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Parker, Chris; Rowley, Peter

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THE 2-MINIMAL SUBGROUPS OF SYMPLECTIC GROUPS

CHRIS PARKER AND PETER ROWLEY

ABSTRACT. For a finite group G, a subgroup P of G is 2-minimal if $B < P$, where $B = N_G(S)$ for some Sylow 2-subgroup S of G, and B is contained in a unique maximal subgroup of P. Here we give a detailed and explicit description of all the 2-minimal subgroups for finite symplectic groups defined over a field of odd characteristic.

1. INTRODUCTION

In [10] the 2-minimal subgroups of linear and unitary groups are classified – it is the purpose of this paper to classify such subgroups for the finite symplectic groups. If the symplectic groups are defined in characteristic 2, then their 2-minimal subgroups are the well-known (and well understood) minimal parabolic subgroups. Thus we only consider the odd characteristic case here. We shall continue to use the notation introduced in [10], and also refer the reader to Section 1 of [10] for a wider discussion on p-minimal subgroups, p a prime. Before stating our main theorem we give a quick review of some of the frequently used definitions and notation, beginning with the definition of a p-minimal subgroup. Suppose that G is a finite group. Let p be a prime, S a Sylow p-subgroup of G and $B = N_G(S)$. A subgroup P of G which properly contains B is called a *p-minimal subgroup* of G (with respect to B) if B is contained in a unique maximal subgroup of P. Put

$$\mathcal{M}(G, B) = \{P \mid B < P \leq G \text{ and } P \text{ is } p\text{-minimal}\}.$$

It turns out, provided $G \neq B$, that G is generated by its p-minimal subgroups. It is the set $\mathcal{M}(G, B)$ that we shall study when $p = 2$. As is the case for the linear and unitary groups, here 2-minimal subgroups of monomial groups in their various guises contribute subgroups to $\mathcal{M}(G, B)$. So we must say a few words about monomial groups and their 2-minimal subgroups. Those of principal interest are wreath products $H = E \wr X$ where E is cyclic of odd order and $X \cong \mathrm{Sym}(n)$, the symmetric group of degree n. Let $T \in \mathrm{Syl}_2(H)$ and let F denote the base group of H. Then $\mathcal{M}(H, N_H(T))$ comprises of three different types of 2-minimal subgroups, namely the toral, linker and fuser 2-minimal subgroups. These three sets of 2-minimal subgroups are denoted, respectively, by $\mathcal{T}(H, N_H(T)), \mathcal{L}(H, N_H(T))$ and $\mathcal{F}(H, N_H(T))$. All of these groups depend on the action of T on Ω (where $X = \mathrm{Sym}(\Omega)$) with the toral 2-minimal subgroups also having in-put data relating to $|E|$ and the action of T on F. We also note that the images of the groups in $\mathcal{L}(H, N_H(T))$, respectively $\mathcal{F}(H, N_H(T))$, in the symmetric group H/F are just the linkers, respectively, fusers in $\mathrm{Sym}(n)$ (see [7]). On the other hand, the toral 2-minimal subgroups are of the form TR where R is a subgroup of F. The definition of these groups may be found in Section 2, and Theorems 4.5 and 2.8 provide the monomial type 2-minimal subgroups which we require. They are described in Definitions 2.4, 2.7 giving the sets of 2-minimal subgroups $\mathcal{L}(G, B)$, $\mathcal{F}(G, B)$ and $\mathcal{T}(G, B)$.

As in [10], the fact that p-minimal subgroups behave well with respect to direct products and quotients is important here. Indeed, in Lemma 4.10, we see that this controlled behaviour even extends to certain wreath products.

It is well known that $\mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$, and the 2-minimal subgroups of these groups are given in Theorem 4.1. The description of these 2-minimal subgroups fractures into a myriad of subcases

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depending on various congruence conditions on q. Some of this complexity feeds through to the general $\mathrm{Sp}_n(q)$ situation, courtesy of the set $\mathcal{X}_2$ which consists of certain 2-minimal subgroups of $\mathrm{Sp}_2(q)$. This set and its companion $\mathcal{X}_4$, via which $\mathrm{Sp}_4(q)$ also leaves its mark, are introduced in Definitions 4.12 and 4.13. These subsets spawn sets of 2-minimal $\mathcal{M}_{\mathcal{X}_2}(G,B)$ and $\mathcal{M}_{\mathcal{X}_4}(G,B)$ defined in Definitions 4.12 and 4.13.

Our main theorem can now be stated as follows.

Theorem 1.1. *Suppose that $q = p^a$ is odd, $n \geq 2$ is even and $G = \mathrm{Sp}_n(q)$. Let $S \in \mathrm{Syl}_2(G)$, $B = N_G(S)$. Then*

$$\mathcal{M}(G,B) \;\;=\;\; \mathcal{F}(G,B) \cup \mathcal{L}(G,B) \cup \mathcal{T}(G,B) \cup \mathcal{M}_{\mathcal{X}_2}(G,B) \cup \mathcal{M}_{\mathcal{X}_4}(G,B).$$

We emphasize that in all cases the structure of P in Theorem 1.1 is known, and in fact explicit matrices can be written down to describe these groups. The use of equal signs in our results is meant to highlight this point via the explicit decomposition of the group action on the natural symplectic space.

We now summarize the contents of this paper. Apart from two general results on p-minimal subgroups at the end of the section, Section 2 is primarily concerned with the 2-minimal subgroups of the monomial groups mentioned earlier. Thus we begin with a discussion of Sylow 2-subgroups and 2-minimal subgroups of symmetric groups. This leads to the main result on 2-minimal subgroups of monomial groups, stated as Theorem 2.5. Building on this theorem an analogous result is proved in Theorem 2.8 for wreath products of dihedral groups with symmetric groups. Section 3, distils the results of Kantor [5], Liebeck and Saxl [8] and Maslova [9] so as to list (up to conjugacy) the maximal subgroups of $\mathrm{Sp}_n(q)$ of odd index. It is in these subgroups, of course, where the proper 2-minimal subgroups are to be found. Then a subgroup $L_k = \mathrm{Sp}_{n/k}(q) \wr \mathrm{Sym}(k)$ for $\mathrm{Sp}_n(q)$ is introduced, followed by Lemma 3.2 which gives the structure of the normalizer of a Sylow 2-subgroup of $\mathrm{Sp}_n(q)$. Section 4 begins with Theorem 4.1 and then the remainder of the section is devoted to the proof of Theorem 1.1. After examining the case of $\mathrm{Sp}_4(q)$ in Lemma 4.2, we define C which, depending on the congruence of q, is either a direct product of quaternion groups of order 8, or is a homocyclic group of odd order. We then see in Lemma 4.7 that, for $P \in \mathcal{M}(G,B)$, where P acts irreducibly on the natural symplectic module, either $P \leq N_G(C)$ or $P \leq L_k$ for some k. We note that $N_G(C)$ is a monomial type subgroup of G. In fact $\mathcal{M}(N_G(C),B) = \mathcal{F}(G,B) \cup \mathcal{L}(G,B) \cup \mathcal{T}(G,B)$. Employing Lemma 4.10, in Lemma 4.11 the possibilities for $P \in \mathcal{M}(G,B) \setminus \mathcal{M}(N_G(C),B)$ are now laid bare, from which Theorem 1.1 follows. By way of illustrating Theorem 1.1, in Section 5 we display in detail the 2-minimal subgroups for $\mathrm{Sp}_{10}(q)$ for all odd q.

Our group theoretic notation is standard as given, for example in [1].

