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DOI:
10.1112/mtk.12064

Citation for published version (Harvard):
Maleva, O & Villanueva-Segovia, C 2021, 'Best constants for Lipschitz quotient mappings in polygonal norms', *Mathematika*, vol. 67, no. 1, pp. 116-144. https://doi.org/10.1112/mtk.12064

Mathematika 67 (2021) 116–144 doi:10.1112/mtk.12064

BEST CONSTANTS FOR LIPSCHITZ QUOTIENT MAPPINGS IN POLYGONAL NORMS

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Abstract. We investigate the relation between the maximum cardinality N of the level sets of a Lipschitz quotient mapping of the plane and the ratio between its Lipschitz and co-Lipschitz constants, with respect to the polygonal norms, and establish that bounds of $1/N$ previously shown to be sharp for Euclidean norm stay sharp for polygonal n-norms if and only if n is not divisible by 4.

§1. *Introduction.* This paper is motivated by a desire to understand how the properties of Lipschitz quotient mappings, in particular cardinality of point preimages, depend on the norm of the space. For a pair of two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, we consider mappings $f\colon X \to Y$ which are Lipschitz and satisfy an additional, 'dual' property of being co-Lipschitz. Namely, a mapping f is Lipschitz quotient if there exist $0 < c \leqslant L < \infty$ such that $B_{cr}(f(x)) \subseteq f(B_r(x)) \subseteq B_{Lr}(f(x))$ for any $x \in X$ and all $r > 0$. If only left (or right) inclusion is satisfied for all $x \in X$, the mapping f is called c-co-Lipschitz (respectively, L-Lipschitz). The infimum of all such L is called the Lipschitz constant of f, denoted by $\mathrm{Lip}(f)$; and the supremum of all such c is called the co-Lipschitz constant of f, denoted by $\text{co-Lip}(f)$. Here $B_\rho(z)$ denotes the open ball centred at z of radius ρ. We also consider pointwise versions of Lipschitzness and co-Lipschitzness.

Definition 1.1. A map $f\colon (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is called pointwise Lipschitz at $x \in X$, if there exists an $L > 0$ such that the L-Lipschitz condition for f at x is satisfied in some open ball centred at x, i.e. if there exists an $R > 0$ such that $f(B_r^X(x)) \subseteq B_{Lr}^Y(f(x))$ for all $r \in (0, R)$. For any such L, we will say that f is pointwise L-Lipschitz at x. Similarly, we say that f is pointwise co-Lipschitz at x if there exist $c > 0$ and $R > 0$ such that $B_{cr}^Y(f(x)) \subseteq f(B_r^X(x))$ for all $r \in (0, R)$. For any such c, we then say that f is pointwise c-co-Lipschitz at x.

The notion of co-Lipschitz mappings was originally introduced in [**3, 4, 10**] but their first systematic study should be attributed to [**1, 5**], where the authors reached very significant results concerning the structure of such mappings. The results support the intuition that Lipschitz quotient mappings are a non-linear analogue of linear quotient mappings between Banach spaces. One particular feature that will be important for us is point preimages under Lipschitz quotient mappings. Under a linear quotient mapping $X \to Y$, each point preimage is an affine subspace of X of dimension $\dim(X) - \dim(Y)$. It is shown in [**6**] that for Lipschitz quotient mappings with constants c and L close enough to each other, point preimages cannot be $(d+1)$-dimensional, where $d = \dim(X) - \dim(Y)$. However, without the constraint on c and L, it is shown in [**2**] that a Lipschitz quotient mapping $\mathbb{R}^3 \to \mathbb{R}^2$ may have a point preimage which contains a 2-dimensional plane. It is still an open question whether point

Received 11 June 2020, published online 23 November 2020.
MSC (2020): 26A16 (primary), 46B20 (secondary).
First author acknowledges the support of the EPSRC grant EP/N027531/1. Second author acknowledges the support of CONACYT.

preimages under a Lipschitz quotient mapping f between finite-dimensional spaces of equal dimension $n \geqslant 3$ can be 1-dimensional.

It is, however, a strong result of [5] that every Lipschitz quotient mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ can be viewed as a reparameterization of a complex polynomial in the sense that there are a homeomorphism h of the plane and a polynomial P of one complex variable such that $f = P \circ h$, see Theorem 2.4 below. This immediately implies that point preimages are finite, without any assumption on the constants. Moreover, see Corollary 2.5, this shows that any Lipschitz quotient mapping of the plane is N-fold for some $N \geqslant 1$, in the following sense.

Definition 1.2. Let $N \geqslant 1$ be a positive integer. A mapping $f\colon X \to Y$ is called N-fold, if $\max_{y \in Y} \# f^{-1}(y) = N$, in other words, the cardinality of preimage of each $y \in Y$ does not exceed N and there is at least one point for which it is exactly N.

From [6, 7] it transpires that the number N is dictated by the ratio c/L of co-Lipschitz and Lipschitz constants of f. Conceptually, N appears to represent the number of connected components of a point preimage under a Lipschitz quotient mapping; see [8, 9] where a sharp upper bound on such a number is obtained as a function of c/L for mappings $\mathbb{R}^2 \to \mathbb{R}^1$, where the preimage is not 0-dimensional. In our case, it follows from [6, Theorem 2] and [7, Theorem 1] that if an N-fold Lipschitz quotient mapping $f\colon (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ is considered in the case when the two norms coincide, $\|\cdot\|_X = \|\cdot\|_Y = \|\cdot\|$, then $N \leqslant L/c$. In the case of the Euclidean norm $|\cdot|$, this upper bound is sharp: the N-fold Lipschitz quotient mappings f_N defined by $f_N(re^{i\theta}) = re^{iN\theta}$ satisfy $N = L/c$.

