathbf{P}_{N_2} v\|_{X^{0, \frac{1}{2}+}}.$$

(ii) *Let $m, n \in \mathbb{Z}$ such that $|m + n|, |m - n| \geq 2$. Then, we have*

$$\|\Pi_m u \Pi_n v\|_{L^2_{x,t}(\mathbb{R}^2)} \lesssim \frac{1}{\sqrt{|m + n| |m - n|}} \|\Pi_m u\|_{X^{0, \frac{1}{2}+}} \|\Pi_n v\|_{X^{0, \frac{1}{2}+}}.$$

In [27], Tao presented a proof of local well-posedness of mKdV (1.1) in $H^{\frac{1}{4}}(\mathbb{R})$ based on the Fourier restriction norm method by establishing the following trilinear estimate.

Lemma 2.3 (Corollary 6.3 in [27]). *Given small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\|\partial_x(u_1 u_2 u_3)\|_{X^{\frac{1}{4}, -\frac{1}{2}+2\varepsilon}} \leq C_\varepsilon \prod_{j=1}^3 \|u_j\|_{X^{\frac{1}{4}, \frac{1}{2}+\varepsilon}}. \tag{2.8}$$

In [27], the estimate (2.8) was stated with $-\frac{1}{2} + \varepsilon$ for the temporal regularity b on the left-hand side. It is, however, easy to see that the result also holds true with $-\frac{1}{2} + 2\varepsilon$.

3. Proof of Theorems 1.1 and 1.2.

3.1. Trilinear estimate. In view of the linear estimates in Lemma 2.1, local well-posedness of (1.1) (Theorem 1.1) follows from a standard contraction argument once we prove the following trilinear estimate.

Proposition 3. *Let $s \geq \frac{1}{4}$ and $2 \leq p < \infty$. Then, given small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\|u_1 \bar{u}_2 \partial_x u_3\|_{X_p^{s, -\frac{1}{2}+2\varepsilon}([0, T])} \leq C_\varepsilon \prod_{j=1}^3 \|u_j\|_{X_p^{s, \frac{1}{2}+\varepsilon}([0, T])} \tag{3.1}$$

for any $T > 0$.

We present the proof of Proposition 3 in the remaining part of this section. By a standard reduction, it suffices to prove (3.1) without the time restriction. Noting that the resonance relation $\tau = \xi^3$ is invariant under $(\tau, \xi) \mapsto (-\tau, -\xi)$, it suffices to prove

$$\|u_1 u_2 \partial_x u_3\|_{X_p^{s, -\frac{1}{2}+2\varepsilon}} \lesssim \prod_{j=1}^3 \|u_j\|_{X_p^{s, \frac{1}{2}+\varepsilon}}. \tag{3.2}$$

Furthermore, by the triangle inequality: $\langle \xi \rangle \lesssim \langle \xi_1 \rangle \langle \xi_2 \rangle \langle \xi_3 \rangle$ under $\xi_1 + \xi_2 + \xi_3 + \xi = 0$, it suffices to prove (3.2) for $s = \frac{1}{4}$. Then, by duality, (3.1) follows once we prove

$$\begin{aligned} \left| \iint_{\mathbb{R} \times \mathbb{R}} u_1 u_2 \partial_x u_3 \langle \partial_x \rangle^{\frac{1}{4}} v dx dt \right| &= \left| \int_{\substack{\xi_1 + \xi_2 + \xi_3 + \xi = 0 \\ \tau_1 + \tau_2 + \tau_3 + \tau = 0}} \langle \xi \rangle^{\frac{1}{4}} \xi_3 \prod_{j=1}^3 \widehat{u}_j(\xi_j, \tau_j) \widehat{v}(\xi, \tau) \right| \\ &\lesssim \prod_{j=1}^3 \|u_j\|_{X_p^{\frac{1}{4}, \frac{1}{2}+\varepsilon}} \|v\|_{X_{p'}^{0, \frac{1}{2}-2\varepsilon}}. \end{aligned} \tag{3.3}$$

In the following, we use $\xi_{\max}, \xi_{\text{med}}, \xi_{\min}$ to denote the rearrangement of ξ_1, ξ_2, ξ_3 such that $|\xi_{\max}| \geq |\xi_{\text{med}}| \geq |\xi_{\min}|$. Under $\xi_1 + \xi_2 + \xi_3 + \xi = 0$, we have $|\xi| \lesssim |\xi_{\max}|$. In the following, we apply dyadic decompositions $|\xi_j| \sim N_j$ and $|\xi| \sim N$ for dyadic $N_j, N \geq 1$. In this case, we also use the notation: $N_{\max} \sim |\xi_{\max}|$, $N_{\text{med}} \sim |\xi_{\text{med}}|$, and $N_{\min} \sim |\xi_{\min}|$.

We prove Proposition 3 by separately considering the following four cases:

- (i) Trivial cases,
- (ii) Non-resonant case: $N_{\max} \gg N_{\text{med}}$,
- (iii) Semi-resonant case: $N_{\max} \sim N_{\text{med}} \gg N_{\min}$,
- (iv) Resonant case: $N_{\max} \sim N_{\min}$.

As we see below, the main difficulty appears in the resonant case (iv). Before going into the details of the proof, we introduce a few more notations. We use σ and σ_j to denote modulations given by

$$\sigma = \tau - \xi^3 \quad \text{and} \quad \sigma_j = \tau_j - \xi_j^3$$

for $j = 1, 2, 3$. We also set

$$\sigma_{\max} = \max(|\sigma|, |\sigma_1|, |\sigma_2|, |\sigma_3|).$$

For conciseness of the presentation, we use the following (slightly abusive) shorthand notations:

$$u_N = \mathbf{P}_N u \quad \text{and} \quad u_n = \Pi_n u,$$

where \mathbf{P}_N is the Littlewood-Paley projector and Π_n is as in (2.1). We only use the capitalized variables to denote dyadic numbers and hence there is no confusion.

Remark 3. By slightly modifying the proof, we can easily extend (3.1) to $1 \leq p < 2$. Note that the proof in this case is easier than that of Proposition 3 since $\ell^p(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$. Furthermore, we can also establish

$$\begin{aligned} & \|u_1 \bar{u}_2 \partial_x u_3\|_{X_p^{s, -\frac{1}{2}+2\epsilon}([0, T])} \\ & \leq C_\epsilon \min_{j=1,2,3} \left(\|u_j\|_{X_p^{s, \frac{1}{2}+\epsilon}([0, T])} \prod_{\substack{k=1 \\ k \neq j}}^3 \|u_k\|_{X^{s, \frac{1}{2}+\epsilon}([0, T])} \right) \end{aligned} \tag{3.4}$$

for $s \geq \frac{1}{4}$ and $1 \leq p < 2$. The tame estimate (3.4) allows us to prove local well-posedness of (1.1) in $M_s^{2,p}(\mathbb{R})$ for $s \geq \frac{1}{4}$ and $1 \leq p < 2$, where the local existence time depends only on the H^s -norm of initial data. In particular, this allows us to prove global well-posedness of (1.1) in $M_s^{2,p}(\mathbb{R})$ for $s \geq \frac{1}{4}$ and $1 \leq p < 2$. See Appendix of [25] for such an argument. Since the required modification is straightforward, we omit details.

