On the P~!-theorem

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On the $\tilde{P}!$-Theorem

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Abstract. The purpose of this paper is to show that the exceptional possibilities in the main theorem of [3] do not occur. This then strengthens that theorem.

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In [3] the authors proved the $\tilde{P}!$-Theorem (the P-tilde uniqueness Theorem). The aim of the present contribution is to strengthen this theorem by removing what appeared to be an exception to the central statement, that is, that $\tilde{P}$ is unique. We first establish some terminology so that we can explain the result.

Let G be a finite group and p be a fixed prime. The normalizer of a nontrivial p-subgroup of G is called a *p-local* subgroup of G. The finite group X is of *characteristic* p if

$$C_X(O_p(X)) \le O_p(X)$$

and G is of *local characteristic* p if every p-local subgroup of G is of characteristic p. The group G is of *parabolic characteristic* p if every p-local subgroup of p'-index in G is of characteristic p. We denote the set of subgroups $L \le G$ containing a given subgroup X and satisfying $C_G(O_p(L)) \le O_p(L)$ by $\mathcal{L}_G(X)$ and the set of maximal p-local subgroups containing X by $\mathcal{M}_G(X)$. For $S \in \mathrm{Syl}_p(G)$, the set of subgroups $P \in \mathcal{L}_G(S)$ such that $O_p(P) \ne S$ and S is contained in a unique maximal subgroup of P is written as $\mathcal{P}_G(S)$. Observe that the members of $\mathcal{P}_G(S)$ have the property that $P = O^{p'}(P)$.

For any $L \in \mathcal{L}_G(1)$, Y_L is the largest elementary abelian normal p-subgroup of L satisfying

$$O_p(L/C_L(Y_L)) = 1.$$

Such a subgroup always exists. In the arguments in this paper we have $Y_L = \Omega_1(Z(O_p(L)))$ in all cases, but in general it can be the case that $Y_L < \Omega_1(Z(O_p(L)))$.

Fix $S \in \mathrm{Syl}_p(G)$, $\tilde{C} \in \mathcal{M}_G(N_G(\Omega_1(Z(S))))$ and put

$$Q = O_p(\tilde{C}).$$

Then $\tilde{C} = N_G(Q)$. For $X \in \mathcal{L}_G(Q)$, we set

$$X^\circ = \langle Q^g \mid g \in G \text{ with } Q^g \le X \rangle.$$

The group G satisfies *Q-uniqueness* if and only if $C_G(x) \le \tilde{C}$ for every $1 \ne x \in C_G(Q)$. A consequence of Q-uniqueness when $C_G(Q) \le Q$ is that G is of parabolic characteristic p. Since the appearance of [3], subgroups Q which enjoy the Q-uniqueness property and have $C_G(Q) \le Q$ have more commonly been called *large* subgroups of G. The work in [4] starts the study of groups with a large subgroup and together with [5] the $\tilde{P}$!-Theorem controls some of the p-local structure of groups with a large subgroup which have local characteristic p.

We say that G is a $\mathcal{K}_p$-group if the simple sections of all the p-local subgroups of G are known simple groups.

In [3], the hypothesis of the $\tilde{P}$!-Theorem is:

- G is a $\mathcal{K}_p$-group of local characteristic p;
- G satisfies Q-uniqueness;
- there exists $P \in \mathcal{P}_G(S)$ such that $P \not\le \tilde{C}$; and
- $Y_M \le Q$ for every $M \in \mathcal{M}_G(P)$.

The $\tilde{P}$!-Theorem asserts that there exists at most one $\tilde{P} \in \mathcal{P}_G(S)$ such that $\tilde{P} \not\le N_G(P^\circ)$ and $\langle P, \tilde{P} \rangle \in \mathcal{L}_G(P)$, or some very special and precisely described situation holds. The purpose of this paper is to further investigate this special configuration and to prove

Main Theorem. *Under the assumptions of the $\tilde{P}$!-Theorem, there exists at most one $\tilde{P} \in \mathcal{P}_G(S)$ such that $\tilde{P} \not\le N_G(P^\circ)$ and $\langle P, \tilde{P} \rangle \in \mathcal{L}_G(P)$.*

Notice that the hypothesis of the $\tilde{P}$!-Theorem doesn't say anything about potential members $X \in \mathcal{P}_G(S)$ with the property that $X \not\le N_G(P^\circ)$ and $O_p(\langle P, X \rangle) = 1$. Such configurations are designated as the *rank 2 case.*

The Main Theorem will follow from a proposition, which we state in a moment. In fact, we will prove our proposition under a broader hypothesis than that of the $\tilde{P}$!-Theorem as we anticipate that as the theory develops an analogue of the $\tilde{P}$!-Theorem will be proved for groups with a large p-subgroup (Q-uniqueness and $C_G(Q) \le Q$) and that the local characteristic p requirement can be dropped.