A natural question then is whether the same happens to other norms, i.e. whether for any norm $\|\cdot\|$ on $\mathbb{R}^2$ and any $N \geqslant 1$, there exists an N-fold Lipschitz quotient mapping $f\colon (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ with constants L and c such that $N = L/c$. In the present paper, we answer this question in the negative and describe a large class of norms for which there are no such mappings. To describe these, we consider the norms whose unit balls are given by regular polygons with n sides, and refer to them as polygonal norms.

We show that if n is divisible by 4, then any $2 \leqslant N$-fold Lipschitz quotient mapping in a polygonal n-norm satisfies $N < L/c$, see Theorem 5.13, whereas all remaining polygonal norms, with n congruent 2 modulo 4, actually possess N-fold Lipschitz quotient mappings with $N = L/c$, see Theorem 3.19, Theorem 4.3 and Corollary 4.5.

In Definition 3.12 below, we generalise the above basic example of 'Euclidean' N-fold Lipschitz quotient mappings with $N = L/c$ to the case of polygonal norms and call such mappings N-fold winding. The idea of N-fold winding maps has already been introduced in [7, Section 3] where it was claimed that for *all* such mappings, with all n (and in particular, $n = 4$ which corresponds to the ℓ^1-norm on the plane), it holds $c/L = 1/N$. The present paper hence shows that this claim is only correct in cases when n is not divisible by 4. Moreover, in Proposition 5.14, we obtain an upper bound for N (strictly less than L/c) for the N-fold winding map in the case when the number of sides is divisible by 4.

§2. *Preliminaries.* We start by quoting a number of important statements that we are going to use in this paper. The following two statements about Lipschitz mappings are standard. Below we also follow [2] to prove Lemma 2.3, which is similar to Lemma 2.2 but is about co-Lipschitz mappings.

LEMMA 2.1. *Let X, Y be normed spaces and $f\colon X \to Y$ a continuous mapping. If f is a Lipschitz mapping on $A \subseteq X$ with Lipschitz constant L, then f is L-Lipschitz on $\overline{A}$.*

LEMMA 2.2. *Let X, Y be normed spaces, $U \subseteq X$ be open and convex and $L > 0$. If $f: X \to Y$ is pointwise L-Lipschitz at x for every $x \in U$, then $f|_U$ is L-Lipschitz on U.*

LEMMA 2.3. *Let $c > 0$. If $f: (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, |||\cdot|||)$ is continuous and is pointwise c-co-Lipschitz at every $x \in \mathbb{R}^2$, then f is (globally) c-co-Lipschitz on $(\mathbb{R}^2, \|\cdot\|)$.*

Proof. Observe first that given any $x \in \mathbb{R}^2$ if $R > 0$ is the radius of the ball centred at x in which the pointwise c-co-Lipschitz property is satisfied at x and $|||y - f(x)||| < cR$, then for $r_{y,f(x)} = \frac{1}{c}|||y - f(x)|||$, it holds

$$y \in \overline{B}_{cr_{y,f(x)}}(f(x)) \subseteq f(\overline{B}_{r_{y,f(x)}}(x)) \tag{2.1}$$

(as f is continuous, compact sets are mapped to compact sets, so co-Lipschitz property can be used for closed balls).

We now show that f is a c-co-Lipschitz mapping on $(\mathbb{R}^2, \|\cdot\|)$. Let $x_0 \in \mathbb{R}^2$, $r > 0$ and $y_0 \in B_{cr}(f(x_0))$. Consider the line segment $\mathcal{L} = (f(x_0), y_0]$ joining $f(x_0)$ with y_0, and let

$$A = \{z \in \mathcal{L} : y \in f(\overline{B}_{r_{y,f(x_0)}}(x_0)) \ \forall\, y \in (f(x_0), z]\}.$$

As f is pointwise c-co-Lipschitz at x_0, by (2.1) we have $A \neq \emptyset$. It is enough to show that A is both open and closed in $\mathcal{L}$, as then $y_0 \in \mathcal{L} = A$, hence, as $r_{y_0,f(x_0)} = \frac{1}{c}|||y_0 - f(x_0)||| < r$, it holds $y_0 \in f(\overline{B}_{r_{y_0,f(x_0)}}(x_0)) \subseteq f(B_r(x_0))$, which finishes the proof. It remains to note that A is open because f is pointwise c-co-Lipschitz at each point $z \in A$, and A is closed because f is continuous. □

The following statement is a consequence of [**5**, Theorem 2.8].

THEOREM 2.4. *Let $f: (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, |||\cdot|||)$ be a Lipschitz quotient mapping. Then $f = P \circ h$, where h is a homeomorphism of $\mathbb{R}^2$ and P is a monic polynomial (viewing $\mathbb{R}^2$ as a complex plane $\mathbb{C}$).*

COROLLARY 2.5. *Any Lipschitz quotient mapping $f: (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, |||\cdot|||)$ is N-fold for some $N \geqslant 1$.*

Note that any homeomorphism of the plane may either preserve or reverse the orientation. We use this to make the following observation.