3.2. Trivial cases. We first consider two trivial cases:

$$(i) \quad |\xi_{\max}| \lesssim 1 \quad \text{and} \quad (ii) \quad \langle \sigma_{\max} \rangle \gg \langle \xi_{\max} \rangle^{10}. \tag{3.5}$$

(i) Suppose $|\xi_{\max}| \lesssim 1$. In this case, we have $|\xi| \lesssim 1$. Then, by Hölder’s inequality, Bernstein’s inequality (2.2), (2.3), (2.4) followed by (2.5) and (2.6), we have

$$\begin{aligned} & \sum_{N_{\max}, N \lesssim 1} N^{\frac{1}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_{N_1} u_{N_2} \partial_x u_{N_3} v_N dx dt \right| \\ & \lesssim \sum_{N_{\max}, N \lesssim 1} \|u_{N_1}\|_{L_{t,x}^2} \|u_{N_2}\|_{L_{x,t}^\infty} \|u_{N_3}\|_{L_{x,t}^\infty} \|v_N\|_{L_{x,t}^2} \end{aligned}$$

$$\begin{aligned} &\lesssim \sum_{N_{\max}, N \lesssim 1} \|u_{N_1}\|_{L^2_{x,t}} \|u_{N_2}\|_{L^\infty_t L^2_x} \|u_{N_3}\|_{L^\infty_t L^2_x} \|v_N\|_{L^2_{x,t}} \\ &\lesssim \sum_{N_{\max}, N \lesssim 1} \left(\prod_{j=1}^3 \|u_{N_j}\|_{X^{0, \frac{1}{2}+}} \right) \|v_N\|_{X^{0, \frac{1}{2}-}} \\ &\lesssim \sum_{N_{\max}, N \lesssim 1} \left(\prod_{j=1}^3 \|u_{N_j}\|_{X^{0, \frac{1}{2}+}_{p'}} \right) \|v_N\|_{X^{0, \frac{1}{2}-}_{p'}}. \end{aligned}$$

By summing over dyadic blocks $N_1, N_2, N_3, N \lesssim 1$, we obtain (3.3).

(ii) Next, we suppose $\langle \sigma_{\max} \rangle \gg \langle \xi_{\max} \rangle^{10}$. In the following, we consider the case $\langle \sigma_1 \rangle = \langle \sigma_{\max} \rangle$, The other cases follow from a similar argument. By Hölder’s and Bernstein’s inequalities, the definition (2.3), and (2.4), we have

$$\begin{aligned} &\sum_{\substack{N_1, N_2, N_3, N \geq 1 \\ \text{dyadic}}} N^{\frac{1}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_{N_1} u_{N_2} \partial_x u_{N_3} v_N dx dt \right| \\ &\lesssim \sum_{N_1, N_2, N_3, N \geq 1} N_{\max}^{\frac{5}{4}} \|u_{N_1}\|_{L^2_{x,t}} \|u_{N_2}\|_{L^\infty_{x,t}} \|u_{N_3}\|_{L^\infty_{x,t}} \|v_N\|_{L^2_{x,t}} \\ &\lesssim \sum_{N_1, N_2, N_3, N \geq 1} N_{\max}^{\frac{9}{4}} \|u_{N_1}\|_{L^2_{x,t}} \|u_{N_2}\|_{L^\infty_t L^2_x} \|u_{N_3}\|_{L^\infty_t L^2_x} \|v_N\|_{L^2_{x,t}} \\ &\lesssim \sum_{N_1, N_2, N_3, N \geq 1} N_{\max}^{\frac{9}{4}} \langle \sigma_1 \rangle^{-\frac{1}{2}-} \|u_{N_1}\|_{X^{0, \frac{1}{2}+}} \left(\prod_{j=2}^3 \|u_{N_j}\|_{X^{0, \frac{1}{2}+}} \right) \|v_N\|_{X^{0, \frac{1}{2}-}} \end{aligned}$$

By applying the lower bound (3.5) together with (2.5) and (2.6),

$$\begin{aligned} &\lesssim \sum_{N_1, N_2, N_3, N \geq 1} N_{\max}^{\frac{9}{4}} N_{\max}^{-\frac{7}{2}-\frac{3}{p}-} \left(\prod_{j=1}^3 \|u_{N_j}\|_{X^{0, \frac{1}{2}+}_{p'}} \right) \|v_N\|_{X^{0, \frac{1}{2}-}_{p'}} \\ &\lesssim \sum_{N_1, N_2, N_3, N \geq 1} \left(\prod_{j=1}^3 N_j^{0-} \|u_{N_j}\|_{X^{0, \frac{1}{2}+}_{p'}} \right) N^{0-} \|v_N\|_{X^{0, \frac{1}{2}-}_{p'}}. \end{aligned}$$

By summing over dyadic blocks $N_1, N_2, N_3, N \geq 1$, we obtain (3.3).

Therefore, we assume that

$$|\xi_{\max}| \gg 1 \quad \text{and} \quad \langle \sigma_{\max} \rangle \lesssim \langle \xi_{\max} \rangle^{10} \tag{3.6}$$

in the following.

Remark 4. In the arguments above, we first established bounds in terms of the standard $X^{s,b}$ -norms and then applied (2.5) and (2.6) to replaced it by the $X^{s,b}_p$ -norms. More precisely, we used

$$\|u_N\|_{X^{s,b}} \lesssim \max(N^{\frac{1}{2}-\frac{1}{p}}, 1) \|u_N\|_{X^{s,b}_p} \tag{3.7}$$

and

$$\sum_{\substack{N \geq 1 \\ \text{dyadic}}} N^{-\varepsilon} \|u_N\|_{X^{s,b}_p} \lesssim \|u\|_{X^{s,b}} \tag{3.8}$$

for any $\varepsilon > 0$. We use the same strategy in the following.

3.3. Non-resonant case: $N_{\max} \gg N_{\text{med}} \geq N_{\min}$. Without loss of generality,⁶ suppose that $N_1 \gg N_2 \geq N_3$. The other cases can be treated by a similar consideration. In this case, we have $N \sim N_1$. Then, by Corollary 1 and (3.6), we have

$$\begin{aligned} & \sum_{N_1 \sim N \gg N_2 \geq N_3} N^{\frac{1}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_{N_1} u_{N_2} \partial_x u_{N_3} v_N dx dt \right| \\ & \lesssim \sum_{N_1 \sim N \gg N_2 \geq N_3} N^{\frac{5}{4}} \|u_{N_1} u_{N_2}\|_{L^2_{x,t}} \|u_{N_3} v_N\|_{L^2_{x,t}} \\ & \lesssim \sum_{N_1 \sim N \gg N_2 \geq N_3} N^{-\frac{3}{4}} \left(\prod_{j=1}^3 \|u_{N_j}\|_{X^{0, \frac{1}{2}+}} \right) \|v_N\|_{X^{0, \frac{1}{2}+}} \\ & \lesssim \sum_{N_1 \sim N \gg N_2 \geq N_3} N^{-\frac{3}{4}+} \left(\prod_{j=1}^3 \|u_{N_j}\|_{X^{0, \frac{1}{2}+}} \right) \|v_N\|_{X^{0, \frac{1}{2}-}} \end{aligned}$$

By applying (3.7) and (3.8),

$$\begin{aligned} & \lesssim \sum_{N_1 \sim N \gg N_2 \geq N_3} N^{-\frac{1}{2}-\frac{1}{p}+} (N_2 N_3)^{\frac{1}{4}-\frac{1}{p}} \left(\prod_{j=1}^3 \|u_{N_j}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \right) \|v_N\|_{X^{0, \frac{1}{2}-}} \\ & \lesssim \sum_{N_1 \sim N \gg N_2 \geq N_3} N^{-\frac{1}{p}+} \left(\prod_{j=1}^3 \|u_{N_j}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \right) \|v_N\|_{X^{0, \frac{1}{2}-}} \\ & \lesssim \prod_{j=1}^3 \|u_j\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|v\|_{X^{0, \frac{1}{2}-}}, \end{aligned} \tag{3.9}$$

provided $p < \infty$.

3.4. Semi-resonant case: $N_{\max} \sim N_{\text{med}} \gg N_{\min}$. We proceed as in the non-resonant case. The frequency separation allows us to use the bilinear estimate (Corollary 1) twice, gaining two derivatives. Without loss of generality, suppose that $N_1 \sim N_2 \gg N_3$. The other cases can be treated by a similar consideration. We distinguish two cases according to the relation between N and N_{\max} .

First, suppose that $N \ll N_{\max}$. Then, by Corollary 1, we have

$$\begin{aligned} & \sum_{N_1 \sim N_2 \gg N_3, N} N^{\frac{1}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_{N_1} u_{N_2} \partial_x u_{N_3} v_N dx dt \right| \\ & \lesssim \sum_{N_1 \sim N_2 \gg N_3, N} N^{\frac{5}{4}} \|u_{N_1} u_{N_3}\|_{L^2_{x,t}} \|u_{N_2} v_N\|_{L^2_{x,t}} \\ & \lesssim \sum_{N_1 \sim N_2 \gg N_3, N} N^{\frac{3}{4}} \left(\prod_{j=1}^3 \|u_{N_j}\|_{X^{0, \frac{1}{2}+}} \right) \|v_N\|_{X^{0, \frac{1}{2}+}}. \end{aligned}$$

The rest follows as in (3.9).

