Matrix roots in the max-plus algebra

Jones, Dan

DOI:
10.1016/j.laa.2021.08.008

Matrix Roots in the Max-Plus Algebra

Daniel Jones*

23/08/2021

Abstract. We define positive integer roots of finite matrices in the 2×2 case. Where possible, we try to generalise results to finite $n \times n$ matrices.

Keywords. matrix; max-algebra; roots; powers

AMS subject classifications. 15A15; 15A16; 15A80

1 Introduction

There are links between finite matrices in the max-plus setting and positive matrices in the classical linear algebra setting. Positive matrices are of interest in the study of stochastic and Markov matrices, see [1] and [2], respectively. In the classical world, it is often the case that positive matrices have many roots but there are so-called principal (unique) roots which are of particular interest and defined using diagonalisation or, more generally, Jordan normal form and by considering only positive roots of eigenvalues. These unique principal roots (should they exist) are analogous to positive real roots of positive real numbers. In this context, for example, $-\sqrt{2}$ is a real number whose square is equal to 2, whereas $\sqrt{2}$ is considered the square root of 2 (the principal root). The spectral properties of matrices (multiplicities / distinct number of algebraic eigenvalues) are closely related to the number of roots of A. For irreducible matrices with m eigenvalues, there are typically p^m distinct pth roots [1].

A similar phenomenon is observed in this paper, which is primarily concerned with finite 2×2 matrices in the tropical (max-plus) world. We observe that spectral properties of these 2×2 matrices (multiplicities of the greatest algebraic eigenvalues) reveal information about the number of roots as we define them. We also observe a direct link between the principal tropical roots of finite 2×2 matrices for which the diagonal terms are the same and the principal roots of real numbers in the classical algebra. In [2], infinitely divisible matrices are defined for stochastic matrices, namely matrices for which arbitrary integer roots exist. In that paper, the problem is related to the question of whether a Markov chain is embeddable. In the tropical setting, we show that a 2×2 finite matrix is infinitely divisible if and only if $\mathrm{d}(A) \geq 0$. We also explore infinite divisibility in the context of general $n \times n$ matrices in the final section and use this to explore a class of idempotent square matrices. Note that it was shown in [3]

*Corresponding author. Inria, École Polytechnique, Paris Saclay, France, danjones1729@hotmail.co.uk. This research was supported by EPSRC grant EP/J00829X/1.

that finding kth roots of Boolean matrices (equivalently kth roots of digraphs) for any fixed positive integer $k \geq 2$ is NP-complete and so we cannot expect that finding roots is a tractable problem for higher dimensions than 2.

2 Preliminaries

For a full introduction to max-plus algebra, see [4].

We assume everywhere that $n \geq 1$ is a natural number and define $N := \{1, ..., n\}$. If $a, b \in \mathbb{R}$ then we set

$$a \oplus b = \max(a, b)$$

and

$$a \otimes b = a + b.$$

The symbol $\otimes$ is often omitted, in the same way the symbol $\times$ is often omitted in classical linear algebra. It will also be useful to define the *dual* operations as follows: for $a, b \in \mathbb{R}$, we set

$$a \oplus' b = \min(a, b)$$

and

$$a \otimes' b = a + b.$$

If $a \in \mathbb{R}$, then the symbol a^{-1} stands for $-a$. The symbol a^k ($k \geq 1$ integer) stands for the iterated product $a \otimes a \otimes \dots$ in which the symbol a stands k times (that is ka in conventional notation). By *max-algebra* (also called "tropical linear algebra") we understand the analogue of linear algebra developed for the pair of operations $(\oplus, \otimes)$, extended to matrices and vectors as in conventional linear algebra. That is, if $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries from $\mathbb{R}$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if

$$c_{ij} = \bigoplus_{k \in N} a_{ik} \otimes b_{kj} = \max_{k \in N}(a_{ik} + b_{kj})$$

for all i and j. If $\alpha \in \mathbb{R}$, then $\alpha \otimes A = (\alpha \otimes a_{ij})$. Given a matrix $A \in \mathbb{R}^{n \times n}$ and a natural number $k \geq 1$, we denote the iterated product of A, k times, by $A^k = \left(a_{ij}^{(k)}\right)$. Note here that a_{ij}^k denotes the kth power of a_{ij}, whereas $a_{ij}^{(k)}$ denotes the (i, j) component in the kth power of A. Define the *min-trace (max-trace) of* A to be the smallest (greatest) of its leading diagonal elements and denoted $\mathrm{Tr}_{\oplus'}(A)$ and $\mathrm{Tr}_{\oplus}(A)$, respectively. That is $\mathrm{Tr}_{\oplus'}(A) = \bigoplus'_{i \in N} a_{ii}$ and $\mathrm{Tr}_{\oplus}(A) = \bigoplus_{i \in N} a_{ii}$. If $A \in \mathbb{R}^{2 \times 2}$, then a' denotes $\mathrm{Tr}_{\oplus'}(A)$ and a denotes $\mathrm{Tr}_{\oplus}(A)$. We denote by P_n the set of permutations on N. The *max-algebraic permanent of* $A \in \mathbb{R}^{n \times n}$ is denoted

$$\mathrm{maper}(A) = \bigoplus_{\sigma \in P_n} \bigotimes_{i \in N} a_{i,\sigma(i)} = \max_{\sigma \in P_n} \sum_{i \in N} a_{i,\sigma(i)}.$$

We respectively define the notions of *positive* and *negative determinant of* A as

$$\det^+(A) = \bigoplus_{\sigma\in P_n:\operatorname{sgn}(\sigma)=1} \bigotimes_{i\in N} a_{i,\sigma(i)} \text{ and } \det^-(A) = \bigoplus_{\sigma\in P_n:\operatorname{sgn}(\sigma)=-1} \bigotimes_{i\in N} a_{i,\sigma(i)},$$

where $\operatorname{sgn}(\sigma) = 1$ if σ is an even permutation and $\operatorname{sgn}(\sigma) = -1$ if σ is an odd permutation. For notational convenience, we also define the quantity

$$\mathrm{d}(A) = \det^+(A)\left[\det^-(A)\right]^{-1}.$$

For $A \in \mathbb{R}^{2\times 2}, \det^+(A) = a'a, \det^-(A) = a_{12}a_{21}$ and $\mathrm{d}(A) = a'aa_{12}^{-1}a_{21}^{-1}$. Note $\det^+(A) \geq \det^-(A)$ if and only if $\mathrm{d}(A) \geq 0$. Given a matrix $M \in \mathbb{R}^{2\times 2}$, define the matrix

$$\ominus(M) := \begin{pmatrix} \sqrt{\mathrm{d}(M)^{-1}} & \sqrt{\mathrm{d}(M)} \\ \sqrt{\mathrm{d}(M)} & \sqrt{\mathrm{d}(M)^{-1}} \end{pmatrix} \circ M, \tag{1}$$

where $\circ$ denotes component-wise max-plus multiplication, or the tropical Hadamard product. For $A \in \mathbb{R}^{n\times n}$, we define the associated weighted digraph $D_A = (N(A), E(A), w)$, where $N(A) = N, E(A) = \{(i,j) : a_{ij} \in \mathbb{R}\}$ and weights are $w(i,j) = a_{ij}$ for $(i,j) \in E(A)$. In this paper we deal only with finite matrices and so the associated weighted digraphs are complete. Suppose that $\sigma = (i_1, \ldots, i_p = i_1)$ is a cycle in D_A, then the *weight* of σ is defined to be

$$w(\sigma, A) = a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{p-1} i_1}.$$

The quantity

$$\mu(\sigma, A) = w(\sigma, A)^{\frac{1}{k}}$$

denotes the *geometric mean* of $\sigma = (i_1, \ldots, i_k, i_1)$. The maximum geometric mean over all possible cycles in D_A is called the *maximum cycle mean* and denoted

$$\lambda(A) = \max_{\sigma} \mu(\sigma, A).$$

The *critical subgraph* of D_A is the subgraph of D_A comprising only cycles σ for which $\mu(\sigma, A) = \lambda(A)$. Such cycles are called *critical cycles.* For finite matrices, the greatest common divisor of the lengths of these critical cycles is the cyclicity of $\lambda(A)$. For finite (and therefore irreducible) matrices A, the maximum cycle mean is the unique geometric eigenvalue of A. That is to say, it is the unique λ for which there exists $x \in \mathbb{R}^n$ such that $Ax = \lambda x$. For more on the eigenproblem, see [5]. For $A \in \mathbb{R}^{2\times 2}$, we have $\lambda(A) = \max\left\{a, \sqrt{a_{12}, a_{21}}\right\}$.

Lemma 1. *Suppose* $a' = a$. *Then* $d(A) \geq 0$ *if and only if* $\lambda(A) = a$.