Hypothesis 1. We have $p = 3$ or 5 and G is a $\mathcal{K}_p$-group which satisfies

(i) Q-uniqueness and $C_G(Q) \le Q$;
(ii) there is $P \in \mathcal{P}_G(S)$ such that $P \not\le \tilde{C}$; and
(iii) there exist $P_1, P_2 \in \mathcal{P}_G(S)$ such that, for $i = 1, 2$, $P_i \not\le N_G(P^\circ)$, $M_i = \langle P, P_i \rangle \in \mathcal{L}_G(P)$ and $O_p(\langle M_1, M_2 \rangle) = 1$. Moreover, for $i = 1, 2$
  (a) $M_i/O_p(M_i) \cong \mathrm{SL}_3(p)$ and $O_p(M_i)/Z(O_p(M_i))$ and $Z(O_p(M_i))$ are natural $\mathrm{SL}_3(p)$-modules for $M_i/O_p(M_i)$ which are dual to each other;
  (b) $Z(O_p(M_i)) \le Q$.

A consequence of Hypothesis 1 (i) is that G is of parabolic characteristic p and not necessarily of local characteristic p. Further in Hypothesis 1 (iii)(b) we only assume that $Y_{M_i} \leq Q$ for $i = 1, 2$, while in the $\tilde{P}$!-Theorem it is assumed that $Y_M \leq Q$ for every $M \in \mathcal{M}_G(P)$.

We will prove

Proposition. *Assume Hypothesis* 1*. Then Q is extraspecial of order p^7 and one of the following holds*

(i) $p = 3$ *and* $F^*(G) \cong \mathrm{M}(22)$ *or* ${}^2\mathrm{E}_6(2)$.

(ii) $p = 5$ *and* $N_G(Q)/Q \cong 4{\cdot}\mathrm{J}_2.2$.

The situation of Proposition(ii) has been treated in [7]. In this case G is shown to be isomorphic to F_1. However there are two problems with the citation which leads us not to use it. The first one is not really serious, the paper is written under the assumption that G is a local $\mathcal{K}$-group whereas here we have the weaker requirement that $\mathcal{K}_5$-group. The second one is more problematic. The paper [7] depends in an essential way on an as yet unpublished paper (in preparation) due to C. Wiedorn and Chr. Parker. Hence for this work we decided not to include the statement $G \cong \mathrm{F}_1$ in our proposition.

Our notation is standard and follows that in familiar texts.

Proof of the Proposition

For the remainder of this article we work under Hypothesis 1.

Lemma 1. *The subgroup Q is weakly closed in S with respect to G and G is of parabolic characteristic p.*

Proof. This follows from the Q-uniqueness property. See [5, (1.6)] and [4, (1.55)(c)]. □

We write, for $i = 1, 2$, $Y_{M_i} = Z(O_p(M_i))$. The next lemma investigates $O_p(M_i)$ and gathers some almost immediate consequences of Hypothesis 1 (iii).

Lemma 2. *For $i = 1, 2$, the following hold*

(i) $Y_{M_i} \leq Q$;

(ii) *if a subgroup of M_i normalizes a subgroup of order p (p^2) in Y_{M_i}, then it normalizes a subgroup of order p^2 (p) in $O_p(M_i)/Y_{M_i}$;*

(iii) $|Z(S)| = p$ *and $Z(S)$ is normalized by P_1 and P_2;*

(iv) $P_i/O_p(P_i) \cong P/O_p(P) \cong \mathrm{SL}_2(p)$;

(v) *$O_p(P_i)/O_p(M_i)$ is a natural $P_i/O_p(P_i)$-module and $O_p(P)/O_p(M_i)$ is a natural $P/O_p(P)$-module;*

(vi) *the action of P_i on Y_{M_i} is uniserial with irreducible factors of dimension* 1 *and* 2 *(with socle of dimension* 1*).*

Proof. Part(i) reiterates Hypothesis 1 (iii)(b).