LEMMA 2.6. *Let $f: (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, |||\cdot|||)$ be a Lipschitz quotient mapping. Then there exist a monic polynomial P and an orientation preserving homeomorphism h such that $(P \circ h)(z) = f(z)$ for all $z \in \mathbb{C}$ or $(P \circ h)(z) = \overline{f(z)}$ for all $z \in \mathbb{C}$.*

Proof. Consider a monic polynomial P_1 and a homeomorphism h_1 such that $P_1 \circ h_1 = f$, the existence of these is guaranteed by Theorem 2.4. If h_1 preserves the orientation, let $P = P_1$ and $h = h_1$. Otherwise, if h_1 reverses the orientation, then set $h(z) = \overline{h_1(z)}$, so that the homeomorphism h preserves the orientation. Denote the coefficients of P_1 by a_k, i.e. $P_1(z) = z^N + \sum_{k=0}^{N-1} a_k z^k$, then put $P(z) = z^N + \sum_{k=0}^{N-1} \overline{a_k} z^k$ so that $\overline{f(z)} = P(h(z))$. □

Remark 2.7. In the present paper, our main concern is the relation between the maximum cardinality N of a point preimage under the Lipschitz quotient mapping $f: \mathbb{R}^2 \to \mathbb{R}^2$ and the ratio c/L of its co-Lipschitz and Lipschitz constants; therefore, we may, and will, without loss of generality, assume that the decomposition of a Lipschitz quotient mapping f under consideration is always given by the monic polynomial P and an orientation preserving homeomorphism h.

The next theorem collects important facts about Lipschitz quotient mappings; it is based on [6, Lemmas 1–3] and assumes that the mapping is decomposed into the composition of a monic polynomial and an orientation preserving homeomorphism. We only mention this condition here; in statements in subsequent sections which use this theorem we will not reiterate this condition.

Recall that $\mathrm{Ind}_0\,\gamma$, the index around 0 of the closed smooth curve γ, is defined as $\frac{1}{2\pi i}\oint_\gamma \frac{\mathrm{d}z}{z}$. Note for future reference that if P is a polynomial of degree N and $\gamma\colon [0,1]\to\mathbb{C}$ is any closed curve with index 1 around 0, such that all roots of P are contained inside γ, which is satisfied, for example, if there is an $R>0$ such that $|\gamma(t)|>R>|z|$ for all $t\in[0,1]$ and all $z\in P^{-1}(0)$, then by Cauchy's Residue theorem, the index of $P\circ\gamma$ around 0 is equal to N.

THEOREM 2.8. *Let* $g\colon (\mathbb{R}^2,\|\cdot\|)\to(\mathbb{R}^2,|||\cdot|||)$ *be an N-fold Lipschitz quotient mapping which can be written as a composition of a monic polynomial and an orientation preserving homeomorphism. Then there exists a positive constant R such that for any $\rho>R$, the following are satisfied:*

(1) $|||g(x)|||\geqslant c(\|x\|-M)$ *for any* $x\in\partial B_\rho^{\|\cdot\|}(0)$*;*

(2) $\mathrm{Ind}_0\, g(\partial B_\rho^{\|\cdot\|}(0))=N$.

Here c is the co-Lipschitz constant of g, $M=\max\{\|z\|\colon g(z)=0\}$ *and in* (2) *the curve* $\partial B_\rho^{\|\cdot\|}(0)$ *is considered as given by its orientation preserving parametrisation going in the counter-clockwise direction around* 0*.*

Proof. Let $g=P\circ h$, where P is a monic polynomial and h is an orientation preserving homeomorphism. Fix $R>M+1$, let $\rho>R$ and $x\in\partial B_\rho^{\|\cdot\|}(0)$. As $0\in\overline{B}_{|||g(x)|||}(g(x))\subseteq g(\overline{B}_{|||g(x)|||/c}(x))$, we conclude that there exists $z\in g^{-1}(0)$ such that $\|x-z\|\leqslant |||g(x)|||/c$. Hence $|||g(x)|||\geqslant c\|x-z\|\geqslant c(\|x\|-\|z\|)\geqslant c(\rho-M)$, and this verifies (1).

Part (2) follows from [6, Lemma 3]. □

We finish this section with the restatements of Lemma 3 and Theorem 1 from [7].

LEMMA 2.9 [7, Lemma 3]. *If* $\Gamma\colon[0;L]\to\mathbb{R}^2$ *is a closed curve with* $\|\Gamma(t)\|\geqslant r$ *for all* $t\in[0,L]$ *and* $\mathrm{Ind}_0\,\Gamma=N$*, then the length of* Γ *in the sense of the* 1*-dimensional Hausdorff measure* $\mathcal{H}^1$ *associated with* $\|\cdot\|$ *is at least* $N\mathcal{H}^1(\partial B_r(0))$.

THEOREM 2.10 [7, Theorem 1]. *If* $f\colon(\mathbb{R}^2,\|\cdot\|)\to(\mathbb{R}^2,\|\cdot\|)$ *is an N-fold L-Lipschitz and c-co-Lipschitz mapping with respect to* $\|\cdot\|$*, then* $c/L\leqslant 1/N$.

§3. *Polygonal norms and winding mappings.* *Definition* 3.1. For $n\geqslant 4$ even integer, a norm in $\mathbb{R}^2$, whose unit ball centred at the origin $B_1^n(0)$ is a regular n-gon, is called a *polygonal n-norm*.

A polygonal n-norm, whose unit ball has a vertex at (1,0), the point on the positive x-axis at $x=1$, will be denoted by $\|\cdot\|_n$.

When n is fixed, we may write $B_r(x)$ instead of $B_r^n(x)$ to simplify the notation.

Remark 3.2. By the above definition, both the ℓ_1-norm and ℓ_∞-norm are polygonal 4-norms. However, only the ℓ_1-norm is denoted by $\|\cdot\|_4$.

Remark 3.3. It is easy to see that all the results about Lipschitz and co-Lipschitz constants of Lipschitz quotient mappings we obtain below for $\|\cdot\|_n$ hold for all polygonal n-norms and, moreover, for all norms whose unit ball is a linear image of a regular n-gon. This is simply because if $U\colon (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|)$ is an isomorphism, then $f\colon (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$ is a Lipschitz quotient mapping if and only if $g(x) = U(f(U^{-1}(x)))$ is a Lipschitz quotient mapping $(\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$, and their respective constants are equal. We use this observation in Corollary 4.5 and Theorem 5.13.