Proof. The result follows immediately since $\sqrt{a'a} = a$. □

Given $A \in \mathbb{R}^{n\times n}$, an *algebraic eigenvalue* of A is a root of the *characteristic max-polynomial* defined $\chi_A(x) = \operatorname{maper}(A \oplus Ix)$, where a *root* is a point of non-differentiability of the polynomial function. The characteristic max-polynomial is composed of piecewise linear functions and the multiplicity of a root is defined as the change in gradient at that root. The greatest algebraic eigenvalue coincides with the maximum cycle mean $\lambda(A)$. It is through this interpretation as an algebraic eigenvalue that we define the multiplicity of $\lambda(A)$. For $A \in \mathbb{R}^{2\times 2}$, $\lambda(A)$ is either a simple eigenvalue (has multiplicity 1) or a double eigenvalue (has multiplicity 2).

Given $A \in \mathbb{R}^{n\times n}$ and $k \geq 1$, the matrix $B \in \mathbb{R}^{n\times n}$ is called a *k root* of A if $B^k = A$. If, in addition, there does not exist a matrix $C \geq B, C \neq B$ such that $C^k = A$, then B is called a *principal k root* of A. We define the equivalence relation $\equiv_k$ so that $B \equiv_k C$ if $B^k = C^k$ and either $B \leq C$ or $C \leq B$. This defines a set of equivalence classes where each class can be represented by the component-wise supremum of its constituent members (the *principal* member). It follows that if there exists a k root of A, then there exists a corresponding principal k root of A. The main achievement of this paper is in identifying all principal k roots of all matrices $A \in \mathbb{R}^{2\times 2}$ for all integers $k \geq 1$.

3 Finite 2×2 matrices

3.1 Matrix Powers

In order to better understand matrix roots of 2×2 matrices of higher orders (cube roots, fourth roots, fifth roots etc.), we first look to better understand matrix powers. If we are examining the powers of a 2×2 matrix B, then it is the relationship between the elements of B which dictate the nature of its natural powers. Useful and well-known in the study of matrix powers is the Cyclicity Theorem, see [6], [7, Theorem 3.9], [8, Theorem 3.109], [9, Theorem 27-6]. Since we will only use this theorem in the context of 2×2 finite matrices, we present here a simplified version.

Theorem 1 (Cyclicity Theorem)**.** *Let $B \in \mathbb{R}^{2\times 2}$ and c be the cyclicity of $\lambda(B)$. Then for k sufficiently large, it holds*

$$B^{k+c} = \lambda(B)^c \otimes B^k.$$

Recall $b' = b_{11} \oplus' b_{22}$ and $b = b_{11} \oplus b_{22}$. We show that it is where 0 stands in relation to the quantities ${b'}^2 b_{12}^{-1} b_{21}^{-1}$ and $b^2 b_{12}^{-1} b_{21}^{-1}$ which dictates the nature of the natural powers of B and we give explicit formulae for those powers. We consider three cases:

1) $b^2 b_{12}^{-1} b_{21}^{-1} < 0$ (see Lemmas 2 and 3).

2) ${b'}^2 b_{12}^{-1} b_{21}^{-1} \leq 0 \leq b^2 b_{12}^{-1} b_{21}^{-1}$ (see Lemma 4).

3) $0 < {b'}^2 b_{12}^{-1} b_{21}^{-1}$ (see Lemmas 5 and 6).

Before the statement of the lemmas, one useful property of 2×2 matrix powers should be noted.

Remark 1. *The position of the smaller of the diagonal elements in B is the position of the smaller of the diagonal elements in B^ℓ for all $\ell \geq 1$. That is to say, $b_{11} \leq b_{22}$ if and only if $b_{11}^{(\ell)} \leq b_{22}^{(\ell)}$.*

Lemma 2. *Let $B \in \mathbb{R}^{2\times 2}$ and $\ell \geq 1$ a natural number. Suppose $b^2 b_{12}^{-1} b_{21}^{-1} < 0$. Then $B^{2\ell} = \lambda^{2\ell-2} B^2$, where $\lambda = \sqrt{b_{12} b_{21}}$. Further, $\mathrm{d}\left(B^{2\ell}\right) > 0$ for all $\ell \geq 1$.*

Lemma 3. *Let $B \in \mathbb{R}^{2\times 2}$ and $\ell \geq 1$ a natural number. Suppose $b^2 b_{12}^{-1} b_{21}^{-1} < 0$. Then $B^{2\ell+1} = \lambda^{2\ell-2} B^3$, where $\lambda = \sqrt{b_{12} b_{21}}$. Further, $\mathrm{d}\left(B^{2\ell+1}\right) < 0$ for all $\ell \geq 1$.*

Lemma 4. *Let $B \in \mathbb{R}^{2\times 2}$ and $\ell \geq 2$ a natural number. Suppose $b'^2 b_{12}^{-1} b_{21}^{-1} \leq 0 \leq b^2 b_{12}^{-1} b_{21}^{-1}$. Then $B^\ell = \lambda^{\ell-2} B^2$, where $\lambda = b$. Further, $\mathrm{d}\left(B^\ell\right) = 0$ for all $\ell \geq 2$. Note $\mathrm{d}(B)$ is not necessarily equal to 0.*

Lemma 5. *Let $B \in \mathbb{R}^{2\times 2}$ and $\ell \geq 1$ a natural number. Suppose $b'^2 b_{12}^{-1} b_{21}^{-1} > 0$ and $b' = b_{11}$. Then*

$$B^\ell = \begin{pmatrix} b'^\ell \oplus b_{12} b_{21} b^{\ell-2} & b_{12} b^{\ell-1} \\ b_{21} b^{\ell-1} & b^\ell \end{pmatrix}.$$

Further, $\mathrm{d}\left(B^\ell\right) \geq 0$ for all $\ell \geq 1$ and $\mathrm{d}\left(B^\ell\right)$ is non-increasing. Additionally, $\mathrm{d}\left(B^\ell\right) = 0$ for ℓ sufficiently large (as soon as $B^{\ell+1} = \lambda B^\ell$, where $\lambda = b$) when $b' < b$ and $\mathrm{d}\left(B^\ell\right) = \mathrm{d}(B) > 0$ for all $\ell \geq 1$ when $b' = b$.

Lemma 6. *Let $B \in \mathbb{R}^{2\times 2}$ and $\ell \geq 1$ a natural number. Suppose $b'^2 b_{12}^{-1} b_{21}^{-1} > 0$ and $b' = b_{22}$. Then*

$$B^\ell = \begin{pmatrix} b^\ell & b_{12} b^{\ell-1} \\ b_{21} b^{\ell-1} & b'^\ell \oplus b_{12} b_{21} b^{\ell-2} \end{pmatrix}.$$

Further, $\mathrm{d}\left(B^\ell\right) \geq 0$ for all $\ell \geq 1$ and $\mathrm{d}\left(B^\ell\right)$ is non-increasing. Additionally, $\mathrm{d}\left(B^\ell\right) = 0$ for ℓ sufficiently large (as soon as $B^{\ell+1} = \lambda B^\ell$, where $\lambda = b$) when $b' < b$ and $\mathrm{d}\left(B^\ell\right) = \mathrm{d}(B) > 0$ for all $\ell \geq 1$ when $b' = b$.

Proof of Lemmas 2 and 3. Suppose $b^2 b_{12}^{-1} b_{21}^{-1} < 0$. In general,

$$B^2 = \begin{pmatrix} b_{12} b_{21} & b_{12} b \\ b_{21} b & b_{12} b_{21} \end{pmatrix} \text{ and } B^3 = \begin{pmatrix} b_{12} b_{21} b & b_{21} b_{12}^2 \\ b_{12} b_{21}^2 & b_{12} b_{21} b \end{pmatrix}. \tag{2}$$

The eigenvalue $\lambda = \sqrt{b_{12} b_{21}}$ is double and the cyclicity of B is 2, which implies generally that $B^{2+\ell} = \lambda^2 B^\ell$ for ℓ sufficiently large. But in the present case, this is true for $\ell \geq 2$ since $B^4 = \lambda^2 B^2$. In particular, by stating the later result we readily get

$$(\forall \ell \geq 1)\, B^{2\ell} = \lambda^{2\ell-2} B^2, B^{2\ell+1} = \lambda^{2\ell-2} B^3.$$

From (2) we see $\mathrm{d}\left(B^2\right) = \left(b_{12} b_{21}\right)^2 \left(b_{12} b_{21} b^2\right)^{-1} = b_{12} b_{21} b^{-2} > 0$. It follows that for $\ell \geq 1$ we have $\mathrm{d}\left(B^{2\ell}\right) = \mathrm{d}\left(B^2\right) > 0$. Similarly, $\mathrm{d}\left(B^3\right) = \left(b_{12} b_{21} b\right)^2 \left(b_{12} b_{21}\right)^{-3} = b^2 b_{12}^{-1} b_{21}^{-1} < 0$. It follows that for $\ell \geq 1$ we have $\mathrm{d}\left(B^{2\ell+1}\right) = \mathrm{d}\left(B^3\right) < 0$. □