By Hypothesis 1 (iii)(a), the $M_i/O_p(M_i)$-modules $O_p(M_i)/Y_{M_i}$ and Y_{M_i} are dual to each other. This immediately yields (ii).

As M_i has characteristic p, $Z(S) \le Y_{M_i}$ for $i = 1, 2$ and so Hypothesis 1 (iii)(a) implies $|Z(S)| = p$ and, as P does not normalise $Z(S)$ by Q-uniqueness, P_1 and P_2 do. This is (iii).

By Hypothesis 1 (iii), $P_i \in \mathcal{P}_{M_i}(S)$ and $M_i/O_p(M_i) \cong \mathrm{SL}_3(p)$. Since the maximal over-groups of $S/O_p(M_i)$ in the group $M_i/O_p(M_i)$ are parabolic subgroups of $M_i/O_p(M_i)$ and $P_i = O^{p'}(P_i)$, it follows that $P_i/O_p(M_i) \cong \mathrm{SL}_2(p)$. Similarly, $P/O_2(P) \cong \mathrm{SL}_2(p)$. This proves (iv).

Part (v) follows from the structure of the parabolic subgroups of the groups $M_i/O_p(M_i) \cong \mathrm{SL}_3(p)$.

Part (vi) is a consequences of the fact that Y_{M_i} is a natural $M_i/O_p(M_i)$-module combined with parts (iii) and (iv). $\square$

We collect together some further properties of P_1 and P_2.

Lemma 3. *For $i = 1, 2$, the following hold*

(i) *$Q \le O_p(P_i)$ and $P_i \le N_G(Q)$;*
(ii) *$O_p(P_i) = QO_p(M_i)$; and*
(iii) *$O_p(M_i) \not\le QO_p(M_{3-i})$ and $O_p(P_1) \ne O_p(P_2)$.*

Proof. Assume that $i \in \{1, 2\}$. Part (i) is a combination of Lemma 2 (iii) and Q-uniqueness.

Because of Lemma 1 and Hypothesis 1 (iii), Q is not contained in $O_p(M_i)$ and so by Lemma 2 (v) $QO_p(M_i)/O_p(M_i)$ is the natural $P_i/O_p(P_i)$-module and $O_p(P_i) = QO_p(M_i)$. This is (ii).

By symmetry it is enough to prove (iv) for $i = 1$. Suppose that $O_p(M_1) \le QO_p(M_2)$. Then by (ii) $O_p(M_1) \le O_p(P_2)$. Thus, again by (ii), $O_p(P_1) = QO_p(M_1) \le O_p(P_2)$ and so $O_p(P_1) = O_p(P_2)$ as $|O_p(P_1)| = |O_p(P_2)|$. Hence to prove (iii) it suffices to show that $O_p(P_1) \ne O_p(P_2)$.

Assume $O_p(P_1) = O_p(P_2)$. We have that $O_p(M_1)O_p(M_2)$ is normalized by P. By Hypothesis 1 (iii) $O_p(M_1) \ne O_p(M_2)$ and so by Lemma 2 (v) $O_p(P) = O_p(M_1)O_p(M_2)$. As $O_p(P_1) = O_p(P_2)$ we see by (iii) that $O_p(P) \le O_p(P_1)$. Since $|O_p(P)| = |O_p(P_1)|$, this yields $O_p(P) = O_p(P_1) = O_p(P_2)$, contrary to Hypothesis 1. This proves (iv). $\square$

Lemma 4. *For $i = 1, 2$, we have*

(i) *for $v \in O_p(M_i) \setminus Y_{M_i}$, $|[\langle v\rangle, O_p(M_i)]| = p^2$; and*
(ii) *if W is a maximal subgroup of $O_p(M_i)$, then $Y_{M_i} = [W, O_p(M_i)]$.*

Proof. (i) For $v \in O_p(M_i) \setminus Y_{M_i}$, $[\langle v\rangle, O_p(M_i)] = |[\langle v\rangle Y_{M_i}, O_p(M_i)]| = p^2$ as the $\mathrm{SL}_3(p)$-modules $O_p(M_i)/Y_{M_i}$ and Y_{M_i} are dual to each other by Hypothesis 1 (iii)(a).