We also fix some further notation.

Notation 3.4. Assume that an even integer $n \geqslant 4$ and a positive integer $N \geqslant 1$ are fixed. Let $V_0 V_1 \dots V_{n-1} = \partial B_1^n(0)$ be the regular n-gon with V_0 being at the point on the positive x-axis at $x = 1$ and vertices going counter-clockwise. In what follows we denote $e_i = V_i$ to be the unit vector in the norm $\|\cdot\|_n$ pointing to the vertex V_i. We will treat the indices indicating the vertex number modulo n, so that $e_n = e_0$, $e_{-1} = e_{n-1}$, etc.

Let $\mathscr{L}_n = n\|e_1 - e_0\|_n$ denote the length of $\partial B_1^n(0)$ in the sense of measuring distances with respect to $\|\cdot\|_n$. The exact value of $\mathscr{L}_n$ can be computed via Lemma 3.6 below.

For each $i \in \{0, \dots, n-1\}$, we divide each of the sides $[V_i, V_{i+1}]$ of $\partial B_1^n(0)$ into N segments of equal Euclidean (and $\|\cdot\|_n$-) length and denote the subdivision points by $v_{i,j}$, where $j \in \{0, \dots, N\}$. Here $v_{i,0} = V_i$ and $v_{i,N} = V_{i+1}$. We let $e_{i,j}$ be the $\|\cdot\|_n$-unit vector pointing to $v_{i,j}$.

For $i \in \{0, \dots, n-1\}$, $j \in \{0, \dots, N-1\}$, we will also consider the open region $\mathcal{R}_{i,j}$ enclosed by the lines $\mathbb{R}_+ e_{i,j}$ and $\mathbb{R}_+ e_{i,j+1}$, where $\mathbb{R}_+ = (0, +\infty)$, and let

$$\mathcal{U}_i = \bigcup_{j=0}^{N-1} \mathcal{R}_{i,j} \cup \bigcup_{j=1}^{N-1} \mathbb{R}_+ e_{i,j}.$$

Effectively, $\mathcal{U}_i$ is the open region enclosed by the lines $\mathbb{R}_+ e_i = \mathbb{R}_+ e_{i,0}$ and $\mathbb{R}_+ e_{i+1} = \mathbb{R}_+ e_{i+1,0}$. We further write $\mathcal{R}_{n,N}(p) = (0, i, j)$ if $p \in \mathbb{R}_+ e_{i,j}$, and $\mathcal{R}_{n,N}(p) = (1, i, j)$ if $p \in \mathcal{R}_{i,j}$. We note that $\mathcal{R}_{n,N}(p)$ is defined for all nonzero $p \in \mathbb{R}^2$. We will use this notation in Lemma 3.14.

For the rest of this section, we will assume that an even integer $n \geqslant 4$ and a positive integer $N \geqslant 1$ are fixed.

Remark 3.5. Before we start working with the polygonal norms, we recall some basic properties of a regular polygon $P = \partial B_r^n(x)$.

(1) The Euclidean length of a side of P is equal to $2r \sin(\pi/n)$.

(2) An apothem of P is a segment joining the centre x with the middle point of a side and it has Euclidean length equal to $r \cos(\pi/n)$.

This remark immediately implies the following statement.

LEMMA 3.6. *The $\|\cdot\|_n$-length of each side of the polygon $\partial B_r^n(0)$ is given by*

(1) $r\|e_1 - e_0\|_n = 2r \tan(\pi/n)$, *if n is divisible by* 4,

(2) $r\|e_1 - e_0\|_n = 2r \sin(\pi/n)$, *if n is not divisible by* 4.

Proof. (1) In this case, an apothem $\mathcal{A}_r$ of $\partial B_r^n(0)$ is parallel to its side, hence by Remark 3.5 (2),

$$\frac{r\|e_1 - e_0\|_n}{\|\mathcal{A}_r\|_n} = \frac{2r \sin(\pi/n)}{r \cos(\pi/n)}.$$

Using $\|\mathcal{A}_r\|_n = r$, we get the desired identity.

(2) This is by Remark 3.5 (1) because in this case a side of $B_r(0)$ is parallel to a segment connecting zero with one of the vertices of $\partial B_r^n(0)$, which clearly has Euclidean length r. $\square$

We now define a notion analogous to polar coordinates in the Euclidean plane.

Definition 3.7. Let $\gamma_1 \colon [0, \mathscr{L}_n] \to \partial B_1^n(0)$ be the 1-Lipschitz parametrization of $\partial B_1^n(0)$, such that $\gamma_1(k\|e_1 - e_0\|_n) = V_k$ for $0 \leqslant k \leqslant n-1$, $\gamma_1(\mathscr{L}_n) = V_0$ and $\mathrm{Ind}_0\, \gamma_1 = 1$.

Given a non-zero point x in the plane, we say that (r, ℓ) are the length coordinates of x under the norm $\|\cdot\|_n$, if $\|x\|_n = r$ and $\ell = (\gamma_1|_{[0,\mathscr{L}_n)})^{-1}(x/r)$. We refer to ℓ as the angle component of the length coordinates of x. The coordinates of the origin are defined to be (0,0). It is clear that each pair $(r, \ell) \in (\mathbb{R}_+ \times [0, \mathscr{L}_n)) \cup \{(0,0)\}$ determines a single point in $\mathbb{R}^2$.