Proof of Lemma 4. Suppose ${b'}^2 b_{12}^{-1} b_{21}^{-1} \leq 0 \leq b^2 b_{12}^{-1} b_{21}^{-1}$. It can be readily shown that $\lambda = b$ is a simple or double eigenvalue depending on whether $b^2 > b_{12} b_{21}$ or not, respectively. In either case, the cyclicity of B is 1 which implies generally that $B^{\ell+1} = \lambda B^{\ell}$ for ℓ sufficiently large. Moreover, in the case $b' = b_{11}$, we get that $(b_{12}, b)^T$ is a right eigenvector, since $B\,(b_{12}, b)^T = \lambda\,(b_{12}, b)^T$. Also note that

$$B^2 = \begin{pmatrix} b_{12}b_{21} & b_{12}b \\ b_{21}b & b^2 \end{pmatrix} = (b_{12}, b)^T\,(b_{21}, b)\,.$$

It follows that for $\ell \geq 2$, we have

$$B^{\ell} = B^{\ell-2}B^2 = B^{\ell-2}\begin{pmatrix} b_{12} \\ b \end{pmatrix}(b_{21}, b) = \lambda^{\ell-2}\begin{pmatrix} b_{12} \\ b \end{pmatrix}(b_{21}, b) = \lambda^{\ell-2}B^2.$$

A similar result is obtained in the case $b' = b_{22}$. Note that $\mathrm{d}\left(B^2\right) = 0$ in both cases and so it follows $(\forall \ell \geq 2)\,\mathrm{d}\left(B^{\ell}\right) = 0$. It is not necessarily true however that $\mathrm{d}\,(B)$ is equal to 0. □

Proof of Lemmas 5 and 6. Suppose $0 < {b'}^2 b_{12}^{-1} b_{21}^{-1}$. It can be shown that $\lambda = b$ is simple or double depending if $b' < b$ or not, respectively. In both cases, the cyclicity of B is 1 which implies generally that $B^{\ell+1} = \lambda B^{\ell}$ for ℓ sufficiently large, in which case B^{ℓ} has the same formula as in Lemma 4. However, contrary to the previous case, this ℓ can be arbitrarily large and depends on the value of $b'b^{-1}$. In fact, it can be shown that it happens when (by a slight abuse of notation)

$$\ell \geq \frac{b_{12}b_{21}\;{b'}^2}{b'b^{-1}} + 2.$$

For the more general formula (which applies either when $b' < b$ or $b' = b$), let us consider the case $b' = b_{11}$ (Lemma 5) and proceed by proof by induction. It is readily seen that

$$B^2 = \begin{pmatrix} {b'}^2 & b_{12}b \\ b_{21}b & b^2 \end{pmatrix} \text{ and } B^3 = \begin{pmatrix} {b'}^3 \oplus b_{12}b_{21}b & b_{12}b^2 \\ b_{21}b^2 & b^3 \end{pmatrix}.$$

For $\ell \geq 2$, let $\mathrm{P}\,(\ell)$ be the statement:

$$\text{“}B^{\ell} = \begin{pmatrix} {b'}^{\ell} \oplus b_{12}b_{21}b^{\ell-2} & b_{12}b^{\ell-1} \\ b_{21}b^{\ell-1} & b^{\ell} \end{pmatrix}.\text{”}$$

Clearly $\mathrm{P}\,(2)$ and $\mathrm{P}\,(3)$ hold. Suppose now $\mathrm{P}\,(\ell)$ holds for some $\ell \geq 2$. Then

$$\begin{aligned} B^{\ell+1} &= \begin{pmatrix} {b'}^{\ell} \oplus b_{12}b_{21}b^{\ell-2} & b_{12}b^{\ell-1} \\ b_{21}b^{\ell-1} & b^{\ell} \end{pmatrix} \otimes \begin{pmatrix} b' & b_{12} \\ b_{21} & b \end{pmatrix} \\ &= \begin{pmatrix} {b'}^{\ell+1} \oplus b'b_{12}b_{21}b^{\ell-2} \oplus b_{12}b_{21}b^{\ell-1} & b_{12}\;{b'}^{\ell} \oplus b_{12}^2 b_{21}b^{\ell-2} \oplus b_{12}b^{\ell} \\ b'b_{21}b^{\ell-1} \oplus b_{21}b^{\ell} & b_{12}b_{21}b^{\ell-1} \oplus b^{\ell+1} \end{pmatrix} \\ &= \begin{pmatrix} {b'}^{\ell+1} \oplus b_{12}b_{21}b^{\ell-1} & b_{12}b^{\ell} \\ b_{21}b^{\ell} & b^{\ell+1} \end{pmatrix} \end{aligned}$$

and so $\mathrm{P}(\ell+1)$ holds. It follows by induction that $\mathrm{P}(\ell)$ holds for all $\ell \geq 2$. The case for $b' = b_{22}$ (Lemma 6) is similar and yields for $\ell \geq 2$:

$$B^{\ell} = \begin{pmatrix} b^{\ell} & b_{12}b^{\ell-1} \\ b_{21}b^{\ell-1} & b'^{\,\ell} \oplus b_{12}b_{21}b^{\ell-2} \end{pmatrix}.$$

In both cases note that $\det^{+}\left(B^{\ell}\right) = \left(b'b\right)^{\ell} \oplus b_{12}b_{21}b^{2\ell-2}$ and $\det^{-}\left(B^{\ell}\right) = b_{12}b_{21}b^{2\ell-2}$ and so $\mathrm{d}\left(B^{\ell}\right) \geq 0$ for all $\ell \geq 2$ and is non-increasing (since it can be shown that $b_{12}b_{21}b^{2\ell-2}$ is growing at a faster rate than $\left(b'b\right)^{\ell}$ in the case $b' < b$). In fact, when $b' < b$, we have $\mathrm{d}\left(B^{2}\right) > 0$ and $\mathrm{d}\left(B^{\ell}\right) = 0$ for ℓ sufficiently large (as soon as $B^{\ell+1} = \lambda B^{\ell}$). Note that when $b' = b$, we have $\det^{+}\left(B^{\ell}\right) = \left(b'b\right)^{\ell} = b^{2\ell}$ for all ℓ and it follows $\mathrm{d}\left(B^{\ell}\right) = \mathrm{d}\left(B\right) > 0$. □

3.2 Odd Roots

Theorem 2. *Let $A \in \mathbb{R}^{2\times 2}$ and $k \geq 1$. If $d(A) \geq 0$, then the unique principal $2k+1$ root is*

$$B = (b_{ij}) = \left(\left(a_{ii} \oplus a_{jj}\right)^{-\frac{2k}{2k+1}} a_{ij}\right).$$

Proof. We consider the cases $\mathrm{d}(A) > 0$ and $\mathrm{d}(A) = 0$ separately and refer in each case to the appropriate lemmas from Lemmas 2 - 6.

- Suppose $\mathrm{d}(A) > 0$ and let us assume $a' = a_{11}$ (the case $a' = a_{22}$ will yield the same result via Lemma 6). It follows by Remark 1 and Lemmas 2 - 6 that the only solution is the one given by Lemma 5: $b'^{\,2} b_{12}^{-1} b_{21}^{-1} > 0$ and

$$B^{2k+1} = \begin{pmatrix} b'^{\,2k+1} \oplus b_{12}b_{21}b^{2k-1} & b_{12}b^{2k} \\ b_{21}b^{2k} & b^{2k+1} \end{pmatrix} = \begin{pmatrix} a' & a_{12} \\ a_{21} & a \end{pmatrix}. \tag{3}$$

  It follows from (3)

$$b = a^{-\frac{2k}{2k+1}} a, b_{12} = a^{-\frac{2k}{2k+1}} a_{12}, b_{21} = a^{-\frac{2k}{2k+1}} a_{21}. \tag{4}$$

  Substituting (4) into the $(1,1)$ component of (3) we see

$$b_{12}b_{21}b^{2k-1} = a_{12}a_{21}a^{\frac{-4k+2k-1}{2k+1}} = a_{12}a_{21}a^{\frac{-(2k+1)}{2k+1}} = a_{12}a_{21}a^{-1} < a'$$

  and so it follows from (3) and cancellation that $b'^{\,2k+1} = a'$, yielding $b' = (a')^{-\frac{2k}{2k+1}} a'$. We conclude

$$B = (b_{ij}) = \left(\left(a_{ii} \oplus a_{jj}\right)^{-\frac{2k}{2k+1}} a_{ij}\right).$$

- Suppose $\mathrm{d}(A) = 0$ and suppose $a' = a_{11}$ (the case $a' = a_{22}$ will yield the same result). It follows by Remark 1 and Lemmas 2 6 that there are only two cases to consider regarding the structure of the corresponding matrix B.

