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Hall, J. I.; Shpectorov, S.

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J. I. Hall · S. Shpectorov

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Abstract We calculate the spectrum of the diagram for each finite 3-transposition group. Such graphs with a given minimum eigenvalue have occurred in the context of compact Griess subalgebras of vertex operator algebras.

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1 Introduction

A 3-transposition group (G, D) is a group G generated by a conjugacy class $D = D^G$ of elements of order 2, such that

$$d, e \in D \implies |de| \in \{1, 2, 3\}.$$

The *diagram* of (G, D) is the graph with vertex set D and edges

$$d \sim e \iff |de| = 3.$$

As a consequence of work by Fischer [5] and later Cuypers and Hall [4] all diagrams for all finite 3-transposition groups are known. In this paper we give the eigenvalues and spectrum of (the adjacency matrix of) each such diagram.

Of particular importance are the minimum eigenvalues, always a negative integer. One result is that for given $-t$ the possibilities for 3-transposition groups with minimum eigenvalue (greater than or) equal to $-t$ are limited. Miyamoto [12] first observed a connection between 3-transposition groups and compact Griess subalgebras found within vertex operator algebras. Particularly relevant for this paper is the work of Matsuo [13,14].

The complement of the diagram is the *codiagram* or *commuting graph*.

J. I. Hall
Department of Mathematics, Michigan State University, 619 Red Cedar Road, East Lansing, MI 48824, USA
E-mail: halljo@msu.edu

S. Shpectorov (✉)
School of Mathematics, University of Birmingham, Watson Building, Edgbaston, Birmingham B15 2TT, UK
E-mail: s.shpectorov@bham.ac.uk

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2 Eigenvalues of graphs

Let X be a nonempty set and (X) a graph with X as vertex set. The adjacency of the edge (x, y) is written $x \sim y$. The (0, 1)-adjacency matrix of the graph will be denoted $\mathrm{AMat}((X))$, and the spectrum of the graph is the (ordered) spectrum of $\mathrm{AMat}((X))$:

$$\mathrm{Spec}((X)) = (\!(\ldots, r_i, \ldots)\!)\,.$$

The all-one vector $\mathbf{1}$ is an eigenvector of $\mathrm{AMat}((X))$ with eigenvalue k if and only if (X) is regular of degree k. In this case, by the Perron-Frobenius Theorem, k is the largest eigenvalue and the corresponding eigenspace has dimension the number of connected components of (X). Hence, when (X) is connected, which will (almost) always be the case in this paper, the eigenspace for k is 1-dimensional, spanned by $\mathbf{1}$.

Furthermore, all other eigenspaces of the regular connected graph (X) are perpendicular to $\mathbf{1}$; that is, they belong to the sum-zero hyperplane of $\mathbb{R}^n$. Such eigenvectors and their associated eigenvalues will be called *restricted*.

For this reason, we will list k first in the spectrum and separate it from the restricted eigenvalues by a semicolon.

The complement of the graph (X) is the graph $[X]$ with the same vertex set but all edges replaced by nonedges and nonedges by edges. Thus

$$\mathrm{AMat}([X]) = J_n - I_n - \mathrm{AMat}((X))\ .$$

where $n = |X|$, I_n is the $n \times n$ identity matrix, and J_n is the $n \times n$ matrix consisting entirely of 1's. (We may drop the subscripts, when n is apparent.) All nonzero vectors of $\mathbb{R}^n$ are eigenvectors of I_n with eigenvalue 1. The all-one vector $\mathbf{1}$ is an eigenvector of J_n with eigenvalue n of multiplicity 1, and the sum-zero hyperplane of $\mathbb{R}^n$ consists of null vectors for J_n—its nonzero vectors are eigenvectors with eigenvalue 0.

We thus have

Proposition 2.1 *If (X) is a regular graph of degree k and the spectrum of (X) is $(\!(k; \ldots, r_i, \ldots)\!)$, then the spectrum of $[X]$ is $(\!(l; \ldots, -1 - r_i, \ldots)\!)$, where $|X| = 1 + k + l$.* □

If M is an $n \times n$ matrix, then $2^{\bullet 1}M$ is the $2n \times 2n$ matrix

$$\begin{bmatrix} M & M \\ M & M \end{bmatrix} = M \otimes J_2\,;$$

and $2^{\bullet h}M$ is the $2^h n \times 2^h n$ matrix that results from repeating this construction h times.

If M is an $n \times n$ matrix, then $3^{\bullet 1}M$ is the $3n \times 3n$ matrix

$$\begin{bmatrix} M & M + I_n & M + I_n \\ M + I_n & M & M + I_n \\ M + I_n & M + I_n & M \end{bmatrix} = (M + I_n) \otimes J_3 - I_{3n}\,;$$

and $3^{\bullet h}M$ is the $3^h n \times 3^h n$ matrix that results from repeating this construction h times.

This nonstandard notation $p^{\bullet h}M$ and the relevance of these matrices will become clear with Lemma 4.2 and the remarks that proceed it.

Proposition 2.2 *Let $v_1(= \mathbf{1}), \ldots, v_i, \ldots, v_n$ be a basis of eigenvectors for the matrix M, the associated spectrum being $(\!(\ldots, r_i, \ldots)\!)$.*

(a) *$2^{\bullet 1}M$ has the basis of eigenvectors*

$$(v_1, v_1)\,,\ \ldots\ ,\ (v_i, v_i)\,,\ (v_i, -v_i)\,,\ \ldots\ ,\ (v_n, -v_n)$$

with associated spectrum $(\!(\ldots, 2r_i, 0, \ldots)\!)$.

(b) *$3^{\bullet 1}M$ has the basis of eigenvectors*

$$\begin{aligned}&(v_1, v_1, v_1)\,,\ (v_1, -v_1, 0)\,,\ \ldots\ ,(v_i, v_i, v_i)\,,\ (v_i, -v_i, 0)\,,\ (v_i, 0, -v_i)\,,\\ &\quad\ldots\ ,\ (v_n, -v_n, 0)\,,\ (v_n, 0, -v_n)\end{aligned}$$

with associated spectrum $(\!(\ldots, 3r_i + 2, -1, -1 \ldots)\!)$. □

If M is an $n \times n$ matrix, then $3 \times M$ is the $3n \times 3n$ matrix

$$\begin{bmatrix} M & J_n & J_n \\ J_n & M & J_n \\ J_n & J_n & M \end{bmatrix} .$$

As the restricted eigenvectors of a regular graph are also eigenvectors for J_n, we find

Proposition 2.3 *Let $v_1(= \mathbf{1}), \ldots, v_i, \ldots, v_n$ be a basis of eigenvectors for the adjacency matrix M of a regular graph of degree k, the associated spectrum being $((k; \ldots, r_i, \ldots))$. Then, $3 \times M$ has the basis of eigenvectors:*

$$(v_1, v_1, v_1), (v_1, -v_1, 0), (v_1, 0, -v_1), \ \ldots\ , (v_i, v_i, v_i), (v_i, -v_i, 0), (v_i, 0, -v_i), \ldots$$
$$\ldots\ , (v_n, v_n, v_n), (v_n, -v_n, 0), (v_n, 0, -v_n)$$

with associated spectrum $((k + 2n; -(n-k), -(n-k), \ \ldots\ , r_i, r_i, r_i, \ \ldots))$. In particular, $-(n-k)$ is an eigenvalue of $3 \times M$. □

The nonstandard matrix notation $3 \times M$ and these results will reappear in Theorem 6.22.

3 Rank 3 and strongly regular graphs

Consider a graph (X) and subgroup G of its automorphism group with the following property:

G is transitive on X, on the set of ordered edges of (X), and on the set of ordered edges of $[X]$.

Assuming that all three sets are nonempty, we say that G acts with *rank* 3 on (X) (and so also on $[X]$) and that (X) and $[X]$ are a complementary pair of *rank* 3 *graphs*. (There is nothing to say if all three are empty. If two are empty, then $|X| = 1$. If one is empty, then (X) and $[X]$ are a complementary pair of a complete and an empty graph, and G is 2-transitive on X; this is *rank* 2 action.)

A *strongly regular graph* is a finite graph (X) with the following strong regularity property:

There are constants k, λ, and μ, such that for $x, y \in X$, the number of common neighbors of x, y is k when $x = y$; λ when $x \sim y$; and μ when $x \nsim y$.

Empty and complete graphs provide the degenerate cases $k = 0$ and $k = n - 1$ of this condition, where

$$|X| = n\,. \tag{3.1}$$

Here we do not include these as strongly regular; that is, we additionally require

$$0 < k < n - 1\,. \tag{3.2}$$

This graph will be connected of diameter 2 unless $\mu = 0$. In that case, the graph is a disjoint union of complete subgraphs K_{k+1}. Its complementary strongly regular graph is then complete multipartite with $\mu = k$. This pair of graphs is *imprimitive*. We shall only be concerned with strongly regular graphs that are not imprimitive—those that are *primitive*.

For us, the basic observation is that a rank 3 graph is strongly regular. A strongly regular graph is, in particular, regular of degree k. One says that the strongly regular graph (X) has *parameters* (n, k, λ, μ). The parameters are thus nonnegative integers with

$$n > k \geq \mu \quad \text{and} \quad k - 1 \geq \lambda\,. \tag{3.3}$$

An elementary calculation shows that if (X) is strongly regular with parameters (n, k, λ, μ), then $[X]$ is also strongly regular, its parameters being

$$(n, k', \lambda', \mu') \quad \text{for} \quad k' = n - k - 1\,,\ \lambda' = n - 2k + \mu - 2\,,\ \text{and}\ \mu' = n - 2k + \lambda\,. \tag{3.4}$$

It is usual to write the codegree as

$$l = k' = n - k - 1\,. \tag{3.5}$$

By counting paths of length two from a fixed vertex

$$\mu l = k(k-1-\lambda)\,. \tag{3.6}$$

Let M be the adjacency matrix of the strongly regular graph (X) with parameters (n, k, λ, μ). Counting all directed paths of length 2 yields

$$M^2 = kI + \lambda M + \mu(J - I - M)$$

whence

$$M^2 + (\mu - \lambda)M + (\mu - k)I = \mu J\,.$$

In particular, the restricted eigenvalues of M are the roots r and s of the monic quadratic polynomial:

$$x^2 + (\mu - \lambda)x + (\mu - k)\,. \tag{3.7}$$

As $\mu \leq k$ the roots r and s are real. We take $s \leq r$ by convention. As $\operatorname{tr} M = 0$ and k is a positive eigenvalue of A, $s < 0$. Also, $-rs = k - \mu \geq 0$, and so the real parameters s and r are restricted by

$$s < 0 \text{ and } 0 \leq r \leq k\,. \tag{3.8}$$

In particular, $s < r$.

The triple (k, r, s) is determined by (k, λ, μ). Conversely (k, r, s) determines (k, λ, μ) via

$$\mu = k + rs \quad \text{and} \quad \lambda = \mu + r + s\,. \tag{3.9}$$

Let r and s have (restricted) multiplicities f and g, respectively. As $r \neq s$, the parameters f, g can be found from

$$1 + f + g = n \quad \text{and} \quad k + fr + gs = \operatorname{tr} M = 0\,. \tag{3.10}$$

Indeed the multiplicities are

$$f = (r-s)^{-1}(-sn + s - k) \quad \text{and} \quad g = (r-s)^{-1}(rn - r + k)\,. \tag{3.11}$$

The fact that f, g must be integers is a strong restriction on possible parameter sets.

Conversely, given the integer f and g, if $f = g$, then $f = g = (n-1)/2$. Therefore

$$k = -fr - gs = -(r+s)(n-1)/2 = (\mu - \lambda)(n-1)/2\,.$$

Since $0 < k < n-1$, it follows that $\mu = \lambda + 1 \neq 0$ and $k = (n-1)/2 = l$. As $k = l$, we have $\mu = k - 1 - \lambda$, hence $k = 2\mu$. Thus $(n, k, \lambda, \mu) = (4t+1, 2t, t-1, t)$ for a suitable integer t, and $r, s = (-1 \pm \sqrt{n})/2$. This is known as the *half case* and will not be of concern here.

In the generic case $f \neq g$, one can solve for r, s from

$$r + s = \lambda - \mu \quad \text{and} \quad fr + gs = -k\,. \tag{3.12}$$

It follows that r, s are rational. As roots of a monic polynomial with integer coefficients, they are also algebraic integers; so they are integral in this case.

The *extended parameter* list for (X) is

$$(n, k, \lambda, \mu\,;\, [r]^f, [s]^g)$$

or

$$(n, k, \lambda, \mu\,;\, \{[r]^f, [s]^g\})$$

when it is not clear which eigenvalue is r and which is s. The corresponding extended parameter list for $[X]$ is

$$(n, l, \lambda', \mu'\,;\, [r']^g, [s']^f)$$

or

$$(n, l, \lambda', \mu' ; \{[r']^g, [s']^f\})$$

where

$$l = n - k - 1\,,\ \lambda' = n - 2k + \mu - 2\,, \mu' = n - 2k + \lambda \tag{3.13}$$

as before, and additionally

$$r' = -s - 1\,, \quad s' = -r - 1 \tag{3.14}$$

since $\mathrm{AMat}([X]) + \mathrm{AMat}((X)) = J_n - I_n$; see Proposition 2.1.

These sets of parameters are highly redundant, being related by the various equations of this section. All parameters can be determined by various small subsets of the complete parameter list. In particular three parameters are enough when we have

$$n\,; \quad \text{one of } \ k = l' \text{ or } l = k'\,; \quad \text{any one of } \ \lambda, \mu, \lambda', \mu'\,.$$

Of course, the more parameters that can be calculated directly, the easier the remaining calculations will be.

It is also of note that all parameters can be derived from the spectrum

$$((k; [r]^f, [s]^g))\,.$$

We have already seen that the values $\mu = 0$ and $\mu = k$ are special—these are the imprimitive graphs. Indeed, these parameters make the complementary statements that one of (X) or $[X]$ is a nontrivial equivalence relation—a disjoint union of complete subgraphs (of fixed size $m > 1$)—while the other is a complete multipartite graph with all parts of size m. As an important special case when G acts imprimitively with rank 3 on (X) and $[X]$, these form a complementary pair of imprimitive strongly regular graphs.

4 3-transposition diagrams and eigenvalues

The normal set D of the group G is a set of 3-*transpositions* in G if it consists of elements of order 2 with the property:

$$d, e \in D \implies |de| \in \{1, 2, 3\}\,.$$

The study of such sets D and groups G was initiated by Bernd Fischer [5]. Fischer's paper and the later paper [4] of Cuypers and Hall are our basic references on this topic.

If E is a subset of D in G then the *diagram* of E, denoted (E), is the graph with vertex set E and having an edge between the two vertices d, e precisely when $|de| = 3$. The *commuting graph* of E, or *codiagram* of E, is the graph complement $[E]$ of the diagram of E.

There are two cases of primary interest. The first has E some small generating set of G; for instance, the 3-transposition group $\mathrm{Sym}(n+1)$ is the Weyl group $\mathrm{W}(A_n)$ with diagram the n-vertex path A_n. In the second case E is equal to the full class D, and we then abuse terminology by saying that the diagram of D is also the diagram of G.