Definition 3.8. We introduce the following notation for $\ell_1, \ell_2 \in [0, \mathscr{L}_n)$:

$$\rho(\ell_1, \ell_2) = \min\{|\ell_1 - \ell_2|, \mathscr{L}_n - |\ell_1 - \ell_2|\} \tag{3.1}$$

and for any $r_1, r_2 > 0$, call $\rho(\ell_1, \ell_2)$ the difference in angle between points (r_1, ℓ_1) and (r_2, ℓ_2).

In the lemma below, we collect some straightforward facts about the notions introduced above.

LEMMA 3.9. (1) *The function $\rho(\ell_1, \ell_2)$ defined by (3.1) is a metric on* $[0, \mathscr{L}_n)$.

(2) *If $x, y \in \mathbb{R}^2$, $\|x\|_n = \|y\|_n = r > 0$, x and y belong to the same side of $\partial B_r(0)$, and ℓ_x and ℓ_y are the angle components of x and y with respect to $\|\cdot\|_n$, then $\|x - y\|_n = r\rho(\ell_x, \ell_y)$.*

The following simple lemma gives a valuable property of how the norm $\|\cdot\|_n$ is related to length coordinates.

LEMMA 3.10. *Let $x = (r_x, \ell_x)$ be a non-zero point in the plane given in its length coordinates with respect to $\|\cdot\|_n$ and $\sigma > 0$. Then there is a $\phi_x \in (0, \mathscr{L}_n/(2n)]$ such that $\phi_x < \sigma$ and for any non-zero $y = (r_y, \ell_y)$, the following three assertions are satisfied.*

(1) *If $\rho(\ell_x, \ell_y) < \phi_x$, then x/r_x and y/r_y are on the same side of $\partial B_1(0)$ and $\|x - y\|_n \leqslant |r_x - r_y| + r_x\rho(\ell_x, \ell_y)$.*

(2) *If $\|x - y\|_n < d_x = r_x\phi_x/2$, then $\rho(\ell_x, \ell_y) \leqslant 2\|x - y\|_n/r_x < \phi_x$. In this case too, x/r_x and y/r_y are on the same side of $\partial B_1(0)$. Moreover, if x/r_x is not a vertex of $\partial B_1(0)$, then y/r_y is also not a vertex of $\partial B_1(0)$.*

(3) *If $\ell_x \neq 0$ and $\rho(\ell_x, \ell_y) < \phi_x$, then $\ell_y \neq 0$.*

Proof. To define $\phi_x > 0$, consider the following two cases. If $\ell_x/\|e_1 - e_0\|_n$ is not integer, i.e. the point $(1, \ell_x)$ is not a vertex of $\partial B_1(0)$, then set ϕ_x to be half of the minimum of σ and the two $\|\cdot\|_n$-distances between $(1, \ell_x)$ and the two vertices of the side of $\partial B_1(0)$ containing the point $(1, \ell_x)$. If $\ell_x/\|e_1 - e_0\|_n$ is integer, i.e. the point $(1, \ell_x)$ is a vertex of $\partial B_1(0)$, let $\phi_x = \frac{1}{2}\min\{\sigma, \|e_1 - e_0\|_n\}$. In both cases, ϕ_x is a positive value less than or equal to $\mathscr{L}_n/(2n)$ and less than σ. Let $d_x = r_x\phi_x/2$.

Let now $x = (r_x, \ell_x)$ and $y = (r_y, \ell_y)$ be two non-zero points. Let $z = (r_x/r_y)y = (r_x, \ell_y)$, then $\|y - z\|_n = |r_x - r_y|$.

Note that if $\rho(\ell_x, \ell_y) < \phi_x$, then by the definition of ϕ_x the points $x/r_x = (1, \ell_x)$ and $y/r_y = (1, \ell_y)$ are on the same side of $\partial B_1(0)$ and, moreover, (3) is satisfied. Hence the points $x = (r_x, \ell_x)$ and $z = (r_x, \ell_y)$ are on the same side of $\partial B_{r_x}(0)$, and so, by Lemma 3.9 (2), we have $\|x - z\|_n = r_x \rho(\ell_x, \ell_y)$. We thus conclude that $\|x - y\|_n \leqslant r_x \rho(\ell_x, \ell_y) + |r_x - r_y|$, which verifies (1).

To verify (2), we note that it always holds $\|x - z\|_n \leqslant \|x - y\|_n + |r_x - r_y| \leqslant 2\|x - y\|_n$, so if $\|x - y\|_n < d_x$, then $\|x - z\|_n < 2d_x = r_x \phi_x$, i.e. $\|x/r_x - z/r_x\|_n < \phi_x$. By the choice of ϕ_x, this implies that x and z are on the same side of $\partial B_{r_x}(0)$, and so, by Lemma 3.9 (2), $\rho(\ell_x, \ell_y) = \|x - z\|_n / r_x \leqslant 2\|x - y\|_n / r_x < 2d_x / r_x = \phi_x$. The points x/r_x and y/r_y are on the same side of $B_1(0)$ because x and z are on the same side of $\partial B_{r_x}(0)$. The last assertion of (2) follows from the definition of ϕ_x. □

COROLLARY 3.11. *If $p \in \mathbb{R}^2 \setminus \{0\}$, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in B_\delta(p)$, the difference in angle between points p and y is less than ε.*

Proof. Let $\delta = \min\{\|p\|_n/2, d_p\}$, where d_p is given by Lemma 3.10 (2), using $\sigma = \varepsilon$. Then for any $y \in B_\delta(p)$ we have $y \neq 0$, hence, by Lemma 3.10 (2), $\|p - y\|_n < d_p$ implies $\rho(\ell_p, \ell_y) < \phi_p < \varepsilon$. Here ℓ_p and ℓ_y are the angle components of p and y, respectively, hence the statement of the corollary is proved. □

Our next goal is to define a mapping which behaves in an analogous way to the exponential mapping $f(re^{i\theta}) = re^{iN\theta}$, but relative to the polygonal n-norm. We subsequently show that in the case when n is not divisible by 4, this mapping realises the desired ratio $1/N$, see Theorem 3.19 and Theorem 4.3.