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2. Preliminary results on 2-minimal subgroups

Just as in [10] we shall encounter 2-minimal subgroups contained in monomial groups. The type of monomial groups which appear are not just isomorphic to wreath products $E \wr \mathrm{Sym}(n)$ where E is a cyclic group of odd order but also groups $D \wr \mathrm{Sym}(n)$ where D is a dihedral group of order

$2|E|$. Setting $X = \mathrm{Sym}(\Omega)$ where $\Omega = \{1, \dots, n\}$, we first describe a Sylow 2-subgroup T of X following [4, Satz 15.3, p. 378]. Write

$$n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r} \text{ where } n_1 > n_2 > \cdots > n_r \geq 0$$

and put $I = \{1, \dots, r\}$. Then T has r orbits on Ω, $\Omega_1, \dots, \Omega_r$ with $|\Omega_i| = 2^{n_i}$, $i \in I$. We set notation so that $\Omega_1 = \{1, \dots, 2^{n_1}\}$, then $\Omega_i = \{m_i, \dots, \sum_{j=1}^{i} 2^{n_j}\}$ where $m_1 = 1$ and, for $i \geq 2$, $m_i = 1 + \sum_{j=1}^{i-1} 2^{n_j}$ is the minimal integer in Ω_i. Further, we have

$$T = T_{n_1} \times T_{n_2} \times \cdots \times T_{n_r}$$

where, for $i \in I$, $T_{n_i} \in \mathrm{Syl}_2(\mathrm{Sym}(\Omega_i))$ is the iterated wreath product of n_i copies of T_1 the cyclic group of order 2. We next introduce two types of subgroups of X. Let $i \in I$. For $j \in \{1, \dots, n_i - 1\}$, let $\Sigma_{n_i;j}$ be the collection of T-invariant block systems of Ω_i consisting of sets of size 2^k where $k \in \{0, \dots, n_i\} \setminus \{j\}$, and define

$$X(n_i; j) = \mathrm{Stab}_{\mathrm{Sym}(\Omega_i)}(\Sigma_{n_i;j}) \times (\prod_{\ell \in I \setminus \{i\}} T_{n_\ell}).$$

For $i, j \in I$, with $i < j$ (so $n_j < n_i$) set $\Lambda_{n_i+n_j} = \Omega_i \cup \Omega_j$. Let Γ_i be the collection of all T-invariant block systems on Ω_i and Γ_j the collection of all T-invariant block systems on Ω_j. We define $\Sigma_{n_i+n_j}$ to be the collection of T-invariant systems of subsets of $\Lambda_{n_i+n_j}$ which are the union of one block system from Γ_i and one from Γ_j with the proviso that the blocks of the two chosen block systems have equal numbers of elements. Then, the second type of subgroup that we require is

$$X(n_i + n_j) = \mathrm{Stab}_{\mathrm{Sym}(\Lambda_{n_i+n_j})}(\Sigma_{n_i+n_j}) \times (\prod_{k \in I \setminus \{i,j\}} T_{n_k}).$$

Notice that $\mathrm{Stab}_{\mathrm{Sym}(\Lambda_{n_i+n_j})}(\Sigma_{n_i+n_j}) \cong T_{n_j} \wr \mathrm{Sym}(2^{n_i - n_j} + 1)$. These subgroups contain $T = N_X(T)$ and in [7, Theorem 1.1], it is shown that

Theorem 2.1. *Suppose that $X = \mathrm{Sym}(\Omega)$ and $T \in \mathrm{Syl}_2(X)$. Then*

$$\mathcal{M}(X, T) = \{X(n_i; j), X(n_k + n_\ell) \mid i, k, \ell \in I, k < \ell \text{ and } j \in \{1, \dots n_i - 1\}\}.$$

Now we look at $H = E \wr \mathrm{Sym}(n) = E \wr X$, where E is cyclic of odd order. Taking F to be the base group of H, we have that F is isomorphic to a direct product of n copies of E and so we write

$$F = \langle e_1, \dots, e_n \rangle$$

with X permuting the generators of F naturally. Let $j \in I$. Then we define

$$D_{n_j} = \langle e_i \mid i \in \Omega_j \rangle,$$

$$Z_{n_j} = C_{D_{n_j}}(T_{n_j}) = \langle \prod_{\ell \in \Omega_j} e_\ell \rangle$$

and note that T normalizes D_{n_j} and centralizes Z_{n_j}. We have

$$C_F(T) = \prod_{j \in I} Z_{n_j} \text{ and } N_H(T) = TC_F(T).$$

Definition 2.2. *For $i \in I$ and $j \in \{1, \dots, n_i - 1\}$,*

$$P(n_i; j) = X(n_i; j) C_F(T).$$

And for $i, k \in I$ with $i < k$,

$$P(n_i + n_k) = X(n_i + n_k) \langle C_F(T)^{X(n_i + n_k)} \rangle.$$

Plainly $P(n_i;j)$ and $P(n_i+n_k)$ are subgroups of H which contain $N_H(T)$.

Let $\Pi(|E|)$ be the set of all prime powers, excluding 1, which divide $|E|$. Take $s\in\Pi(|E|)$ a prime, let s^b be the largest power of s in $\Pi(|E|)$ and put $\bar{s}=|E|/s^b$. Then, for $\alpha\in\Omega$ and $s^c\in\Pi(|E|)$, set

$$u_\alpha = e_\alpha^{\bar{s}}, \text{ and } w_\alpha = u_\alpha^{s^{b-c}}$$

and note that $\langle u_\alpha \mid \alpha\in\Omega\rangle \in \mathrm{Syl}_s(F)$ and w_α has order s^c.

For $j\in I$, $s^c\in\Pi(|E|)$, $1\le k\le n_j$

$$\begin{aligned} U(n_j;s^c;k) &= \langle (\sum_{i=m_j}^{m_j+2^{n_j-k}} w_i - w_{2^{n_j-k}+i})^t \mid t\in T_{n_j}\rangle \\ &= \langle (\sum_{i=m_j}^{m_j+2^{n_j-k}} w_i - w_{2^{n_j-k}+i})^t \mid t\in T\rangle. \end{aligned}$$

Notice that $U(n_j;s^c;k)\le D_{n_j}$ and is normalized by T.

Definition 2.3. *For $j\in I, s^c\in\Pi(|E|)$ and $1\le k\le n_j$,*

$$T(n_j;s^c;k)=U(n_j;s^c;k)N_H(T).$$

Finally we describe the *toral*, *linker* and *fuser* 2-minimal subgroups of H.

Definition 2.4.

(i) $\mathcal{F}(H,N_H(T))=\{P(n_i+n_j)\mid i,j\in I, i<j\}$;

(ii) $\mathcal{L}(H,N_H(T))=\{P(n_i;j)\mid i\in I, j\in\{1,\dots,n_i-1\}\}$ *and*

(iii) $\mathcal{T}(H,N_H(T))=\{T(n_i;s^c;j)\mid i\in I, s^c\in\Pi(|E|)$ *and* $1\le j\le n_i\}$.

In [10, Section 4] explicit examples are presented of these subgroups of H. See also Section 5 of this paper. One of the main theorems from [10] is as follows.

Theorem 2.5. *Suppose that $H=E\wr \mathrm{Sym}(n)$ where $n\ge 2$ and E is a cyclic group of odd order. Then*

$$\mathcal{M}(H,N_H(T))=\mathcal{F}(H,N_H(T))\cup\mathcal{L}(H,N_H(T))\cup\mathcal{T}(H,N_H(T)).$$

Proof. See [10, Theorem 4.12]. □

One of our applications of Theorem 2.5 is to wreath products of dihedral groups with a symmetric group. Here to make the notation match up, the symmetric group we need is $\mathrm{Sym}(n/2)$ for n even. We start with the following technical observation.