Theorem 4.1 *(a) If H is a subgroup of G, then $D \cap H = \emptyset$ or $D \cap H$ is a normal set of 3-transpositions in H. If N is a normal subgroup of G, then $D \subset N$ or the nontrivial elements of DN/N form a normal subset of 3-transpositions in G/N.*

(b) Let D_i, for $i \in I$, be the connected components of (D). Then each D_i is a conjugacy class of 3-transpositions in the group $G_i = \langle D_i \rangle$. Furthermore, the normal subgroup $\langle D \rangle$ of G is the central product of its subgroups G_i.

(c) If $G = \langle D \rangle$ then, for $d \in D \setminus \mathrm{Z}(G)$, each coset $d\,\mathrm{Z}(G)$ meets D only in d.

The first two parts of the theorem are Fischer's basic Inheritance Properties [5, (1.2)]. The second of these allows us to focus on the case $G = \langle D \rangle$ for the conjugacy class D of 3-transpositions. In this situation we say that (G, D) is a 3-*transposition group*.

The third part of the theorem is embedded in Fischer's [5, Lemma (2.1.1)] and is also in [4, Lemma 3.16].

We say that the two 3-transposition groups (G_1, D_1) and (G_2, D_2) have the same *central type* (usually abbreviated to *type*) provided $G_1/Z(G_1)$ and $G_2/Z(G_2)$ are isomorphic as 3-transposition groups. Theorem 4.1(c) tells us that the 3-transposition properties of groups sharing a central type are essentially the same. In particular the two 3-transposition groups have the same type if and only if they have isomorphic diagrams (D_1) and (D_2).

A consequence of the work by Fischer [5] and later Cuypers and Hall [4] is the classification up to isomorphism of all diagrams for all finite 3-transposition groups.[1] In Sect. 6 we shall give the eigenvalues and spectrum of (the adjacency matrix of) each such diagram. If $(\!(\ldots, r_i, \ldots)\!)$ is the spectrum of $\mathrm{AMat}((D))$, we also say that $(\!(\ldots, r_i, \ldots)\!)$ is the spectrum of (G, D) and G.

As D is a conjugacy class of the 3-transposition group (G, D), its diagram (D) is connected. Thus the degree k of the diagram is an eigenvalue of multiplicity one (associated with the eigenvector $\mathbf{1}$) and will be listed first in the spectrum. The remaining eigenvalues are restricted.

For the 3-transposition group (G, D) we write $p^{\bullet h}$, with $p \in \{2, 3\}$, for a normal p-subgroup N with $|D \cap dN| = p^h$ for all $d \in D$. We call this the *shape* of N.

Lemma 4.2 *Let (G, D) be a 3-transposition group with normal subgroup N of shape $p^{\bullet h}$ and 3-transposition quotient (H, E) for $H = G/N$ and $E = DN/N$. The adjacency matrix of the diagram (G) $(= (D))$ for the group $G = p^{\bullet h}H$ is the matrix $p^{\bullet h}M$ of Proposition 2.2, where M is the adjacency matrix of the diagram (H) $(= (E))$ for H.*

Proof Let $d, e \in D$. If $p = 2$, then the 2^h vertices of $(dN \cap D)$ admit no edges, while if $p = 3$ the subdiagram $(dN \cap D)$ of size 3^h is complete. In both cases, if de has order 2 then there are no edges between $(dN \cap D)$ and $(eN \cap D)$, while if de has order 3, all possible edges between $(dN \cap D)$ and $(eN \cap D)$ occur. □

For fixed H and $p^{\bullet h}$, there may be 3-transposition groups G of distinct central type with N of type $p^{\bullet h}$ and $G/N = H$, so that they have the same diagram.

Proposition 2.2 yields:

Corollary 4.3 *If the spectrum of (H, E) is $(\!(k; \ldots, r_i, \ldots)\!)$ then:*

(a) *the spectrum of $G = 2^{\bullet 1}H$ is $(\!(2k; 0, \ldots, 2r_i, 0, \ldots)\!)$, and for each $r_i \notin \{k, 0\}$, the multiplicity of $2r_i$ for G is equal to that of r_i for H;*

(b) *the spectrum of $G = 3^{\bullet 1}H$ is $(\!(3k + 2; -1, -1, \ldots, 3r_i + 2, -1, -1 \ldots)\!)$, and for each $r_i \notin \{k, -1\}$, the multiplicity of $3r_i + 2$ for G is equal to that of r_i for H.* □

The first part of the corollary appears in Matsuo's original papers on vertex operator algebras [13, Lemma 4.1.3] and [14, §5].

As a first example, the 3-transposition group $\mathrm{Sym}(2)$ has diagram adjacency matrix $M = [0]$ with unique eigenvalue $k = 0$. Therefore, $\mathrm{Sym}(3) = 3^{\bullet 1}\,\mathrm{Sym}(2)$ has diagram adjacency matrix:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

with spectrum $(\!(3 \cdot 0 + 2; -1, -1)\!) = (\!(2; -1, -1)\!)$ and $\mathrm{Sym}(4) = 2^{\bullet 1}\,\mathrm{Sym}(3)$ has spectrum

$$\begin{aligned}(\!(2 \cdot 2; 0, 2 \cdot (-1), 0, 2 \cdot (-1), 0)\!) &= (\!(4; 0, -2, 0, -2, 0)\!) \\ &= (\!(4; [-2]^2, [0]^3)\!) = (\!(4; [-2]^2, [0]^\star)\!)\,.\end{aligned}$$

Here, again, we use the convention that $[t]^c$ indicates an eigenvalue t of multiplicity c. We also introduce the notation $[t]^\star$ to indicate that the eigenvalue t has multiplicity equal to whatever is required for the total multiplicity to be the size n.

We can continue in this fashion, so that

$$\mathrm{SU}_3(2)' = 3^{\bullet 2}\,\mathrm{Sym}(2) = 3^{\bullet 1}(3^{\bullet 1}\,\mathrm{Sym}(2))$$

[1] Beware: nonisomorphic groups may have the same diagram.

has spectrum

$$((8; -1, -1, -1, -1, -1, -1, -1, -1)) = ((8; [-1]^8)) = ((8; [-1]^\star))$$

while

$$\mathrm{W}(D_4) = 2^{\bullet 2}\,\mathrm{Sym}(3) = 2^{\bullet 1}(2^{\bullet 1}\,\mathrm{Sym}(3))$$

has spectrum

$$((8; 0, -4, 0, -4, 0, 0, 0, 0, 0, 0)) = ((8; [-4]^2, [0]^9)) = ((8; [-4]^2, [0]^\star)).$$

Iteration of the previous corollary yields:

Corollary 4.4 *Let the* 3*-transposition group* (H, E) *have spectrum*

$$((k; \ldots, [r_i]^{m_i}, \ldots))$$

and size is $n_H = |E| = 1 + \sum_i m_i$.

(a) A 3*-transposition group* $G = 2^{\bullet h}H$*, for* $h \geq 1$*, with class* $D_G = 2^{\bullet H}E$ *has size*

$$n_G = |D_G| = 2^h n_H$$

and spectrum

$$((2^h k; \ldots, [2^h r_i]^{m_i}, \ldots, [0]^\star)).$$

(b) A 3*-transposition group* $F = 3^{\bullet h}H$*, for* $h \geq 1$*, with class* $D_F = 3^{\bullet H}E$ *has size*

$$n_F = |D_F| = 3^h n_H$$

and spectrum

$$((3^h(k+1) - 1; \ldots, [3^h(r_i+1) - 1]^{m_i}, \ldots, [-1]^\star)).$$

□

Note that in (a) one of the r_i may be zero, in which case the expected tail multiplicity $(2^h - 1)n_H$ should be combined with the multiplicity m_i of $2^h r_i = 0$; this explains the exponent $\star$, which indicates a multiplicity that is whatever is needed to exhaust all eigenvalues. Similarly, in (b) one of r_i may be -1 and then the expected tail multiplicity $(3^h - 1)n_H$ is added to the multiplicity m_i of $3^h(r_i + 1) - 1 = -1$.

5 Classifications of 3-transposition groups

Fischer's [5] main theorem on 3-transposition groups is:

Theorem 5.1 *Let* (G, D) *be a finite* 3*-transposition group with no nontrival normal solvable subgroup. Then, the group* G *has exactly one of the central types below. Furthermore, for each* G *the generating class* D *is uniquely determined up to an automorphism of* G.

I2. $\mathrm{Sym}(m)$*, all* $m \geq 5$*;*
I3. $\mathrm{O}^\epsilon_{2m}(2)$*,* $\epsilon = \pm$*, all* $m \geq 3$*,* $(m, \epsilon) \neq (3, +)$*;*
I4. $\mathrm{Sp}_{2m}(2)$*, all* $m \geq 3$*;*
I5. ${}^+\Omega^\epsilon_m(3)$*,* $\epsilon = \pm$*, all* $m \geq 6$*;*
I6. $\mathrm{SU}_m(2)$*, all* $m \geq 4$*;*
I7. $\mathrm{Fi}_{22}, \mathrm{Fi}_{23}, \mathrm{Fi}_{24}, \mathrm{P}\Omega^+_8(2) : \mathrm{Sym}(3), \mathrm{P}\Omega^+_8(3) : \mathrm{Sym}(3)$.

The notation is that of [4] and will be discussed in the next section.

No example appears twice in the theorem. Apparent omissions within it, in the first table of the next section, and throughout the paper are explained by the following coincidences.

Lemma 5.2 *(a)* $1 = {}^+\Omega^+_1(3)$*;*

(b) $\mathrm{Sym}(2) \simeq {}^{+}\Omega_1^{-}(3) \simeq {}^{+}\Omega_2^{+}(3) \simeq \mathrm{Z}_2$*:*
(c) $1 \neq \mathrm{O}_3(\mathrm{Sym}(3))$*;* $\mathrm{Sym}(3) \simeq \mathrm{O}_2^{-}(2) \simeq \mathrm{Sp}_2(2) \simeq \mathrm{SU}_2(2)$*;*
(d) $1 \neq \mathrm{O}_2(\mathrm{Sym}(4))$*;* $\mathrm{Sym}(4) \simeq {}^{+}\Omega_3^{-}(3)$*;*
(e) $\mathrm{Sym}(5) \simeq \mathrm{O}_4^{-}(2)$*;*
(f) $\mathrm{Sym}(6) \simeq \mathrm{Sp}_4(2) \simeq {}^{+}\Omega_4^{-}(3)$*;*
(g) $\mathrm{Sym}(8) \simeq \mathrm{O}_6^{+}(2)$*;*
(h) $\mathrm{O}_6^{-}(2) \simeq {}^{+}\Omega_5^{+}(3)$*;*
(i) $2 \times \mathrm{SU}_4(2) \simeq {}^{+}\Omega_5^{-}(3)$*;*
(j) $1 \neq \mathrm{O}_2(G)$ *for* $G \in \{\mathrm{O}_4^{+}(2), {}^{+}\Omega_2^{-}(3), {}^{+}\Omega_3^{+}(3), {}^{+}\Omega_4^{+}(3)\}$*;*
(k) $1 \neq \mathrm{O}_3(\mathrm{SU}_3(2)')$.

Proof All these can be found in [4, §2]. □

Fischer's theorem was extended in [4]. A consequence of the main theorem of that paper is:

Theorem 5.3 *Let* (G, D) *be a finite* 3*-transposition group. Then, for integral* m *and* h*, the group* G *has one of the central types below. Furthermore, for each* G *the generating class* D *is uniquely determined up to an automorphism of* G.

PR1. $3^{\bullet h} : \mathrm{Sym}(2)$*, all* $h \geq 1$*;*
PR2(a). $2^{\bullet h} : \mathrm{Sym}(m)$*, all* $h \geq 0$*, all* $m \geq 4$*;*
PR2(b). $3^{\bullet h} : \mathrm{Sym}(m)$*, all* $h \geq 1$*, all* $m \geq 4$*;*
PR2(c). $3^{\bullet h} : 2^{\bullet 1} : \mathrm{Sym}(m)$*, all* $h \geq 1$*, all* $m \geq 4$*;* ***PR2(d).*** $4^{\bullet h} : 3^{\bullet 1} : \mathrm{Sym}(m)$*, all* $h \geq 1$*, all* $m \geq 4$*;*
PR3. $2^{\bullet h} : \mathrm{O}_{2m}^{\epsilon}(2)$*,* $\epsilon = \pm$*, all* $h \geq 0$*, all* $m \geq 3$*,* $(m, \epsilon) \neq (3, +)$*;*
PR4. $2^{\bullet h} : \mathrm{Sp}_{2m}(2)$*, all* $h \geq 0$*, all* $m \geq 3$*;*
PR5. $3^{\bullet h} {}^{+}\Omega_m^{\epsilon}(3)$*,* $\epsilon = \pm$*, all* $h \geq 0$*, all* $m \geq 5$*;*
PR6. $4^{\bullet h} \mathrm{SU}_m(2)'$*, all* $h \geq 0$*, all* $m \geq 3$*;*
PR7(a-e). $\mathrm{Fi}_{22}, \mathrm{Fi}_{23}, \mathrm{Fi}_{24}, \mathrm{P}\Omega_8^{+}(2) : \mathrm{Sym}(3), \mathrm{P}\Omega_8^{+}(3) : \mathrm{Sym}(3)$*;*
PR8. $4^{\bullet h} : (3 \cdot {}^{+}\Omega_6^{-}(3))$*, all* $h \geq 1$*;*
PR9. $3^{\bullet h} : (2 \times \mathrm{Sp}_6(2))$*, all* $h \geq 1$*;*
PR10. $3^{\bullet h} : (2 \cdot \mathrm{O}_8^{+}(2))$*, all* $h \geq 1$*;*
PR11. $3^{\bullet 2h} : (2 \times \mathrm{SU}_5(2))$*, all* $h \geq 1$*;*
PR12. $3^{\bullet 2h} : 4^{\bullet 1} : \mathrm{SU}_3(2)'$*, all* $h \geq 1$.

The notation **Ik** and **PRk** of the two theorems and the tables of the next section comes from [4], where the first suggests that the groups act **I**rreducibly on their natural modules, while the second says that more general examples arise from **P**arabolic subgroups of the irreducible examples—specifically their subgroups generated by **R**eflections or transvections, as appropriate.

In the theorem (and elsewhere) $A : B$ indicates a split group extension with normal subgroup A, while $A \cdot B$ is a nonsplit group extension with normal subgroup A and quotient B. The related notation AB indicates that A is normal with quotient B, but the extension may or may not be split. Extensions are left-adjusted, so in $A : B : C$, the normal subgroup $A : B$ is split by C, while $A : B$ has A normal and split by B.

Neither the actual structure of the normal p-subgroup nor the splitting of the extension affect the shape of the normal subgroup and so the diagram. This allows us in the theorem to bundle the exotic cases **PR13-19** from [4] under the corresponding generic cases **PR5-6** (where both split and nonsplit group extensions may occur).