– If B satisfies the condition of Lemma 4, then

$$B^{2k+1} = b^{2k-1} \begin{pmatrix} b_{12}b_{21} & b_{12}b \\ b_{21}b & b^2 \end{pmatrix} = \begin{pmatrix} a' & a_{12} \\ a_{21} & a \end{pmatrix}. \tag{5}$$

It follows from (5)

$$b = a^{-\frac{2k}{2k+1}} a, b_{12} = a^{-\frac{2k}{2k+1}} a_{12}, b_{21} = a^{-\frac{2k}{2k+1}} a_{21}. \tag{6}$$

Note

$$b^{2k-1} b_{12} b_{21} = a^{\frac{2k-1}{2k+1}} a_{12} a_{21} a^{\frac{-4k}{2k+1}} = a_{12} a_{21} a^{\frac{-(2k+1)}{2k+1}} = a_{12} a_{21} a^{-1} = a'.$$

It follows from Lemma 4 and the definition of b' that

$$\begin{aligned} b' &\leq \min\left\{\sqrt{b_{12}b_{21}}, b\right\} \\ &= \min\left\{a^{-\frac{2k}{2k+1}}\sqrt{a_{12}a_{21}}, a^{-\frac{2k}{2k+1}} a\right\} \\ &= \min\left\{a^{-\frac{2k}{2k+1}}\sqrt{a'a}, a^{-\frac{2k}{2k+1}} a\right\} \\ &= a^{-\frac{2k}{2k+1}}\sqrt{a'a}. \end{aligned}$$

We conclude that the component-wise supremum over all B such that ${b'}^2 b_{12}^{-1} b_{21}^{-1} \leq 0 \leq b^2 b_{12}^{-1} b_{21}^{-1}, B^{2k+1} = A$ and satisfy the condition of Lemma 4 is given by

$$B_1 = a^{-\frac{2k}{2k+1}} \begin{pmatrix} \sqrt{a'a} & a_{12} \\ a_{21} & a \end{pmatrix}$$

which is also a solution.

– If B satisfies the condition of Lemma 5, then

$$B^{2k+1} = \begin{pmatrix} {b'}^{2k+1} \oplus b_{12}b_{21}b^{2k-1} & b_{12}b^{2k} \\ b_{21}b^{2k} & b^{2k+1} \end{pmatrix} = \begin{pmatrix} a' & a_{12} \\ a_{21} & a \end{pmatrix}. \tag{7}$$

It follows from (7)

$$b = a^{-\frac{2k}{2k+1}} a, b_{12} = a_{12} a^{-\frac{2k}{2k+1}}, b_{21} = a_{21} a^{-\frac{2k}{2k+1}}. \tag{8}$$

From (8) we see

$$b_{12} b_{21} b^{2k-1} = a_{12} a_{21} a^{-1} = a'$$

and so ${b'}^{2k+1} \leq a'$ is sufficient. We conclude that the component-wise supremum over all B such that ${b'}^2 b_{12}^{-1} b_{21}^{-1} > 0, B^{2k+1} = A$ and satisfy the condition of Lemma 5 is given by

$$B_2 = (b_{ij}) = \left((a_{ii} \oplus a_{jj})^{-\frac{2k}{2k+1}} a_{ij}\right)$$

which is also a solution.

We conclude by noting that the only distinction between B_1 and B_2 is the $(1,1)$ component. It can be shown that

$$(B_1)_{11} = a^{-\frac{2k}{2k+1}}\sqrt{a'a} \leq (a')^{-\frac{2k}{2k+1}}\, a' = (B_2)_{11}$$

and so the component-wise supremum over all matrices B such that $B^{2k+1} = A$ is given by

$$B = (b_{ij}) = \left((a_{ii} \oplus a_{jj})^{-\frac{2k}{2k+1}}\, a_{ij}\right).$$

The result follows. □

Theorem 3. *Let $A \in \mathbb{R}^{2\times 2}$ and $k \geq 1$. If $\mathrm{d}(A) < 0$, then there exists a matrix B such that $B^{2k+1} = A$ if and only if $a' = a$. Further, if $a' = a$, then the unique principal $2k+1$ root is given by*

$$B = (\sqrt{a_{12}a_{21}})^{-\frac{2k}{2k+1}}\, A.$$

Proof. Suppose $\mathrm{d}(A) < 0$. It follows from Lemmas 2 - 6 and (2) that the only solution is obtained through Lemma 3: $a' = a$ and $B^{2k+1} = \lambda^{2k-2}B^3$, where $\lambda = \sqrt{b_{12}b_{21}}$. By referring to (2), we see

$$\begin{pmatrix} a & a_{12} \\ a_{21} & a \end{pmatrix} = (b_{12}b_{21})^k \begin{pmatrix} b & b_{12} \\ b_{21} & b \end{pmatrix},$$

yielding a system of three linear equations in three variables. It can be shown

$$\begin{aligned} b &= a\,(\sqrt{a_{12}a_{21}})^{-\frac{2k}{2k+1}} \\ b_{12} &= a_{12}\,(\sqrt{a_{12}a_{21}})^{-\frac{2k}{2k+1}} \\ b_{21} &= a_{21}\,(\sqrt{a_{12}a_{21}})^{-\frac{2k}{2k+1}}. \end{aligned}$$

Since $b' \leq b$ we conclude the component-wise supremum over all B such that $B^{2k+1} = A$ is given by

$$B = (\sqrt{a_{12}a_{21}})^{-\frac{2k}{2k+1}}\, A$$

which is also a solution. □

3.3 Even Roots

Theorem 4. *Let $A \in \mathbb{R}^{2\times 2}$ and $k \geq 1$. If $\mathrm{d}(A) \geq 0$, then there is a unique principal $2k$ root B_1 satisfying $\mathrm{d}(B_1) \geq 0$ given by*

$$B_1 = (b_{ij}) = \left((a_{ii} \oplus a_{jj})^{-\frac{2k-1}{2k}}\, a_{ij}\right). \tag{9}$$

Further, there is an additional principal $2k$ root B_2 if and only if $a' = a$ and $\mathrm{d}(A) \neq 0$. In this case, B_2 is the unique principal $2k$ root satisfying $\mathrm{d}(B_2) < 0$ and is given by

$$B_2 = a^{-\frac{2k-1}{2k}} \begin{pmatrix} \sqrt{a_{12}a_{21}} & a\sqrt{a_{12}a_{21}^{-1}} \\ a\sqrt{a_{21}a_{12}^{-1}} & \sqrt{a_{12}a_{21}} \end{pmatrix}. \tag{10}$$

Also, when $a' = a$ and $\mathrm{d}(A) = 0$, we have $B_1 = B_2$.

Proof. We consider the cases $\mathrm{d}(A) > 0$ and $\mathrm{d}(A) = 0$ separately and refer in each case to the appropriate lemmas from Lemmas 2 - 6.

- Suppose $\mathrm{d}(A) > 0$ and let us suppose $a' = a_{11}$ (the case $a' = a_{22}$ will yield the same result via Lemma 6). It follows by Remark 1 and Lemmas 2 - 6 that there are only two solutions (from Lemmas 2 and 5) regarding the structure of the corresponding matrix B.
  - If B satisfies the condition of Lemma 2, then
$$B^{2k} = (b_{12}b_{21})^{k-1}\begin{pmatrix} b_{12}b_{21} & b_{12}b \\ b_{21}b & b_{12}b_{21} \end{pmatrix} = \begin{pmatrix} a' & a_{12} \\ a_{21} & a \end{pmatrix}.$$
It follows $a' = a$ and it can be shown that
$$\begin{aligned} b &= \sqrt{a_{12}a_{21}}\,a^{-\frac{2k-1}{2k}} \\ b_{12} &= a\sqrt{a_{12}a_{21}^{-1}}\,a^{-\frac{2k-1}{2k}} \\ b_{21} &= a\sqrt{a_{21}a_{12}^{-1}}\,a^{-\frac{2k-1}{2k}}. \end{aligned}$$
It follows
$$B = \begin{pmatrix} \lambda & a\sqrt{a_{12}a_{21}^{-1}}\,a^{-\frac{2k-1}{2k}} \\ a\sqrt{a_{21}a_{12}^{-1}}\,a^{-\frac{2k-1}{2k}} & \mu \end{pmatrix},$$
where $\lambda \oplus \mu = \sqrt{a_{12}a_{21}}\,a^{-\frac{2k-1}{2k}}$. Therefore the component-wise supremum over all matrices B such that $B^{2k} = A$ and satisfy the condition of Lemma 2 is given by
$$B_2 = a^{-\frac{2k-1}{2k}}\begin{pmatrix} \sqrt{a_{12}a_{21}} & a\sqrt{a_{12}a_{21}^{-1}} \\ a\sqrt{a_{21}a_{12}^{-1}} & \sqrt{a_{12}a_{21}} \end{pmatrix}$$
and B_2 itself is a solution of $B^{2k} = A$.
  - If B satisfies the condition of Lemma 5, then
$$B^{2k} = \begin{pmatrix} b'^{\,2k} \oplus b_{12}b_{21}b^{2k-2} & b_{12}b^{2k-1} \\ b_{21}b^{2k-1} & b^{2k} \end{pmatrix} = \begin{pmatrix} a' & a_{12} \\ a_{21} & a \end{pmatrix}.$$
It follows
$$\begin{aligned} b &= a^{-\frac{2k-1}{2k}}\,a \\ b_{12} &= a_{12}a^{-\frac{2k-1}{2k}} \\ b_{21} &= a_{21}a^{-\frac{2k-1}{2k}}. \end{aligned}$$