Lemma 2.6. *Suppose that D is a dihedral group of twice odd order, and let E be the cyclic group of index 2 in D. Assume that n is even, $Y=\mathrm{Sym}(n/2)$ and T_1 is the cyclic group of order 2. Let $X_1=T_1\wr Y$ identified as a subgroup of $X=\mathrm{Sym}(n)$, $W=E\wr X_1\le E\wr X$, F be the base group of W and J be the base group of X_1. Then $D\wr Y\cong[F,J]X_1$.*

Proof. Recall, if H, K and L are groups and $\phi: H\to K$ is a homomorphism, then ϕ can be used to define a homomorphism $\hat{\phi}: H\wr L\to K\wr L$. Hence we first show that $E\wr T_1$ maps homomorphically onto D. For this we let e be a generator of E and $D=\langle e,s\rangle$ for some involution $s\in D$. A typical element of $E\wr T_1$ can be written as $(e^j,e^k)t^\ell$ where $t\in T_1$ and j,k,ℓ are integers. We let θ map $(e^j,e^k)t^\ell$ to $e^{j-k}s^\ell$. Since

$$(e^j,e^k)t^\ell(e^p,e^q)t^r=(e^j,e^k)(e^p,e^q)^{t^\ell}t^{\ell+r}=\begin{cases}(e^{j+p},e^{k+q})t^{\ell+r} & \ell=0\\ (e^{j+q},e^{k+p})t^{\ell+r} & \ell=1\end{cases}$$

this map is a homomorphism and it is plainly onto.

From θ we construct $\hat{\theta}: W \to D \wr Y$. The kernel of $\hat{\theta} = C_F(J)$. Moreover, as $W = C_F(J)[F,J]X_1$ and $C_F(J) \cap [F,J]X_1 = 1$, we have $D \wr Y \cong [F,J]X_1$. This proves the lemma. $\square$

Letting $L = D \wr X$ and identifying L with $[F,J]X_1$, we now list a collection of subgroup of $[F,J]X_1$ which contain $T \in \mathrm{Syl}_2(L) (\subseteq \mathrm{Syl}_2(X_1))$ where X_1 is identified as a subgroup of X.

Definition 2.7.

$$\begin{aligned} \mathcal{F}(L,T) &= \{X(n_i+n_j) \mid i,j \in I, i<j\}; \\ \mathcal{L}(L,T) &= \{X(n_i;j) \mid i \in I, j \in \{2,\ldots,n_i-1\}\} \textit{ and} \\ \mathcal{T}(L,T) &= \{U(n_i;s^c;n_i)T \mid i \in I, \textit{ and } s^c \in \Pi(|E|)\}. \end{aligned}$$

Note that $|\mathcal{T}(L,T)| = r|\Pi(|E|)|$.

Theorem 2.8. *Suppose that D is a dihedral group of twice odd order. Let $L = D \wr \mathrm{Sym}(n/2)$, $T \in \mathrm{Syl}_2(L)$ and identify L with $[F,J]X_1$. Then $T = N_L(T)$ and*

$$\mathcal{M}(L,T) = \mathcal{F}(L,T) \cup \mathcal{L}(L,T) \cup \mathcal{T}(L,T).$$

Proof. We have $X_1 = T_1 \wr \mathrm{Sym}(n/2) \le X$ with notation chosen so that $T \le X_1$. We identify $L = [F,J]X_1 \le H$ where $H = E \wr X$. Since $C_F(T) \le C_F(J)$ and $C_F(J) \cap [F,J] = 1$, we have

$$N_L(T) = N_H(T) \cap L = C_F(T)T \cap L = T.$$

Suppose that $P \in \mathcal{M}(L,T)$. Then, by Lemma 2.10, either $P \le [F,J]T$ or $PF/F \in \mathcal{M}(LF/F, TF/F)$. In the former case, P centralizes $C_F(T)$ and $PC_F(T) \in \mathcal{M}(H, N_H(T))$. Thus $PC_F(T) \in \mathcal{T}(H, N_H(T))$ and so $PC_F(T) = U(n_i;s^c;k)C_F(T)T$ for some $i \in I$, $s^c \in \Pi(|E|)$ and $1 \le k \le n_i$ by Theorem 2.5. If $k \ne n_i$, then J centralizes $U(n_i;s^c;k)$ and so $P \le C_F(J)T$ which means that $P \le C_F(J)T \cap L = T$, a contradiction. Hence

$$P = U(n_i;s^c;n_i)TC_F(T).$$

Because T acts irreducibly on $U(n_i;s^c;n_i)/\Phi(U(n_i;s^c;n_i))$ by [10, Lemma 4.1 (iv)], we have

$$U(n_i;s^c;n_i) \le [F,J] \le L.$$

Using the Dedekind modular law gives

$$\begin{aligned} P &= P(L \cap C_F(T)) = L \cap (PC_F(T)) \\ &= L \cap (U(n_i;s^c;n_i)TC_F(T)) = U(n_i;s^c;n_i)T(L \cap C_F(T)) \\ &= U(n_i;s^c;n_i)T. \end{aligned}$$

Thus $P \in \mathcal{T}(L,T)$.

Assume now that $PF/F \in \mathcal{M}(LF/F, TF/F)$. Then

$$PF/F \cong P/(P \cap F) = P/(P \cap [F,J]) \cong P[F,J]/[F,J] \le L/[F,J].$$

Since L normalizes $[F,J]J$ and $J \le P$, $P = N_P(J)(P \cap F)$. Because $T(P \cap [F,J]) < P$, the fact that P is 2-minimal implies $P = N_P(J)$. Now $C_{[F,J]}(J) = 1$, so we know $P \le N_L(J) = X_1$. Therefore $P \in \mathcal{M}(X_1,T)$ and the result follows from Theorem 2.1 as $P \le 2 \wr \mathrm{Sym}(n/2)$. $\square$

For the final two lemmas of this section, G is a finite group, p is a prime, $S \in \mathrm{Syl}_p(G)$ and $B = N_G(S)$.

Lemma 2.9. *Suppose $G = KL$ where K and L are normal subgroups of G with $K \cap L = 1$ and let $P \in \mathcal{M}(G,B)$. Assume that neither K nor L is p-closed. Then either $P \cap K \in \mathcal{M}(K, B \cap K)$ or $P \cap L \in \mathcal{M}(L, B \cap L)$.*

Proof. See Lemma 3.13 of [10]. $\square$

Lemma 2.10. *Suppose that K is a normal subgroup of G and $P \in \mathcal{M}(G,B)$. Then either*

(i) *$P \in \mathcal{M}(BK, B)$; or*
(ii) *$PK/K \in \mathcal{M}(G/K, BK/K)$ and $P \in \mathcal{M}(N_G(S \cap K), B)$.*

In particular, $\mathcal{M}(G, B) = \mathcal{M}(BK, B) \cup \mathcal{M}(N_G(S \cap K), B)$.

Proof. See Lemma 3.8 of [10]. □

3. Symplectic groups

We begin this section with a description of the maximal subgroups of odd index in symplectic groups.

Theorem 3.1. *Suppose that $G = \mathrm{Sp}_n(q)$ where $n \geq 4$ and $q = p^a$, p a prime, is odd. Let V be the natural symplectic $\mathrm{GF}(q)$-module. If H is a maximal subgroup of G of odd index, then one of the following holds.*

(i) *$H \cong \mathrm{Sp}_n(q_0)$ where $q_0^c = q$ and c is an odd prime.*
(ii) *H is the stabilizer of a non-degenerate proper subspace of V.*
(iii) *H is the stabilizer of an orthogonal decomposition of $V = \bigoplus V_i$ into an orthogonal sum of isometric non-degenerate subspaces V_i of dimension ℓ where $\ell = 2^b \geq 2$.*
(iv) *$n = 4$, $q = p \equiv 3, 5 \pmod 8$ and $H \sim 2_-^{1+4}.\mathrm{Alt}(5)$.*

Furthermore, in cases (i) and (iv) there is a unique conjugacy class of such subgroups.