In Theorem 5.3, we have rewritten shapes $2^{\bullet 2h}$ as $(4)^{\bullet h}$ when all the nontrivial composition factors in the normal subgroup those factors are naturally $\mathbb{F}_4$-modules for the quotient.

The only three repetitions on the list are $3^{\bullet 2} : \mathrm{Sym}(2) \simeq \mathrm{SU}_3(2)'$ appearing under both **PR1** and **PR6**; ${}^{+}\Omega_5^{+}(3) \simeq \mathrm{O}_6^{-}(2)$ appearing in the $h = 0$ cases of both **PR5** and **PR3**; and ${}^{+}\Omega_5^{-}(3) \simeq 2 \times \mathrm{SU}_4(2)$ appearing in the $h = 0$ cases of both **PR5** and **PR6**.

Under **PR2(a)** the groups $2^{\bullet h} : \mathrm{Sym}(3)$, for $h \geq 1$, have the same central type as $2^{\bullet h-1} : \mathrm{Sym}(4)$. Other apparent absences are justified by Lemma 5.2.

6 Case analysis of spectra

In Theorem 5.3, each choice of parameters in each part yields a unique diagram which admits a 3-transposition group (and perhaps many). In this section, we calculate the size (number of vertices) and spectrum of the diagram in each case. These are collected in two tables—one for the Irreducible examples of Theorem 5.1 and a second for the Parabolic Reflection examples of Theorem 5.3. The second of these essentially comes from combining the first with Corollary 4.4.

As Fischer noted [5, Theorem 3.3.5], in each case of Theorem 5.1 (except for the triality groups $\mathrm{P}\Omega_8^+(2):\mathrm{Sym}(3)$ and $\mathrm{P}\Omega_8^+(3):\mathrm{Sym}(3)$), the permutation representation of G acting on D by conjugation is primitive of rank 3. Therefore, the corresponding spectrum obeys all the conditions discussed in Sect. 3.

The redundancy of the parameter sets is of aid here. We have $n = |D|$. As the codiagram is the commuting graph of D, we also have

$$k = |\,\mathrm{C}_D(d)| - 1$$

for $d \in D$, where $\mathrm{C}_D(d)\backslash\{d\}$ is the noncentral normal set (indeed conjugacy class) of 3-transpositions in the subgroup $\mathrm{C}_G(d)$. Similarly if $e \in \mathrm{C}_D(d)\backslash\{d\}$ and $c \in D\backslash\,\mathrm{C}_D(d)$, then

$$\lambda' = |\,\mathrm{C}_D(d,e)| - 2 \quad \text{and} \quad \mu' = |\,\mathrm{C}_D(d,c)|$$

count the 3-transpositions of the subgroups $\mathrm{C}_G(d,e)$ and $\mathrm{C}_G(d,c)$.

We have seen in Theorems 5.1 and 5.3 that in the pair (G, D) the group G determines the generating class D uniquely up to an automorphism. Therefore, we may abuse notation by writing (G) for the diagram in place of (D).

Most of the results given here could be extracted from the literature—for instance [10] and [1]—although the notation varies enough that translation into the form we desire can be difficult. We have recalculated everything (to our own satisfaction) but only outline the paths taken.

The first table gives the extended parameters $(n, k, \lambda, \mu\,; \{[r]^f, [s]^g\})$ of the rank 3 (strongly regular) codiagrams $[G]$ and diagrams (G). Note the set notation for the eigenvalues and their multiplicities. This is because in some cases the roles of r (positive eigenvalue) and s (negative eigenvalue) may switch depending on the value of m. In these cases we use d and e for multiplicities to avoid misleading the reader.

The second table gives the size n and spectrum $((k; \ldots, [r_i]^{m_i}, \ldots))$ of all diagrams (G). The eigenvalue in bold is the minimum eigenvalue. This will be of relevance in Sect. 7.

In Theorem 5.3 we have restricted parameters to minimize repetition of examples. In the second table we reverse that decision, enlarging the parameter sets to a natural level of generality. In particular, unless otherwise stated, h can be any nonnegative integer.

6.1 Moufang case

This is the situation in which the diagram (D) is a complete graph. That is, there are no D-subgroups of G isomorphic to $\mathrm{Sym}(4)$. The terminology comes from a connection with commutative Moufang loops of exponent 3; see [4].

For $h \geq 0$, let N_h be an elementary abelian 3-group of order 3^h. Furthermore, let d be an element of order 2 that acts on N_h as inversion. Then for $G_h = N_h{:}\langle d\rangle$ and $D_h = d^{N_h}$, the pair (G_h, D_h) is a 3-transposition group $3^{\bullet h}:\mathrm{Sym}(2)$ of Moufang type **PR1**. Conversely, every finite 3-transposition group with complete diagram arises as $N:\mathrm{Sym}(2)$ for some normal 3-subgroup N. (Appropriate N exist with arbitrarily large nilpotence class.)

By Corollary 4.4:

Theorem 6.1 ***PR1****: the diagram* $(3^{\bullet h}:\mathrm{Sym}(2))$ *for* $h \geq 0$ *has size*

$$n = 3^h$$

and spectrum

$$((3^h - 1; -1, -1, \ldots, -1, -1, -1)) = ((3^h - 1; [-1]^{-1+3^h}))\,.$$

□

The fundamental 3-transposition groups $\mathbb{Z}_2$ and $\mathrm{Sym}(3)$ occur here as $h = 0$ and $h = 1$.

Graph	**Extended parameters** $(n, k, \lambda, \mu\,;\{[r]^f, [s]^g\})$
I2	
[Sym(m)], $m \geq 4$	$\left(\binom{m}{2}, \binom{m-2}{2}, \binom{m-4}{2}, \binom{m-3}{2}\,;\{[1]^{m(m-3)/2}, [-m+3]^{m-1}\}\right)$
(Sym(m)), $m \geq 4$	$\left(\binom{m}{2},\ 2(m-2),\ m-2\,,4\,;\{[m-4]^{m-1}, [-2]^{m(m-3)/2}\}\right)$
I3	
$[\mathrm{O}^{\epsilon}_{2m}(2)]$	$\big((2^{2m-1}-\epsilon 2^{m-1},\ 2^{2m-2}-1,\ 2^{2m-3}-2,\ 2^{2m-3}+\epsilon 2^{m-2}\,;$
$\epsilon=\pm,\ m\geq 2$	$\{[\epsilon 2^{m-2}-1]^{(2^{2m}-4)/3}, [-\epsilon 2^{m-1}-1]^{(2^m-\epsilon 1)(2^{m-1}-\epsilon 1)/3}\}\big)$
$(\mathrm{O}^{\epsilon}_{2m}(2))$	$\big(2^{2m-1}-\epsilon 2^{m-1},\ 2^{2m-2}-\epsilon 2^{m-1},\ 2^{2m-3}-\epsilon 2^{m-2},\ 2^{2m-3}-\epsilon 2^{m-1}\,;$
$\epsilon=\pm,\ m\geq 2$	$\{[\epsilon 2^{m-1}]^{(2^m-\epsilon 1)(2^{m-1}-\epsilon 1)/3}, [-\epsilon 2^{m-2}]^{(2^{2m}-4)/3}\}\big)$
I4	
$[\mathrm{Sp}_{2m}(2)]$	$\big(2^{2m}-1\,,\ 2^{2m-1}-2,\ 2^{2m-2}-3,\ 2^{2m-2}-1\,;$
$m\geq 2$	$\{[2^{m-1}-1]^{2^{2m-1}+2^{m-1}-1}, [-2^{m-1}-1]^{2^{2m-1}-2^{m-1}-1}\}\big)$
$(\mathrm{Sp}_{2m}(2))$	$\big(2^{2m}-1\,,\ 2^{2m-1},\ 2^{2m-2},\ 2^{2m-2}\,;$
$m\geq 2$	$\{[2^{m-1}]^{2^{2m-1}-2^{m-1}-1}, [-2^{m-1}]^{2^{2m-1}+2^{m-1}-1}\}\big)$
I5	
$[{}^{+}\Omega^{\epsilon}_{m}(3)]$	$\big((3^{m-1}-\epsilon 3^{(m-1)/2})/2\,,\ (3^{m-2}+\epsilon 3^{(m-3)/2})/2,$
odd $m\geq 5$	$(3^{m-3}+\epsilon 3^{(m-3)/2})/2\,,\ (3^{m-3}+\epsilon 3^{(m-3)/2})/2\,;$
$\epsilon=\pm$	$\{[3^{(m-3)/2}]^g\,,\ [-3^{(m-3)/2}]^f\}\big)$
$({}^{+}\Omega^{\epsilon}_{m}(3))$	$\big((3^{m-1}-\epsilon 3^{(m-1)/2})/2\,,\ 3^{m-2}-2\epsilon 3^{(m-3)/2}-1\,,$
odd $m\geq 5$	$2(3^{m-3}-\epsilon 3^{(m-3)/2}-1)\,,\ 2(3^{m-3}-\epsilon 3^{(m-3)/2})\,;$
$\epsilon=\pm$	$\{[3^{(m-3)/2}-1]^f\,,\ [-3^{(m-3)/2}-1]^g\}\big)$
with	$f=(3^{m-1}-1-(\epsilon-1)(3^{(m-1)/2}-1))/4$
	$g=(3^{m-1}-1-(\epsilon+1)(3^{(m-1)/2}+1))/4$
$[{}^{+}\Omega^{\epsilon}_{m}(3)]$	$\big((3^{m-1}-\epsilon 3^{(m-2)/2})/2\,,\ (3^{m-2}-\epsilon 3^{(m-2)/2})/2\,,$
even $m\geq 6$	$(3^{m-3}+\epsilon 3^{(m-4)/2})/2\,,\ (3^{m-3}-\epsilon 3^{(m-2)/2})/2\,;$
$\epsilon=\pm$	$\{\,[\epsilon 3^{(m-2)/2}]^d\,,\ [-\epsilon 3^{(m-4)/2}]^e\,\}\big)\,,$
$({}^{+}\Omega^{\epsilon}_{m}(3))$	$\big((3^{m-1}-\epsilon 3^{(m-2)/2})/2\,,\ 3^{m-2}-1\,,$
even $m\geq 6$	$2(3^{m-3}-1)\,,\ 2(3^{m-3}+\epsilon 3^{(m-4)/2})\,;$
$\epsilon=\pm$	$\{\,[-\epsilon 3^{(m-2)/2}-1]^d\,,\ [\epsilon 3^{(m-4)/2}-1]^e\,\}\big)$
with	$d=(3^{m/2}-\epsilon)(3^{(m-2)/2}-\epsilon)/8$
	$e=(3^m-9)/8$
I6	
$[\mathrm{SU}_m(2)]$	$\big((2^{2m-1}-(-2)^{m-1}-1)/3\,,\ 4(2^{2m-5}-(-2)^{m-3}-1)/3\,,$
$m\geq 4$	$3+16(2^{2m-9}-(-2)^{m-5}-1)/3\,,\ (2^{2m-5}-(-2)^{m-3}-1)/3\,;$
	$\{[(-2)^{m-3}-1]^d, [(-2)^{m-2}-1]^e\}\big)$
$(\mathrm{SU}_m(2))$	$(2^{2m-1}-(-2)^{m-1}-1)/3\,,\ 2^{2m-3}\,,$
$m\geq 4$	$3(2^{2m-5})+(-2)^{m-3}\,,\ 3(2^{2m-5})\,;$
	$\{[-(-2)^{m-3}]^d, [-(-2)^{m-2}]^e\}\big)$
with	$d=8(2^{2m-3}-(-2)^{m-2}-1)/9$
	$e=4(2^{2m-3}-7(-2)^{m-3}-1)/9$
I7	
$[\mathrm{Fi}_{22}]$	$\big(3510, 693, 180, 126\,;\{[63]^{429}, [-9]^{3080}\}\big)$
(Fi_{22})	$\big(3510, 2816, 2248, 2304\,;\{[8]^{3080}, [-64]^{429}\}\big)$
$[\mathrm{Fi}_{23}]$	$\big(31671, 3510, 693, 351\,;\{[351]^{782}, [-9]^{30888}\}\big)$
(Fi_{23})	$\big(31671, 28160, 25000, 25344\,;\{[8]^{30888}, [-352]^{782}\}\big)$
$[\mathrm{Fi}_{24}]$	$\big(306936, 31671, 3510, 3240\,;\{[351]^{57477}, [-81]^{249458}\}\big)$
(Fi_{24})	$\big(306936, 275264, 246832, 247104\,;\{[80]^{249458}, [-352]^{57477}\}\big)$

6.2 Symmetric cases

Theorem 6.2 *(a) For $m \geq 4$ the codiagram* [Sym(m)] *has extended parameters*

$$\left(\binom{m}{2}, \binom{m-2}{2}, \binom{m-4}{2}, \binom{m-3}{2}\,;[1]^{m(m-3)/2}, [-m+3]^{m-1}\right).$$