Next, consider the case $N \sim N_{\max}$. In this case, we have $|\xi_1 + \xi_2 + \xi| = |\xi_3| \ll N \sim N_{\max}$. Hence, we must have $\xi_1 \xi_2 < 0$, $\xi_1 \xi < 0$, or $\xi_2 \xi < 0$. Without loss of

⁶Since the derivative falls on the third factor on the left-hand side of (3.3), there is no symmetry among frequencies ξ_1, ξ_2 , and ξ_3 . However, we simply bound this derivative by the largest frequency in the following and thus we may pretend that there is symmetry among frequencies. The same comment applies in the following.

generality, suppose that $\xi_1 \xi_2 < 0$. (The proofs for the other cases are similar.) We then have $|\xi - \xi_3| |\xi + \xi_3| \sim N_{\max}^2$ and

$$|\xi_1 - \xi_2| |\xi_1 + \xi_2| = |\xi_1 - \xi_2| |\xi + \xi_3| \sim N_{\max}^2.$$

Hence, by Lemma 2.2, we have

$$\begin{aligned} & \sum_{N_1 \sim N_2 \sim N \gg N_3} N^{\frac{1}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_{N_1} u_{N_2} \partial_x u_{N_3} v_N dx dt \right| \\ & \lesssim \sum_{N_1 \sim N_2 \sim N \gg N_3} N_{\max}^{\frac{5}{4}} \|u_{N_1} u_{N_2}\|_{L^2} \|u_{N_3} v_N\|_{L^2} \\ & \lesssim \sum_{N_1 \sim N_2 \sim N \gg N_3} N_{\max}^{-\frac{3}{4}} \left(\prod_{j=1}^3 \|u_{N_j}\|_{X^{0, \frac{1}{2}+}} \right) \|v_N\|_{X^{0, \frac{1}{2}+}}. \end{aligned} \tag{3.10}$$

The rest follows as in (3.9).

3.5. Resonant case. In this case, we have $N_1 \sim N_2 \sim N_3$. Without loss of generality, we may further assume that $N_1 \sim N$, since, otherwise, i.e. $N_1 \gg N$, the proof can be reduced to (3.10) with the roles of N and N_3 switched.

Hence, we assume that

$$N_1 \sim N_2 \sim N_3 \sim N \tag{3.11}$$

in the following. This case requires more careful analysis and we need to use the unit-cube decomposition:

$$u = \sum_{n \in \mathbb{Z}} u_n = \sum_{n \in \mathbb{Z}} \Pi_n u.$$

Given $n \in \mathbb{Z}$, we set $I_n = [n, n + 1)$.

- **Case 1:** We first consider the case $|\xi_i - \xi_j| \geq |\xi_i + \xi_j|$ for some pair (i, j) .

Without loss of generality, we assume $(i, j) = (1, 2)$. In the next two subcases, we treat the case $|\xi_1 + \xi_2| \lesssim 1$.

Subcase 1.1: $|\xi_1 + \xi_2| \lesssim 1$ and $\min(|\xi_1 - \xi_3|, |\xi_1 + \xi_3|) \lesssim 1$.

We only consider the case where $|\xi_1 - \xi_3| \lesssim 1$, since the proof for the case $|\xi_1 + \xi_3| \lesssim 1$ is similar. Suppose that $\xi_1 \in I_n = [n, n + 1)$. Then, we have

$$\xi_2 = -n + O(1), \quad \xi_3 = n + O(1), \quad \text{and} \quad \xi = -n + O(1).$$

Hence, we need to estimate the following expression:

$$\sum_{n \in \mathbb{Z}} \sum_{j, k, \ell = O(1)} \langle n \rangle^{\frac{1}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_n u_{-n+j} \partial_x u_{n+k} v_{-n+\ell} dx dt \right|.$$

For simplicity of the presentation, we only consider the “diagonal” case, i.e. $j = k = \ell = 0$ in the following. By Lemma 2.3 and Hölder’s inequality in n , we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \langle n \rangle^{\frac{1}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_n u_{-n} \partial_x u_n v_{-n} dx dt \right| & \lesssim \sum_{n \in \mathbb{Z}} \langle n \rangle^{\frac{1}{4}} \|u_n u_{-n} \partial_x u_n\|_{X^{0, -\frac{1}{2}+2\varepsilon}} \|v_{-n}\|_{X^{0, \frac{1}{2}-2\varepsilon}} \\ & \lesssim \sum_{n \in \mathbb{Z}} \|u_n\|_{X^{\frac{1}{4}, \frac{1}{2}+\varepsilon}}^3 \|v_n\|_{X^{0, \frac{1}{2}-2\varepsilon}} \\ & \lesssim \|u\|_{X_{3p}^{\frac{1}{4}, \frac{1}{2}+\varepsilon}}^3 \|v\|_{X_{p'}^{0, \frac{1}{2}-2\varepsilon}} \\ & \lesssim \|u\|_{X_p^{\frac{1}{4}, \frac{1}{2}+\varepsilon}}^3 \|v\|_{X_{p'}^{0, \frac{1}{2}-2\varepsilon}} \end{aligned}$$

for sufficiently small $\varepsilon > 0$. This is the only case where we need to be precise about the temporal regularities.

Subcase 1.2: $|\xi_1 + \xi_2| \lesssim 1$ and $|\xi_1 \pm \xi_3| \gg 1$.

Suppose that $\xi_1 \in I_n$ and $\xi_3 \in I_m$. In view of (3.11), we have $|n| \sim |m|$. Moreover, we have $\xi_2 \in I_{-n+j}$ and $\xi \in I_{-m+k}$ for $j, k = O(1)$. As in Subcase 1.1, we only estimate the contribution from $j = k = 0$. Without loss of generality, we assume that $|m + n| \geq |m - n|$. By Corollary 1 with $|m \pm n| \gg 1$, $|m| \sim |n| \gg 1$, and (3.6), we have

$$\begin{aligned} & \sum_{\substack{m, n \in \mathbb{Z} \\ m \neq n}} |m|^{\frac{1}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_n u_{-n} \partial_x u_m v_{-m} dx dt \right| \leq \sum_{\substack{m, n \in \mathbb{Z} \\ m \neq n}} |m|^{\frac{5}{4}} \|u_m u_n\|_{L^2} \|u_{-n} v_{-m}\|_{L^2} \\ & \lesssim \sum_{\substack{m, n \in \mathbb{Z} \\ m \neq n}} \frac{|m|^{\frac{5}{4}}}{|m - n| |m + n|} \|u_n\|_{X^{0, \frac{1}{2}+}} \|u_m\|_{X^{0, \frac{1}{2}+}} \|u_{-n}\|_{X^{0, \frac{1}{2}+}} \|v_{-m}\|_{X^{0, \frac{1}{2}+}} \\ & \lesssim \sum_{\substack{m, n \in \mathbb{Z} \\ m \neq n}} \frac{|n|^{\frac{1}{2}+}}{|m - n| |n + m|} \|u_n\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|u_m\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|u_{-n}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|v_m\|_{X^{0, \frac{1}{2}-}} \\ & \lesssim \sum_{\substack{m, n \in \mathbb{Z} \\ m \neq n}} \frac{1}{|m - n| \langle n \rangle^{\frac{1}{2}-}} \|u_n\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|u_m\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|u_{-n}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|v_m\|_{X^{0, \frac{1}{2}-}} \end{aligned}$$

By applying (2.7) and (2.3),

$$\begin{aligned} & \lesssim \left\| \|u_n\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|u_{-n}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \right\|_{\ell_n^{\frac{p}{2}}} \left\| \|u_m\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|v_m\|_{X^{0, \frac{1}{2}-}} \right\|_{\ell_m^{\frac{p}{p-2}}} \\ & \lesssim \|u\|_{X_p^{\frac{1}{4}, \frac{1}{2}+}}^3 \|v\|_{X_{p'}^{0, \frac{1}{2}-}}. \end{aligned}$$

In the next three subcases, we treat the case $|\xi_1 + \xi_2| \gg 1$.