Note that

$$b_{12}b_{21}b^{2k-2} = a_{12}a_{21}a^{\frac{-2k+1-2k+1+2k-2}{2k}} = a_{12}a_{21}a^{-1} < a'$$

and so $b' = (a')^{-\frac{2k-1}{2k}}\, a'$. We conclude

$$B = B_1 = (b_{ij}) = \left((a_{ii} \oplus a_{jj})^{-\frac{2k-1}{2k}}\, a_{ij}\right).$$

- Suppose $\mathrm{d}\,(A) = 0$ and let us suppose $a' = a_{11}$ (the case $a' = a_{22}$ will yield the same result). Note that we do not need to consider Lemma 5 since if $2k$ is sufficiently large so that $\mathrm{d}\left(B^{2k}\right) = 0$, then we obtain the formula of Lemma 4 anyway. By considering Lemmas 2 - 6 we see the only case to consider is Lemma 5 and the proof in this case is similar to the $\mathrm{d}\,(A) = 0$ case in the proof of Theorem 2 and yields the component-wise supremum over all matrices B such that $B^{2k} = A$ and satisfy the condition of Lemma 5 is given by

$$\tilde{B} = a^{-\frac{2k-1}{2k}} \begin{pmatrix} \sqrt{a_{12}a_{21}} & a_{12} \\ a_{21} & a \end{pmatrix}. \tag{11}$$

  which is also a solution.

We note that the only distinction between B_1 in (9) and $\tilde{B}$ in (11) is the $(1,1)$ component. It can be shown that

$$(B_1)_{11} = (a')^{-\frac{2k-1}{2k}}\, a' \geq a^{-\frac{2k-1}{2k}}\sqrt{a_{12}a_{21}} = \left(\tilde{B}\right)_{11}$$

and so B_1 is the component-wise supremum of B_1 and $\tilde{B}$. The result follows. □

Corollary 1. *Let $A \in \mathbb{R}^{2\times 2}$ and $k \geq 1$. If $d\,(A) > 0$ and $a' = a$, then there are two and only two principal $2k$ roots of A. These matrices are B_1 and B_2 (see (9) and (10)) and satisfy $d\,(B_1) > 0$ and $d\,(B_2) < 0$, respectively.*

Theorem 5. *Let $A \in \mathbb{R}^{2\times 2}$ and $k \geq 1$. If $d\,(A) < 0$, then there does not exist a matrix B such that $B^{2k} = A$.*

Proof. By referring to Lemmas 2 - 3, we see that for any matrix $B \in \mathbb{R}^{2\times 2}$ and $k \geq 1$, we have $\mathrm{d}\left(B^{2k}\right) \geq 0$. The result follows. □

3.4 Concluding Remarks and Observations on 2×2 Matrices

The following remark aims to make clear the relationship between the two principal $2k$ roots of A (when they exist).

Remark 2. *Suppose $a' = a$ and $k \geq 1$. If $d\,(A) \geq 0$, then $\lambda\,(A)$ is a double eigenvalue and there are only two principal $2k$ roots of A. Namely*

$$B_1 = \lambda\,(A)^{-\frac{2k-1}{2k}}\, A$$
$$B_2 = \ominus\left(\lambda\,(A)^{-\frac{2k-1}{2k}}\, A\right).$$

Further, if $d\,(A) = 0$, then $B_1 = B_2$ is a repeated principal $2k$ root.

Corollary 2. *Suppose $a' = a$ and $k \geq 1$. There exists a matrix B such that $B^k = A$ if and only if*

$$B = \lambda(A)^{-\frac{k-1}{k}} A$$

satisfies $B^k = A$.

Corollary 3. *Suppose $a' \neq a$ and $k \geq 1$. The maximum cycle mean is a simple eigenvalue and there exists a matrix B such that $B^k = A$ if and only if $\mathrm{d}(A) \geq 0$. Further, in this case the unique principal k root is given by*

$$B = (b_{ij}) = \left((a_{ii} \oplus a_{jj})^{-\frac{k-1}{k}} a_{ij}\right).$$

Remark 3. *There are parallels between tropical matrix roots of 2×2 matrices A for which $a' = a$ in the max-plus algebra and classical roots of real numbers in the classical linear algebra, summarised in table 1.*

tropical matrix roots of $A \in \mathbb{R}^{2\times 2}$ when $a' = a$	**classical real roots of $t \in \mathbb{R}$**
$\mathrm{d}(A) \geq 0$	$t \geq 0$
$\sqrt[2k+1]{A}$ is the unique principal $2k+1$ root of A. $\mathrm{d}\left(\sqrt[2k+1]{A}\right) \geq 0$	$\sqrt[2k+1]{t}$ is the unique real number r such that $r^{2k+1} = t$. $\sqrt[2k+1]{t} \geq 0$.
$\sqrt[2k]{A}$ and $\ominus \sqrt[2k]{A}$ are the only two principal $2k$ roots of A. $\mathrm{d}\left(\sqrt[2k]{A}\right) \geq 0$. $\mathrm{d}\left(\ominus \sqrt[2k]{A}\right) \leq 0$. $\sqrt[2k]{A} = \ominus \sqrt[2k]{A}$ iff $\mathrm{d}(A) = 0$.	$\sqrt[2k]{t}$ and $-\sqrt[2k]{t}$ are the only two real numbers r such that $r^{2k} = t$. $\sqrt[2k]{t} \geq 0$. $-\sqrt[2k]{t} \leq 0$. $\sqrt[2k]{t} = -\sqrt[2k]{t}$ iff $t = 0$.
$\mathrm{d}(A) < 0$	$t < 0$
$\sqrt[2k+1]{A}$ is the unique principal $2k+1$ root of A. $\mathrm{d}\left(\sqrt[2k+1]{A}\right) < 0$.	$\sqrt[2k+1]{t}$ is the unique real number r such that $r^{2k+1} = t$. $\sqrt[2k+1]{t} < 0$.
There does not exist a matrix B such that $B^{2k} = A$.	There does not exist a real number r such that $r^{2k} = t$.

Table 1: Parallels between tropical matrix roots of 2×2 matrices with constant term on the leading diagonal and classical real roots of real numbers.

Based on table 1, it seems natural to define the matrix $\sqrt[k]{A}$ in the following way for matrices A.

Definition 1. *Let $A \in \mathbb{R}^{2\times 2}$ and let $k \geq 1$.*

- *If $a' = a$,*

$$\sqrt[k]{A} := \lambda(A)^{-\frac{k-1}{k}} A.$$

- *If $a' \neq a$,*

$$\sqrt[k]{A} := (b_{ij}) = \left((a_{ii} \oplus a_{jj})^{-\frac{k-1}{k}}\, a_{ij}\right).$$

Observations made for 2×2 matrices suggest an extension for matrices of higher dimension.

Example 1. *Let*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note $(\forall i)\, a_{ii} = 1$ and $d(A) = 1 > 0$. Also $\lambda(A)^{-\frac{1}{2}} = -\frac{3}{4}$ and so we might hope that the matrix B defined as

$$B := -\frac{3}{4}A = \begin{pmatrix} 1/4 & 1/4 & -3/4 \\ 5/4 & 1/4 & 1/4 \\ -3/4 & -3/4 & 1/4 \end{pmatrix}$$

acts as a square root of the matrix A but it is not the case, thus showing that we cannot generalise our results on 2×2 matrices to higher dimensions quite so easily.

4 A generalisation to special types of $n \times n$ matrices

In this section, $A \in \mathbb{R}^{n\times n}$ is a matrix such that

$$(\forall i)\,(\forall j)\,(\forall t)\, t \neq i, j; \quad a_{ij}a_{tt} \geq a_{it}a_{tj}. \tag{12}$$

This is essentially the condition that certain 2×2 minors (those which touch the diagonal but may not be principal) are non-negative in the sense that their positive determinant is greater than or equal to their negative determinant. This is weaker than the condition that all the 2×2 minors are nonnegative, a condition which is shown to be equivalent to total positivity of the matrix A, see [10]. Let $k \geq 1$ be fixed and for the remainder of this section define

$$B = (b_{ij}) = \left(a_{ij}\,(a_{ii} \oplus a_{jj})^{-\frac{k-1}{k}}\right),$$

so that

$$(\forall i)\, b_{ii} = a_{ii}^{\frac{1}{k}}.$$

Theorem 6.

$$(\forall 1 \leq \ell \leq k)\, B^{\ell} = \left(b_{ij}^{(\ell)}\right) = \left(a_{ij}\,(a_{ii} \oplus a_{jj})^{-\frac{k-\ell}{k}}\right).$$

The following lemma may be useful in the proof of Theorem 6.