(b) For $m \geq 4$ the diagram (Sym(m)) *has extended parameters*

$$\left(m(m-1)/2\,,\ 2(m-2)\,,\ m-2\,,4\,;[m-4]^{m-1}, [-2]^{m(m-3)/2}\right)$$

Label	**Diagram** (G)
Size n	**Spectrum** $((k; \dots, [r_i]^{m_i}, \dots))$
PR1	$(3^{\bullet h} : \mathrm{Sym}(2))$
3^h	$((3^h - 1; [\mathbf{-1}]^{-1+3^h}))$
PR2(a)	$(2^{\bullet h} : \mathrm{Sym}(m))$
$m \geq 4 : \ 2^{h-1} m(m-1)$	$((2^{h+1}(m-2); [2^h(m-4)]^{m-1}, [0]^\star, [\mathbf{-2^{h+1}}]^{m(m-3)/2}))$
PR2(b)	$(3^{\bullet h} : \mathrm{Sym}(m))$
$m \geq 4, h \geq 1 : \ 3^h m(m-1)/2$	$((3^h(2m-3) - 1; [3^h(m-3) - 1]^{m-1}, [-1]^\star, [\mathbf{-3^h - 1}]^{m(m-3)/2}))$
PR2(c)	$(3^{\bullet h} : 2^{\bullet 1} : \mathrm{Sym}(m))$
$m \geq 4, h \geq 1 : \ 3^h m(m-1)$	$((3^h(4m-7) - 1; [3^h(2m-7) - 1]^{m-1}, [3^h - 1]^{m(m-1)/2}, [-1]^\star, [\mathbf{-3^{h+1} - 1}]^{m(m-3)/2}))$
PR2(d)	$(4^{\bullet h} : 3^{\bullet 1} : \mathrm{Sym}(m))$
$m \geq 4, h \geq 1 : \ 3(2^{2h-1}) m(m-1)$	$((4^h(6m-10); [4^h(3m-10)]^{m-1}, [0]^\star, [-4^h]^{m(m-1)}, [\mathbf{-4^{h+1}}]^{m(m-3)/2}))$
PR3	$(2^{\bullet h} : \mathrm{O}^\epsilon_{2m}(2))$
$m \geq 3, \ \epsilon = + : \ 2^h(2^{2m-1} - 2^{m-1})$	$((2^h(2^{2m-2} - 2^{m-1}); \ [2^{h+m-1}]^{(2^m-1)(2^{m-1}-1)/3}, [0]^\star, [\mathbf{-2^{h+m-2}}]^{(2^{2m}-4)/3},))$
$m \geq 2, \ \epsilon = - : \ 2^h(2^{2m-1} + 2^{m-1})$	$((2^h(2^{2m-2} + 2^{m-1}); \ [2^{h+m-2}]^{(2^{2m}-4)/3}, [0]^\star, [\mathbf{-2^{h+m-1}}]^{(2^m+1)(2^{m-1}+1)/3}))$
PR4	$(2^{\bullet h} : \mathrm{Sp}_{2m}(2))$
$m \geq 2, \ 2^h(2^{2m} - 1)$	$((2^{2m-1+h}; [2^{m-1+h}]^{2^{2m-1}-2^{m-1}-1}, [0]^\star, [\mathbf{-2^{h+m-1}}]^{2^{2m-1}+2^{m-1}-1}))$
PR5 (*and***PR13-16**)	$(3^{\bullet h} {}^{+}\Omega^\epsilon_m(3))$
odd $m \geq 5$	$((3^h(3^{m-2} - 2\epsilon 3^{(m-3)/2}) - 1 ; \ [3^{(m-3)/2+h} - 1]^f , \ [-1]^\star, [\mathbf{-3^{(m-3)/2+h} - 1}]^g))$
$\epsilon = + : \ 3^h(3^{m-1} - 3^{(m-1)/2})/2$	for $f = (3^{m-1} - 1)/4$ and $g = (3^{m-1} - 1 - 2(3^{(m-1)/2} + 1))/4$
$\epsilon = - : \ 3^h(3^{m-1} + 3^{(m-1)/2})/2$	for $f = (3^{m-1} - 1 + 2(3^{(m-1)/2} - 1))/4$ and $g = (3^{m-1} - 1)/4$
even $m \geq 6, \ \epsilon = + :$	$((3^{m-2+h} - 1 ; \ [3^{(m-4)/2+h} - 1]^f , \ [-1]^\star, [\mathbf{-3^{(m-2)/2+h} - 1}]^g))$
$3^h(3^{m-1} - 3^{(m-2)/2})/2$	for $f = (3^m - 9)/8$ and $g = (3^{m/2} - 1)(3^{(m-2)/2} - 1)/8$
even $m \geq 6, \ \epsilon = - :$	$((3^{m-2+h} - 1 ; \ [3^{(m-2)/2+h} - 1]^f , [-1]^\star, [\mathbf{-3^{(m-4)/2+h} - 1}]^g))$
$3^h(3^{m-1} + 3^{(m-2)/2})/2$	for $f = (3^{m/2} + 1)(3^{(m-2)/2} + 1)/8$ and $g = (3^m - 9)/8$
PR6 (*and***PR17-19**)	$(4^{\bullet h} \, \mathrm{SU}_m(2)', \quad m \geq 3$
even $m \geq 4 : \ 4^h(2^{2m-1} - 1 + 2^{m-1})/3$	$((2^{2h+2m-3}; [2^{2h+m-3}]^f, [0]^\star, [\mathbf{-2^{2h+m-2}}]^g))$
	for $f = 8(2^{2m-3} - 1 - 2^{m-2})/9$ and $g = 4(2^{2m-3} - 1 + 7(2^{m-3}))/9$
odd $m \geq 3 : \ 4^h(2^{2m-1} - 1 - 2^{m-1})/3$	$((2^{2h+2m-3}; [2^{2h+m-2}]^f, [0]^\star, [\mathbf{-2^{2h+m-3}}]^g))$
	for $f = 4(2^{2m-3} - 1 - 7(2^{m-3}))/9$ and $g = 8(2^{2m-3} - 1 + 2^{m-2})/9$
PR7(a)	(Fi_{22})
3510	$((2816 ; [8]^{3080}, [\mathbf{-64}]^{429}))$
PR7(b)	(Fi_{23})
31671	$((28160 ; [8]^{30888}, [\mathbf{-352}]^{782}))$
PR7(c)	(Fi_{24})
306936	$((275264 ; [80]^{249458}, [\mathbf{-352}]^{57477}))$
PR7(d)	$(\mathrm{P}\Omega^+_8(2) : \mathrm{Sym}(3))$
360	$((296; [8]^{105}, [-4]^{252}, [\mathbf{-64}]^2))$
PR7(e)	$(\mathrm{P}\Omega^+_8(3) : \mathrm{Sym}(3))$
3240	$((2888; [8]^{2457}, [-28]^{780}, [\mathbf{-352}]^2))$
PR8	$(4^{\bullet h} : (3 \cdot {}^{+}\Omega^-_6(3)))$
$h \geq 1 : \ 126(4^h)$	$((5(4^{h+2}); [2^{2h+3}]^{35}, [0]^\star, [\mathbf{-4^{h+1}}]^{90}))$
PR9	$(3^{\bullet h} : (2 \times \mathrm{Sp}_6(2))$
$h \geq 1 : \ 63(3^h)$	$((11(3^{h+1}) - 1; [5(3^h) - 1]^{27}, [-1]^\star, [\mathbf{-3^{h+1} - 1}]^{35}))$
PR10	$(3^{\bullet h} : (2 \cdot \mathrm{O}^+_8(2)))$
$h \geq 1 : \ 120(3^h)$	$((19(3^{h+1}) - 1; [3^{h+2} - 1]^{35}, [-1]^\star, [\mathbf{-3^{h+1} - 1}]^{84}))$
PR11	$(3^{\bullet 2h} : (2 \times \mathrm{SU}_5(2)))$
$h \geq 1 : \ 165(3^{2h})$	$((43(3^{2h+1}) - 1; [3^{2h+2} - 1]^{44}, [-1]^\star, [\mathbf{-3^{2h+1} - 1}]^{120}))$
PR12	$(3^{\bullet 2h} : 4^{\bullet 1} : \mathrm{SU}_3(2)')$
$h \geq 1 : \ 36(3^{2h})$	$((11(3^{2h+1}) - 1; [3^{2h} - 1]^{27}, [-1]^\star, [\mathbf{-3^{2h+1} - 1}]^8))$

and spectrum

$$((2(m-2); [m-4]^{m-1}, [-2]^{m(m-3)/2})) .$$

Proof This is well known, but it is also easy to calculate the basic parameters of $[\mathrm{Sym}(m)]$ using 3-transposition properties:

(i) $n = \binom{m}{2}$: the 3-transposition class is $D = (1, 2)^{\mathrm{Sym}(m)}$.

(ii) $k' = \binom{m-2}{2}$: $C_{\mathrm{Sym}(m)}((1,2))$ has type $\mathrm{Sym}(m-2)$.
(iii) $\lambda' = \binom{m-4}{2}$: $C_{\mathrm{Sym}(m)}((1,2),(3,4))$ has type $\mathrm{Sym}(m-4)$.
(iv) $\mu' = \binom{m-3}{2}$: $C_{\mathrm{Sym}(m)}((1,2),(2,3))$ has type $\mathrm{Sym}(m-3)$. □

Corollary 4.4 gives directly:

Proposition 6.3 ***PR2(a)****: the diagram* $(2^{\bullet h} : \mathrm{Sym}(m))$ *with* $m \geq 4$ *and* $h \geq 0$ *has size*

$$n = 2^{h-1}m(m-1)$$

and spectrum

$$((2^{h+1}(m-2); [2^h(m-4)]^{m-1}, [-2^{h+1}]^{m(m-3)/2}, [0]^\star)) .$$

□

Proposition 6.4 ***PR2(b):*** *the diagram* $(3^{\bullet h} : \mathrm{Sym}(m))$*, with* $m \geq 4$ *and* $h \geq 0$*, has size*

$$n = 3^h m(m-1)/2$$

and spectrum

$$((3^h(2m-3)-1; [3^h(m-3)-1]^{m-1}, [-3^h-1]^{m(m-3)/2}, [-1]^\star)) .$$

□

Proposition 6.5 ***PR2(c):*** *the diagram* $(3^{\bullet h} : 2^{\bullet 1} : \mathrm{Sym}(m))$*, with* $m \geq 4$ *and* $h \geq 0$*, has size*

$$n = 3^h m(m-1)$$

and spectrum

$$((-1+3^h(4m-7); [3^h(2m-7)-1]^{m-1}, [-3^{h+1}-1]^{m(m-3)/2}, [3^h-1]^{m(m-1)/2}, [-1]^\star)) .$$

Proof Apply Corollary 4.4 to the diagram $(2^{\bullet 1} : \mathrm{Sym}(m))$, which has size

$$n = m(m-1)$$

and spectrum

$$((4(m-2); [2(m-4)]^{m-1}, [-4]^{m(m-3)/2}, [0]^{m(m-1)/2})) .$$

□

Proposition 6.6 ***PR2(d):*** *the diagram* $(2^{\bullet 2h} : 3^{\bullet 1} : \mathrm{Sym}(m))$*, with* $m \geq 4$ *and* $h \geq 0$*, has size*

$$n = 3(2^{2h-1})m(m-1)$$

and spectrum

$$((4^h(6m-10); [4^h(3m-10)]^{m-1}, [-4^{h+1}]^{m(m-3)/2}, [-4^h]^{m(m-1)}, [0]^\star)) .$$

Proof Apply Corollary 4.4 to the diagram $(3^{\bullet 1} : \mathrm{Sym}(m))$, which has size

$$n = 3m(m-1)/2$$

and spectrum

$$((6m-10; [3m-10]^{m-1}, [-4]^{m(m-3)/2}, [-1]^{m(m-1)})) .$$

□

6.3 Polar space cases

For us a finite polar space graph $[X]$ has as vertex set X the isotropic[2] 1-spaces for a nondegenerate reflexive sesquilinear form f_i on a finite space $V_i = \mathbb{F}_q^i$ with edges given by perpendicularity. In our context, the form f_i is either symplectic over $\mathbb{F}_2$ or hermitian over $\mathbb{F}_4$. By Witt's Theorem, the corresponding isometry group acts with rank 3 (or less). There are exactly two types of 2-spaces spanned by isotropic vectors—totally isotropic 2-spaces with $q+1$ pairwise perpendicular isotropic 1-subspaces and hyperbolic 2-spaces with s pairwise nonperpendicular isotropic 1-subspaces. The hyperbolic 2-spaces are precisely the nondegenerate 2-spaces containing an isotropic 1-space. A vertex is either adjacent to all those of a given totally isotropic 2-space or exactly one.

Let $s_i = 1 + k_i + l_i$ be the number of isotropic 1-spaces in V_i. (So $s = s_2$.) The decomposition $V_i = V_2 \perp V_{i-2}$ can be used to calculate the parameters. This yields recursions for the degree of $[X]$

$$k_i' = l_i = qs_{i-2} \tag{6.1}$$

and its codegree

$$l_i' = k_i = (s_2 - 1)q^{i-2}, \tag{6.2}$$

hence

$$s_i = 1 + (s_2 - 1)q^{i-2} + qs_{i-2}\,. \tag{6.3}$$

Here we initialize with $s_1 = 0$ (as nondegenerate 1-spaces contain no isotropic vectors), but s_2 will depend upon the type of form under consideration. A further consequence of the decomposition is

$$\mu_i' = s_{i-2}\,. \tag{6.4}$$

Therefore, we have the three parameters s_i, k_i', and μ_i', from which it is (at least in principal) easy to calculate all parameters of $[X]$ and (X) using the identities of Sect. 3. The additional identity

$$\lambda_i' = (q-1) + q^2 s_{i-4}\,, \tag{6.5}$$

can be seen within $\langle x, y\rangle^\perp$, where $\langle x\rangle$ and $\langle y\rangle$ are distinct perpendicular isotropic 1-spaces. This is because $\langle x, y\rangle^\perp/\langle x, y\rangle \cong V_{i-4}$.

6.3.1 Symplectic over $\mathbb{F}_2$

The nondegenerate form $f = f_{2m}$ above is symplectic on V_{2m} if it is bilinear with all 1-spaces isotropic: $f(x, x) = 0$ for all $x \in V_{2m}$. Its polar graph is denoted $[\mathrm{Sp}_{2m}(q)]$.

In the special case of symplectic polar spaces over $\mathbb{F}_2$ the corresponding transvection isometries D form a class of 3-transpositions in the full isometry group $G = \mathrm{Sp}_{2m}(2)$ with the codiagram $[D] = [X] = [\mathrm{Sp}_{2m}(2)]$. In this case $s_i = n_i$, $i = 2m$, $q = 2$, and $n_2 = 1 + 2 = 3$.

Theorem 6.7 *(a) For $m \geq 2$ the codiagram $[\mathrm{Sp}_{2m}(2)]$ has extended parameters*

$$\big(2^{2m} - 1\,,\ 2^{2m-1} - 2,\ 2^{2m-2} - 3,\ 2^{2m-2} - 1\,;\\ [2^{m-1} - 1]^{2^{2m-1}+2^{m-1}-1}, [-2^{m-1} - 1]^{2^{2m-1}-2^{m-1}-1}\big)\,.$$

(b) For $m \geq 2$ the diagram $(\mathrm{Sp}_{2m}(2))$ has extended parameters

$$\big(2^{2m} - 1\,,\ 2^{2m-1},\ 2^{2m-2},\ 2^{2m-2}\,;\, [2^{m-1}]^{2^{2m-1}-2^{m-1}-1}, [-2^{m-1}]^{2^{2m-1}+2^{m-1}-1}\big)$$

and spectrum

$$\big((2^{2m-1};\ [2^{m-1}]^{2^{2m-1}-2^{m-1}-1}, [-2^{m-1}]^{2^{2m-1}+2^{m-1}-1})\big)\,.$$

Proof (i) $n = n_{2m} = 2^{2m} - 1$: all 1-spaces are isotropic.

[2] More generally, singular.

(ii) by (6.1) $k = k_{2m} = l'_{2m} = (n_2 - 1)q^{i-2} = 2^{2m-1}$.
(iii) by (6.4) $\mu' = \mu'_{2m} = n_{2m-2} = 2^{2m-2} - 1$.
(iv) by (6.5) $\lambda' = (q-1) + q^2 n_{2m-4} = 1 + 4(2^{2m-4} - 1) = 2^{2m-2} - 3$. □

Corollary 4.4 gives immediately:

Proposition 6.8 ***PR4:*** *the diagram* $(2^{\bullet h} : \mathrm{Sp}_{2m}(2))$ *with* $m \geq 2$ *and* $h \geq 0$ *has size*

$$n = 2^h(2^{2m} - 1)$$

and spectrum

$$((2^{2m-1+h}; [2^{m-1+h}]^{2^{2m-1}-2^{m-1}-1}, [-2^{m-1+h}]^{2^{2m-1}+2^{m-1}-1}, [0]^\star)) .$$

□

Proposition 6.9 ***PR9:*** *the diagram* $(3^{\bullet h} : (2 \times \mathrm{Sp}_6(2))$ *with* $h \geq 0$ *has size*

$$n = 63(3^h)$$

and spectrum

$$((11(3^{h+1}) - 1; [5(3^h) - 1]^{27}, [-3^{h+1} - 1]^{35}, [-1]^\star)) .$$

Proof Apply Corollary 4.4 to the diagram $(\mathrm{Sp}_6(2)) = (2 \times \mathrm{Sp}_6(2))$, which has extended parameters

$$\left(63\,,\ 32,\ 16,\ 16\,;\, [4]^{27}, [-4]^{35}\right) .$$

□

6.3.2 Unitary over $\mathbb{F}_4$

For finite unitary polar graphs we must have $q = t^2$ for some prime power t. The nondegenerate form $f = f_m$ is hermitian (or unitary) on V_m if it is biadditive with

$$f(ax, by) = af(x, y)b^t$$

and

$$f(x, y) = f(y, x)^t$$

for all $x, y \in V_m$ and $a, b \in \mathbb{F}_q$. Its polar graph is denoted $[\mathrm{SU}_{2m}(t)]$.