(ii) Let W be a maximal subgroup of $O_p(M_i)$. As $[WY_{M_i}, O_p(M_i)] = [W, O_p(M_i)]$, we may as well assume that $Y_{M_i} \le W$ for otherwise the result is true. Then Hypothesis 1 (iii)(a) implies $[W, O_p(M_i)]$ is normalized by a parabolic subgroup of M_i which normalizes a subgroup of Y_{M_i} of order p. Since $|[W, O_p(M_i)]| \ge p^2$ by (i), we must have $[W, O_p(M_i)] = Y_{M_i}$. $\square$

Set $C = C_G(Z(S))$, $\overline{C} = C/Q$ and

$$H = \langle P_1, P_2 \rangle \leq \tilde{C}.$$

By Lemma 2 (iv), $P_1 = O^{p'}(P_1)$ and $P_2 = O^{p'}(P_2)$ and, by Lemma 2 (iii), P_i centralize $Z(S)$. Thus $H = O^{p'}(H)$ and $H \leq C$.

Lemma 5. *The following properties hold:*

(i) *Q is extraspecial of order p^7 and $Z(Q) = Z(S)$; and*
(ii) *H acts irreducibly on $Q/Z(Q)$.*

Proof. By Lemma 2 (i), for $i = 1, 2$, we know that $Y_{M_i} \leq Q$.

We first show that for $i = 1, 2$,

$$|QO_p(M_i)/O_p(M_i)| = p^2, |O_p(M_i) \cap Q| = p^5 \text{ and } |Q| = p^7. \tag{5.1}$$

That $|QO_p(M_i)/O_p(M_i)| = p^2$ follows directly from Lemma 3 (ii). Using $O_p(M_i)/Y_{M_i}$ and Y_{M_i} are dual to each other as $\mathrm{SL}_3(p)$-modules and P_i normalizes $Z(S)$, we now have $|[Q, O_p(M_i)]Y_{M_i}/Y_{M_i}| = p^2$ and, since $Y_{M_i} \leq Q$, we conclude that $|Q \cap O_p(M_i)| = p^5$ because Lemma 3 (iii). This proves (5.1).

We now show that H acts irreducibly on $Q/Z(Q)$ and Q is extraspecial.

Assume that $V < Q$ is normalized by H. If $V \not\leq O_p(M_1)$, then we obtain $VO_p(M_1) = QO_p(M_1)$ and $[V, O_p(M_1)]Y_{M_1}/Y_{M_1}$ has order p^2. Lemma 4 (ii) implies that

$$Y_{M_1} = [[V, O_p(M_1)]Y_{M_1}, O_p(M_1)] = [V, O_p(M_1), O_p(M_1)] \leq V.$$

We conclude that $V = Q$ from (5.1), a contradiction. Thus $V \leq O_p(M_1)$ and similarly $V \leq O_p(M_2)$. Hence, using Hypothesis 1 (iii)(a)

$$V \leq O_p(M_1) \cap O_p(M_2) = Y_{M_1}Y_{M_2}$$

which has order p^4. If $V \not\leq Y_{M_i}$ for some $i = 1, 2$, then $Y_{M_1}Y_{M_2} = VY_{M_i}$ is normalized by P_i and P, a contradiction. Thus $V \leq Y_{M_1} \cap Y_{M_2}$. As $Y_{M_1} \cap Y_{M_2}$ is normalized by P, we deduce that $V \leq Z(S)$ and (ii) is proved. Therefore, $Z(S) = [Y_P, Q] \leq Q'$ and Q is non-abelian. It follows that $Q' = \Phi(Q) = Z(Q)$ and H acts irreducibly on $Q/Z(Q)$. Hence (i) holds. $\square$

We collect a few facts which follow from Lemma 5 which will assist with the identification of H/Q.

Lemma 6. (i) *$\overline{C}$ embeds into $\mathrm{Sp}_6(p)$.*

(ii) *A Sylow p-subgroup $\overline{S}$ of $\overline{C}$ is elementary abelian of order p^2.*
(iii) *For $i = 1, 2$, $O_p(M_i)Q/Q$ has order p and does not act quadratically on $Q/Z(Q)$.*
(iv) *C contains H, $\overline{P}_i \cong \mathbb{Z}_p \times \mathrm{SL}_2(p)$, $i = 1, 2$, where the subgroups isomorphic to $\mathrm{SL}_2(p)$ induce on each of the three P_i-chief factors in $Q/Z(Q)$ a natural $\mathrm{SL}_2(p)$-module. In particular, the involution in $\overline{P}_i$ inverts $Q/Z(Q)$ and so is in $Z(\mathrm{Out}(Q))$.*

Proof. (i) follows directly from Lemma 5 (i) and [8]. As $|S| = p^9$, (ii) is also obvious.