Lemma 7. *Let $a, b \in \mathbb{R}$ and suppose $r, s \geq 0$. Then*

$$(a \oplus b)^{r+s} \geq a^r b^s.$$

Proof of Theorem 6. Let $\mathrm{P}(\ell)$ be the statement

$$\text{“}B^\ell = \left(b_{ij}^{(\ell)}\right) = \left(a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell-k}{k}}\right)\text{”}$$

for $\ell \leq k$. Clearly $\mathrm{P}(1)$ holds. Let $1 \leq \ell \leq k-1$ and suppose $P(\ell)$ holds, that is

$$(\forall i)\,(\forall j)\, b_{ij}^{(\ell)} = a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell-k}{k}} \Rightarrow (\forall i)\, b_{ii}^{(l)} = a_{ii}^{\frac{\ell}{k}}.$$

We show that $P(\ell+1)$ holds also.

- Let $i \in \mathbb{N}$. Then

$$\begin{aligned} b_{ii}^{(\ell+1)} &= \bigoplus_{t \in N} b_{it}^{(\ell)} b_{ti} \\ &= b_{ii}^{(\ell)} b_{ii} \oplus \bigoplus_{t \neq i} b_{it}^{(\ell)} b_{ti} \\ &= a_{ii}^{\frac{\ell}{k}} a_{ii} a_{ii}^{\frac{1-k}{k}} \oplus \bigoplus_{t \neq i} a_{it}\left(a_{ii} \oplus a_{tt}\right)^{\frac{\ell-k}{k}} a_{ti}\left(a_{tt} \oplus a_{ii}\right)^{\frac{1-k}{k}} \\ &= a_{ii}^{\frac{\ell+1}{k}} \oplus \bigoplus_{t \neq i} a_{it} a_{ti}\left(a_{ii} \oplus a_{tt}\right)^{\frac{\ell+1-2k}{k}}. \end{aligned}$$

  Let $t \neq i$ be fixed. We claim that

$$a_{ii}^{\frac{\ell+1}{k}} \geq a_{it} a_{ti}\left(a_{ii} \oplus a_{tt}\right)^{\frac{\ell+1-2k}{k}}.$$

  To see this, first observe

$$\left(a_{ii} \oplus a_{tt}\right)^{\frac{2k-\ell-1}{k}} \geq a_{ii}^{\frac{k-\ell-1}{k}} a_{tt}^{\frac{k}{k}} \tag{13}$$

  by Lemma 7 (note $k - \ell - 1 \geq 0$). We then have

$$\begin{aligned} & a_{ii}^{\frac{\ell+1}{k}} \geq a_{it} a_{ti}\left(a_{ii} \oplus a_{tt}\right)^{\frac{\ell+1-2k}{k}} \\ \Leftrightarrow\ & a_{ii}^{\frac{\ell+1}{k}} \overbrace{\left(a_{ii} \oplus a_{tt}\right)^{\frac{2k-\ell-1}{k}}}^{\geq a_{ii}^{\frac{k-\ell-1}{k}} a_{tt}^{\frac{k}{k}} \text{ by (13)}} \geq a_{it} a_{ti} \\ \Leftarrow\ & a_{ii}^{\frac{\ell+1}{k}} a_{ii}^{\frac{k-\ell-1}{k}} a_{tt}^{\frac{k}{k}} \geq a_{it} a_{ti} \\ \Leftrightarrow\ & a_{ii} a_{tt} \geq a_{it} a_{ti}, \end{aligned}$$

  which holds by (12). It follows

$$b_{ii}^{(\ell+1)} = a_{ii}^{\frac{\ell+1}{k}} = a_{ii}\left(a_{ii} \oplus a_{ii}\right)^{\frac{\ell+1-k}{k}},$$

  as required.

- Let $i, j \in N, i \neq j$. First observe

$$a_{ii}^{\frac{\ell}{k}} \oplus a_{jj}^{\frac{1}{k}} \left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell-1}{k}} = \left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell}{k}}. \tag{14}$$

We then have

$$\begin{aligned} b_{ij}^{(\ell+1)} =& \bigoplus_{t \in N} b_{it}^{(\ell)} b_{tj} \\ =& b_{ii}^{(\ell)} b_{ij} \oplus b_{ij}^{(\ell)} b_{jj} \oplus \bigoplus_{t \neq i,j} b_{it}^{(\ell)} b_{tj} \\ =& a_{ii}^{\frac{\ell}{k}} a_{ij} \left(a_{ii} \oplus a_{jj}\right)^{\frac{1-k}{k}} \oplus a_{ij} \left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell-k}{k}} a_{jj}^{\frac{1}{k}} \oplus \bigoplus_{t \neq i,j} b_{it}^{(\ell)} b_{tj} \\ =& a_{ij} \left(a_{ii} \oplus a_{jj}\right)^{\frac{1-k}{k}} \overbrace{\left[a_{ii}^{\frac{\ell}{k}} \oplus a_{jj}^{\frac{1}{k}} \left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell-1}{k}}\right]}^{=(a_{ii} \oplus a_{jj})^{\frac{\ell}{k}} \text{ by (14)}} \oplus \bigoplus_{t \neq i,j} b_{it}^{(\ell)} b_{tj} \\ =& a_{ij} \left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell+1-k}{k}} \oplus \bigoplus_{t \neq i,j} a_{it} \left(a_{ii} \oplus a_{tt}\right)^{\frac{\ell-k}{k}} a_{tj} \left(a_{tt} \oplus a_{jj}\right)^{\frac{1-k}{k}}. \end{aligned}$$

Let $t \neq i, j$ be fixed. We claim that

$$a_{ij} \left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell+1-k}{k}} \geq a_{it} \left(a_{ii} \oplus a_{tt}\right)^{\frac{\ell-k}{k}} a_{tj} \left(a_{tt} \oplus a_{jj}\right)^{\frac{1-k}{k}}. \tag{15}$$

To see this, first observe

$$\left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell+1-k}{k}} \left(a_{ii} \oplus a_{tt}\right)^{\frac{k-\ell}{k}} \left(a_{tt} \oplus a_{jj}\right)^{\frac{k-1}{k}} \geq a_{tt}. \tag{16}$$

This follows since

$$\begin{aligned} & \left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell+1-k}{k}} \left(a_{ii} \oplus a_{tt}\right)^{\frac{k-\ell}{k}} \left(a_{tt} \oplus a_{jj}\right)^{\frac{k-1}{k}} \\ \geq & \left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell+1-k}{k}} a_{ii}^{\frac{k-\ell-1}{k}} a_{tt}^{\frac{1}{k}} a_{tt}^{\frac{k-1}{k}} \\ = & \left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell+1-k}{k}} a_{ii}^{\frac{k-\ell-1}{k}} a_{tt} \\ =& a_{ii}^{\frac{\ell+1-k}{k}} a_{ii}^{\frac{k-\ell-1}{k}} a_{tt} \oplus a_{jj}^{\frac{\ell+1-k}{k}} a_{ii}^{\frac{k-\ell-1}{k}} a_{tt} \\ =& a_{tt} \oplus a_{jj}^{\frac{\ell+1-k}{k}} a_{ii}^{\frac{k-\ell-1}{k}} a_{tt} \\ \geq & a_{tt}, \end{aligned}$$

where the first inequality holds by two uses of Lemma 7 (note that $k - \ell - 1 \geq 0$).

We can now show (15) holds, as follows:

$$\begin{aligned} & a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell+1-k}{k}} \geq a_{it}\left(a_{ii} \oplus a_{tt}\right)^{\frac{\ell-k}{k}} a_{tj}\left(a_{tt} \oplus a_{jj}\right)^{\frac{1-k}{k}} \\ \Leftrightarrow & a_{ij} \overbrace{\left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell+1-k}{k}}\left(a_{ii} \oplus a_{tt}\right)^{\frac{k-\ell}{k}}\left(a_{tt} \oplus a_{jj}\right)^{\frac{k-1}{k}}}^{\geq a_{tt} \text{ by } (16)} \geq a_{it} a_{tj} \\ \Leftarrow & a_{ij} a_{tt} \geq a_{it} a_{tj}, \end{aligned}$$

which holds by (12).

Therefore

$$b_{ij}^{(\ell+1)} = a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell+1-k}{k}}.$$

We conclude $P(\ell+1)$ holds and the result follows by induction. □

Corollary 4.

$$B^k = A.$$

Proof.

$$B^k = \left(b_{ij}^{(k)}\right) = \left(a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{\frac{k-k}{k}}\right) = \left(a_{ij}\right) = A.$$

□

Example 2. $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ *satisfies (12). For* $k = 3$ *we have*

$$B = (b_{ij}) = \left(a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{\frac{-2}{3}}\right) = \begin{pmatrix} \frac{1}{3} & -\frac{4}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{4}{3} & \frac{1}{3} \end{pmatrix}.$$

We then see that

$$B^2 = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{4}{3} & \frac{4}{3} \\ -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} \text{ and } B^3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} = A,$$

as expected.