In the special case of unitary polar spaces over $\mathbb{F}_4$ the corresponding transvection isometries D form a class of 3-transpositions in the isometry group $G = \mathrm{SU}_m(2)$ with the codiagram $[D] = [X] = [\mathrm{SU}_m(2)]$. In this case $s_i = n_i$, $i = m$, $q = 4$, $t = 2$, and $n_2 = 1 + 2 = 3$.

Theorem 6.10 *For* $m \geq 3$ *set*

$$d = 8(2^{2m-3} - 1 - (-2)^{m-2})/9$$

and

$$e = 4(2^{2m-3} - 1 - 7(-2)^{m-3})/9 .$$

(a) For $m \geq 3$ *the codiagram* $[\mathrm{SU}_m(2)]$ *has extended parameters*

$$\begin{aligned}&\big((2^{2m-1} - 1 - (-2)^{m-1})/3\,,\ 2^2(2^{2m-5} - 1 - (-2)^{m-3})/3\,,\\ &\lambda' = 3 + 16(2^{2m-9} - (-2)^{m-5} - 1)/3\,,\ \mu' = (2^{2m-5} - 1 - (-2)^{m-3})/3\,;\\ &\{[r']^g, [s']^f\} = \{[(-2)^{m-3} - 1]^d, [(-2)^{m-2} - 1]^e\}\big) ,\end{aligned}$$

(b) For $m \geq 3$ the diagram $(\mathrm{SU}_m(2))$ has extended parameters

$$\big((2^{2m-1}-1-(-2)^{m-1})/3\,,\ 2^{2m-3}\,,$$
$$\lambda = 3(2^{2m-5}) + (-2)^{m-3}\,,\ \mu = 3(2^{2m-5})\,;$$
$$\{[r]^f, [s]^g\} = \{[-(-2)^{m-3}]^d, [-(-2)^{m-2}]^e\}\big)\,,$$

and spectrum

$$(\!(2^{2m-3}; [-(-2)^{m-3}]^d, [-(-2)^{m-2}]^e)\!)\,.$$

Proof (i) $n = n_m = (2^{2m-1}-1-(-2)^{m-1})/3$: the recursion of (6.3)

$$n_i = 1 + (n_2 - 1)q^{i-2} + qn_{i-2}$$

is initialized by

$$n_1 = 0 = (2-1-1)/3 = (2^{2-1}-1-(-2)^0)/3$$

and

$$n_2 = 3 = (8-1+2)/3 = (2^{4-1}-1-(-2)^{2-1})/3\,.$$

(ii) $k = l' = l'_m = (n_2-1)q^{m-2} = (3-1)4^{m-2}$.

(iii) $\mu' = n_{m-2} = (2^{2m-5}-1-(-2)^{m-3})/3$. □

Proposition 6.11 ***PR6****: the diagram $(4^{\bullet h}\,\mathrm{SU}_m(2)')$ for all $h \geq 0$ and all $m \geq 3$ has size*

$$n = 4^h(2^{2m-1}-1-(-2)^{m-1})/3$$

and spectrum

$$(\!(2^{2m-3+2h}; [-(-2)^{m-3+2h}]^d, [-(-2)^{m-2+2h}]^e, [0]^\star)\!)$$

where

$$d = 8(2^{2m-3}-1-(-2)^{m-2})/9$$

and

$$e = 4(2^{2m-3}-1-7(-2)^{m-3})/9\,.$$

□

Proposition 6.12 ***PR11:*** *the diagram $(3^{\bullet 2h}{:}(2 \times \mathrm{SU}_5(2)))$, for $h \geq 0$, has size*

$$n = 165(3^{2h})$$

and spectrum

$$(\!(129(3^{2h})-1; [3^{2h+2}-1]^{44}, [-3^{2h+1}-1]^{120}, [-1]^\star)\!)\,.$$

□

Proof Apply Corollary 4.4 to the diagram $(\mathrm{SU}_5(2)) = (2 \times \mathrm{SU}_5(2))$, which has size 165 and spectrum

$$(\!(128; [8]^{44}, [-4]^{120})\!)\,.$$

□

Proposition 6.13 ***PR12:*** *the diagram $(3^{\bullet 2h}{:}4^{\bullet 1}{:}\,\mathrm{SU}_3(2)')$, for $h \geq 0$, has size*

$$36(3^{2h})$$

and spectrum

$$(\!(33(3^{2h})-1; [3^{2h}-1]^{27}, [-3^{2h+1}-1]^8, [-1]^\star)\!)\,.$$

Proof Apply Corollary 4.4 to the diagram $(4^{\bullet 1}{:}\,\mathrm{SU}_3(2)')$, which has size

$$n = 36$$

and spectrum

$$(\!(32; [0]^{27}, [-4]^8)\!)\,.$$

□

6.4 Nonsingular orthogonal cases over $\mathbb{F}_2$

Let $V_{2m} = \mathbb{F}_2^{2m}$ admit the nondegenerate symplectic form $f = f_{2m}$. An associated quadratic $q_{2m}^\epsilon = q$ is a map $q: V_{2m} \longrightarrow \mathbb{F}_2$, such that

$$f(x, y) = q(x + y) + q(x) + q(y)$$

for all $x, y \in V_{2m}$. The vectors x with $q(x) = 0$ are singular and those with $q(x) = 1$ are nonsingular. Each of the two types of symplectic 2-spaces resolves into two types of orthogonal 2-spaces. A totally isotropic 2-space is either totally singular (q is identically 0) or is defective—it has exactly two nonsingular vectors. A symplectic hyperbolic 2-space is either orthogonal hyperbolic—a unique nonsingular vector—or is asingular—its only singular vector is 0. Thus the isometry type of a 2-space is uniquely determined by the number of nonsingular vectors it contains—respectively, 0, 2, 1, 3.

Up to isometry, the form q has one of two types denoted by the Witt sign ϵ, equal to $+ = +1$ or $- = -1$ depending upon whether maximal totally singular spaces have dimension m or $m - 1$. The corresponding diagram $(\mathrm{O}_{2m}^\epsilon(2))$ has as vertices the nonsingular 1-spaces $\langle x\rangle \in V_{2m}^\epsilon$ with two adjacent when not perpendicular. That is, $(\mathrm{O}_{2m}^\epsilon(2))$ is the subgraph of $(\mathrm{Sp}_{2m}(2))$ induced on the set of 1-spaces that are nonsingular for q, and correspondingly for $[\mathrm{O}_{2m}^\epsilon(2)]$.

The symplectic transvections centered at nonsingular vectors form a generating conjugacy class[3] D of 3-transpositions in the corresponding orthogonal group $\mathrm{O}_{2m}^\epsilon(2)$.

Theorem 6.14 *(a) For $m \geq 1$ the codiagram $[\mathrm{O}_{2m}^\epsilon(2)]$ has extended parameters*

$$\big(2^{2m-1} - \epsilon 2^{m-1},\ 2^{2m-2} - 1\,,\ 2^{2m-3} - 2\,,\ 2^{2m-3} + \epsilon 2^{m-2};$$
$$\{[\epsilon 2^{m-2} - 1]^{(2^{2m}-4)/3}, [-\epsilon 2^{m-1} - 1]^{(2^m-\epsilon 1)(2^{m-1}-\epsilon 1)/3}\}\big)\,.$$

(b) For $m \geq 1$ the diagram $(\mathrm{O}_{2m}^\epsilon(2))$ has extended parameters

$$\big(2^{2m-1} - \epsilon 2^{m-1},\ 2^{2m-2} - \epsilon 2^{m-1},\ 2^{2m-3} - \epsilon 2^{m-2},\ 2^{2m-3} - \epsilon 2^{m-1}\,;$$
$$\{[\epsilon 2^{m-1}]^{(2^m-\epsilon 1)(2^{m-1}-\epsilon 1)/3}, [-\epsilon 2^{m-2}]^{(2^{2m}-4)/3}\}\big)\,,$$

and spectrum

$$((2^{2m-2} - \epsilon 2^{m-1};\ [\epsilon 2^{m-1}]^{(2^m-\epsilon 1)(2^{m-1}-\epsilon 1)/3}, [-\epsilon 2^{m-2}]^{(2^{2m}-4)/3}))\,.$$

Proof (i) $n_{2m}^\epsilon = 2^{2m-1} - \epsilon 2^{m-1}$: initialize with $n_2^+ = 1$ and $n_2^- = 3$. As V_2^+ contains a unique nonsingular vector, the decomposition $V_{2m}^\epsilon = V_2^+ \perp V_{2m-2}^\epsilon$ gives the recursion

$$n_{2m}^\epsilon = 3n_{2m-2}^\epsilon + (2^{2m-2} - n_{2m-2}^\epsilon) = 2^{2m-2} + 2n_{2m-2}^\epsilon\,,$$

and the result follows.

(ii) $k' = 2^{2m-2} - 1$: consider $[\mathrm{O}_{2m}^\epsilon(2)]$ as an induced subgraph of $[\mathrm{Sp}_{2m}(2)]$. For the nonsingular vector x, a 2-space containing x and in $x^\perp$ must be defective—of its two 1-spaces not containing x, one is singular and one is nonsingular. Therefore

$$k'_{[\mathrm{O}_{2m}^\epsilon(2)]} = \frac{1}{2}k'_{[\mathrm{Sp}_{2m}(2)]} = \frac{1}{2}(2^{2m-1} - 2)\,.$$

(iii) $\lambda' = 2^{2m-3} - 2$: again consider $[\mathrm{O}_{2m}^\epsilon(2)]$ as an induced subgraph of $[\mathrm{Sp}_{2m}(2)]$. In calculating λ' for the symplectic case, and more generally for the polar cases over $\mathbb{F}_q$ as in equation 6.5, we found

$$\lambda'_{[\mathrm{Sp}_{2m}(2)]} = (q - 1) + q^2 n_{2m-4} = 1 + 4(2^{2m-4} - 1) = 2^{2m-2} - 3\,,$$

counting the $q-1$ remaining isotropic 1-spaces of the totally isotropic 2-space $\langle x, y\rangle$ plus the q^2 additional 1-spaces of each 3-space in $\langle x, y\rangle^\perp$ on $\langle x, y\rangle$, these enumerated by the 1-spaces of $\langle x, y\rangle^\perp/\langle x, y\rangle$ of

[3] More accurately, D is a normal set in the orthogonal group. However, the only case in which it is not a generating conjugacy class is $\mathrm{O}_4^+(2)$, where these transvections generate a proper normal subgroup $\mathrm{Sym}(3) \times \mathrm{Sym}(3)$.

dimension 4 less. Here we must restrict ourselves to nonsingular vectors, so the isotropic spaces through x and y are defective. The only singular vectors in $\langle x, y\rangle$ are x and y, and each 3-space in $\langle x, y\rangle^{\perp}$ on $\langle x, y\rangle$ has exactly two additional nonsingular vectors. Therefore, the count becomes

$$\lambda'_{[\mathrm{O}^{\epsilon}_{2m}(2)]} = 0 + 2(2^{2m-4} - 1) = 2^{2m-3} - 2\,.$$

(iv) $\mu' = 2^{2m-3} + \epsilon 2^{m-2}$: the decomposition $V^{\epsilon}_{2m} = V_2^{-} \perp V^{-\epsilon}_{2m-2}$ gives $\mu' = n^{-\epsilon}_{2m-2}$. □

Proposition 6.15 ***PR3:*** *the diagram* $(2^{\bullet h} : \mathrm{O}^{\epsilon}_{2m}(2))$ *for* $h \geq 0$, $\epsilon = \pm$, *and* $m \geq 1$ *has size*

$$n = 2^h(2^{2m-1} - \epsilon 2^{m-1})$$

and spectrum

$$((2^h(2^{2m-2} - \epsilon 2^{m-1});\ [\epsilon 2^{h+m-1}]^{(2^m-\epsilon 1)(2^{m-1}-\epsilon 1)/3}, [-\epsilon 2^{h+m-2}]^{(2^{2m}-4)/3}, [0]^{\star}))\,.$$

□

Proposition 6.16 ***PR10:*** *the diagram* $(3^{\bullet h} : (2 \cdot \mathrm{O}^{+}_{8}(2)))$ *for* $h \geq 0$ *has size*

$$n = 120(3^h)$$

and spectrum

$$((57(3^h) - 1;\ [3^{h+2} - 1]^{35}, [-3^{h+1} - 1]^{84}, [-1]^{\star}))\,.$$

Proof Apply Corollary 4.4 to the diagram $(\mathrm{O}^{+}_{8}(2)) = (2 \cdot \mathrm{O}^{+}_{8}(2))$, which has extended parameters

$$\left(120, 56, 28, 24\,;\, [8]^{35}, [-4]^{84}\right)\,.$$

□

6.5 Nonsingular orthogonal cases over $\mathbb{F}_3$

Let $V = V_m = \mathbb{F}_3^m$ admit the nondegenerate symmetric bilinear (that is, orthogonal) form f. The diagonal of f yields the quadratic form $q\colon V \longrightarrow \mathbb{F}_3$ given by

$$q(x) = f(x, x)\,.$$

Conversely, f can be reconstructed from q via

$$f(x, y) = -q(x + y) + q(x) + q(y)\,.$$

In this context, the isotropic vectors (those x with $q(x) = f(x, x) = 0$) are called singular. As described at the beginning of Sect. 6.3, the singular (= isotropic) 1-spaces form the vertex set of a polar space graph which is strongly regular and indeed rank 3 (by Witt's Theorem). There are two types of nonsingular 1-spaces $\langle x\rangle$; those with $q(x) = 1$ are called $+$-spaces and x is a $+$-vector; those with $q(x) = -1$ are $-$-spaces and x is a $-$-vector.

There are two parameters of interest for the nondegenerate space V—its discriminant and its Witt index. The discriminant δ is the determinant of any Gram matrix for the space. It is either $+1 = +$ or $-1 = -$ and determines V up to isometry. Concretely, V has discriminant $+1$ if and only if it possesses an orthonormal basis.