As $|[Q, O_p(M_i)]Y_{M_i}/Y_{M_i}| = p^2$, Lemma 4 (ii) implies

$$Y_{M_1} = [Q, O_p(M_1), O_p(M_1)].$$

Hence $|O_p(M_1)Q/Q| = p$ and acts cubically on $Q/Z(Q)$.

We know $\overline{P_i} \cong \mathbb{Z}_p \times \mathrm{SL}_2(p)$. Further $\overline{P_i}$ induces a natural module on $Q/(Q \cap O_p(M_i))$, on $(Q \cap O_p(M_i))/Y_{M_i}$ and on $Y_{M_i}/Z(Q)$ as well. This yields (iv). $\square$

Recall that

$$|\mathrm{Sp}_6(3)| = 2^{10} \cdot 3^9 \cdot 5 \cdot 7 \cdot 13 \text{ and } |\mathrm{Sp}_6(5)| = 2^{10} \cdot 3^4 \cdot 5^9 \cdot 7 \cdot 13 \cdot 31.$$

As $\mathrm{Sp}_2(p) \wr \mathrm{Sym}(3)$ is a subgroup of $\mathrm{Sp}_6(p)$ and contains a Sylow 2-subgroup of $\mathrm{Sp}_6(p)$, we see that a Sylow 2-subgroup of $\mathrm{Sp}_6(p)$ is isomorphic to $\mathrm{Q}_8 \times (\mathrm{Q}_8 \wr \mathbb{Z}_2)$ (recall $p \in \{3, 5\}$).

In what follows we consider $\overline{C}$ as a subgroup of $\mathrm{Sp}_6(p) \cong O^2(\mathrm{Out}(Q))$.

Lemma 7. *If $E(\overline{C}) \neq 1$, then $p = 5$, $E(\overline{C}) \cong 2{\cdot}\mathrm{J}_2$ and $E(\overline{C})$ acts irreducibly on $Q/Z(Q)$.*

Proof. Suppose that L is a component of $\overline{C}$. We first demonstrate

$$\overline{H} \text{ normalizes } L. \tag{7.1}$$

Otherwise, as $O^{p'}(H) = H$, $L^{\overline{H}}$ contains at least three components of $\overline{C}$. In particular, by Burnside's Theorem there are at least two odd primes, which divide the order of $\overline{C}$ by a third power. By Lemma 6 (ii) neither of them is equal to p. This contradicts the order of $\mathrm{Sp}_6(p)$. This proves (7.1).

$$L \text{ acts irreducibly on } Q/Z(Q). \tag{7.2}$$

Suppose false. Since L is normal in $L\overline{H}$ and, by Lemma 5 (ii), $\overline{H}$ acts irreducibly on $Q/Z(Q)$, $C_{Q/Z(Q)}(L) = 1$. Hence, by Clifford's Theorem, as an L-module, $Q/Z(Q)$ is either a direct sum of two 3-dimensional irreducible submodules or of three 2-dimensional irreducible submodules. Suppose the first possibility holds. If $p = 3$, then L is isomorphic to a subgroup of $\mathrm{SL}_3(3)$ and, as $\mathrm{SL}_3(3)$ is a minimal simple group, we obtain $L \cong \mathrm{SL}_3(3)$. This contradicts Lemma 6 (ii). Hence $p = 5$ and we have $L \not\cong \mathrm{SL}_3(5)$ again by Lemma 6 (ii). From the subgroup structure of $\mathrm{SL}_3(5)$ and the irreducibility of L as a subgroup of $\mathrm{SL}_3(5)$, we have that $L \cong \Omega_3(5) \cong \mathrm{PSL}_2(5)$. Since, by Lemma 6 (iv), $\overline{P}_i \cong \mathbb{Z}_5 \times \mathrm{SL}_2(5)$ for $i = 1, 2$, we deduce from Lemma 6 (ii) and as $\overline{O^5(P_i)} \not\leq L$ that $\overline{P_i} \cap L = \overline{S} \cap L = O_5(\overline{P_i})$. Hence $O_5(\overline{P_1}) = O_5(\overline{P_2})$ contrary to Lemma 3 (iii).