Proof. See [5], [8] or [9] (note that the possibility $q = p \equiv 5 \pmod 8$ is missing in (iv) from case (21) of Theorem 1 of [9], see [2]). □

For $k \geq 2$ even and dividing n as in case (iii) of Theorem 3.1, we write $m = n/k$ and decompose V into an orthogonal sum of m non-degenerate subspaces each of dimension k. Thus

$$V = V_1 \perp \cdots \perp V_m$$

with $\dim V_i = k$. For $1 \leq i \leq m$, we define $K_i = \mathrm{Sp}(V_i)$ and put

$$F_k = K_1 \times \cdots \times K_m.$$

We define L_k to be the subgroup of G which preserves the decomposition of V while permuting the factors. This gives

$$L_k = \mathrm{Sp}(V_1) \wr \mathrm{Sym}(m)$$

and F_k is the base group of L_k. We next examine some properties of the Sylow 2-subgroups of $\mathrm{Sp}_n(q)$ that we shall need.

Lemma 3.2. *Suppose that $G = \mathrm{Sp}_n(q)$ where q is odd and write $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$ with $n_1 > n_2 > \cdots > n_r > 0$. Let $S \in \mathrm{Syl}_2(G)$ and $B = N_G(S)$. Then*

(i) *when $q \equiv 1, 7 \pmod 8$, $B = S$;*
(ii) *when $q \equiv 3, 5 \pmod 8$, B/S is elementary abelian of order 3^r; and*
(iii) *we may choose S so that*

$$S \leq B \leq L_2 = \mathrm{SL}_2(q) \wr \mathrm{Sym}(n/2).$$

Proof. For part (i) and (ii) see [6] and for (iii) consult [3]. □

Because of Lemma 3.2(ii), the case $k = 2$ is of special interest. Write $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$ with $n_1 > n_2 > \cdots > n_r > 0$, and let $T \in \mathrm{Syl}_2(\mathrm{Sym}(n/2))$, where we take $\mathrm{Sym}(n/2)$ to be the permutation matrix which permutes the non-degenerate 2-spaces $V_1, \ldots, V_{n/2}$ used to define L_2. Then

$$T = T_{n_1-1} \times T_{n_2-1} \times \cdots \times T_{n_r-1}$$

where $T_\ell \in \mathrm{Syl}_2(\mathrm{Sym}(2^\ell))$. This leads to a corresponding decomposition $B = B_{n_1-1} \times B_{n_2-1} \times \cdots \times B_{n_r-1}$. (Such decompositions occur in [10] - see Sections 1 and 5 there.)

4. The 2-minimal subgroups of Symplectic Groups

Throughout this section we have $G = \mathrm{Sp}_n(q)$, where n is even and $q = p^a$ is odd. We continue to write the 2-adic decomposition of n as $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$. Let V be the natural symplectic module for G. Also we fix $S \in \mathrm{Syl}_2(G)$ and set $B = N_G(S)$.

In Theorem 4.1, the superscript [2] indicates that there are two G-conjugacy classes of the given group. Also, if ℓ is a positive integer, ℓ_2 will denote the largest 2-power dividing ℓ, $\ell_{2'} = \ell/\ell_2$ and, we recall, $\Pi(\ell)$ is the set of all odd prime powers greater than 1 which divide ℓ. We remark that in the next theorem all the described extensions have quaternion Sylow 2-subgroups.

Theorem 4.1. *Suppose that $G = \mathrm{Sp}_2(q)$ with $q = p^a$ odd.*

(i) *If $q \equiv 3, 5 \pmod 8$ and $p \neq 3, 5$, then one of the following holds:*

(a) $q \equiv \pm 11, \pm 19 \pmod{40}$ *and*
$$\mathcal{M}(G,B) = \{2^{\cdot}\mathrm{Alt}(5)^{[2]}, \mathrm{Sp}_2(p^{s^t}) \mid s^t \in \Pi(a)\}; \text{ or}$$

(b) $q \not\equiv \pm 11, \pm 19 \pmod{40}$ *and*
$$\mathcal{M}(G,B) = \{\mathrm{Sp}_2(p^{s^t}) \mid s^t \in \Pi(a) \cup \{1\}\}.$$

(ii) *If $q \equiv 3, 5 \pmod 8$ and $p = 3$, then*
$$\mathcal{M}(G,B) = \{\mathrm{Sp}_2(3^{s^t}) \mid s^t \in \Pi(a)\}.$$

(iii) *If $q \equiv 3, 5 \pmod 8$ and $p = 5$, then*
$$\mathcal{M}(G,B) = \{\mathrm{Sp}_2(5^{s^t}) \mid s^t \in \Pi(a) \cup \{1\}\}.$$

(iv) *If $q \equiv 1 \pmod 8$, then one of the following holds:*

(a) $a_2 > 2$ *or* $a_2 = 2$ *and* $q \equiv 1 \pmod{16}$,
$$\mathcal{M}(G,B) = \mathcal{M}(2^{\cdot}\mathrm{Dih}(q-1), B) \cup \{2^{\cdot}\mathrm{PGL}_2(p^{a_2/2})^{[2]}\};$$

(b) $p = 5$, $a_2 = 2$ *and*
$$\mathcal{M}(G,B) = \mathcal{M}(2^{\cdot}\mathrm{Dih}(q-1), B) \cup \{2^{\cdot}\mathrm{Sym}(5)^{[2]}\} \cup \{2^{\cdot}\mathrm{Sym}(4)^{[2]}\};$$

(c) $p = 3$, $a_2 = 2$ *and*
$$\mathcal{M}(G,B) = \{2^{\cdot}\mathrm{PGL}_2(3)^{[2]}\};$$

(d) $a_2 = 2$ *and* $q \equiv 9 \pmod{16}$ *with* $p > 5$,
$$\mathcal{M}(G,B) = \mathcal{M}(2^{\cdot}\mathrm{Dih}(q-1), B) \cup \{2^{\cdot}\mathrm{Sym}(4)^{[2]}\};$$

(e) $q \equiv 1 \pmod{16}$, $a_2 = 1$,
$$\mathcal{M}(G,B) = \mathcal{M}(2^{\cdot}\mathrm{Dih}(q-1), B) \cup \{\mathrm{Sp}_2(p)\}; \text{ or}$$

(f) $q \equiv 9 \pmod{16}$, $a_2 = 1$,
$$\mathcal{M}(G,B) = \mathcal{M}(2^{\cdot}\mathrm{Dih}(q-1), B) \cup \{2^{\cdot}\mathrm{Sym}(4)^{[2]}\}.$$

(v) *If $q \equiv 7 \pmod 8$, then one of the following holds:*

(a) $q \equiv 7 \pmod{16}$,
$$\mathcal{M}(G,B) = \mathcal{M}(2^{\cdot}\mathrm{Dih}(q+1), B) \cup \{2^{\cdot}\mathrm{Sym}(4)^{[2]}\}; \text{ or}$$

(b) $q \equiv 15 \pmod{16}$,
$$\mathcal{M}(G,B) = \mathcal{M}(2^{\cdot}\mathrm{Dih}(q+1), B) \cup \{\mathrm{Sp}_2(p)\}.$$

In particular, $O^{2'}(P) = P$ for all 2-minimal subgroups P of G.

Proof. Since $\mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$, this is just [10, Theorem 13.2]. □

We next consider the 2-minimal subgroups of $\mathrm{Sp}_4(q)$ as they appear explicitly in the general picture for all the symplectic groups just as $n = 4$ does in the case of $\mathrm{GL}^{\epsilon}_n(q)$.