The Witt index (introduced in the previous section for $\mathbb{F}_2$-spaces) is the maximum dimension of a totally singular subspace (q identically 0). In even dimension $m = 2a$ the Witt index is either a or $a - 1$, and (again as before) we attach the Witt sign ϵ equal to $+ = +1$ or $- = -1$ in these respective cases. In odd dimension $m = 2a + 1$, the Witt index is always a. In even dimension m always

$$\delta\epsilon = -1^{\binom{m+1}{2}}\,,$$

and we use this identity to define the sign ϵ for odd dimension m as well. The space V may then be denoted ${}_\delta V_m^\epsilon$, which is sometimes abbreviated to ${}_\delta V_m$ or V_m^ϵ, since once m has be fixed, the parameters δ and ϵ determine each other.

For odd m an equivalent geometric definition is that V_m has sign ϵ when it is isometric to $x^\perp$ for a $+$-vector x in the even dimensional V_{m+1}^ϵ.[4]

As before, in the polar space of ${}_\delta V_m^\epsilon$ there are exactly two types of 2-spaces spanned by singular (= isotropic) vectors: the totally singular 2-spaces with $q+1=4$ pairwise perpendicular singular 1-subspaces and the hyperbolic 2-spaces with $s_2=2$ nonperpendicular singular 1-subspaces plus a $+$-space and a perpendicular $-$-space. The hyperbolic 2-spaces have type ${}_- V_2^+$.

The 2-spaces spanned by nonsingular vectors have three types. The only nondegenerate example is the asingular space ${}_+V_2^-$, which is spanned by a pair of perpendicular $+$-spaces and a pair of perpendicular $-$-spaces. The two degenerate examples are the $+$-tangent spaces—consisting of a singular radical of dimension 1 and three $+$-spaces—and the similar $-$-tangent spaces.

By Witt's Theorem again, the full isometry group of ${}_\delta V_m^\epsilon$ has rank 3 on the $+$-spaces.[5] The reflections with centers of $+$-type form a normal set of 3-transpositions, commuting pairs of reflections corresponding to asingular 2-spaces ${}_+V_2^-$ and noncommuting pairs to $+$-tangent spaces and their three pairwise nonperpendicular $+$-spaces. We consider the 3-transposition groups (G, D): the group $G = {}^+_\delta\Omega_m^\epsilon(3) = {}^+\Omega_m^\epsilon(3) = {}^+_\delta\Omega_m(3)$ (with $m \geq 4$) is that subgroup of the full isometry group generated by the reflection class D having centers of $+$-type.

Proposition 6.17 *For $m \geq 1$, the number of singular 1-spaces in ${}_\delta V_m^\epsilon$ is*

$$ {}_\delta s_m^\epsilon = \begin{cases} \frac{1}{2}(3^{m-1}-1) & \text{for } m \text{ odd;} \\ \frac{1}{2}(3^{m-1}-1)+\epsilon 3^{(m-2)/2} & \text{for } m \text{ even.} \end{cases} $$

Proof As in (6.3), the decomposition $V_m^\epsilon = V_2^+ \perp V_{m-2}^\epsilon$ implies

$$ s_m^\epsilon = 1 + 3^{m-2} + 3s_{m-2}^\epsilon \,. $$

When initialized with

$$ s_1^\epsilon = 0\,,\ s_2^+ = 2\,,\ s_2^- = 0\,, $$

the result follows. □

Theorem 6.18 (a) *For odd $m \geq 5$ the codiagram $[{}^+\Omega_m^\epsilon(3)]$ has extended parameters*

$$ \begin{aligned} &\big((3^{m-1}-\epsilon 3^{(m-1)/2})/2\,,\ k' = (3^{m-2}+\epsilon 3^{(m-3)/2})/2, \\ &\lambda' = \mu' = (3^{m-3}+\epsilon 3^{(m-3)/2})/2\,; \\ &[r']^g = [3^{(m-3)/2}]^g\,,\ [s']^f = [-3^{(m-3)/2}]^f\big)\,, \end{aligned} $$

and the diagram $({}^+\Omega_m^\epsilon(3))$ has extended parameters

$$ \begin{aligned} &\big((3^{m-1}-\epsilon 3^{(m-1)/2})/2\,,\ k = 3^{m-2}-2\epsilon 3^{(m-3)/2}-1\,, \\ &\lambda = 2(3^{m-3}-\epsilon 3^{(m-3)/2}-1)\,,\ \mu = 2(3^{m-3}-\epsilon 3^{(m-3)/2})\,; \\ &[r]^f = [3^{(m-3)/2}-1]^f\,,\ [s]^g = [-3^{(m-3)/2}-1]^g\big)\,, \end{aligned} $$

where

$$ f = (3^{m-1}-1-(\epsilon-1)(3^{(m-1)/2}-1))/4 $$

and

$$ g = (3^{m-1}-1-(\epsilon+1)(3^{(m-1)/2}+1))/4\,. $$

[4] Our convention for ϵ is that of [3,4]. See [3] for a discussion and a comparison with other conventions from the literature. Our choice differs from that of Brouwer [1], where $\delta\epsilon = -1^{\binom{m}{2}}$. With Brouwer's convention, for odd m and x a $+$-vector within V_m, the even dimensional $x^\perp$ is isometric to V_{m-1}^ϵ.

[5] Similar remarks hold for the $-$-spaces, but this only leads to examples isomorphic to the ones being discussed.

(b) *For even $m \geq 4$ the codiagram $[{}^{+}\Omega_m^{\epsilon}(3)]$ has extended parameters*

$$\begin{aligned}&\big((3^{m-1}-\epsilon 3^{(m-2)/2})/2\,,\ k'=(3^{m-2}-\epsilon 3^{(m-2)/2})/2\,,\\ &\lambda'=(3^{m-3}+\epsilon 3^{(m-4)/2})/2\,,\ \mu'=(3^{m-3}-\epsilon 3^{(m-2)/2})/2\,;\\ &\{[r']^g,[s']^f\}=\{\,[\epsilon 3^{(m-2)/2}]^d\,,\ [-\epsilon 3^{(m-4)/2}]^e\,\}\big)\,,\end{aligned}$$

and the diagram $({}^{+}\Omega_m^{\epsilon}(3))$ has extended parameters

$$\begin{aligned}&\big((3^{m-1}-\epsilon 3^{(m-2)/2})/2\,,\ k=3^{m-2}-1\,,\\ &\lambda=2(3^{m-3}-1)\,,\ \mu=2(3^{m-3}+\epsilon 3^{(m-4)/2})\,;\\ &\{[r]^f,[s]^g\}=\{\,[-\epsilon 3^{(m-2)/2}-1]^d\,,\ [\epsilon 3^{(m-4)/2}-1]^e\,\}\big)\,,\end{aligned}$$

where

$$d=(3^{m/2}-\epsilon)(3^{(m-2)/2}-\epsilon)/8$$

and

$$e=(3^m-9)/8\,.$$

Proof Some of the calculations work better in terms of δ, while for others ϵ may be preferred. As ϵ is the canonical parameter in even dimension, we state the final results in terms of it, remembering that always

$$\delta\epsilon=-1^{\binom{m+1}{2}}\,.$$

Some rules-of-thumb for a fixed δ:

if m is even then dropping to $m-1$ does not change ϵ, while if m is odd then dropping to $m-1$ changes ϵ to $-\epsilon$; thus any drop by 2 changes ϵ to $-\epsilon$;

(i) ${}_{\delta}k_m^{\epsilon}=2\,{}_{\delta}s_{m-1}=\begin{cases}2s_{m-1}^{\epsilon}=3^{m-2}-1 & \text{for } m \text{ even};\\ 2\,s_{m-1}^{-\epsilon}=3^{m-2}-1-2\epsilon 3^{(m-3)/2} & \text{for } m \text{ odd}.\end{cases}$
In the decomposition ${}_{\delta}V_m={}_{+}V_1\perp{}_{\delta}V_{m-1}$, every $+$-tangent on $\langle x\rangle={}_{+}V_1$ is spanned by x and the unique singular 1-space of the tangent, which belongs to $x^{\perp}$. The remaining two $+$-spaces of the tangent are adjacent to $\langle x\rangle$ in the diagram.

(ii)

$$\begin{aligned}{}_{\delta}(k')_m^{\epsilon}&={}_{\delta}n_{m-1}\\ &=n_{m-1}^{\epsilon}=\frac{1}{2}(3^{m-2}-\epsilon 3^{(m-2)/2}) && \text{for } m \text{ even};\\ &=n_{m-1}^{-\epsilon}=\frac{1}{2}(3^{m-2}+\epsilon 3^{(m-3)/2}) && \text{for } m \text{ odd};\\ {}_{\delta}n_m^{\epsilon}&=1+{}_{\delta}k_m+{}_{\delta}k'_m=1+2\,{}_{\delta}s_{m-1}+{}_{\delta}n_{m-1}\\ &=\frac{1}{2}(3^{m-1}-\epsilon 3^{(m-1)/2}) && \text{for } m \text{ odd};\\ &=\frac{1}{2}(3^{m-1}-\epsilon 3^{(m-2)/2}) && \text{for } m \text{ even}.\end{aligned}$$

The identity ${}_{\delta}k'_m={}_{\delta}n_{m-1}$ follows directly from ${}_{\delta}V_m={}_{+}V_1\perp{}_{\delta}V_{m-1}$. Initialization of the recursion is provided by

$${}_{-}n_1^{+}=0\,,\ {}_{+}n_1^{-}=1\,,\ {}_{-}n_2^{+}=1\,,\ {}_{+}n_2^{-}=2\,.$$

(iii)

$$\begin{aligned} {}_\delta(\lambda')^\epsilon_m &= {}_\delta n_{m-2} = n^{-\epsilon}_{m-2} \\ &= \frac{1}{2}(3^{m-3} + \epsilon 3^{(m-3)/2}) && \text{for } m \text{ odd}; \\ &= \frac{1}{2}(3^{m-3} + \epsilon 3^{(m-4)/2}) && \text{for } m \text{ even}. \end{aligned}$$

The identity ${}_\delta(\lambda')^\epsilon_m = {}_\delta n_{m-2}$ follows directly from ${}_\delta V_m = {}_+V_2 \perp {}_\delta V_{m-2}$.

(iv) The parameters we have found so far are enough to calculate all remaining ones using the identities of Sect. 3. Some are also geometrically evident. Consider the decomposition

$${}_\delta V_m = {}_+V_1 \perp {}_-V_2^+ \perp {}_{-\delta} V_{m-3}\,.$$

Let x be a $+$-vector spanning ${}_+V_1$. If z is a nonzero singular vector in the hyperbolic ${}_-V_2^+$ then the 2-space $\langle x, z\rangle$ is a $+$-tangent, and within it $y = x + z$ spans a $+$-space not perpendicular to x. This leads to

$${}_\delta\mu'_m = 3\,{}_{-\delta}n_{m-3} \quad \text{and} \quad {}_\delta\lambda_m = {}_\delta s_{m-1} + 3\,{}_{-\delta}s_{m-3}\,.$$

□

Proposition 6.19 *(a)* ***PR5:*** *the diagram* $(3^{\bullet h}\,{}^+\Omega^\epsilon_m(3))$ *for odd* $m \geq 5$, $\epsilon = \pm$, *and* $h \geq 0$ *has size*

$$n = 3^h(3^{m-1} - \epsilon 3^{(m-1)/2})/2$$

and spectrum

$$((3^{m-2+h} - 2\epsilon 3^{(m-3)/2+h} - 1\,;\ [3^{(m-3)/2+h} - 1]^f\,,\ [-1]^\star,\ [-3^{(m-3)/2+h} - 1]^g))$$

where

$$f = (3^{m-1} - 1 - (\epsilon - 1)(3^{(m-1)/2} - 1))/4$$

and

$$g = (3^{m-1} - 1 - (\epsilon + 1)(3^{(m-1)/2} + 1))/4\,.$$

(b) ***PR5:*** *the diagram* $(3^{\bullet h}\,{}^+\Omega^\epsilon_m(3))$ *for even* $m \geq 6$, $\epsilon = \pm$, *and* $h \geq 0$ *has size*

$$n = 3^h(3^{m-1} - \epsilon 3^{(m-2)/2})/2$$

and spectrum

$$((3^{m-2+h} - 1\,;\ [-\epsilon 3^{(m-2)/2+h} - 1]^d\,,\ [-1]^\star,\ [\epsilon 3^{(m-4)/2+h} - 1]^e))$$

where

$$d = (3^{m/2} - \epsilon)(3^{(m-2)/2} - \epsilon)/8$$

and

$$e = (3^m - 9)/8\,.$$

□

Proposition 6.20 ***PR8:*** *the diagram* $(4^{\bullet h}{:}(3 \cdot {}^+\Omega^-_6(3)))$ *for* $h \geq 0$ *has size*

$$n = 126(4^h)$$

and spectrum

$$((80(4^h)\,;\ [8(4^h)]^{35},\ [-4^{h+1}]^{90},\ [0]^\star))\,.$$

Proof Apply Corollary 4.4 to the diagram $({}^+\Omega^-_6(3)) = (3 \cdot {}^+\Omega^-_6(3))$, which has extended parameters

$$\left(126, 80, 52, 48\,;\ [8]^{35}, [-4]^{90}\right)\,.$$

□

6.6 Sporadic cases

Theorem 6.21 (a) *The codiagram* [Fi_{22}] *has extended parameters*

$$\left(3510, 693, 180, 126\,;\, [63]^{429}, [-9]^{3080}\right).$$

(b) ***PR7(a):*** *The diagram* (Fi_{22}) *has extended parameters*

$$\left(3510, 2816, 2248, 2304\,;\, [8]^{3080}, [-64]^{429}\right)$$

and spectrum

$$((2816\,;\, [8]^{3080}, [-64]^{429})).$$

(c) *The codiagram* [Fi_{23}] *has extended parameters*

$$\left(31671, 3510, 693, 351\,;\, [351]^{782}, [-9]^{30888}\right).$$

(d) ***PR7(b):*** *The diagram* (Fi_{23}) *has extended parameters*

$$\left(31671, 28160, 25000, 25344\,;\, [8]^{30888}, [-352]^{782}\right)$$

and spectrum

$$((28160\,;\, [8]^{30888}, [-352]^{782}))$$

(e) *The codiagram* [Fi_{24}] *has extended parameters*

$$\left(306936, 31671, 3510, 3240\,;\, [351]^{57477}[-81]^{249458}, \right).$$

(f) ***PR7(c):*** *The diagram* (Fi_{24}) *has extended parameters*

$$\left(306936, 275264, 246832, 247104\,;\, [80]^{249458}, [-352]^{57477}\right)$$

and spectrum

$$((275264\,;\, [80]^{249458}, [-352]^{57477})).$$

Proof See [5] or [10] for the basic parameters. The extended parameters can then be calculated as in Sect. 3 and are also given in [1].