Definition 3.12. For an integer $N \geqslant 1$, we define the N-fold winding mapping for the norm $\|\cdot\|_n$, using the length coordinates, as $f_n^N(r, \ell) = (r, N\ell \pmod{\mathscr{L}_n})$. Here $t \pmod{\mathscr{L}_n}$, $t \in \mathbb{R}$, is equal to the value $s \in [0, \mathscr{L}_n)$ such that $(t - s)/\mathscr{L}_n \in \mathbb{Z}$, as usual.

Of course, if $N = 1$, then $f_n^N = f_n^1$ is the identity mapping, so $c/L = 1$. Hence it only makes sense to study N-fold winding mappings for $N \geqslant 2$.

LEMMA 3.13. *Let $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant N-1$, $s = (Ni + j) \pmod{n}$, $r > 0$. Then the following properties are satisfied:*

(1) $f_n^N(\mathcal{R}_{i,j}) = \mathcal{U}_s$ (see Notation 3.4) *and* $f_n^N(\mathbb{R}e_{i,j}) = \mathbb{R}e_s$;
(2) $f_n^N(re_{i,j} + t(e_{i+1} - e_i)) = re_s + Nt(e_{s+1} - e_s)$ *and* $f_n^N(re_{i,j+1} - t(e_{i+1} - e_i)) = re_{s+1} - Nt(e_{s+1} - e_s)$, *for any $t \in [0, r/N]$;*
(3) *if $\|p_1\|_n = \|p_2\|_n = r$, the difference in angle between p_1 and p_2 is smaller than $\frac{\mathscr{L}_n}{nN}$ and the points $f_n^N(p_1)$ and $f_n^N(p_2)$ are on the same side of $\partial B_r(0)$, then there exist $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant N-1$ such that $p_1, p_2 \in \overline{\mathcal{R}}_{i,j}$;*
(4) *if $q_1, q_2 \in [re_s, re_{s+1}]$, $p_1 \in [re_{i,j}, re_{i,j+1})$, $f_n^N(p_1) = q_1$ and $q_2 = q_1 + t'(e_{s+1} - e_s)$ for $t' \in [0, r]$, then $p_2 = p_1 + t'/N(e_{i+1} - e_i) \in [re_{i,j}, re_{i,j+1})$ and $f_n^N(p_2) = q_2$;*
(5) *if $\|p_1\|_n = \|p_2\|_n = r$ and $p_1, p_2 \in \overline{\mathcal{R}}_{i,j}$, then $\|f_n^N(p_1) - f_n^N(p_2)\|_n = N\|p_1 - p_2\|_n$;*
(6) *if $p_1, p_2 \in \mathcal{R}_{i,j}$ are such that $p_1 = r'e_{i,j} + t'(e_{i+1} - e_i)$ and $p_2 = r'e_{i,j+1} - t'(e_{i+1} - e_i)$, then $r' = \|p_1\|_n = \|p_2\|_n > 0$ and $t' \in (0, r'/N)$;*
(7) *if $p_1, p_2 \in \mathcal{U}_i$ are such that $p_1 = r'e_i + t'(e_{i+1} - e_i)$ and $p_2 = r'e_{i+1} - t'(e_{i+1} - e_i)$, then $r' = \|p_1\|_n = \|p_2\|_n > 0$ and $t' \in (0, r')$.*

Proof. (1) This follows from the fact that the angle component ℓ of a point $x \in \mathcal{R}_{i,j}$ satisfies $i\frac{\mathscr{L}_n}{n} + j\frac{\mathscr{L}_n}{nN} < \ell < i\frac{\mathscr{L}_n}{n} + (j+1)\frac{\mathscr{L}_n}{nN}$, i.e.

$$\frac{\mathscr{L}_n}{nN}(iN + j) < \ell < \frac{\mathscr{L}_n}{nN}(iN + (j+1)),$$

thus the angle component of $f_n^N(x)$ belongs to $\left(\frac{\mathscr{L}_n}{n}(iN+j), \frac{\mathscr{L}_n}{n}(iN+(j+1))\right)$, modulo $\mathscr{L}_n$, which means $f_n^N(x) \in \mathcal{U}_s$.

For $x \in \mathbb{R}e_{i,j}$, its angle component satisfies $\ell = \frac{\mathscr{L}_n}{nN}(Ni + j)$, thus $f_n^N(x) \in \mathbb{R}e_s$.

(2) Let $x = re_{i,j} + t(e_{i+1} - e_i)$ and $y = re_{i,j+1} - t(e_{i+1} - e_i)$. Since $t \in [0, \frac{r}{N}]$, we see that $x, y \in [re_{i,j}, re_{i,j+1}] \subseteq \overline{\mathcal{R}}_{i,j}$, so $\|f_n^N(x)\|_n = \|x\|_n = \|f_n^N(y)\|_n = \|y\|_n = r$ and $f_n^N(x), f_n^N(y) \in \overline{\mathcal{U}}_s$ by (1), hence $f_n^N(x), f_n^N(y) \in [re_s, re_{s+1}]$. Furthermore, $f_n^N(re_{i,j}) = re_s$ and $f_n^N(re_{i,j+1}) = re_{s+1}$, hence f_n^N is linear on $[re_{i,j}, re_{i,j+1}]$ and the desired identities are satisfied.