Lemma 4.2. *Suppose that $G \cong \mathrm{Sp}_4(q)$. If $P \in \mathcal{M}(G, B)$, then either*

(i) *P is contained in $\mathrm{Sp}_2(q) \wr 2$ which preserves a decomposition of V into a perpendicular sum of two non-degenerate subspaces; or*
(ii) *$q \equiv 3, 5 \pmod 8$ and P has shape $2^{1+4}_{-}.\mathrm{Alt}(5)$ and is unique up to conjugacy containing B; or*
(iii) *$q \equiv 1, 7 \pmod 8$ and $P = \mathrm{Sp}_4(p^{a_2})$ and is unique up to conjugacy containing B.*

In particular, in cases (ii) and (iii), $O^{2'}(P) = P$.

Proof. Recall that, if Y_1, Y_2 are subgroups of G which both contain B, then either $Y_1 = Y_2$ or Y_1 and Y_2 are not G-conjugate. By Theorem 3.1, B is contained in a maximal subgroup M of G with $M = \mathrm{Sp}_2(q) \wr 2$.

Suppose that G is 2-minimal. Then, as $B \le M$, Theorem 3.1 (i) forces $G = \mathrm{Sp}_4(p^{a_2})$. Moreover, if $q \equiv 3, 5 \pmod 8$, then $|B| = 2^7.3$ and B is contained in a subgroup of shape $2^{1+4}_{-}.\mathrm{Alt}(5)$ by Theorem 3.1 (iv) and this subgroup is not contained in M. This contradicts $G \in \mathcal{M}(G, B)$. Therefore, if $P = G$, then $G \cong \mathrm{Sp}_4(p^{a_2})$ and $p \equiv 1, 7 \pmod 8$. Conversely, if these conditions hold, then Theorem 3.1 implies that M is the unique maximal subgroup of G which contains M.

Now suppose that $P \in \mathcal{M}(G, B)$ and assume that a is chosen minimally such that the statement of the lemma does not hold. If $P = G$, then case (iii) holds, a contradiction. Hence $P < G$. Let K be a maximal subgroup of G that contains P. Then $K \ne \mathrm{Sp}_2(p^{a/c})$ with c an odd prime and $K \ne M$. Since S acts irreducibly on the natural symplectic space, the only other possibility is that $q = p$ and $K \sim 2^{1+4}_{-}{\cdot}\mathrm{Alt}(5)$, but then B is maximal in K and so $P = K$, a contradiction. □

We now define a subgroup of G which will play a similar role to that played by A in the investigation of the linear and unitary groups, see Section 5 of [10]. Recall, from Lemma 3.2 (ii) that $L_2 \le G$ and we may assume

$$S \le B \le L_2 = \mathrm{SL}_2(q) \wr \mathrm{Sym}(n/2).$$

Put $S_0 = F_2 \cap S$. Also let C be the Thompson subgroup of S_0 generated by maximal order abelian subgroups of S_0. Then either $q \equiv 3, 5 \pmod 8$ and $C = S_0$ is a direct product of $n/2$ quaternion groups Q_8 or $q \equiv 1, 7 \pmod 8$ and C is a homocyclic subgroup of S_0 of exponent $(q \pm 1)_2 \ge 8$. We write $C = C_1 \times \cdots \times C_{n/2}$ where, for $1 \le i \le n/2$, C_i is a cyclic group of order $(q \pm 1)_2$ of order at least 8 or is a quaternion group of order 8 with C_i contained in the ith factor of the base group of L_2.

Just as in the proof of Lemma 5.2 in [10] we may prove the following lemma.

Lemma 4.3. *The following hold:*

(i) *if $q \equiv 1 \pmod 8$, then $N_G(C)/C \cong \mathrm{Dih}(2(q-1)_{2'}) \wr \mathrm{Sym}(n/2)$;*
(ii) *if $q \equiv 7 \pmod 8$, then $N_G(C)/C \cong \mathrm{Dih}(2(q+1)_{2'}) \wr \mathrm{Sym}(n/2)$; and*
(iii) *if $q \equiv 3, 5 \pmod 8$, then $N_G(C)/C \cong 3 \wr \mathrm{Sym}(n/2)$.*

Proof. Set $W_i = [V, C_i]$ for $1 \le i \le n/2$. Then $\dim W_i = 2$ and we have an orthogonal decomposition

$$V = [V, C] = W_1 \oplus \cdots \oplus W_{n/2}.$$

These 2-dimensional spaces are permuted naturally by $\mathrm{Sym}(n/2)$. Since the C_i are the maximal subgroups of C with 2-dimensional commutators, we infer that $N_G(C)$ is as described. □

Lemma 4.4. *Suppose that $C \le R \le S$. Then C is weakly closed in S and $N_G(R) \le N_G(C)$.*

Proof. Again set $W_i = [V, C_i]$ for $1 \le i \le n/2$. Then, as in Lemma 4.3, $\dim W_i = 2$ and

$$V = [V, C] = W_1 \oplus \cdots \oplus W_{n/2}$$

permuted naturally by $\mathrm{Sym}(n/2)$.

Suppose that $g \in G$ and $C^g \le S$ and $C^g \ne C$. Then, without loss, $Y = C_1^g \not\le C$. If $Y \cap F_2 \ne 1$, then, as $[V, Y \cap F_2] = [V, C_i^g]$ has dimension 2, we have $Y \cap F_2 \le K_j = \mathrm{Sp}(W_j)$ for some $1 \le j \le n/2$. Since $W_k = [V, Y \cap F_2] = [V, Y]$, we deduce that $Y \le F_2$ and thus $Y \le K_j$. But then $Y \le C$. Hence $Y \cap F_2 = 1$. Let $y \in Y$ have order 4. Then y transitively permutes four of the subgroups K_i, $1 \le i \le n/2$. This means $\dim[V, y] \ge 6$, which is a contradiction. This proves that C is weakly closed in S and the lemma follows from this. □

Our next lemma describes all the 2-minimal subgroups of $N_G(C)$. Because of Lemma 4.3, the description of these subgroups fluctuate with the congruence of q modulo 8. From the structure of $N_G(C)$ given in Lemma 4.3, we have $N_G(C)/C \cong D \wr \mathrm{Sym}(n/2)$ where D is a dihedral group of order $2(q-1)_{2'}$ if $q \equiv 1 \pmod 8$, $2(q+1)_{2'}$ if $q \equiv 7 \pmod 8$ and otherwise has order 3 by Lemma 4.3. Using Theorem 2.5 when $q \equiv 3, 5 \pmod 8$ and Theorem 2.8 when $q \equiv \pm 1 \pmod 8$, we obtain

Theorem 4.5. *The 2-minimal subgroups of $N_G(C)$ are*

$$\mathcal{M}(N_G(C), B) = \mathcal{T}(N_G(C), B) \cup \mathcal{F}(N_G(C), B) \cup \mathcal{L}(N_G(C), B).$$

Observe that when $q \equiv 3, 5 \pmod 8$, the structure of the members of $\mathcal{T}(N_G(C), B)$ is considerably different to the case when $q \equiv \pm 1 \pmod 8$. Because of Theorem 4.5 the emphasis is now to discover the 2-minimal subgroups which are not contained in $N_G(C)$.

For the next two lemmas we assume that $n \ge 6$.