While we do not repeat these calculations, the basic parameters for the codiagram (= commuting graph) appear naturally within Fischer's 3-transposition theory. Fischer [5] attacked the classification by induction, noting that the 3-transposition group (G, D) is essentially determined by the codiagrams of two if its "local" 3-transposition subgroups:

$$K_G = \langle \mathrm{C}_D(d) \rangle \quad \text{and} \quad M_G = \langle \mathrm{C}_D(d, c) \rangle$$

for $d, c \in D$ with $|dc| = 3$. Fischer used this local data to reconstruct the global group (G, D).

This is particularly relevant for us, since

$$k' = |\,\mathrm{C}_D(d) \backslash \{d\}| = |[K_G]|$$

and

$$\mu' = |\,\mathrm{C}_D(d, c)| = |[M_G]|\,.$$

The additional 3-transposition subgroup $L_G = \langle \mathrm{C}_D(d, e) \rangle$ with $|de| = 2$ (naturally found as the subgroup K_{K_G} of K_G) yields

$$\lambda' = |\,\mathrm{C}_D(d, e) \backslash \{d, e\}| = |[L_G]|\,.$$

The local parameters k', μ', λ' then allow us, using (3.6), to calculate the global parameter

$$n = 1 + k' + l' = 1 + k' + k'(k' - 1 - \lambda')/\mu' .$$

For each pair of 3-transposition group $\tilde{K}$ supplied with 3-transposition subgroup $\tilde{M}$, Fischer looked for a 3-transposition group G with $[K_G] = [\tilde{K}]$ and $[M_G] = [\tilde{M}]$. Everything went smoothly, producing the symmetric and classical groups which we have discussed; but there was one loose-end—the pair

$$\mathrm{PSU}_6(2) = \tilde{K} \geq \tilde{M} = {}^+\Omega_6^-(3) .$$

This special case led to a tower of 3-transposition groups—those that are sporadic. Specifically Fischer found the $([K], [M])$-tower:

$$\begin{aligned}
&([\mathrm{PSU}_6(2)], [{}^+\Omega_6^-(3)]) = ([K_{\mathrm{Fi}_{22}}], [M_{\mathrm{Fi}_{22}}]) ,\\
&([\mathrm{Fi}_{22}], [{}^+\Omega_7^+(3)]) = ([K_{\mathrm{Fi}_{23}}], [M_{\mathrm{Fi}_{23}}]) ,\\
&([\mathrm{Fi}_{23}], [\mathrm{P}\Omega_8^+(3) : \mathrm{Sym}(3)]) = ([K_{\mathrm{Fi}_{24}}], [M_{\mathrm{Fi}_{24}}]) .
\end{aligned}$$

This construction of the sporadic examples aids in the identification and calculation of their basic parameters. In the table, $[K_G]$ occurs on the line above $[G]$ and $[L_G]$ two lines above. We have: Here the global

$[G]$	$\{n$,	k' ,	$\lambda'\}$				μ'	$[M_G]$
$[\mathrm{PSU}_6(2)]$				*693*	180	51	45	$[\mathrm{SU}_4(2)]$
$[\mathrm{Fi}_{22}]$			*3510*	693	180		126	$[{}^+\Omega_6^-(3)]$
$[\mathrm{Fi}_{23}]$		*31671*	3510	693			351	$[{}^+\Omega_7^+(3)]$
$[\mathrm{Fi}_{24}]$	*306936*	31671	3510				3240	$[\mathrm{P}\Omega_8^+(3) : \mathrm{Sym}(3)]$

parameters (in italics) can be calculated from the local parameters. Initialization is provided by the values for $[\mathrm{SU}_6(2)] = [\mathrm{PSU}_6(2)]$ found in Theorem 6.10[6] ; the remaining M_G were identified as part of Fischer's induction. For the size n of $[\mathrm{Fi}_{22}]$ we calculate

$$1 + 693 + 693(693 - 1 - 180)/126 = 3510$$

as claimed. The others are similar. □

Theorem 6.22 (a) ***PR7(d):*** *the diagram* $(\mathrm{P}\Omega_8^+(2) : \mathrm{Sym}(3))$, *has size*

$$n = 360$$

and spectrum

$$((296; [-64]^2, [8]^{105}, [-4]^{252})) .$$

(b) ***PR7(e):*** *the diagram* $(\mathrm{P}\Omega_8^+(3) : \mathrm{Sym}(3))$, *has size*

$$n = 3240$$

and spectrum

$$((2888; [-352]^2, [8]^{2457}, [-28]^{780})) .$$

[6] Although the codiagrams $[\mathrm{PSU}_6(2)]$ and $[\mathrm{SU}_6(2)]$ are equal, we use the first notation here, because $K_{\mathrm{Fi}_{22}}/\langle d \rangle$ is isomorphic to $\mathrm{PSU}_6(2)$. Recall that central type is a coarser equivalence relation than isomorphism.

Proof By Theorem 6.14 the diagram $(\mathrm{O}_8^+(2)) = (\mathrm{P}\Omega_8^+(2) : \mathrm{Sym}(2))$ has extended parameters

$$\left(120, 56, 28, 24 ; [8]^{35}, [-4]^{84}\right) .$$

As in Proposition 2.3, (the adjacency matrix for) the diagram $(\mathrm{P}\Omega_8^+(2) : \mathrm{Sym}(3))$ is $3 \times (\mathrm{O}_8^+(2))$, so it has size $3(120) = 360$ and spectrum

$$\begin{aligned} &((56 + 2(120); [-(120 - 56)]^2, [8]^{3(35)}, [-4]^{3(84)})) \\ &= ((296; [-64]^2, [8]^{105}, [-4]^{252})) . \end{aligned}$$

Similarly $(\mathrm{P}\Omega_8^+(3) : \mathrm{Sym}(3))$ is $3 \times ({}^+\Omega_8^+(3))$, where the diagram $({}^+\Omega_8^+(3))$ $(= (\mathrm{P}\Omega_8^+(3) : \mathrm{Sym}(2)))$ has, by Theorem 6.18, extended parameters

$$\left(1080, 728, 484, 504 ; [8]^{819}, [-28]^{260}\right) .$$

Proposition 2.3 again applies to give the spectrum

$$\begin{aligned} &((728 + 2(1080); [-(1080 - 728)]^2, [8]^{3(819)}, [-28]^{3(260)})) \\ &= ((2888; [-352]^2, [8]^{2457}, [-28]^{780})) . \end{aligned}$$

□

7 Diagram minimum eigenvalues

Miyamoto [12] associated 3-transposition groups with the Griess algebras of certain vertex operator algebras of OZ-type. In that context, the minimum eigenvalue of the diagram for the group is important, particularly those with minimum eigenvalue greater than or equal to -8. Classification of the associated groups and Griess algebras was pursued by Miyamoto and Kitazume [11] and Matsuo [13,14]. Similar issues arise for minimum eigenvalue at least -64, and that was the initial motivation for the current paper.

7.1 Compact Matsuo and Griess algebras

Let η be an element of $\mathbb{R}$ not equal to 0 or 1. A real *Matsuo algebra* for the eigenvalue η is a commutative algebra $M = \bigoplus_{a \in \mathcal{A}} \mathbb{R}a$ with basis $\mathcal{A} = \{a_i \mid i \in I\}$ of idempotents a_i (called *axes*) and having the property that any two $a, b \in \mathcal{A}$ generate one of the subalgebras

(i) $1A = \mathbb{R}$ with $a = b$;
(ii) $2B = \mathbb{R}^2 = \mathbb{R}a \oplus \mathbb{R}b$ with $ab = 0$;
(iii) $3C(\eta) = \mathbb{R}a \oplus \mathbb{R}b \oplus \mathbb{R}c$ with $xy = \frac{\eta}{2}(x + y - z)$ for $\{x, y, z\} = \{a, b, c\} = (\mathbb{R}a \oplus \mathbb{R}b \oplus \mathbb{R}c) \cap \mathcal{A}$.

On M we define the symmetric bilinear form $\langle \cdot \mid \cdot \rangle$ given by, respectively,

$$\begin{aligned} 1A : &\quad \langle a \mid a \rangle = 1 ; \\ 2B : &\quad \langle a \mid b \rangle = 0 ; \\ 3C(\eta) : &\quad \langle a \mid b \rangle = \frac{\eta}{2} . \end{aligned}$$

We say that the algebra M is *compact* if the associated form is positive definite.

Matsuo algebras were introduced [13–15], because certain compact Matsuo algebras arise as the Griess algebras of compact vertex operator algebras of OZ type, as noted by Miyamoto [12]. A classification of all such Griess algebras is desirable.

The crucial observation, due to Miyamoto, is that in the Griess algebra case, for each axis $a \in \mathcal{A}$, the permutation τ_a of $\mathcal{A}$ given by

$$\begin{aligned} 1A : &\quad \tau_a(a) = a ; \\ 2B : &\quad \tau_a(b) = b ; \end{aligned}$$

$$3C(\eta): \quad \tau_a(b) = c\,.$$

is an automorphism and indeed $\{\tau_{a_i} \mid i \in I\}$ is a normal set of 3-transpositions in the automorphism group of M. It is enough to consider the case in which $D = \{\tau_{a_i} \mid i \in I\}$ is a class of 3-transpositions in the group $G = \langle D\rangle \leq \mathrm{Aut}(M)$.

This property of Griess and Matsuo algebras was seen in [8] to characterize[7] the *axial algebras of Jordan type* η. In that case the symmetric form $\langle\cdot\,|\,\cdot\rangle$ is *associative* in that

$$\langle xy\,|\,z\rangle = \langle x\,|\,yz\rangle$$

for all $x, y, z \in M$. An important property of every associative form is that its radical R is an ideal of the algebra M.

The *Gram matrix* of the form with respect to $\mathcal{A}$ is

$$I + \frac{\eta}{2}H\,,$$

where H is the adjacency matrix of the diagram (D), so the compact axial algebra M of Jordan type η must have

$$1 + \frac{\eta}{2}\rho > 0$$

hence

$$\rho > -\frac{2}{\eta}\,,$$

where ρ is the minimum eigenvalue of (D). In the case $\rho = -\frac{2}{\eta}$, the algebra M is positive semidefinite and its axial quotient M/R is again compact (which is to say, positive definite). The quotient is thus also a candidate to be a compact Griess algebra.

Initial interest focuses on the eigenvalues

$$\eta = \frac{1}{4} \quad \text{and} \quad \eta = \frac{1}{32}$$

since these are the eigenvalues associated with the Monster algebra of Griess [7] as embedded in the Moonshine vertex operator algebra of Frenkel, Lepowsky, and Meurman [6]. Correspondingly, we are interested in the minimum eigenvalues

$$\rho \geq -8 \quad \text{and} \quad \rho \geq -64\,.$$

The Griess algebra case for the eigenvalue $\eta = \frac{1}{4}$ was investigated by Kitazume and Miyamoto [11] and Matsuo [13,14]. The classification is due to Matsuo:

Theorem 7.1 *A compact Griess algebra for the eigenvalue $\eta = \frac{1}{4}$ exists if and only if the associated finite 3-transposition group has one of the central types below. In each case the algebra is uniquely determined as the corresponding Matsuo algebra (modulo the radical of its form when the minimum eigenvalue is $\rho = -8$).*

(a) $\rho = -1:$ $\mathrm{Sym}(3)$;
(b) $\rho = -2^{r+1}$ *with* $0 \leq r \leq 2:$ $(2^{m-1})^r : \mathrm{Sym}(m)$ *for each* $m \geq 4$;
(c) one of the nine individual groups
  (i) $\rho = -4:$ $\mathrm{O}_6^-(2)$; $\mathrm{O}_8^+(2)$; $\mathrm{Sp}_6(2)$;
  (ii) $\rho = -8:$ $2^6 : \mathrm{O}_6^-(2)$; $\mathrm{O}_8^-(2)$; $2^8 : \mathrm{O}_8^+(2)$; $\mathrm{O}_{10}^+(2)$; $2^6 : \mathrm{Sp}_6(2)$; $\mathrm{Sp}_8(2)$.

Matsuo's proof comes in three pieces:

(*i*) Properties of Griess algebras prove that (G, D) must have the central type of a 3-transposition subgroup of some $\mathrm{Sp}_{2n}(2)$. (See [14, Prop. 1].)
(*ii*) Identification of the 3-transposition groups of symplectic type with minimum eigenvalue $\rho \geq -8$. (These are precisely the groups of the theorem, as can be verified from the table in Sect. 6 or the results of the Subsection 7.3.)
(*iii*) For each qualifying group, checking that the appropriate Matsuo algebra (quotient) is indeed a Griess algebra for some vertex operator algebra.

[7] This remark is somewhat inaccurate in the special case $\eta = \frac{1}{2}$. See [8,9] for precision.

7.2 Classification by minimum eigenvalue

The second table of Sect. 6 provides the minimum eigenvalue in bold for the diagram of each finite 3-transposition groups and so has several useful consequences. In parallel to Matsuo's result Theorem 7.1 we have:

Theorem 7.2 *There are nondecreasing, nonnegative integral valued functions $S(t)$ and $I(t)$ defined on $2 \leq t \in \mathbb{Z}^+$, such that the diagram (D) of a finite conjugacy class D of 3-transpositions has minimum eigenvalue $\rho_{\min} \geq -t$ if and only if the corresponding 3-transposition group (G, D) belongs to one of the following central type classes:*

(a) infinitely many groups $3^u{:}2$ of Moufang type (the case $\rho_{\min} = -1$);
(b) $S(t)$ distinct groups $N : \mathrm{Sym}(m)$ for each $m \geq 4$;
(c) $I(t)$ individual examples. □

For a given $\eta \leq \frac{2}{t}$ let $S^\eta(t)$ and $I^\eta(t)$ be the corresponding functions counting those 3-transposition groups realized by some Griess algebra for the eigenvalue η. Clearly $0 \leq S^\eta(t) \leq S(t)$ and $0 \leq I^\eta(t) \leq I(t)$. Matsuo's Theorem 7.1 and the results of the next subsection give

$$3 = S^{\frac{1}{4}}(8) \leq S(8) = 4 \quad \text{and} \quad 9 = I^{\frac{1}{4}}(8) \leq I(8) = 14\,.$$

The differences are caused by Matsuo's proven restriction to symplectic type. For $\rho \geq -8$ three of the four symmetric families as in Theorem 7.2(b) have symplectic type ($\mathrm{W}_3(\tilde{A}_{m-1})$ does not), and Matsuo showed that all lead to Griess algebras. Similarly exactly nine of the 14 individual groups counted by $I(8)$ are of symplectic type, and they too produce Griess algebras. Almost by definition, Sym(3) is the only 3-transposition group that is simultaneously of Moufang and symplectic type, and this is reflected in the stark difference between Theorem 7.1(a) and Theorem 7.2(a).

For the case $\eta = \frac{1}{32}$, hence $\rho \geq -64$, the next section reveals

$$S(64) = 13 \quad \text{and} \quad I(64) = 90\,.$$

Very little is known about the corresponding $S^{\frac{1}{32}}(64)$ and $I^{\frac{1}{32}}(64)$. Chen and Lam [2] have shown that $\mathrm{SU}_3(2)'$ can be realized for $\eta = \frac{1}{32}$. In particular Matsuo's restriction to symplectic type will not be available in this case.