Subase 1.3: $|\xi_1 + \xi_2| \gg 1$ and $|\xi_i - \xi_j| \lesssim 1$ for some $(i, j) \neq (1, 2)$.

Without loss of generality, we may assume $(i, j) = (1, 3)$. Suppose that $|\xi - \xi_3| \lesssim 1$. Then, we need to show

$$\sum_{n \in \mathbb{Z}} \langle n \rangle^{\frac{1}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_n u_{-3n} \partial_x u_n v_n dx dt \right| \lesssim \|u\|_{X_p^{\frac{1}{4}, \frac{1}{2}+}}^3 \|v\|_{X_{p'}^{0, \frac{1}{2}-}},$$

which can be easily obtained by repeating the argument in Subcase 1.1. Hence, we assume that $|\xi - \xi_3| \gg 1$ in the following.

Suppose that $\xi_1 \in I_n$ and $\xi \in I_m$. In view of (3.11), we have $|n| \sim |m|$. Moreover, we have $\xi_3 \in I_{n+j}$ and $\xi_2 \in I_{-m-2n-k}$ for $j, k = O(1)$. As above, we only estimate the contribution from $j = k = 0$. By the triangle inequality, we have $\max(|\xi - \xi_3|, |\xi + \xi_3|) \gtrsim |m| \sim |n| \gg 1$. In the following, we only consider the case $|\xi - \xi_3| \sim |m|$ since the other case follows in a similar manner. Moreover, since $|\xi_1 - \xi_2| \geq |\xi_1 + \xi_2|$, we conclude that $|m + 3n| \sim |m|$. Hence, by Corollary 1, (3.6), and (2.7), we have

$$\sum_{\substack{m, n \in \mathbb{Z} \\ m \neq n}} |m|^{\frac{1}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_n u_{-m-2n} \partial_x u_n v_m dx dt \right|$$

$$\begin{aligned}
 &\lesssim \sum_{\substack{m,n \in \mathbb{Z} \\ m \neq n}} |m|^{\frac{5}{4}} \|u_n u_{-m-2n}\|_{L^2} \|u_n v_m\|_{L^2} \\
 &\lesssim \sum_{\substack{m,n \in \mathbb{Z} \\ m \neq n}} \frac{|m|^{\frac{1}{2}+}}{|m+n| \sqrt{|m-n||m+3n|}} \|u_n\|_{X^{\frac{1}{4}, \frac{1}{2}+}}^2 \|u_{-m-2n}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|v_m\|_{X^{0, \frac{1}{2}-}} \\
 &\lesssim \|u\|_{X^{\frac{1}{4}, \frac{1}{2}+}}^2 \sum_{\substack{m,n \in \mathbb{Z} \\ m \neq n}} \frac{1}{|m+(m-2n)| \langle m \rangle^{\frac{1}{2}-}} \|u_m\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|v_{m-2n}\|_{X^{0, \frac{1}{2}-}} \\
 &\lesssim \|u\|_{X^{\frac{1}{4}, \frac{1}{2}+}}^3 \|v\|_{X^{0, \frac{1}{2}-}}.
 \end{aligned}$$

Subcase 1.4: $|\xi_1 + \xi_2| \gg 1$ and $|\xi_i + \xi_j| \lesssim 1$ for some $(i, j) \neq (1, 2)$.

We can proceed as in Subcase 1.3 above and thus we omit details.

Subcase 1.5: $|\xi_1 + \xi_2| \gg 1$ and $|\xi_i \pm \xi_j| \gg 1$ for all $(i, j) \neq (1, 2)$.

By assumption, we have $|\xi_1 - \xi_2| \geq |\xi_1 + \xi_2|$ and hence we have $|\xi_i \pm \xi_j| \gg 1$ for all $i \neq j$. Recalling that

$$\sigma + \sigma_1 + \sigma_2 + \sigma_3 = 3(\xi_1 + \xi_2)(\xi_2 + \xi_3)(\xi_1 + \xi_3)$$

under $\xi_1 + \xi_2 + \xi_3 + \xi = 0$ and $\tau_1 + \tau_2 + \tau_3 + \tau = 0$, we have

$$\langle \sigma_{\max} \rangle \gtrsim |\xi_1 + \xi_2| |\xi_2 + \xi_3| |\xi_1 + \xi_3|. \tag{3.12}$$

Without loss of generality, we assume that $\langle \sigma_1 \rangle = \langle \sigma_{\max} \rangle$. By Bernstein’s inequalities, (3.12), and Corollary 1 with (3.6), we have

$$\begin{aligned}
 &\sum_{n_1+n_2+n_3+n=O(1)} |n|^{\frac{1}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_{n_1} u_{n_2} \partial_x u_{n_3} v_n dx dt \right| \\
 &\lesssim \sum_{n_1+n_2+n_3+n=O(1)} |n|^{\frac{5}{4}} \|u_{n_1}\|_{L^2_{x,t}} \|u_{n_2}\|_{L^\infty_x L^2_t} \|u_{n_3} v_n\|_{L^2_{x,t}} \\
 &\lesssim \sum_{n_1+n_2+n_3+n=O(1)} \frac{|n|^{\frac{5}{4}+}}{\sqrt{|n_1+n_2||n_1+n_3||n_2+n_3|} \sqrt{|n_3-n||n_3+n|}} \\
 &\quad \times \left(\prod_{j=1}^3 \|u_{n_j}\|_{X^{0, \frac{1}{2}+}} \right) \|v_n\|_{X^{0, \frac{1}{2}-}}.
 \end{aligned} \tag{3.13}$$

By the triangle inequality, we have $\max(|n_3 - n|, |n_3 + n|) \geq |n|$ and

$$\max(|n_1 + n_3|, |n_2 + n_3|) \geq |n_1 - n_2| \gtrsim |n|.$$

In the following, we only consider the case $|n_1 + n_3| \sim |n_3 - n| \gtrsim |n|$. Then, we have

$$\begin{aligned}
 \text{LHS of (3.13)} &\lesssim \sum_{n_1+n_2+n_3+n=O(1)} \frac{|n|^{-\frac{1}{2}+}}{\sqrt{|n_1+n_2||n_2+n_3||n_3+n|}} \\
 &\quad \times \left(\prod_{j=1}^3 \|u_{n_j}\|_{X^{0, \frac{1}{2}+}} \right) \|v_n\|_{X^{0, \frac{1}{2}-}} \\
 &\lesssim \sup_{n_3, n} \left(\sum_{n_2} \frac{|n_2|^{-\frac{1}{2}+}}{\sqrt{|n_2+n_3|}} \|u_{-n_2-n_3-n}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|u_{n_2}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \right)
 \end{aligned}$$

$$\times \sum_{n_3, n} \frac{1}{|n_3 + n| \langle n \rangle^{0+}} \|u_{n_3}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|v_n\|_{X^{0, \frac{1}{2}-}}$$

By applying Hölder’s inequality in n_2 and (2.7),

$$\lesssim \|u\|_{X_p^{\frac{1}{4}, \frac{1}{2}+}}^3 \|v\|_{X_p^{0, \frac{1}{2}-}},$$

provided that $p < \infty$.

• **Case 2:** $|\xi_i - \xi_j| \leq |\xi_i + \xi_j|$ for all i, j .