Lemma 4.6. *Suppose that $P \in \mathcal{M}(G, B)$. Then one of the following holds.*

(i) *$P \in \mathcal{M}(\mathrm{Sp}(U) \times \mathrm{Sp}(U^\perp), B)$ for U a B-invariant non-degenerate subspaces of V such that $V = U \oplus U^\perp$.*
(ii) *$n = 2^b m$ for some $m \ge 1$ and $P \in \mathcal{M}(L_{2^b}, B)$.*

Proof. This follows from Theorem 3.1, as $n \ge 6$ the 2-minimal subgroups contained in a subfield subgroup are already accounted for in (i) and (ii). □

Lemma 4.7. *Assume that $P \in \mathcal{M}(G, B)$ and P acts irreducibly on V. Then either*

(i) *$P \in \mathcal{M}(N_G(C), B)$; or*
(ii) *$P \in \mathcal{M}(F_{2^b} T_m, B)$ where $n = 2^{m+b}$.*

Proof. By Lemma 4.6, $P \in \mathcal{M}(L_{2^b}, B)$ for some $b \ge 1$. It follows from Lemma 2.10 that either $P \in \mathcal{M}(F_{2^b} B, B)$ or $P = N_P(S \cap F_{2^b})$. Since $S \cap F_{2^b}$ contains C, in the latter case (i) holds by Lemma 4.4. Thus we may suppose that $P \le F_{2^b} B$. Now the fact that P operates irreducibly on V and $F_{2^b} B = F_{2^b} S$ together imply that $n/2^b$ is a power of 2. Thus (ii) holds. □

The next definition and the two following lemmas are needed in Lemma 4.11 to analyse the wreath product subgroup case.

Definition 4.8. *Let r be a prime, A be a group which acts on the group J, $R \in \mathrm{Syl}_r(J)$ and $P \in \mathcal{M}(J, N_J(R))$. Then P is A-*immutable *provided that for all $\alpha \in A$, $P^\alpha \in \mathcal{M}(J, N_J(R))$ implies $P^\alpha = P$. We say that J is A-immutable provided all the members of $\mathcal{M}(J, N_J(R))$ are A-immutable.*

Lemma 4.9. *Suppose that W is a non-degenerate subspace of V and assume that $K, L \leq G$ satisfy $W = [V, K] = [V, L]$. Then L and K are G-conjugate if and only if they are $\mathrm{Sp}(W)$-conjugate where $\mathrm{Sp}(W)$ is considered as a subgroup of G.*

Proof. Let $g \in G$ be such that $K^g = L$. Then

$$W = [V, L] = [V, K^g] = [V, K]^g = W^g,$$

and so $g \in N_G(W)$. Since W is non-degenerate, $N_G(W) = N_G(W^\perp) = \mathrm{Sp}(W) \times \mathrm{Sp}(W^\perp)$. Because K and L centralize $W^\perp$, we deduce that $K, L \leq \mathrm{Sp}(W)$ and then that they are conjugate in $\mathrm{Sp}(W)$. $\square$

Lemma 4.10. *Assume that $\dim V = n = 2^a$. If $1 \leq b \leq a-1$, and $P \in \mathcal{M}(F_{2^b}S, B)$, then there exists $Q \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))$ such that $P = \langle O^{2'}(Q), B\rangle$. Furthermore, if $Q \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))$, then $P = \langle O^{2'}(Q), B\rangle \in \mathcal{M}(F_{2^b}S, B)$.*

Proof. Let $m = 2^{a-b}$ and, as usual, write

$$F_{2^b} = K_1 \times \cdots \times K_m.$$

Then S acts transitively on $\{K_1, \ldots, K_m\}$. The main step is to show that K_1 is $N_B(K_1)$-immutable. First of all note that, from the decomposition of V which defines F_k, $V_1 = [V, K_1]$ and, as $\dim V_1 = 2^b$, $V_1 = [V, S \cap K_1]$. Now let $g \in N_B(K_1)$ and $Y \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))$, then

$$[V, Y] = [V, S \cap K_1] = [V, Y^g].$$

Since $V_1 = [V, K_1] = [V, S \cap K_1]$ is a non-degenerate subspace of V, applying Lemma 4.9 yields Y and Y^g are conjugate in K_1. Since $Y \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))$, we have $Y = Y^g$. Therefore K_1 is $N_B(K_1)$-immutable. Let $Y \in \mathcal{M}(K_1, N_{K_1}(S \cap K_1))$ and let π_1 be the projection map from F_k to K_1. Since $\pi_1(B \cap F_k) = N_{K_1}(S \cap K_1)$, the lemma follows from [10, Lemma 3.15]. $\square$

Lemma 4.11. *Suppose that $P \in \mathcal{M}(G, B)$. Then one of the following holds.*

(i) *$P \in \mathcal{M}(N_G(C), B)$;*

(ii) *there exists $i \in \{1, \ldots, r\}$ with $n_i \geq 1$ such that*

$$P \in \mathcal{M}(\mathrm{Sp}_2(q) \wr T_{n_i - 1} \times \prod_{j \neq i} B_{n_j - 1}, B)$$

and a 2-minimal subgroup Q of $\mathrm{Sp}_2(q)$ such that $P = \langle Q, B\rangle \cong Q \wr T_{n_i-1} \times \prod_{j\neq i} B_{n_j-1}$; or

(iii) *there exists $i \in \{1, \ldots, r\}$ with $n_i \geq 2$ such that*

$$P \in \mathcal{M}(\mathrm{Sp}_4(q) \wr T_{n_i - 2} \times \prod_{j \neq i} B_{n_j - 1}, B)$$

and a 2-minimal subgroup Q of $\mathrm{Sp}_4(q)$ such that $P = \langle Q, B\rangle \cong Q \wr T_{n_i-2} \times \prod_{j\neq i} B_{n_j-1}$.

Proof. We argue by induction on $\dim V$. The result certainly is true if $n \leq 4$ as then either (ii) or (iii) holds by Lemma 4.2. So we may suppose $n \geq 6$. Assume that $P \notin \mathcal{M}(N_G(C), B)$. Suppose that P does not act irreducibly on V. Then $P \in \mathcal{M}(\mathrm{Sp}(U) \times \mathrm{Sp}(U^\perp), B)$ for some proper B-invariant subspace U of V. Let $L = \mathrm{Sp}(U)$ and $K = \mathrm{Sp}(U^\perp)$. Then by Lemma 2.9 we may as well suppose that, $P = (P \cap K) \times (B \cap L)$ with $P \cap K \in \mathcal{M}(K, B \cap K)$. The lemma then follows by induction. Hence we may now assume that P acts irreducibly on V. Lemma 4.7 implies that either (i) holds or P is contained in $\mathcal{M}(\mathrm{Sp}_{2^b}(q) \wr T_{n/2^b}, B)$. Suppose the latter possibility holds and choose b minimal subject to this containment. Applying Lemma 4.10 we get that $P = \langle O^{2'}(Q), B\rangle$ for some 2-minimal subgroup Q in $\mathrm{Sp}_{2^b}(q)$. Using the minimality of b and the fact that $P \notin \mathcal{M}(N_G(C), B)$, Lemma 4.6 and then 4.7 indicate that $b \leq 2$. Finally, we note that for 2-minimal subgroups Q of $\mathrm{Sp}_2(q)$ and $\mathrm{Sp}_4(q)$, Theorem 4.1 and Lemma 4.2, give $Q = O^{2'}(Q)$ and so we conclude that $P = \langle Q, B\rangle$. This proves the lemma. $\square$

Because of Lemma 4.11, to complete the inventory of 2-minimal subgroups of G we just need to know the candidates for Q. These have been presented in Theorem 4.1 and Lemma 4.2 and we list them here again in a more convenient way.