(3) Assume that such i, j do not exist. This implies, because of the condition on angle difference between p_1 and p_2, that at least one of p_1 or p_2, say p_1, belongs to $\mathcal{R}_{i_1,j_1}$ for some i_1, j_1; then $p_2 \notin \overline{\mathcal{R}}_{i_1,j_1}$. Hence, by (1), we have $f_n^N(p_1) \in \mathcal{U}_s$ and $f_n^N(p_1) \notin \overline{\mathcal{U}}_s$, a contradiction.

(4) This follows from statement (2) of the present lemma and linearity of f_n^N on $[re_{i,j}, re_{i,j+1}]$.

(5) For such p_1, p_2 there are $t_1, t_2 \in [0, r/N]$ such that $p_k = re_{i,j} + t_k(e_{i+1} - e_i)$, $k = 1, 2$. Then (2) implies the statement.

(6) Let $r = \|p_1\|_n$, then $p_1 = re_{i,j} + t(e_{i+1} - e_i)$ with $r > 0$ and $0 < t < r/N$. Together with the expression given in this part, we get $(r - r')e_{i,j} = (t' - t)(e_{i+1} - e_i)$. As the vectors $e_{i,j}$ and $(e_{i+1} - e_i)$ are not collinear, this implies $r' = r > 0$ and $t' = t \in (0, r/N) = (0, r'/N)$. The proof for p_2 is analogous.

(7) Similarly to (6), let $p_1 = re_i + t(e_{i+1} - e_i)$ where $r = \|p_1\|_n$ and $0 < t < r$. Then $r' = r > 0$ and $t' = t \in (0, r) = (0, r')$. The proof for p_2 is analogous. □

The following lemma describes images of some of the vertices of a small $\|\cdot\|_n$-neighbourhood of a non-zero point p under the N-fold winding mapping f_n^N.

LEMMA 3.14. *Let p be a non-zero point in the plane. Then there exists $\Delta = \Delta_p > 0$, defined by (3.2) and (3.3), such that whenever $\delta \in (-\Delta, \Delta)$, the following are satisfied.*

If $\mathcal{R}_{n,N}(p) = (1, i, j)$, i.e. $p \in \mathcal{R}_{i,j}$ for some $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant N-1$, then for $s = (Ni + j) \pmod n$ it holds that

(1) *for $\mathcal{A} = \{p + \delta e_i, p + \delta e_{i+1}, p + \delta e_{i,j}, p + \delta e_{i,j+1}\}$ we have $\mathcal{A} \subseteq B_\Delta(p) \subseteq \mathcal{R}_{i,j}$ and $f_n^N(\mathcal{A}) \subseteq \mathcal{U}_s$,*

(2) *$\|x\|_n = \|p\|_n + \delta$ for any $x \in \mathcal{A}$,*

(3) *$f_n^N(p + \delta e_{i,j}) = f_n^N(p) + \delta e_s$,*

(4) *$f_n^N(p + \delta e_{i,j+1}) = f_n^N(p) + \delta e_{s+1}$.*

If $\mathcal{R}_{n,N}(p) = (0, i, j)$, i.e. $p \in \mathbb{R}_+ e_{i,j}$ for some $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant N-1$, then (3) holds true, again for $s = (Ni + j) \pmod n$.

Proof. Let $\Delta = \Delta(p) > 0$ be such that:

$$B_\Delta(p) \subseteq \mathcal{R}_{i,j}, \text{ if } p \in \mathcal{R}_{i,j}; \tag{3.2}$$

$$\Delta = \|p\|_n/2, \text{ if } p \in \mathbb{R}e_{i,j}. \tag{3.3}$$

Let $\delta \in (-\Delta, \Delta)$. For brevity, we write f instead of f_n^N. Let $r_0 = \|p\|_n$.

Assume first that $\mathcal{R}_{n,N}(p) = (0, i, j)$, i.e. $p \in \mathbb{R}_+ e_{i,j}$, and let $s = (Ni + j) \pmod n$. Notice that $f(re_{i,j}) = re_s$ for any $r > 0$, so as $p \in \mathbb{R}_+ e_{i,j}$, we have $p = r_0 e_{i,j}$, then $f(p + \delta e_{i,j}) = f((r_0 + \delta)e_{i,j}) = (r_0 + \delta)e_s$, so (3) is satisfied. Here $r_0 + \delta > 0$ follows from the choice of Δ in (3.3).

Consider now the case $\mathcal{R}_{n,N}(p) = (1, i, j)$, i.e. $p \in \mathcal{R}_{i,j}$. In this case, for $\delta \in (-\Delta, \Delta)$ we have $\mathcal{A} \subseteq B_\Delta(p) \subseteq \mathcal{R}_{i,j}$, hence by Lemma 3.13 (1), $f(\mathcal{A}) \subseteq \mathcal{U}_s$; this verifies (1).