In this case, all ξ_j ’s for $j = 1, 2, 3$ have the same sign. Thus, we have $|\xi \pm \xi_j| \gtrsim |\xi_{\max}|$ for $j = 1, 2, 3$. Indeed, from $\xi_1 + \xi_2 + \xi_3 + \xi = 0$ and (3.11), we have

$$|\xi + \xi_j| = \left| \sum_{k \in \{1, 2, 3\} \setminus \{j\}} \xi_k \right| \sim |\xi_{\max}|.$$

and see that ξ has the opposite sign from ξ_j , $j = 1, 2, 3$, thus yielding $|\xi - \xi_j| \gtrsim |\xi_j| \sim |\xi_{\max}|$. Moreover, from (3.12), (3.11), and the fact that ξ_j ’s for $j = 1, 2, 3$ have the same sign, we have

$$\langle \sigma_{\max} \rangle \gtrsim |\xi_{\max}|^3. \tag{3.14}$$

We first consider the case $\sigma_j = \sigma_{\max}$ for some $j = 1, 2, 3$. Without loss of generality, we assume that $\sigma_2 = \sigma_{\max}$. By Hölder’s and Bernstein’s inequalities, (2.3), (3.14), and Lemma 2.2 with $|\xi \pm \xi_3| \gtrsim |\xi_{\max}|$, we have

$$\begin{aligned} & \sum_{N_{\max} \sim N_{\min} \sim N} N^{\frac{5}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_{N_1} u_{N_2} u_{N_3} v_N dx dt \right| \\ & \lesssim \sum_{N_{\max} \sim N_{\min} \sim N} N^{\frac{7}{4}} \|u_{N_1}\|_{L_t^\infty L_x^2} \|u_{N_2}\|_{L_{x,t}^2} \|u_{N_3} v_N\|_{L_{x,t}^2} \\ & \lesssim \sum_{N_{\max} \sim N_{\min} \sim N} N^{\frac{1}{4}-} \|u_{N_1}\|_{X^{0, \frac{1}{2}+}} \|u_{N_2}\|_{X^{0, \frac{1}{2}+}} \|u_{N_3} v_N\|_{L_{x,t}^2} \\ & \lesssim \sum_{N_{\max} \sim N_{\min} \sim N} N^{-\frac{3}{4}+} \left(\prod_{j=1}^3 \|u_{N_j}\|_{X^{0, \frac{1}{2}+}} \right) \|v_N\|_{X^{0, \frac{1}{2}-}}. \end{aligned}$$

Then, the rest follows as in (3.9).

In the following, we assume that $\sigma = \sigma_{\max}$. The proof for this case is more involved and thus we split it into several subcases.

Subcase 2.1: $\sigma = \sigma_{\max}$ and $|\xi_i - \xi_j| \lesssim 1$ for some $i \neq j$.

Without loss of generality, we may assume $|\xi_1 - \xi_2| \lesssim 1$. Suppose that $\xi_1 \in I_n$ and $\xi_3 \in I_m$. Then, we have $\xi_2 \in I_{n+j}$ and $\xi_3 \in I_{-m-2n-k}$ for $j, k = O(1)$. In the following, we only estimate the contribution from $j = k = 0$:

$$\sum_{\substack{m, n \in \mathbb{Z} \\ |n| \sim |m|}} |n|^{\frac{5}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_n u_n u_m v_{-m-2n} dx dt \right|. \tag{3.15}$$

We first consider the case $|\xi_1 - \xi_3| \lesssim 1$. In this case, we can further reduce (3.15) to the following diagonal case:

$$\sum_{n \in \mathbb{Z}} |n|^{\frac{5}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_n u_n u_n v_{-3n} dx dt \right|. \tag{3.16}$$

By Hölder’s inequality, Bernstein’s inequality (2.2) and (3.14), we have

$$\begin{aligned}
 (3.16) &\lesssim \sum_{n \in \mathbb{Z}} |n|^{\frac{5}{4}} \|u_n\|_{L_t^\infty L_x^2}^2 \|u_n\|_{L_{x,t}^2} \|v_{-3n}\|_{L_{x,t}^2} \\
 &\lesssim \sum_{n \in \mathbb{Z}} |n|^{-\frac{1}{4}+} \|u_n\|_{X^{0, \frac{1}{2}+}}^3 \|v_{3n}\|_{X^{0, \frac{1}{2}-}}.
 \end{aligned}$$

Then, the rest follows from Hölder’s inequality in n .

Next, we consider the case $|\xi_1 - \xi_3| \gg 1$. In this case, we have $|m+n| \geq |m-n| \gg$

1. By Hölder’s and Bernstein’s inequalities, (3.14), Corollary 1, we have

$$\begin{aligned}
 (3.15) &\lesssim \sum_{m,n \in \mathbb{Z}} |n|^{\frac{5}{4}} \|u_m u_n\|_{L_{x,t}^2} \|u_n\|_{L_t^\infty L_x^2} \|v_{-m-2n}\|_{L_{x,t}^2} \\
 &\lesssim \sum_{m,n \in \mathbb{Z}} \frac{|n|^{-\frac{1}{4}+}}{\sqrt{|m-n||m+n|}} \|u_n\|_{X^{0, \frac{1}{2}+}}^2 \|u_m\|_{X^{0, \frac{1}{2}+}} \|v_{-m-2n}\|_{X^{0, \frac{1}{2}-}} \\
 &\lesssim \|u\|_{X_{\infty}^{\frac{1}{4}, \frac{1}{2}+}}^2 \sum_{m,n \in \mathbb{Z}} \frac{1}{|n + (-m - 2n)|^{\frac{1}{2}} \langle n \rangle^{1-}} \|u_n\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|v_{-m-2n}\|_{X^{0, \frac{1}{2}-}} \\
 &\sim \|u\|_{X_{\infty}^{\frac{1}{4}, \frac{1}{2}+}}^2 \sum_{m,n \in \mathbb{Z}} \frac{1}{|n + (-m - 2n)| \langle n \rangle^{\frac{1}{2}-}} \|u_n\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|v_{-m-2n}\|_{X^{0, \frac{1}{2}-}}.
 \end{aligned}$$

Then, the rest follows from (2.7).

Subcase 2.2: $\sigma = \sigma_{\max}$ and $|\xi_i - \xi_j| \gg 1$ for all $i \neq j$.

Since all ξ_j ’s have the same sign, we have $|\xi_i + \xi_j| \sim |\xi_i| \sim |\xi_{\max}|$. Then, by Hölder’s and Bernstein’s inequalities, (3.14), and Corollary 1 with $|n_1 \pm n_2| \gg 1$, we have

$$\begin{aligned}
 &\sum_{n_1+n_2+n_3+n=O(1)} |n|^{\frac{5}{4}} \left| \int_{\mathbb{R} \times \mathbb{R}} u_{n_1} u_{n_2} u_{n_3} v_n dx dt \right| \\
 &\lesssim \sum_{n_1+n_2+n_3+n=O(1)} \langle n \rangle^{\frac{5}{4}} \|u_{n_1} u_{n_2}\|_{L_{x,t}^2} \|u_{n_3}\|_{L_t^\infty L_x^2} \|v_n\|_{L_{x,t}^2} \\
 &\lesssim \sum_{n_1+n_2+n_3+n=O(1)} \langle n \rangle^{-\frac{1}{4}+} \|u_{n_1} u_{n_2}\|_{L_{x,t}^2} \|u_{n_3}\|_{X^{0, \frac{1}{2}+}} \|v_n\|_{X^{0, \frac{1}{2}-}} \\
 &\lesssim \sum_{n_1+n_2+n_3+n=O(1)} \frac{\langle n \rangle^{-1+}}{\sqrt{|n_1 - n_2||n_1 + n_2|}} \left(\prod_{j=1}^3 \|u_{n_j}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \right) \|v_n\|_{X^{0, \frac{1}{2}-}}.
 \end{aligned} \tag{3.17}$$

By noting $|n_1 + n_2| \sim |n_3 + n| \sim |n| \sim |n_1|$ and applying Hölder’s inequality in n_1 and (2.7), we have

LHS of (3.17)

$$\begin{aligned}
 &\lesssim \sum_{n_1+n_2+n_3+n=O(1)} \frac{\langle n_1 \rangle^{-\frac{1}{2}+} \langle n \rangle^{0-}}{\sqrt{|n_1 - n_2||n_3 + n|}} \left(\prod_{j=1}^3 \|u_{n_j}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \right) \|v_n\|_{X_{2,2}^{0, \frac{1}{2}-}} \\
 &\lesssim \sup_{n_3, n} \left(\sum_{n_1} \frac{\langle n_1 \rangle^{-\frac{1}{2}+}}{\sqrt{\langle 2n_1 + n + n_3 \rangle}} \|u_{n_1}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|u_{-n_1-n_2-n}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \right) \\
 &\quad \times \left(\sum_{n_3, n} \frac{\langle n \rangle^{0-}}{|n_3 + n|} \|u_{n_3}\|_{X^{\frac{1}{4}, \frac{1}{2}+}} \|v_n\|_{X^{0, \frac{1}{2}-}} \right)
 \end{aligned}$$

$$\lesssim \|u\|_{X_p^{\frac{1}{4}, \frac{1}{2}+}}^3 \|v\|_{X_{p'}^{0, \frac{1}{2}-}},$$

provided that $p < \infty$.