Definition 4.12. *The set $\mathcal{X}_2$ consists of 2-minimal subgroups of $\mathrm{Sp}_2(q)$ which are not toral. Its members are describe in Table 1 and*

$$\mathcal{M}_{\mathcal{X}_2}(G,B) = \{Q \wr T_{n_i-1} \times \prod_{j\in I\setminus\{i\}} B_{n_j-1} \mid i \in I, Q \in \mathcal{X}_2\}.$$

$\mathcal{X}_2$	conditions on q
$q \equiv 3,5 \pmod 8$	
$\{2{\cdot}\mathrm{Alt}(5)^{[2]}, \mathrm{Sp}_2(p^{s^t}) \mid s^t \in \Pi(a)\}$	$q \equiv \pm 11, \pm 19 \pmod{40}$
$\{\mathrm{Sp}_2(p^{s^t}) \mid s^t \in \Pi(a) \cup \{1\}\}$	$q \equiv \pm 3, \pm 5, \pm 13 \pmod{40}$, $p \neq 3,5$
$\{\mathrm{Sp}_2(3^{s^t}) \mid s^t \in \Pi(a)\}$	$p = 3, a_2 = 1$
$\{\mathrm{Sp}_2(5^{s^t}) \mid s^t \in \Pi(a) \cup \{1\}\}$	$p = 5, a_2 = 1$
$q \equiv 1 \pmod 8$	
$\{2{\cdot}\mathrm{PGL}_2(p^{a_2/2})^{[2]}\}$	$a_2 > 2$
$\{2{\cdot}\mathrm{PGL}_2(p^{a_2/2})^{[2]}\}$	$a_2 = 2$ and $q \equiv 1 \pmod{16}$
$\{2{\cdot}\mathrm{PGL}_2(5)^{[2]}, 2{\cdot}\mathrm{Sym}(4)^{[2]}\}$	$a_2 = 2$ and $p = 5$
$\{2{\cdot}\mathrm{PGL}_2(3)^{[2]}\}$	$a_2 = 2$ and $p = 3$
$\{2{\cdot}\mathrm{Sym}(4)^{[2]}\}$	$a_2 = 2$ and $q \equiv 9 \pmod{16}$, $p > 5$
$\{\mathrm{Sp}_2(p)\}$	$a_2 = 1$ and $q \equiv 1 \pmod{16}$
$\{2{\cdot}\mathrm{Sym}(4)^{[2]}\}$	$a_2 = 1$ and $q \equiv 9 \pmod{16}$
$q \equiv 7 \pmod 8$	
$\{2{\cdot}\mathrm{Sym}(4)^{[2]}\}$	$q \equiv 7 \pmod{16}$
$\{\mathrm{Sp}_2(p)\}$	$q \equiv 15 \pmod{16}$

Table 1: The description of $\mathcal{X}_2$

Definition 4.13.

$$\mathcal{X}_4 = \begin{cases} \{\mathrm{Sp}_4(p^{a_2})\} & q \equiv 1,7 \pmod 8 \\ \{2_-^{1+4}.\mathrm{Alt}(5)\} & q \equiv 3,5 \pmod 8 \end{cases}$$

and

$$\mathcal{M}_{\mathcal{X}_4}(G,B) = \{Q \wr T_{n_i-2} \times \prod_{j\in I\setminus\{i\}} B_{n_j-1} \mid i \in I, n_i \geq 2, Q \in \mathcal{X}_4\}.$$

Finally, we set $\mathcal{F}(G,B) = \mathcal{F}(N_G(C),B)$, $\mathcal{L}(G,B) = \mathcal{L}(N_G(C),B)$ and $\mathcal{T}(G,B) = \mathcal{T}(N_G(C),B)$. Then applying Lemma 4.11 provides a complete description of the 2-minimal subgroups in the symplectic groups, and have now proved Theorem 1.1.

5. An Example

In the following example, if X is a group with a given name, we shall use X^n to denote the direct product of $n \geq 2$ copies of X.

Example 5.1. *Suppose that $G \cong \mathrm{Sp}_{10}(q)$ where $q = p^a$ is odd.*

Case $\mathbf{q \equiv 3,5}$ mod 8.

In this case we have

$$B \cong (\mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8 \times \mathrm{Q}_8).(3 \times T_2) \times \mathrm{Q}_8.(3 \times T_0)$$

where we recall that T_0 is the trivial group (we have included this in the first instance to illustrate the more general situation). In this case $C = \mathrm{Q}_8^5$ and then, by Theorem 2.5 applied with $N_G(C)/C \cong 3 \wr \mathrm{Sym}(5)$,

$$\begin{aligned}
\mathcal{L}(N_G(C), B) &= \left\{\mathrm{Q}_8^5{:}(3 \times \mathrm{Sym}(4) \times 3)\right\};\\
\mathcal{F}(N_G(C), B) &= \left\{\mathrm{Q}_8^5{:}3^5 : \mathrm{Sym}(5)\right\} \text{ and}\\
\mathcal{T}(N_G(C), B) &= \{\mathrm{Q}_8^5{:}(3 \times 3^2 \times 3)(T_2 \times T_0), \mathrm{Q}_8^5{:}(3 \times 3_- \times 3)(T_2 \times T_0)\}
\end{aligned}$$

where 3_- indicates a group of order 3 inverted by T_2. We define $\mathcal{M}_{\mathcal{X}_2}(G, B)$ as in Table 2 and

$$\mathcal{M}_{\mathcal{X}_4}(G, B) = \{2_-^{1+4}.\mathrm{Alt}(5) \wr T_1 \times \mathrm{Q}_8 : 3\}.$$

Conditions	$\mathcal{M}_{\mathcal{X}_2}(G,B)$
$q \equiv \pm 11, \pm 19 \pmod{40}$	$(2{\cdot}\mathrm{Alt}(5) \wr T_2 \times \mathrm{Q}_8 : 3)^{[2]}$
$s^t \in \Pi(a)$	$\mathrm{Sp}_2(p^{s^t}) \wr T_2 \times \mathrm{Q}_8 : 3$
	$(\mathrm{Q}_8^4 : (3 \times T_2) \times 2{\cdot}\mathrm{Alt}(5))^{[2]}$
	$\mathrm{Q}_8^4 : (3 \times T_2) \times \mathrm{Sp}_2(p^{s^t})$
$q \pm 3, \pm 5, \pm 13 \pmod{40}$	$\mathrm{Sp}_2(p^{s^t}) \wr T_2 \times \mathrm{Q}_8 : 3$
$p \notin \{3,5\}$, $s^t \in \Pi(a) \cup \{1\}$	$\mathrm{Q}_8^4 : (3 \times T_2) \times \mathrm{Sp}_2(p^{s^t})$
$p = 3$ and $a_2 = 1$	$\mathrm{Sp}_2(3^{s^t}) \wr T_2 \times \mathrm{Q}_8 : 3$
$s^t \in \Pi(a)$	$\mathrm{Q}_8^4 : (3 \times T_2) \times \mathrm{Sp}_2(3^{s^t})$
$p = 5$ and $a_2 = 1$	$\mathrm{Sp}_2(5^{s^t}) \wr T_2 \times \mathrm{Q}_8 : 3$
$s^t \in \Pi(a) \cup \{1\}$	$\mathrm{Q}_8^4 : (3 \times T_2) \times \mathrm{Sp}_2(5^{s^t})$

Table 2: Definition of $\mathcal{M}_{\mathcal{X}_2}(G, B)$, $q \equiv 3, 5 \pmod 8$

Then we have

$$\mathcal{M}(G, B) = \mathcal{L}(N_G(C), B) \cup \mathcal{F}(N_G(C), B) \cup \mathcal{T}(N_G(C), B) \cup \mathcal{M}_{\mathcal{X}_2}(G, B) \cup \mathcal{M}_{\mathcal{X}_4}(G, B).$$

Case $\mathbf{q \equiv 1, 7 \bmod 8}$.