Hence, as an L-module, $Q/Z(Q)$ is isomorphic to a direct sum of three natural $\mathrm{SL}_2(5)$-modules and, in particular, $\overline{S} \cap L$ acts quadratically on $Q/Z(Q)$. Since $P_1 \neq P_2$ and $P_i = O^5(P_i)S$ for $i = 1, 2$, $O^5(P_1) \neq O^5(P_2)$. Therefore we may assume that $L \neq \overline{O^5(P_1)}$ and that $O^5(P_1)$ induces inner automorphisms on L by conjugation. Furthermore, as $|\overline{S}| = 5^2$ by Lemma 6 (ii), $\overline{S} \leq L\overline{O^5(P_1)}$. As $\langle (\overline{S} \cap L)^{\overline{P}_1} \rangle$ is normalized by $\overline{P}_1$ and $\overline{O^5(P_1)} \not\leq L$, we see $\overline{S} \cap L = \overline{O_5(P_1)}$. Hence $\overline{O_5(P_1)} = \overline{O_5(M_1)}$ acts quadratically on $Q/Z(Q)$, contrary to Lemma 6. We conclude that L acts irreducibly on $Q/Z(Q)$ and (7.2) holds.

By (7.2) and Schur's Lemma, $\mathrm{End}_L(Q/Z(Q))$ is a division ring and so, as $|\mathrm{End}_L(Q/Z(Q))|$ is finite, $\mathrm{End}_L(Q/Z(Q))$ is a field by Wedderburn's little theorem. In particular, $C_{\overline{C}}(L)$ is contained in the subfield F which is generated by $C_{\overline{C}}(L)$ over the prime field $\mathrm{GF}(p)$ and $C_{\overline{C}}(L)$ is a cyclic p'-group and therefore $E(\overline{C}) = L$. Furthermore, we have

$$m_p(\mathrm{Aut}_{\overline{C}}(L)) = 2. \tag{7.3}$$

In particular by Lemma 6 (iv)

$$\text{If } m_p(L) = 2, \text{ then } \overline{H} \leq L, Z(L) \neq 1 \text{ and contains } Z(\mathrm{Sp}_6(p)). \tag{7.4}$$

Suppose that $L/Z(L)$ is a sporadic group. Then, by [2, Lemma 5.1], $L \cong 2{\cdot}\mathrm{J}_2$ and by considering the order of $\mathrm{Sp}_6(3)$, we obtain $p = 5$. Thus $E(\overline{C}) \cong 2{\cdot}\mathrm{J}_2$ in this case and this is the recorded outcome.

Assume next that $L/Z(L) \cong \mathrm{Alt}(n)$ with $n \geq 7$. Then, by [2, Lemma 4.1], $n = 7$ and so $p = 3$. As the action is defined over $\mathrm{GF}(3)$, [2, Lemma 4.2] implies that $L \cong \mathrm{Alt}(7)$, a contradiction to (7.3) and (7.4).

Assume that $L/Z(L)$ is of Lie type in characteristic not p. Then we may apply [2, Lemma 3.1]. This yields $L/Z(L) \cong \mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(13)$, $\mathrm{PSL}_2(5)$ $(p = 3)$, $\mathrm{PSL}_2(9)$ $(p = 5)$ or $\mathrm{PSL}_3(4)$. As $m_p(\mathrm{Aut}_{\overline{C}}(L)) = 2$ by (7.3), we have $L/Z(L) \cong \mathrm{PSL}_3(4)$ and $p = 3$. By (7.4), $L \cong 2{\cdot}\mathrm{PSL}_3(4)$. Now $P_1 \leq L$ and, as centralizers of 3-elements in $L/Z(L)$ are 3-groups, this contradicts Lemma 6 (iv) because $\overline{P_1} \leq L$.