Example 3. $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ *does not satisfy (12) since* $a_{21}a_{33} < a_{23}a_{31}$. *If we define*

$$B = (b_{ij}) = \left(a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{\frac{-2}{3}}\right) = \begin{pmatrix} \frac{1}{3} & -\frac{4}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix},$$

then

$$B^2 = \begin{pmatrix} \frac{2}{3} & 0 & \frac{2}{3} \\ 1 & \frac{4}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \text{ and } B^3 = \begin{pmatrix} 1 & \frac{2}{3} & 1 \\ \frac{5}{3} & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \neq A.$$

Interestingly, the matrix B^3 is "off" in exactly two positions. We can also check that the matrix A violates exactly two of the conditions of Theorem 6. In particular, $a_{21}a_{33} < a_{23}a_{31}$ and $a_{12}a_{33} < a_{13}a_{32}$. All other conditions are satisfied. We have the following remark.

Remark 4. *In the proof of Theorem 6, we fix i, j and then prove $b_{ij}^{(\ell)}$ is as expected. Not all conditions of the theorem are used however. To prove $b_{ij}^{(\ell)} = a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell-k}{k}}$ we require only*

$$(\forall t \neq i, j)\, a_{ij}a_{tt} \geq a_{it}a_{tj}. \tag{17}$$

We call (17) the *root constraints* for (i, j). We summarise with the following corollary.

Corollary 5. *Let $A \in \mathbb{R}^{n\times n}$ and $k \geq 2$ an integer. Define the matrix*

$$B = (b_{ij}) = \left(a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{\frac{1-k}{k}}\right)$$

and let i, j be fixed. If

$$(\forall t \neq i, j)\, a_{ij}a_{tt} \geq a_{it}a_{tj},$$

then

$$(\forall \ell \geq 1)\, b_{ij}^{(\ell)} = a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{\frac{\ell-k}{k}}.$$

It follows that if a matrix $A \in \mathbb{R}^{n\times n}$ satisfies the root constraints for (i, j) for most pairs (i, j), then B may serve as a good approximation (in some sense) to a kth root of A. This could prove an interesting direction for future research.

5 Idempotent matrices

We discuss the connection between matrix roots and idempotent matrices. Firstly, suppose $A \in \mathbb{R}^{n\times n}$ satisfies (12), which we recall here:

$$(\forall i)\,(\forall j)\,(\forall t)\, t \neq i, j; \quad a_{ij}a_{tt} \geq a_{it}a_{tj}.$$

Define

$$A\,(k) := \left(a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{\frac{1-k}{k}}\right), \text{ for } k \geq 2.$$

Remark 5.

$$\lim_{k\to\infty} A\,(k) = \left(a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{-1}\right) =: \overline{A}$$

and the diagonal elements of $\overline{A}$ are zero.

Proposition 1. *Let $A \in \mathbb{R}^{n\times n}$ satisfying the root conditions (12). Then $\overline{A}$, as defined in Remark 5, is an idempotent matrix.*

Proof. For ease of notation, denote $\overline{A}$ by B. It is sufficient to show $B^2 = B$. Denote $B^2 = \left(b_{ij}^{(2)}\right)$ and let $i, j \in N$. Then

$$\begin{aligned} b_{ij}^{(2)} &= \bigoplus_{t\in N} b_{it}b_{tj} \\ &= b_{ii}b_{ij} \oplus b_{ij}b_{jj} \oplus \bigoplus_{t\in N, t\neq i,j} b_{it}b_{tj} \\ &= b_{ij} \oplus \bigoplus_{t\in N, t\neq i,j} b_{it}b_{tj}. \end{aligned}$$

Let $t \in N, t \neq i, j$. It suffices to show

$$b_{it}b_{tj} \leq b_{ij}. \tag{18}$$

Note that (18) holds if and only if

$$a_{it}\left(a_{ii} \oplus a_{tt}\right)^{-1} a_{tj}\left(a_{tt} \oplus a_{jj}\right)^{-1} \leq a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{-1}. \tag{19}$$

We consider cases, as follows.

- $a_{tt} \geq a_{ii} \oplus' a_{jj}$. Without loss of generality,

$$a_{tt} \geq a_{ii} \tag{20}$$

(the case $a_{tt} \geq a_{jj}$ is similar). Then, (19) holds if and only if

$$\begin{aligned} & a_{it}a_{tt}^{-1}a_{tj}\left(a_{tt} \oplus a_{jj}\right)^{-1} \leq a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{-1} \\ \Leftrightarrow & a_{it}a_{tj}\left(a_{tt} \oplus a_{jj}\right)^{-1} \leq a_{ij}a_{tt}\left(a_{ii} \oplus a_{jj}\right)^{-1} \\ \Leftarrow & \begin{cases} a_{it}a_{tj} \leq a_{ij}a_{tt} \\ a_{tt} \oplus a_{jj} \geq a_{ii} \oplus a_{jj} \end{cases} \\ \Leftarrow & (12) \text{ and } (20). \end{aligned}$$

-

$$a_{tt} < a_{ii} \oplus' a_{jj}. \tag{21}$$

Then, (19) holds if and only if

$$\begin{aligned} & a_{it}a_{ii}^{-1}a_{tj}a_{jj}^{-1} \leq a_{ij}\left(a_{ii} \oplus a_{jj}\right)^{-1} \\ \Leftrightarrow & a_{it}a_{tj} \leq a_{ij}\left(a_{ii} \oplus' a_{jj}\right) \\ \Leftarrow & (12) \text{ and } (21) \end{aligned}$$

since

$$a_{it}a_{tj} \leq a_{ij}a_{tt} < a_{ij}\left(a_{ii} \oplus' a_{jj}\right).$$

It follows that $\overline{A}$ is idempotent. □

We have seen that given a matrix $A \in \mathbb{R}^{n\times n}$ satisfying the root conditions (12), there is a corresponding matrix, namely the matrix $\overline{A}$, which is idempotent and has all diagonal entries equal to 0. An interesting question is: "given such an idempotent matrix B, can we describe a matrix A such that A satisfies the root conditions (12) and $\overline{A} = B$?"

Note that, necessarily, an idempotent matrix has non-positive diagonal entries but it is not necessary in general for all diagonal entries to be 0. Therefore we are not considering here all idempotent matrices.

Lemma 8. *Let $B \in \mathbb{R}^{n\times n}$ be an idempotent matrix such that $(\forall i)\, b_{ii} = 0$. Then*

$$(\forall i)\,(\forall j)\,(\forall t)\; t \neq i,j, b_{it}b_{tj} \leq b_{ij}.$$

Proof. Let $i,j,t \in N, t \neq i,j$. Then

$$\begin{aligned} b_{ij}^{(2)} &= \bigoplus_{t\in N} b_{it}b_{tj} \\ &= b_{ii}b_{ij} \oplus b_{ij}b_{jj} \oplus \bigoplus_{t\in N, t\neq i,j} b_{it}b_{tj} \\ &= b_{ij} \oplus \bigoplus_{t\in N, t\neq i,j} b_{it}b_{tj}. \end{aligned}$$

It follows by idempotency of B that $(\forall t \neq i,j)\, b_{it}b_{tj} \leq b_{ij}$. □

Theorem 7. *Let $B \in \mathbb{R}^{n\times n}$ be an idempotent matrix such that $(\forall i)\, b_{ii} = 0$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ satisfy*

$$(\forall i)\,(\forall j)\,(\forall t)\; t \neq i,j; \lambda_{ij}\left(b_{ij}^{-1}b_{it}b_{tj}\right) \leq \lambda_t \leq \lambda'_{ij}\left(b_{ij}b_{it}^{-1}b_{tj}^{-1}\right),$$

where $\lambda_{ij} := \lambda_i \oplus \lambda_j$ and $\lambda'_{ij} := \lambda_i \oplus' \lambda_j$. Then the matrix A defined by

$$(\forall i)\,(\forall j)\, a_{ij} := b_{ij}\lambda_{ij}$$

satisfies the root conditions (12) and is such that $\overline{A} = B$.

Proof. Suppose now there exists a matrix A such that $\overline{A} = B$. Denote for all $i, \lambda_i = a_{ii}$ and define $\lambda_{ij} := \lambda_i \oplus \lambda_j$. Similarly, define $\lambda'_{ij} := \lambda_i \oplus' \lambda_j$. Since $(\forall i)\,(\forall j)\, \overline{a}_{ij} = a_{ij}\lambda_{ij}^{-1}$, we have

$$(\forall i)\,(\forall j)\, a_{ij} = b_{ij}\lambda_{ij}. \tag{22}$$

The matrix A should satisfy the root conditions (12). Let $i,j,t \in N, t \neq i,j$. Using (22) and $(\forall i)\, b_{ii} = 0$, we see (12) holds if and only if

$$b_{ij}\lambda_{ij}\lambda_t b_{it}^{-1}\lambda_{it}^{-1}b_{tj}^{-1}\lambda_{tj}^{-1} \geq 0. \tag{23}$$

Denote by T the left hand side of (23). We consider six cases, as follows.