This completes the proof of Proposition 3 and hence the proof of Theorem 1.1.

3.6. Persistence of regularity. We conclude this section by presenting the proof of global well-posedness (Theorem 1.2). When $\frac{1}{4} \leq s < 1 - \frac{1}{p}$, global well-posedness immediately follows from the local well-posedness in Theorem 1.1 together with the global-in-time a priori bound (1.6) in Proposition 1. In the following, we briefly discuss the situation for $s \geq 1 - \frac{1}{p}$. In this case, the proof is based on combining the global-in-time a priori bound (1.6) in Proposition 1 on the $M_{\frac{1}{4}}^{2,p}$ -norms of solutions and a persistence-of-regularity argument.

With the notations from the previous subsections, we have $|\xi| \lesssim |\xi_{\max}|$. Hence, by slightly modifying the proof of Proposition 3, we obtain

$$\| |u|^2 \partial_x u \|_{X_p^{s, -\frac{1}{2}+2\varepsilon}([0, T])} \leq C_\varepsilon \|u\|_{X_p^{\frac{1}{4}, \frac{1}{2}+\varepsilon}([0, T])}^2 \|u\|_{X_p^{s, \frac{1}{2}+\varepsilon}([0, T])} \tag{3.18}$$

for any $s \geq \frac{1}{4}$, any $T > 0$, and for small $\varepsilon > 0$.

Let $u_0 \in M_s^{2,p}(\mathbb{R})$ for some $s \geq \frac{1}{4}$ and $2 \leq p < \infty$. Since $u_0 \in M_{\frac{1}{4}}^{2,p}(\mathbb{R})$, there exists a unique global solution $u \in C(\mathbb{R}; M_{\frac{1}{4}}^{2,p}(\mathbb{R}))$ to (1.1) with $u|_{t=0} = u_0$. We need to check that u indeed lies in the correct space $C(\mathbb{R}; M_s^{2,p}(\mathbb{R}))$. In view of the global-in-time a priori bound (1.6), there exists small local existence time

$$\delta \sim (1 + \|u_0\|_{M_{\frac{1}{4}}^{2,p}})^{-\theta} > 0 \tag{3.19}$$

for some $\theta > 0$ such that a standard contraction argument in $X_p^{\frac{1}{4}, \frac{1}{2}+\varepsilon}(I)$ can be applied on any interval I of length δ . Moreover, with $I = [t_0, t_0 + \delta]$, we have

$$\|u\|_{X_p^{\frac{1}{4}, \frac{1}{2}+\varepsilon}(I)} \leq C_0 \|u(t_0)\|_{M_{\frac{1}{4}}^{2,p}} \tag{3.20}$$

for some absolute constant $C_0 > 0$. Then, from the Duhamel formula, Lemma 2.1 (with $b = \frac{1}{2} + \varepsilon$ and $b' = -\frac{1}{2} + 2\varepsilon$), (3.18), and (3.20), we obtain

$$\begin{aligned} \|u\|_{X_p^{s, \frac{1}{2}+\varepsilon}(I)} &\lesssim \|u(t_0)\|_{M_s^{2,p}} + \delta^\varepsilon \|u\|_{X_p^{\frac{1}{4}, \frac{1}{2}+\varepsilon}(I)}^2 \|u\|_{X_p^{s, \frac{1}{2}+\varepsilon}(I)} \\ &\lesssim \|u(t_0)\|_{M_s^{2,p}} + \delta^\varepsilon \|u(t_0)\|_{M_{\frac{1}{4}}^{2,p}}^2 \|u\|_{X_p^{s, \frac{1}{2}+\varepsilon}(I)}. \end{aligned} \tag{3.21}$$

In particular, from (2.4) and (3.21), we conclude that there exists an absolute constant $C_1 > 0$ such that

$$\sup_{t \in [t_0, t_0 + \delta]} \|u(t)\|_{M_s^{2,p}} \leq C_1 \|u(t_0)\|_{M_s^{2,p}} \tag{3.22}$$

for any $t_0 \in \mathbb{R}$. Then, by iterating the local argument with (3.19), we conclude from (3.22) that

$$\sup_{t \in [0, T]} \|u(t)\|_{M_s^{2,p}} \leq C^{(1 + \|u_0\|_{M_{\frac{1}{4}}^{2,p}})^\theta T} \|u_0\|_{M_s^{2,p}}$$

for any $T > 0$. This proves global well-posedness of (1.1) in $M_s^{2,p}(\mathbb{R})$ for $s \geq 1 - \frac{1}{p}$.

4. **On the failure of local uniform continuity below $H^{\frac{1}{4}}(\mathbb{R})$.** In this section, we present the proof of Proposition 2. In particular, by adapting the argument in [19] to the modulation space setting, we prove the following statement.

Lemma 4.1. *Suppose that (s, p) satisfies one of the following conditions: (i) $2 \leq p \leq \infty$ and $0 \leq s < \frac{1}{4}$ or (ii) $2 \leq p < \infty$ and $-\frac{1}{p} < s < 0$. There exist two sequences $\{u_{0,n}\}_{n \in \mathbb{N}}$ and $\{\tilde{u}_{0,n}\}_{n \in \mathbb{N}}$ in $\mathcal{S}(\mathbb{R})$ such that*

- (a) $u_{0,n}$ and $\tilde{u}_{0,n}$ are uniformly bounded in $M_s^{2,p}(\mathbb{R})$,
- (b) $\lim_{n \rightarrow \infty} \|u_{0,n} - \tilde{u}_{0,n}\|_{M_s^{2,p}} = 0$,
- (c) Let u_n and \tilde{u}_n be the solutions to the focusing mKdV (1.1) (with the + sign) with initial data $u_n|_{t=0} = u_{0,n}$ and $\tilde{u}_n|_{t=0} = \tilde{u}_{0,n}$, respectively. Then, there exists $c > 0$ such that

$$\liminf_{n \rightarrow \infty} \|u_n(T) - \tilde{u}_n(T)\|_{M_s^{2,p}} \geq c > 0$$

for any $T > 0$.

In [19], Kenig-Ponce-Vega proved Lemma 4.1 for $p = 2$ by using explicit soliton solutions with parameters (see (4.2) below). In the following, we use exactly the same explicit soliton solutions to show an analogous instability in the modulation space setting.

Let

$$Q(x) = \operatorname{sech}(x). \tag{4.1}$$

Then, Q solves the ODE: $-Q + Q'' + 2Q^3 = 0$ and hence

$$-Q' + Q''' + 6Q^2Q' = 0.$$

With $Q_\lambda(x) = \lambda Q(\lambda x)$, define $u_{N,\lambda}$ by

$$u_{N,\lambda}(x, t) = \frac{1}{\sqrt{6}} e^{it(N^3 - 3N\lambda^2) + iNx} Q_\lambda(x + 3N^2t - \lambda^2t) \tag{4.2}$$

for $N, \lambda > 0$. Then, it is easy to check that $u_{N,\lambda}$ is a solution to (1.1) with $u_{N,\lambda}|_{t=0} = \frac{1}{\sqrt{6}} e^{iNx} Q_\lambda$ for any $N, \lambda > 0$. Recalling that

$$\widehat{Q}_\lambda(\xi) = \widehat{Q}\left(\frac{\xi}{\lambda}\right) = \pi \operatorname{sech}\left(\frac{\pi\xi}{2\lambda}\right),$$

we have

$$\widehat{Q}_\lambda(\xi) \sim e^{-\frac{\pi|\xi|}{2\lambda}}. \tag{4.3}$$

In particular, when $\lambda \gg 1$, it follows from (4.2) that $\widehat{u}_{N,\lambda}(\xi, t)$ is highly concentrated around $|\xi| \sim N$. See (4.7) below.