Suppose $\overline{P}_1 \leq L$. Then Lemma 6 (iv) implies $L/Z(L)$ cannot be of Lie type in characteristic p as in such groups p-local subgroups are of characteristic p. Thus, if $L/Z(L)$ is of Lie type in characteristic p, then $\overline{P}_1 \not\leq L$ and so $m_p(L) = 1$. This shows $p = 5$ and $L/Z(L) \cong \mathrm{PSL}_2(5)$. This contradicts (7.3) and proves the lemma. $\square$

Lemma 8. *If $E(\overline{C}) = 1$, then $p = 3$.*

Proof. Suppose $p = 5$. Then just by considering the order of $\mathrm{Sp}_6(5)$ and noting that $|O_3(\overline{C})| \leq 3^3$, we see that $\overline{H}$ must centralize $O_r(\overline{C})$ for each prime $r \neq 2$. Hence $\overline{P}_1$ induces by conjugation $\mathbb{Z}_5 \times \mathrm{PSL}_2(5)$ on $O_2(\overline{C})$. As $Z(\mathrm{Sp}_6(5)) \leq O_2(\overline{C})$, this implies that $|O_2(\overline{C})| \geq 2^9$ and so $|O_2(\overline{C})\overline{P}_1|_2 \geq 2^{11}$, which contradicts $|\mathrm{Sp}_6(5)|_2 = 2^{10}$. This proves the lemma. $\square$

Lemma 9. *If $E(\overline{C}) = 1$, then $F^*(G) \cong \mathrm{M}(22)$ or ${}^2\mathrm{E}_6(2)$.*

Proof. Suppose $E(\overline{C}) = 1$. By Lemma 8, we have $p = 3$. As for odd primes $r \geq 5$, the Sylow r-subgroups of $\overline{C}$ are cyclic, we see that $\overline{P_1}'$ centralizes $O_r(\overline{C})$. Thus $O_2(\overline{C}) \neq Z(\mathrm{Sp}_6(3))$. Set $\langle z \rangle = Z(\mathrm{Sp}_6(3))$.

Suppose there is a non-trivial $x \in \overline{S}$ such that $O_2(\overline{C})$ is centralized by x. Then $\langle x^{\overline{P_1}} \rangle$ centralizes $O_2(\overline{C})$ and so either $x \in O_3(\overline{P_1})$ or $O_2(\overline{P_1})$ centralizes $O_2(\overline{C})$.

Assume that $O_2(\overline{P}_1)$ centralizes $O_2(\overline{C})$. Since $O_2(\overline{P_1})$ centralizes $O_r(\overline{C})$ for $r \geq 5$, we have $O_2(\overline{P_1}) \leq C_{\overline{C}}(F^*(\overline{C})) \leq F^*(\overline{C})$, a contradiction as $O_2(\overline{P_1})$ is non-abelian. Hence $x \in O_3(\overline{P_1})$. Since $O_3(P_1) \neq O_3(P_2)$ by Lemma 3 (iii), we derive a contradiction using P_2.

Hence $\overline{S}$ acts faithfully on $O_2(\overline{C})$ and, in particular, $O_2(\overline{C})/\Phi(O_2(\overline{C}))$ has order at least 2^4. Put $V = \Omega_1(Z(O_2(\overline{C})))$ and recall that $z \in V$ by Lemma 6 (iv). Since $O_2(\overline{C})$ is a 2-group, it cannot act irreducibly on $Q/Z(Q)$. Therefore, $O_2(\overline{C})$ leaves invariant a 2-space W. If $O_2(\overline{C})$ acts faithfully on W, then $O_2(\overline{C})$ embeds into $\mathrm{GL}_2(3)$ and this contradicts $|O_2(\overline{C})/\Phi(O_2(\overline{C}))| \geq 2^4$. Thus $C_{O_2(\overline{C})}(W) \neq 1$ and, in particular, $|V| \geq 2^2$. Assume that $F \leq V$ has order 4 and contains z. Then, by coprime action and using the determinant of every element in $\overline{C}$ is 1, there exists $t \in F$ with $[Q/Z(Q), t]$ of order 3^2. If $V = F$, then $\overline{C}$ centralizes V and also leaves $[Q/Z(Q), t]$ invariant. However, $\overline{C}$ acts irreducibly on $Q/Z(Q)$ by Lemma 5 (ii) and so $|V| \geq 2^3$. Coprime action now implies $|V| = 2^3$ and that $Q/Z(Q)$ is a direct sum of 3 pairwise perpendicular 2-spaces W_1, W_2 and W_3 for $O_2(\overline{C})$ and these spaces are permuted transitively by $\overline{C}$ by Lemma 5 (ii). As $|V| = 8$, we now know $C_{\overline{S}}(V)$ is non-trivial and $C_{\overline{S}}(V)$ acts quadratically on W_1, W_2 and W_3. Since $\overline{S}$ contains elements which do not act quadratically on $Q/Z(Q)$ by Lemma 6 (iii), $[V, \overline{S}] \neq 1$. Hence $C_{\overline{S}}(V)$ has order 3. As $|O_2(\overline{C})/\Phi(O_2(\overline{C}))| \geq 2^4$, the fact that $|V| = 2^3$ implies $O_2(\overline{C})$ is non-abelian and acts on W_1, W_2 and W_3. Since each W_i is non-degenerate, we have that $O_2(\overline{C})$ is isomorphic to a subgroup of $\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8$ and, as $\overline{S}$ permutes $\{W_1, W_2, W_3\}$ transitively, $C_{\overline{S}}(V)$ acts non-trivially on each Q_8 factor. Hence $|O_2(\overline{C})| = 2^5$, 2^7 or 2^9. If $|O_2(\overline{C})/V| = 2^2$, then $O_2(\overline{C})\overline{S} \cong \mathrm{Alt}(4) \times \mathrm{SL}_2(3)$. But in this group there is a unique subgroup isomorphic to Q_8, which is centralized by some element of order three. As $O_2(\overline{C})\overline{S} = O_2(\overline{C})\overline{P}_1 = O_2(\overline{C})\overline{P}_2$, this again implies the contradiction $P_1 = P_2$. Hence $|O_2(\overline{C})/Z(O_2(L))| = 2^4$ or 2^6. Now application of [6, Theorem 1.3 and Theorem 1.4] shows that $F^*(G) \cong {}^2\mathrm{E}_6(2)$ or $\mathrm{M}(22)$. □