- Suppose $\lambda_i \geq \lambda_j \geq \lambda_t$. Then

$$\begin{aligned} T &= b_{ij}\lambda_i\lambda_t b_{it}^{-1}\lambda_i^{-1}b_{tj}^{-1}\lambda_j^{-1} \\ &= b_{ij}\lambda_t b_{it}^{-1}b_{tj}^{-1}\lambda_j^{-1} \end{aligned}$$

and

$$b_{ij}\lambda_t b_{it}^{-1}b_{tj}^{-1}\lambda_j^{-1} \geq 0 \Leftrightarrow \lambda_t \geq \lambda_j \left(b_{ij}^{-1}b_{it}b_{tj}\right).$$

- Suppose $\lambda_i \geq \lambda_t \geq \lambda_j$. Then

$$\begin{aligned} T &= b_{ij}\lambda_i\lambda_t b_{it}^{-1}\lambda_i^{-1}b_{tj}^{-1}\lambda_t^{-1} \\ &= b_{ij}b_{it}^{-1}b_{tj}^{-1} \geq 0, \end{aligned}$$

where the last inequality holds by Lemma 8.

- Suppose $\lambda_j \geq \lambda_i \geq \lambda_t$. Then

$$\begin{aligned} T &= b_{ij}\lambda_j\lambda_t b_{it}^{-1}\lambda_i^{-1}b_{tj}^{-1}\lambda_j^{-1} \\ &= b_{ij}\lambda_t b_{it}^{-1}\lambda_i^{-1}b_{tj}^{-1} \end{aligned}$$

and

$$b_{ij}\lambda_t b_{it}^{-1}\lambda_i^{-1}b_{tj}^{-1} \geq 0 \tag{24}$$
$$\Leftrightarrow \lambda_t \geq \lambda_i \left(b_{ij}^{-1}b_{it}b_{tj}\right). \tag{25}$$

- Suppose $\lambda_j \geq \lambda_t \geq \lambda_i$. Then

$$\begin{aligned} T &= b_{ij}\lambda_j\lambda_t b_{it}^{-1}\lambda_t^{-1}b_{tj}^{-1}\lambda_j^{-1} \\ &= b_{ij}b_{it}^{-1}b_{tj}^{-1} \geq 0, \end{aligned}$$

where the last inequality holds by Lemma 8.

- Suppose $\lambda_t \geq \lambda_i \geq \lambda_j$. Then

$$\begin{aligned} T &= b_{ij}\lambda_i\lambda_t b_{it}^{-1}\lambda_t^{-1}b_{tj}^{-1}\lambda_t^{-1} \\ &= b_{ij}\lambda_i b_{it}^{-1}\lambda_t^{-1}b_{tj}^{-1} \end{aligned}$$

and

$$b_{ij}\lambda_i b_{it}^{-1}\lambda_t^{-1}b_{tj}^{-1} \geq 0 \Leftrightarrow \lambda_t \leq \lambda_i \left(b_{ij}b_{it}^{-1}b_{tj}^{-1}\right). \tag{26}$$

- Suppose $\lambda_t \geq \lambda_j \geq \lambda_i$. Then

$$\begin{aligned} T &= b_{ij}\lambda_j\lambda_t b_{it}^{-1}\lambda_t^{-1}b_{tj}^{-1}\lambda_t^{-1} \\ &= b_{ij}\lambda_j b_{it}^{-1}\lambda_t^{-1}b_{tj}^{-1} \end{aligned}$$

and

$$b_{ij}\lambda_j b_{it}^{-1}\lambda_t^{-1}b_{tj}^{-1} \geq 0 \Leftrightarrow \lambda_t \leq \lambda_j \left(b_{ij}b_{it}^{-1}b_{tj}^{-1}\right). \tag{27}$$

Note that in the case $\lambda_t \leq \lambda'_{ij}$ and by considering (24) and (25), we have $T \geq 0$ if and only if

$$\lambda'_{ij} \left(b_{ij}^{-1} b_{it} b_{tj}\right) \leq \lambda_t \leq \lambda'_{ij},$$

which is implied by the stronger condition

$$\lambda_{ij} \left(b_{ij}^{-1} b_{it} b_{tj}\right) \leq \lambda_t \leq \lambda'_{ij}. \tag{28}$$

Similarly, in the case $\lambda_t \geq \lambda_{ij}$ and by considering (26) and (27), we have $T \geq 0$ if and only if

$$\lambda_{ij} \leq \lambda_t \leq \lambda_{ij} \left(b_{ij} b_{it}^{-1} b_{tj}^{-1}\right),$$

which is implied by the stronger condition

$$\lambda_{ij} \leq \lambda_t \leq \lambda'_{ij} \left(b_{ij} b_{it}^{-1} b_{tj}^{-1}\right). \tag{29}$$

Since we don't have an a priori knowledge of the ordering of the λ elements, we may combine (28) and (29) to obtain the sufficient condition in the lemma statement. □

Remark 6. *Note that the sufficient condition in Theorem 7 has the solution* $(\forall i)\, \lambda_i = \lambda$ *for any* $\lambda \in \mathbb{R}$. *Indeed, if* $\lambda \in \mathbb{R}$ *is fixed, then it is easy to check* $\overline{\lambda B} = B$.

Remark 7. *The sufficient condition in Theorem 7 is a system of dual inequalities and the solution set can be described in terms of a generating matrix in polynomial time. See [4].*

The following example shows that less trivial solutions also exist. Note the use of classical linear notation for ease of understanding.

Example 4. *Consider the matrix*

$$B = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

We wish to find a matrix A *of the form*

$$A = \begin{pmatrix} \lambda_1 & \lambda_{12} - 1 \\ \lambda_{12} - 1 & \lambda_2 \end{pmatrix}.$$

The sufficient condition of Theorem 7 gives

$$\begin{cases} \lambda_1 - 2 \leq \lambda_2 \leq \lambda_1 + 2 \\ \lambda_2 - 2 \leq \lambda_1 \leq \lambda_2 + 2 \end{cases}$$
$$\Leftrightarrow \|\lambda_1 - \lambda_2\| \leq 2.$$

One non-trivial solution is $\lambda_1 = 1, \lambda_2 = 0$, *which yields*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Indeed, $det^+(A) \geq det^-(A)$ *and* $\overline{A} = B$.

6 Summary

We defined principal k roots (for a fixed positive integer k) for 2×2, finite matrices (when such roots exist). We also explicitly described when such roots do and do not exist and also related the number of roots to the multiplicity of the maximum cycle mean. We were able to generalise the formula for the kth root of a matrix to the general $n \times n$ case, provided the matrix satisfied some conditions on some of its 2×2 minors. Corollary 5 motivated a question about approximations of matrix roots - which may be useful considering finding exact roots for $n \times n$ matrices is NP-complete in the Boolean case. Some open questions are the following:

- Does there exist a matrix $A \in \mathbb{R}^{n \times n}$ and a natural number $k \geq 2$ for which A does not satisfy the root conditions (12) but there exists a matrix $B \in \mathbb{R}^{n \times n}$ such that $B^k = A$?
- Can we extend the ideas of this paper to rational matrix powers, negative matrix powers and matrix polynomials?
- In the classical linear algebra, considering matrices over $\mathbb{C}^{n \times n}$ allows us to find matrix roots with complex entries, is there a corresponding way to extend the tropical algebra to find more roots, especially in the 2×2 case?

References

[1] N. J. Higham and L. Lin, *On pth roots of stochastic matrices*, LAA, 435 (3), pp. 448-463, 2011.

[2] A. Van-Brunt, *Infinitely divisible nonnegative matrices, M-matrices, and the embedding problem for finite state stationary Markov Chains*, LAA, 541, pp. 163-176, 2018.

[3] M. Kutz, *The complexity of Boolean matrix root computation*, Theoretical Computer Science 325, pp. 373-390, 2004.

[4] P. Butkovič, *Max-linear Systems: Theory and Algorithms*, Springer Monographs in Mathematics, Springer-Verlag, London, 2010.

[5] R. B. Bapat, D. Stanford and P. van den Driessche, *The eigenproblem in max-algebra*, (DMS-631-IR). University of Victoria, British Colombia, 1993.

[6] S. Sergeev and H. Schneider, *CSR expansions of matrix powers in max algebra*, Transactions of the American Mathematical society, 2009.

[7] B. Heidergott, G. J. Olsder and J. van der Woude, *Max Plus at Work: Modeling and Analysis of Synchronized Systems*, A Course on Max-Plus Algebra, Princeton University Press, 2005.

[8] F. L. Baccelli, G. Cohen, G. J. Olsder and J. P. Quadrat, *Synchronization and Linearity*, John Wiley, Chichester, New York, 1992.

[9] R. A. Cuninghame-Green, *Minimax Algebra*, Lecture Notes in Economics and Mathematical Systems 166, Berlin, Springer, pp. 251-267, 1979.

[10] S. Gaubert, A. Niv, *Tropically totally positive matrices*, Journal of Algebra, 515, pp. 511-544, 2018.