In the following, we first present the argument for $0 \leq s < \frac{1}{4}$. We then discuss the case for $-\frac{1}{p} < s < 0$ in Subsection 4.3.

4.1. On the size of the soliton solutions. Fix $2 \leq p \leq \infty$ and $0 \leq s < \frac{1}{4}$. Given $N \geq 1$, we consider two solutions $u_{N_1,\lambda}$ and $u_{N_2,\lambda}$ of the form (4.2), where

$$\lambda = N^{-2s} \quad \text{and} \quad N_1, N_2 = N + O(1). \tag{4.4}$$

As we see below, we also impose that $|N_1 - N_2| \ll 1$. Furthermore, fix $\theta = \theta(s) > 0$ such that

$$4s - 1 + 2\theta < 0. \tag{4.5}$$

In the following, we estimate the $M_s^{2,p}$ -norms of $u_{N_j,\lambda}$, $j = 1, 2$. Noting that $|\widehat{u}_{N_j,\lambda}(\xi, t)| = |\widehat{u}_{N_j,\lambda}(\xi, 0)|$, the following computation holds uniformly in $t \in \mathbb{R}$. We separately consider the contributions from (i) $|\xi - N| \ll N^\theta$ and (ii) $|\xi - N| \gtrsim N^\theta$. Set

$$u_{N_j,\lambda}^{(1)} = \mathcal{F}_x^{-1}(\mathbf{1}_{|\xi-N| \ll N^\theta} \cdot \widehat{u}_{N_j,\lambda}) \quad \text{and} \quad u_{N_j,\lambda}^{(2)} = u_{N_j,\lambda} - u_{N_j,\lambda}^{(1)}. \tag{4.6}$$

We first consider (ii). Note that when $|\xi - N| \gtrsim N^\theta$ and $|\xi| \gtrsim N$, we have $|\xi - N| \gtrsim |\xi|^\theta$ for small $\theta > 0$. Then, by separately considering the contribution from $|\xi| \ll N$ and $|\xi| \gtrsim N$, it follows from (4.2), (4.3), (4.4), and (4.5) that

$$\begin{aligned} \|u_{N_j,\lambda}^{(2)}(t)\|_{M_s^{2,p}} &\sim \left(\sum_{|n-N| \gtrsim N^\theta} \langle n \rangle^{sp} e^{-\frac{p\pi}{2} N^{2s} |n-N_j|} \right)^{\frac{1}{p}} \\ &\lesssim \left(\sum_{|n| \ll N} \langle n \rangle^{sp} e^{-\frac{p\pi}{2} N^{\theta+2s}} \right)^{\frac{1}{p}} + \left(\sum_{|n| \gtrsim N} \langle n \rangle^{sp} e^{-\frac{p\pi}{2} N^{2s} |n|^\theta} \right)^{\frac{1}{p}} \tag{4.7} \\ &\lesssim e^{-cN^{\theta+2s}} \end{aligned}$$

since $\theta + 2s > 0$. On the other hand, by a change of variables with (4.4) and (4.3), we have

$$\begin{aligned} \|u_{N_j,\lambda}^{(1)}(t)\|_{M_s^{2,p}} &\leq \|u_{N_j,\lambda}^{(1)}(t)\|_{H^s} \\ &\lesssim N^s \left(\int_{|\xi-N| \ll N^\theta} |\widehat{Q}_\lambda(\xi - N_j)|^2 d\xi \right)^{\frac{1}{2}} \tag{4.8} \\ &= \left(\int_{|\xi| \ll N^{\theta+2s}} e^{-\pi|\xi|} d\xi \right)^{\frac{1}{2}} \\ &\sim 1. \end{aligned}$$

By considering the contribution from $|\xi - N| \lesssim 1$, we also see that

$$\|u_{N_j,\lambda}^{(1)}(t)\|_{M_s^{2,p}} \gtrsim 1. \tag{4.9}$$

Hence, from (4.6), (4.7), (4.8), and (4.9), we conclude that

$$\|u_{N_j,\lambda}(t)\|_{M_s^{2,p}} \sim 1 \tag{4.10}$$

for any $t \in \mathbb{R}$, independent of $N, N_1, N_2 \geq 1$.

4.2. On the difference of the soliton solutions. When $t = 0$, we have the following upper bound from [19, (3.5)]:

$$\begin{aligned} \|u_{N_1,\lambda}(0) - u_{N_2,\lambda}(0)\|_{M_s^{2,p}} &\leq \|u_{N_1,\lambda}(0) - u_{N_2,\lambda}(0)\|_{H^s} \\ &\lesssim N^{2s} |N_1 - N_2|. \end{aligned} \tag{4.11}$$

Fix $T > 0$. We establish a lower bound on the $M_s^{2,p}$ -norm of the difference of $u_{N_j,\lambda}(T)$. In view of (4.6) and (4.7), it suffices to consider $u_{N_1,\lambda}^{(1)}(T) - u_{N_2,\lambda}^{(1)}(T)$. As in [19], the main ingredient is separation of the physical supports of the soliton solutions $u_{N_j,\lambda}$, $j = 1, 2$. From (4.2) with (4.1), we see that $u_{N_j,\lambda}(T)$ is concentrated on an interval of length $\sim \lambda^{-1}$ centered at $3N_j^2 T - \lambda^2 T$. Note that these essential supports of $u_{N_j,\lambda}(T)$, $j = 1, 2$ are disjoint, provided that

$$N|N_1 - N_2|T \gg \lambda^{-1} = N^{2s}.$$

In our modulation space setting, however, we need to establish separation of the physical supports of the frequency localized soliton solutions $\Pi_n u_{N_j, \lambda}$, $j = 1, 2$. From (2.1), there exists $\eta \in \mathcal{S}(\mathbb{R})$ such that

$$|\Pi_n^2 u(x)| \leq (|\eta| * |u|)(x)$$

for any $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then, from (4.2) and (4.1), we have

$$\begin{aligned} & \left| \langle \Pi_n u_{N_1, \lambda}(T), \Pi_n u_{N_2, \lambda}(T) \rangle_{L_x^2} \right| = \left| \langle \Pi_n^2 u_{N_1, \lambda}(t), u_{N_2, \lambda}(T) \rangle_{L_x^2} \right| \\ & \lesssim \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\eta(y)| Q_\lambda(x - y + 3N_1^2 T - T\lambda^2) dy \right) Q_\lambda(x + 3N_2^2 T - T\lambda^2) dx \\ & = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\eta(y)| Q_\lambda(x - y) dy \right) Q_\lambda(x + 3(N_2^2 - N_1^2)T) dx \\ & = \iint_{\mathbb{R}} |\eta(\lambda^{-1}y)| Q(x - y) Q(x + 3\lambda(N_2^2 - N_1^2)T) dy dx \\ & \lesssim \iint_{\mathbb{R}} \frac{1}{\langle \lambda^{-1}y \rangle^K} e^{-|x-y|} e^{-|x+3\lambda(N_2^2 - N_1^2)T|} dy dx \\ & \lesssim \frac{1}{N|N_1 - N_2|T} \end{aligned} \tag{4.12}$$

uniformly in $n \in \mathbb{Z}$.