7.3 Minimum eigenvalue $\rho \geq -64$

The table also yields:

Theorem 7.3 *Let (G, D) be a finite 3-transposition group. Then the minimum eigenvalue ρ of its diagram (G) satisfies one of:*

(a) $\rho = -1$ and G has Moufang type $3^u{:}2$.
(b) $\rho = -2^a$ for the positive integer a, with (G, D) being one of:
  *(i) an infinite class of examples with quotient $\mathrm{Sym}(m)$ under **PR2(a)**;*
  *(ii) if a is even and at least 4, an infinite class of examples with quotient $\mathrm{Sym}(m)$ under **PR2(d)**;*
  *(iii) a finite number of classical examples in characteristic 2 under **PR3,4,6,17-19**;*
  *(iv) if a is even, a single mixed characteristic example $(4^6)^h{:}3 \cdot {}^+\Omega_6^-(3)$ with $h = (a-2)/2$ under **PR5,8**;*
  *(v) if $a = 6$ (so $\rho = -64$), the examples Fi_{22} or $\mathrm{P}\Omega_8^+(2) : \mathrm{Sym}(3)$ under **PR7**.*
(c) $\rho = -3^b - 1$ for the positive integer b, with (G, D) being one of:
  *(i) an infinite class of examples with quotient $\mathrm{Sym}(m)$ under **PR2(b)**;*
  *(ii) if b is at least 2, an infinite class of examples with quotient $\mathrm{Sym}(m)$ under **PR2(c)**;*
  *(iii) a finite number of classical examples in characteristic 3 under **PR5,13-16**;*
  *(iv) a finite number of mixed characteristic examples under **PR9-12**.*
*(d) $\rho = -352$ and (G, D) has type Fi_{23}, Fi_{24}, or $\mathrm{P}\Omega_8^+(3) : \mathrm{Sym}(3)$ under **PR7**.* □

Here we list those groups whose diagrams have minimum eigenvalue greater than or equal to -64. As $3^b + 1$ is never a multiple of 8, cases (a) and (b) only overlap at $\rho = -4 = -2^2 = -3^1 - 1$. This case is discussed in Sect. 7.3.2.

We list the actual groups—as detailed in Sect. 2 of [4]—not just their diagrams (although we continue not to distinguish between groups of the same central type unless necessary). Therefore, unlike Theorem 5.3, the exotic cases **PR13-19** from [4] appear separately from other cases. These all involve nonsplitting of certain extensions, and their diagrams are the same as those of groups from earlier in the list all of whose extensions split. For instance, the split extension $4^7 : \mathrm{SU}_7(2)$ of **PR6** and the nonsplit extension $4^7 \cdot \mathrm{SU}_7(2)$ of **PR18** have same diagram, and so they share the minimum eigenvalue -64. Similarly for $\rho = -28$ the groups $(3^5)^2 :{}^+\Omega_5^-(3)$ of **PR5** and $(3^5 \odot 3^5) :{}^+\Omega_5^-(3)$ of **PR13** have the same diagram but differ in that each is the split extension of a subgroup 3^{10} by ${}^+\Omega_5^-(3)$, but the first 3^{10} is, as ${}^+\Omega_5^-(3)$-module, the direct sum $(3^5)^2 = 3^5 \oplus 3^5$ of two copies of the natural module 3^5, while the second is a nonsplit ${}^+\Omega_5^-(3)$-module extension $3^5 \odot 3^5$. (Here we use notation, where a split module extension with submodule A and quotient B is denoted $A \oplus B$, while a nonsplit module extension is $A \odot B$.

The four classes with symmetric quotient mentioned in parts (b) and (c) of the theorem are those of PR2(a), PR2(b), PR2(c), and PR2(d). For the classes PR2(a), PR2(c), and PR2(d) the parameters a or b and m determine the group G uniquely up to central type. That is false for PR2(b) with central type $3^{\bullet h} : \mathrm{Sym}(m)$ whenever $h \geq 3$. In that case, the type of G is that of some $\mathrm{Wr}(B, m)$—the subgroup of the wreathed product of B by $\mathrm{Sym}(m)$ generated by its transpositions [4, p. 162]. Here B can be any group of exponent 3. For $|B| = 3^b$ the group $\mathrm{Wr}(B, m)$ is a 3-transpostion group of central type $3^{\bullet b} : \mathrm{Sym}(m)$ having minimum eigenvalue $-3^b - 1$. As b increases, the number of choices for B increases dramatically. The smallest nontrivial case is $b = 3$, where the only two choices for B are the elementary abelian group 3^3 and the extraspecial group 3^{1+2}. This leads to two different groups with minimum eigenvalue -28, as seen in Sect. 7.3.7.

7.3.1 $\rho = -1$

PR1. The examples are the groups $3^u{:}2$ of Moufang type with complete diagram. (Indeed, any connected regular graph with minimum eigenvalue -1 is complete. Exercise!)

7.3.2 $\rho = -2$

PR2(a). A 3-transposition group has minimum eigenvalue -2 if and only if it is isomorphic to $\mathrm{Sym}(m)$ $(= \mathrm{W}(A_{m-1}))$ for some $m \geq 4$.

7.3.3 $\rho = -4$

The eigenvalue -4 is anomalous, as it can be written -2^2, as in Theorem 7.3(b), and $-3^1 - 1$, as in Theorem 7.3(c). Thus it behaves like a characteristic 2 case and also like a characteristic 3 case. Both parts of the theorem predict two infinite families with symmetric quotient. In the characteristic 2 case these should be **PR2(a)** and **PR2(d)** while in the characteristic 3 case these should be **PR2(b)** and **PR2(c)**. The eigenvalue -4 compromises by choosing **PR2(a)** and **PR2(b)**. We also have the mixed characteristic example ${}^+\Omega_6^-(3)$.

Another mixed characteristic oddity for -4 is that the groups

$$ {}^+\Omega_5^+(3) = \mathrm{O}_6^-(2) \quad \text{and} \quad {}^+\Omega_5^-(3) = 2 \times \mathrm{SU}_4(2) $$

appear twice on the list, once under **PR5** in characteristic 3 and a second time under **PR3** or **PR6**, as appropriate, in characteristic 2.

PR2(a). $2^{m-1} : \mathrm{Sym}(m)$ $(= \mathrm{W}(D_m) = \mathrm{W}_2(\tilde{A}_{m-1}))$ for all $m \geq 4$.
PR2(b). $3^{m-1} : \mathrm{Sym}(m)$ $(= \mathrm{W}_3(\tilde{A}_{m-1}))$ for all $m \geq 4$.
PR3. $\mathrm{O}_6^-(2)$ $(= \mathrm{W}(E_6) = {}^+\Omega_5^+(3))$; $\mathrm{O}_8^+(2)$ $(= \mathrm{W}(E_8)/2)$.
PR4. $\mathrm{Sp}_6(2)$ $(= \mathrm{W}(E_7)/2)$.
PR5. ${}^+\Omega_5^+(3)$ $(= \mathrm{W}(E_6) = \mathrm{O}_6^-(2))$; ${}^+\Omega_5^-(3)$ $(= 2 \times \mathrm{SU}_4(2))$; ${}^+\Omega_6^-(3)$.
PR6. $4^3 : \mathrm{SU}_3(2)'$; $\mathrm{SU}_4(2)$ $(= {}^+\Omega_5^-(3)/2)$; $\mathrm{SU}_5(2)$.

7.3.4 $\rho = -8$

PR2(a). $(2^{m-1})^2 : \mathrm{Sym}(m)$ for all $m \geq 4$.
PR3. $2^6 : \mathrm{O}_6^-(2)$ $(= \mathrm{W}_2(\tilde{E}_6))$; $2^8 : \mathrm{O}_8^+(2)$ $(= \mathrm{W}_2(\tilde{E}_8)/2)$; $\mathrm{O}_8^-(2)$; $\mathrm{O}_{10}^+(2)$.
PR4. $2^6 : \mathrm{Sp}_6(2)$ $(= \mathrm{W}_2(\tilde{E}_7)/2)$; $\mathrm{Sp}_8(2)$.

7.3.5 $\rho = -10$

PR2(b). $(3^{m-1})^2 : \mathrm{Sym}(m)$ for all $m \geq 4$.
PR2(c). $3^m : 2^{m-1} : \mathrm{Sym}(m)$ $(= \mathrm{W}_3(\tilde{D}_m))$ for all $m \geq 4$.
PR5. $3^5 : {}^+\Omega_5^-(3)$ $(= \mathrm{W}_3(\tilde{E}_6)/3)$; $3^5 : {}^+\Omega_5^+(3)$; $3^6 : {}^+\Omega_6^-(3)$; ${}^+\Omega_6^+(3)$; ${}^+\Omega_7^-(3)$; ${}^+\Omega_7^+(3)$; ${}^+\Omega_8^-(3)$.
PR9. $3^7 : (2 \times \mathrm{Sp}_6(2))$ $(= \mathrm{W}_3(\tilde{E}_7))$.
PR10. $3^8 : (2 \cdot \mathrm{O}_8^+(2))$ $(= \mathrm{W}_3(\tilde{E}_8))$.

7.3.6 $\rho = -16$

PR2(a). $(2^{m-1})^3 : \mathrm{Sym}(m)$ for all $m \geq 4$.
PR2(d). $4^m : 3^{m-1} : \mathrm{Sym}(m)$, for all $m \geq 4$.
PR3. $(2^6)^2 : \mathrm{O}_6^-(2)$; $(2^8)^2 : \mathrm{O}_8^+(2)$; $2^8 : \mathrm{O}_8^-(2)$; $2^{10} : \mathrm{O}_{10}^+(2)$; $\mathrm{O}_{10}^-(2)$; $\mathrm{O}_{12}^+(2)$.
PR4. $(2^6)^2 : \mathrm{Sp}_6(2)$; $2^8 : \mathrm{Sp}_8(2)$; $\mathrm{Sp}_{10}(2)$.
PR6. $(4^3)^2 : \mathrm{SU}_3(2)'$; $4^4 : \mathrm{SU}_4(2)$; $4^5 : \mathrm{SU}_5(2)$; $\mathrm{SU}_6(2)$; $\mathrm{SU}_7(2)$.
PR8. $4^6 : (3 \cdot {}^+\Omega_6^-(3))$.

7.3.7 $\rho = -28$

PR2(b). $(3^{m-1})^3 : \mathrm{Sym}(m)$ and $(3^{m-1} \cdot (3^{m-1})^2) : \mathrm{Sym}(m)$ for all $m \geq 4$.
PR2(c). $(3^m)^2 : 2^{m-1} : \mathrm{Sym}(m)$ for all $m \geq 4$.
PR5. $(3^5)^2 : {}^+\Omega_5^-(3)$; $(3^5)^2 : {}^+\Omega_5^+(3)$; $(3^6)^2 : {}^+\Omega_6^-(3)$; $3^6 : {}^+\Omega_6^+(3)$; $3^7 : {}^+\Omega_7^-(3)$; $3^7 : {}^+\Omega_7^+(3)$; $3^8 : {}^+\Omega_8^-(3)$; ${}^+\Omega_8^+(3)$; ${}^+\Omega_9^-(3)$; ${}^+\Omega_9^+(3)$; ${}^+\Omega_{10}^-(3)$.
PR9. $(3^6)^2 : (2 \times \mathrm{Sp}_6(2))$.
PR10. $(3^8)^2 : (2 \cdot \mathrm{O}_8^+(2))$.
PR11. $3^{10} : (2 \times \mathrm{SU}_5(2))$.
PR12. $3^8 : (U : \mathrm{SU}_3(2)')$, $U = 2^{1+6}$, $U' = 2$, $U/U' = 4^3$.
PR13. $(3^5 \odot 3^5) : {}^+\Omega_5^-(3)$.
PR14. $(3^6 \odot 3^6) : (3 \cdot {}^+\Omega_6^-(3))$.
PR15. $3^7 \cdot {}^+\Omega_7^-(3)$.
PR16. $3^8 \cdot {}^+\Omega_8^-(3)$.

7.3.8 $\rho = -32$

PR2(a). $(2^{m-1})^4 : \mathrm{Sym}(m)$ for all $m \geq 4$.
PR3. $(2^6)^3 : \mathrm{O}_6^-(2)$; $(2^8)^3 : \mathrm{O}_8^+(2)$; $(2^8)^2 : \mathrm{O}_8^-(2)$; $(2^{10})^2 : \mathrm{O}_{10}^+(2)$; $2^{10} : \mathrm{O}_{10}^-(2)$; $2^{12} : \mathrm{O}_{12}^+(2)$; $\mathrm{O}_{12}^-(2)$; $\mathrm{O}_{14}^+(2)$.
PR4. $(2^6)^3 : \mathrm{Sp}_6(2)$; $(2^8)^2 : \mathrm{Sp}_8(2)$; $2^{10} : \mathrm{Sp}_{10}(2)$; $\mathrm{Sp}_{12}(2)$.

7.3.9 $\rho = -64$

PR2(a). $(2^{m-1})^5 : \mathrm{Sym}(m)$ for all $m \geq 4$.
PR2(d). $(4^m)^2 : 3^{m-1} : \mathrm{Sym}(m)$, for all $m \geq 4$.
PR3. $(2^6)^4 : \mathrm{O}_6^-(2)$; $(2^8)^4 : \mathrm{O}_8^+(2)$; $(2^8)^3 : \mathrm{O}_8^-(2)$; $(2^{10})^3 : \mathrm{O}_{10}^+(2)$; $(2^{10})^2 : \mathrm{O}_{10}^-(2)$; $(2^{12})^2 : \mathrm{O}_{12}^+(2)$; $2^{12} : \mathrm{O}_{12}^-(2)$; $2^{14} : \mathrm{O}_{14}^+(2)$; $\mathrm{O}_{14}^-(2)$; $\mathrm{O}_{16}^+(2)$.
PR4. $(2^6)^4 : \mathrm{Sp}_6(2)$; $(2^8)^3 : \mathrm{Sp}_8(2)$; $(2^{10})^2 : \mathrm{Sp}_{10}(2)$; $2^{12} : \mathrm{Sp}_{12}(2)$; $\mathrm{Sp}_{14}(2)$.
PR6. $(4^3)^3 : \mathrm{SU}_3(2)'$; $(4^4)^2 : \mathrm{SU}_4(2)$; $(4^5)^2 : \mathrm{SU}_5(2)$; $4^6 : \mathrm{SU}_6(2)$; $4^7 : \mathrm{SU}_7(2)$; $\mathrm{SU}_8(2)$; $\mathrm{SU}_9(2)$.

PR7. Fi_{22}; $\mathrm{P}\Omega_8^+(2):\mathrm{Sym}(3)$.
PR8. $(4^6)^2{:}(3\cdot{}^+\Omega_6^-(3))$.
PR17. $(4^5 \odot 4^5):\mathrm{SU}_5(2)$.
PR18. $4^7\cdot\mathrm{SU}_7(2)$.
PR19. $T:\mathrm{SU}_3(2)'$, $T = 4^{3+(3+3)}$, $\mathrm{Z}(T) = T' = 4^3$, $T/T' = (4^3)^2$.

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