In this situation C is a homocyclic group with $10/2 = 5$ direct factors of order m and $N_G(C)/C \cong \mathrm{Dih}(2\ell) \wr \mathrm{Sym}(5)$ where

$$\begin{aligned}
m &= (q-1)_2, \ell = (q-1)_{2'} \text{ if } q \equiv 1 \pmod 8; \text{ and}\\
m &= (q+1)_2, \ell = (q+1)_{2'} \text{ if } q \equiv 7 \pmod 8.
\end{aligned}$$

The Sylow 2-subgroups of $\mathrm{Sp}_2(q)$ are quaternion of order $2m \geq 16$ and we denote such groups by Q_{2m}. We have

$$B = m^4{\cdot}T_3 \times m{\cdot}T_1 = \mathrm{Q}_{2m} \wr T_2 \times \mathrm{Q}_{2m} \wr T_0.$$

where $L{\cdot}M$ denotes a non-split extension of L by M. From Theorem 2.8 applied to $\mathrm{Dih}(2\ell) \wr \mathrm{Sym}(5)$, we have

$$\begin{aligned}
\mathcal{L}(N_G(C), B) &= \left\{(m^4 \times m){\cdot}(2 \wr \mathrm{Sym}(4) \times 2) = \mathrm{Q}_{2m}^4 \wr \mathrm{Sym}(4) \times \mathrm{Q}_{2m}\right\};\\
\mathcal{F}(N_G(C), B) &= \left\{m^5{\cdot}(2 \wr \mathrm{Sym}(5)) = \mathrm{Q}_{2m}^5 \wr \mathrm{Sym}(5)\right\}; \text{ and}\\
\mathcal{T}(N_G(C), B) &= \{m^5{\cdot}(U(3, s^c, 3)T_3 \times T_1), m^5{\cdot}(T_3 \times (U(1, s^c, 1)T_1)) \mid s^c \in \Pi(\ell)\}.
\end{aligned}$$

We define $\mathcal{M}_{\mathcal{X}_2}(G, B)$ as in Table 3 and

$$\mathcal{M}_{\mathcal{X}_4}(G, B) = \{(\mathrm{Sp}_4(p^{a_2}) \wr T_1) \times \mathrm{Q}_{2m}\}.$$

Conditions	$\mathcal{M}_{\mathcal{X}_2}(G,B)$
$q \equiv 1 \pmod 8$ $a_2 > 2$ or $a_2 = 2$ and $q \equiv 1 \pmod{16}$	$(2{\cdot}\mathrm{PGL}_2(p^{a_2/2}) \wr T_2 \times \mathrm{Q}_{2m})^{[2]}$ $(\mathrm{Q}_{2m}^4 : T_2 \times 2{\cdot}\mathrm{PGL}_2(p^{a_2/2}))^{[2]}$
$q \equiv 1 \pmod 8$ $a_2 = 2$ and $p = 5$	$(2{\cdot}\mathrm{PGL}_2(5) \wr T_2 \times \mathrm{Q}_{16})^{[2]}$ $(2{\cdot}\mathrm{Sym}(4) \wr T_2 \times \mathrm{Q}_{16})^{[2]}$ $(\mathrm{Q}_{16}^4 : T_2 \times 2{\cdot}\mathrm{PGL}_2(5))^{[2]}$ $(\mathrm{Q}_{16}^4 : T_2 \times 2{\cdot}\mathrm{Sym}(4))^{[2]}$
$q \equiv 1 \pmod 8$ $a_2 = 2$ and $p = 3$	$(2{\cdot}\mathrm{PGL}_2(3) \wr T_2 \times \mathrm{Q}_{16})^{[2]}$ $(\mathrm{Q}_{16}^4 : T_2 \times 2{\cdot}\mathrm{PGL}_2(3))^{[2]}$
$q \equiv 1 \pmod 8$ $a_2 = 2$ and $q \equiv 9 \pmod{16}$, $p > 5$; or $a_2 = 1$ and $q \equiv 9 \pmod{16}$	$(2{\cdot}\mathrm{Sym}(4) \wr T_2 \times \mathrm{Q}_{16})^{[2]}$ $(\mathrm{Q}_{16}^4 : T_2 \times 2{\cdot}\mathrm{Sym}(4))^{[2]}$
$q \equiv 1 \pmod 8$ $a_2 = 1$ and $q \equiv 1 \pmod{16}$	$\mathrm{Sp}_2(p) \wr T_2 \times \mathrm{Q}_{2m}$ $\mathrm{Q}_{2m}^4 : T_2 \times \mathrm{Sp}_2(p)$
$q \equiv 7 \pmod 8$ and $q \equiv 7 \pmod{16}$	$(2{\cdot}\mathrm{Sym}(4) \wr T_2 \times \mathrm{Q}_{16})^{[2]}$ $(\mathrm{Q}_{16}^4 : T_2 \times 2{\cdot}\mathrm{Sym}(4))^{[2]}$
$q \equiv 7 \pmod 8$ and $q \equiv 15 \pmod{16}$	$\mathrm{Sp}_2(p) \wr T_2 \times \mathrm{Q}_{2m}$ $\mathrm{Q}_{2m}^4 : T_2 \times \mathrm{Sp}_2(p)$

Table 3: Definition of $\mathcal{M}_{\mathcal{X}_2}(G,B)$, $q \equiv 1, 7 \pmod 8$

Then

$$\mathcal{M}(G,B) \;=\; \mathcal{L}(N_G(C),B) \cup \mathcal{F}(N_G(C),B) \cup \mathcal{T}(N_G(C),B) \cup \mathcal{M}_{\mathcal{X}_2}(G,B) \cup \mathcal{M}_{\mathcal{X}_4}(G,B).$$

References

[1] Aschbacher, M. Finite group theory. Second edition. Cambridge Studies in Advanced Mathematics, 10. Cambridge University Press, Cambridge, 2000.

[2] Bray, John; Holt, Derek; Roney-Dougal, Colva Mary. The maximal subgroups of the low-dimensional finite classical groups. Cambridge University Press, 2013.

[3] Carter, Roger; Fong, Paul. The Sylow 2-subgroups of the finite classical groups. J. Algebra 1 (1964), 139–151.

[4] Huppert, B. Endliche Gruppen. I. Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin-New York, 1967.

[5] Kantor, William M. Primitive permutation groups of odd degree, and an application to finite projective planes. J. Algebra 106 (1987), no. 1, 15–45.

[6] Kondrat'ev, A. S. Normalizers of Sylow 2-subgroups in finite simple groups. (Russian) Mat. Zametki 78 (2005), no. 3, 368–376; translation in Math. Notes 78 (2005), no. 3-4, 338–346.

[7] Lempken, Wolfgang; Parker, Christopher; Rowley, Peter. Minimal parabolic systems for the symmetric and alternating groups. The atlas of finite groups: ten years on (Birmingham, 1995), 149–162, London Mathematical Society Lecture Note Series, 249. Cambridge University Press, Cambridge, 1998.

[8] Liebeck, Martin W.; Saxl, Jan. The primitive permutation groups of odd degree. J. London Math. Soc. (2) 31 (1985), no. 2, 250–264.

[9] Maslova, N. V. Classification of maximal subgroups of odd index in finite simple classical groups, Proceedings of the Steklov Institute of Mathematics, 2009, Suppl. S164-S183.

[10] Parker, Christopher; Rowley Peter. 2-minimal subgroups of monomial, linear and unitary groups. J. Algebra 543 (2020), 1–53.

Chris Parker, School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom

Email address: `c.w.parker@bham.ac.uk`

PETER ROWLEY, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MANCHESTER, OXFORD ROAD, M13 6PL, UNITED KINGDOM

Email address: peter.j.rowley@manchester.ac.uk