Given $N \gg 1$, choose $N_1, N_2 \sim N$ such that

$$|N_1 - N_2| \sim \frac{N^{2s-1+2\theta}}{T}, \tag{4.13}$$

where $\theta > 0$ is as in (4.5). Thus, from the triangle inequality, (4.12), Minkowski's inequality, and (4.9) we have

$$\begin{aligned} & \|u_{N_1, \lambda}^{(1)}(T) - u_{N_2, \lambda}^{(1)}(T)\|_{M_s^{2,p}}^2 \\ & \sim N^{2s} \left(\sum_{|n-N| \ll N^\theta} \|\Pi_n u_{N_1, \lambda}(T) - \Pi_n u_{N_2, \lambda}(T)\|_{L_x^2}^p \right)^{\frac{2}{p}} \\ & = N^{2s} \left(\sum_{|n-N| \ll N^\theta} \left(\|\Pi_n u_{N_1, \lambda}(T)\|_{L_x^2}^2 + \|\Pi_n u_{N_2, \lambda}(T)\|_{L_x^2}^2 \right. \right. \\ & \quad \left. \left. - 2 \operatorname{Re} \langle \Pi_n u_{N_1, \lambda}(t), \Pi_n u_{N_2, \lambda}(t) \rangle_{L_x^2} \right)^{\frac{p}{2}} \right)^{\frac{2}{p}} \\ & \gtrsim \|u_{N_1, \lambda}^{(1)}(T)\|_{M_s^{2,p}}^2 - N^{\frac{2}{p}\theta+2s} N^{-2\theta-2s} \\ & \gtrsim 1 - N^{-2\theta(1-\frac{1}{p})} \sim 1 \end{aligned} \tag{4.14}$$

for any sufficiently large $N \gg 1$. Hence, from (4.7), (4.10), and (4.14), we conclude that

$$\|u_{N_1, \lambda}(T) - u_{N_2, \lambda}(T)\|_{M_s^{2,p}} \sim 1. \tag{4.15}$$

On the other hand, from (4.11) and (4.13) with (4.5), we have

$$\begin{aligned} & \|u_{N_1, \lambda}(0) - u_{N_2, \lambda}(0)\|_{M_s^{2,p}} \sim T^{-1} N^{4s-1+2\theta} \\ & \longrightarrow 0 \end{aligned} \tag{4.16}$$

by taking $N \rightarrow \infty$. Finally, given $n \in \mathbb{N}$, let $N = 2^n$ and set $u_n = u_{N_1(n),\lambda(n)}$ and $\tilde{u}_n = u_{N_2(n),\lambda(n)}$, Lemma 4.1 and hence Proposition 2 follow from (4.10), (4.15), and (4.16), provided $2 \leq p \leq \infty$ and $0 \leq s < \frac{1}{4}$.

4.3. Failure of uniform continuity in negative regularities. In this subsection, we briefly consider the case $s < 0$. With $\lambda = N^{-2s}$ as in (4.4), the estimate (4.10) is no longer true and hence (4.15) fails in this case.

Fix $2 \leq p < \infty$ and $-\frac{1}{p} < s < 0$. In the following, we use a new choice for the parameter λ :

$$\lambda = N^{-ps} \tag{4.17}$$

and let $u_{N_j,\lambda}$, $j = 1, 2$, be the solutions of the form (4.2) with this choice of λ (and $N_1, N_2 \sim N$). We also choose new $\theta = \theta(s, p) > 0$ such that

$$-ps < \theta < 1. \tag{4.18}$$

This imposes the lower bound: $s > -\frac{1}{p}$. Note that $|n - N| \ll N^\theta$ implies $|n| \sim N$ since $\theta < 1$.

By repeating the computation in (4.7), we have

$$\|u_{N_j,\lambda}^{(2)}(t)\|_{M_s^{2,p}} \lesssim e^{-cN^{\theta+ps}} \tag{4.19}$$

thanks to (4.18). On the other hand, from (4.2) and $Q_\lambda(x) = \lambda Q(\lambda x)$, we have

$$\begin{aligned} \|u_{N_j,\lambda}^{(1)}(t)\|_{M_s^{2,p}} &\sim N^s \left(\sum_{|n-N| \ll N^\theta} \left(\int_n^{n+1} |\widehat{Q}_\lambda(\xi - N_j)|^2 d\xi \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &= N^s \lambda^{\frac{1}{2}} \left(\sum_{|n| \ll N^\theta} \left(\int_{\frac{n}{\lambda}}^{\frac{n+1}{\lambda}} |\widehat{Q}(\xi)|^2 d\xi \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &\sim N^s \lambda^{\frac{1}{p}} \left(\sum_{|n| \ll N^\theta} |\widehat{Q}(\frac{n}{\lambda})|^p \lambda^{-1} \right)^{\frac{1}{p}} \end{aligned}$$

By the Riemann sum approximation with (4.17) and (4.18),

$$\sim \|Q\|_{\mathcal{FL}^{0,p}} \sim 1, \tag{4.20}$$

uniformly in large $N \gg 1$. Hence, from (4.6), (4.19), and (4.20), we conclude that

$$\|u_{N_j,\lambda}(t)\|_{M_s^{2,p}} \sim \|u_{N_j,\lambda}^{(1)}(t)\|_{M_s^{2,p}} \sim 1 \tag{4.21}$$

for any $t \in \mathbb{R}$, independent of $N, N_1, N_2 \geq 1$.

Next, we estimate the difference of the soliton solutions as in Subsection 4.2. A direct computation as in [19, (2.10)] shows that

$$\|u_{N_1,\lambda}(0) - u_{N_2,\lambda}(0)\|_{M_s^{2,p}} \lesssim N^s \lambda^{-\frac{1}{2}} |N_1 - N_2|. \tag{4.22}$$

In estimating the difference at time $T > 0$, we once again use the almost orthogonality of the two soliton solutions, provided that

$$N|N_1 - N_2|T \gg \lambda^{-1} = N^{ps}.$$

Given $N \gg 1$, choose $N_1, N_2 \sim N$ such that

$$|N_1 - N_2| \sim \frac{N^{ps-1+\frac{3}{2}\theta}}{T}. \tag{4.23}$$

Then, by proceeding as in (4.14) with (4.21) and (4.12) and choosing $\theta > -ps$ sufficiently close to $-ps$, we obtain

$$\begin{aligned} \|u_{N_1,\lambda}^{(1)}(T) - u_{N_2,\lambda}^{(1)}(T)\|_{M_s^{2,p}}^2 &\gtrsim \|u_{N_1,\lambda}^{(1)}(T)\|_{M_s^{2,p}}^2 - N^{\frac{2}{p}\theta+2s}N^{-ps-\frac{3}{2}\theta} \\ &\gtrsim 1 \end{aligned} \quad (4.24)$$

for all sufficiently large $N \gg 1$. Hence, from (4.19), (4.21), and (4.24), we conclude that

$$\|u_{N_1,\lambda}(T) - u_{N_2,\lambda}(T)\|_{M_s^{2,p}}^2 \sim 1.$$

On the other hand, from (4.22) with (4.17) and (4.23), we have

$$\begin{aligned} \|u_{N_1,\lambda}(0) - u_{N_2,\lambda}(0)\|_{M_s^{2,p}}^2 &\sim T^{-1}N^{s+\frac{3}{2}(\theta+ps)-1} \\ &\rightarrow 0 \end{aligned}$$

by taking $N \rightarrow \infty$ since we chose $\theta > -ps$ sufficiently close to $-ps$. This completes the proof of Lemma 4.1 and hence Proposition 2 when $2 \leq p < \infty$ and $-\frac{1}{p} < s < 0$.

Remark 5. Note that our parameter choices (4.4) for $s \geq 0$ and (4.17) for $s < 0$ agree with those in [19] and [9], respectively. In the following, we provide an intuitive explanation of our choices. Given $f \in \mathcal{S}(\mathbb{R})$, let $f_{N,\lambda}(x) = \lambda e^{iNx}f(\lambda x)$. When $s > 0$, $\lambda = N^{-2s}$ in (4.4) tends to 0 as $N \rightarrow \infty$. This implies that $\widehat{f}_{N,\lambda}$ is highly localized around $|\xi - N| \lesssim \lambda$. Namely, $\widehat{f}_{N,\lambda}$ is essentially supported in one interval $[N - \frac{1}{2}, N + \frac{1}{2}]$, in which case the $M_s^{2,p}$ -norm of $\widehat{f}_{N,\lambda}$ reduces to its H^s -norm (which in turn can be reduced to the L^2 -norm of f). Therefore, the choice $\lambda = N^{-2s}$ from the H^s -theory in [19] is appropriate in this case.

On the other hand, when $s < 0$, $\lambda = N^{-ps}$ tends to ∞ as $N \rightarrow \infty$. Namely, the essential support of $\widehat{f}_{N,\lambda}$ spreads out as $N \rightarrow \infty$. Then, arguing as in (4.20), we see that the $M_s^{2,p}$ -norm of $\widehat{f}_{N,\lambda}$ essentially reduces to the $\mathcal{FL}^{0,p}$ -norm of f , which shows that the choice $\lambda = N^{-ps}$ from the $\mathcal{FL}^{s,p}$ -theory in [9] is appropriate in this case.

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