Proof of the Proposition: By Lemma 5, Q is extraspecial of order p^7. If $E(\overline{C}) = 1$, then part (i) of the proposition follows from Lemma 8 and Lemma 9. If $E(\overline{C}) \neq 1$, then Lemma 7 implies $p = 5$ and $L = E(\overline{C}) \cong 2{\cdot}\mathrm{J}_2$. As M_1 induces $\mathrm{SL}_3(5)$ on Y_{M_1}, we see that $|N_{M_1}(Z(Q)) : C_{M_1}(Z(Q))| = 4$. Hence $|\tilde{C} : C| = 4$. By [2, Lemma 5.4], we have that $N_{\overline{C}}(L) = L$. Furthermore, also by [2, Lemma 5.4], if we consider $U = L/Z(L)$ as a subgroup of $\mathrm{PGL}_6(5)$, then we have that $N_{\mathrm{PGL}_6(5)}(U) = U : 2$ and there is an outer automorphism of order two of U. Hence back in $N_G(Q)/Q$, we have a cyclic group of order 4 which centralizes L and an outer automorphism of order two on L. This shows that $N_G(Q)/Q \cong 4{\cdot}\mathrm{J}_2.2$, which is part (ii) of the proposition. □

Proof of the Main Theorem

We now add to Hypothesis 1 the assumption that G is of local characteristic p.

Assume first that Proposition (i) holds. If $F^*(G) \cong \mathrm{M}(22)$, then by [1, Table 5.3t] there is an element $\rho \in F^*(G)$ of order three with $C_{F^*(G)}(\rho) \cong \langle\rho\rangle \times \mathrm{PSU}_4(3)$. Thus G is not of local characteristic 3, a contradiction.

If $F^*(G) \cong {}^2\mathrm{E}_6(2)$, then by [6, Lemma 7.1] there is an element $\rho \in F^*(G)$ of order three with $C_{F^*(G)}(\rho) \cong \langle\rho\rangle \times \mathrm{PSU}_6(2)$. Again this contradicts the hypothesis that G has local characteristic 3. This proves the Main Theorem when Proposition (i) holds.

Assume that Proposition (ii) holds. Then, by [7, Lemma 3.2], the conjugation action of $\tilde{C}$ on Q induces two orbits on the subgroups of order 5 in $Q \setminus Z(Q)$, one of them is conjugate to $Z(Q)$ in G. Choose R a representative of the other class. Then in the last line of the proof of [7, Proposition 6.2] the authors show that the centralizer of R in G is not characteristic 5, which means that G is not of local characteristic 5. Here and in [7, Lemma 3.2] the $\mathcal{K}$-group assumption is not used. It is just used to give the precise structure of this centralizer, which we do not require. This completes the proof of the Main Theorem. $\square$

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