Note that as $\|p\|_n = r_0$ and $p \in \mathcal{R}_{i,j}$, there is a $t \in (0, r_0/N)$ such that

$$p = r_0 e_{i,j} + t(e_{i+1} - e_i) = r_0 e_{i,j+1} - \Big(\frac{r_0}{N} - t\Big)(e_{i+1} - e_i) \tag{3.4}$$

and

$$p = r_0 e_i + \Big(t + r_0 \frac{j}{N}\Big)(e_{i+1} - e_i) = r_0 e_{i+1} - \Big(r_0 \frac{N-j}{N} - t\Big)(e_{i+1} - e_i). \tag{3.5}$$

We then get from (3.4) that

$$p + \delta e_{i,j} = (r_0 + \delta)e_{i,j} + t(e_{i+1} - e_i);$$

$$p + \delta e_{i,j+1} = (r_0 + \delta)e_{i,j+1} - \Big(\frac{r_0}{N} - t\Big)(e_{i+1} - e_i),$$

and from (3.5) that

$$p + \delta e_i = (r_0 + \delta)e_i + \Big(t + r_0 \frac{j}{N}\Big)(e_{i+1} - e_i);$$

$$p + \delta e_{i+1} = (r_0 + \delta)e_{i+1} - \Big(r_0 \frac{N-j}{N} - t\Big)(e_{i+1} - e_i).$$

As $\mathcal{A} \subseteq \mathcal{R}_{i,j}$, by Lemma 3.13 (6) and (7), we conclude that $\|p + \delta e_{i,j}\|_n = \|p + \delta e_{i,j+1}\|_n = \|p + \delta e_i\|_n = \|p + \delta e_{i+1}\|_n = r_0 + \delta$, which verifies (2) of the present lemma. We also infer that we can use Lemma 3.13 (2) to get from (3.4) and (3.5):

$$f(p) = r_0 e_s + Nt(e_{s+1} - e_s) = r_0 e_{s+1} - N\Big(\frac{r_0}{N} - t\Big)(e_{s+1} - e_s);$$

$$f(p + \delta e_{i,j}) = (r_0 + \delta)e_s + Nt(e_{s+1} - e_s);$$

$$f(p + \delta e_{i,j+1}) = (r_0 + \delta)e_{s+1} - N\Big(\frac{r_0}{N} - t\Big)(e_{s+1} - e_s).$$

Identities (3) and (4) of the present lemma then immediately follow from the above. □

For the next lemma, we use the length coordinates to denote the points on the plane.

LEMMA 3.15. *A sequence $\{x_k = (r_k, \ell_k)\}_{k \geqslant 1}$ of points in the plane converges to $x = (r, \ell) \neq (0, 0)$ under the norm $\|\cdot\|_n$ if and only if $r_k \to r \neq 0$ and $\rho(\ell_k, \ell) \to 0$.*

Proof. The forward implication follows from Corollary 3.11.

Assume now that $x = (r, \ell)$ and for $x_k = (r_k, \ell_k)$ we have $r_k \to r \neq 0$ and $\rho(\ell_k, \ell) \to 0$. Then for (sufficiently large) $k \geqslant k_0$ we have $r_k \neq 0$ and $\rho(\ell_k, \ell) < \phi_x$ (given by Lemma 3.10 (1)). Hence, by Lemma 3.10 (1), for such k we have $0 \leqslant \|x_k - x\|_n \leqslant |r_k - r| + r_x \rho(\ell_k, \ell) \to 0$. □

Using Lemma 3.15, we immediately get the following.

COROLLARY 3.16. *The N-fold winding mapping f_n^N for the norm $\|\cdot\|_n$ is continuous.*

Proof. The mapping $f_n^N\colon (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$ is continuous at all $x \neq 0$ by Lemma 3.15. To check continuity at $x = 0$, it is enough to notice that $\|f_n^N(y)\|_n = \|y\|_n$ for all $y \in \mathbb{R}^2$. □

Our aim now is to show that for any $n \geqslant 4$, the N-fold winding mapping f_n^N is 1-co-Lipschitz under the norm $\|\cdot\|_n$. We will do so in Theorem 3.19. Note that the value of the Lipschitz constant of f_n^N is not always N (as in the case of the exponential mapping $f(re^{i\theta}) = re^{iN\theta}$ for the Euclidean norm), see Theorem 4.3 and Proposition 5.14. We first introduce some notation we will be using.

Definition 3.17. Consider again $\mathbb{R}^2$ equipped with $\|\cdot\|_n$. Recall the Notation 3.4 for $e_0, \ldots, e_{n-1}$. For each $\varepsilon > 0$, $\delta \in (-\varepsilon, \varepsilon)$, $p \in \mathbb{R}^2$ and $i = 0, 1, \ldots, n-1$, let

- $\lambda(\varepsilon, \delta, p, i) \geqslant 0$ be the unique $\lambda \geqslant 0$ such that $p + \delta e_i + \lambda(e_{i+1} - e_i) \in \partial B_\varepsilon(p)$.
- $\lambda^*(\varepsilon, \delta, p, i) \geqslant 0$ be the unique $\lambda^* \geqslant 0$ such that $p + \delta e_i - \lambda^*(e_{i+1} - e_i) \in \partial B_\varepsilon(p)$.

Here and below, the balls $B_\varepsilon(p)$ are considered with respect to the norm $\|\cdot\|_n$.

Note that the existence and uniqueness of values $\lambda(\varepsilon, \delta, p, i)$ and $\lambda^*(\varepsilon, \delta, p, i)$ follows from the fact that $p + \delta e_i$ belongs to the open ball $B_\varepsilon(p)$ for any $|\delta| < \varepsilon$. We also let $\lambda(\varepsilon, \varepsilon, p, i) = \lambda^*(\varepsilon, -\varepsilon, p, i) = \varepsilon$ and $\lambda(\varepsilon, -\varepsilon, p, i) = \lambda^*(\varepsilon, \varepsilon, p, i) = 0$.

The following technical lemma summarises the basic properties of $\lambda(\varepsilon, \delta, p, i)$ and $\lambda^*(\varepsilon, \delta, p, i)$.

LEMMA 3.18. *In the notation of Definition 3.17, the following assertions are true for any $\varepsilon > 0$.*

(1) *For any $\delta \in [-\varepsilon, \varepsilon]$ the values of $\lambda(\varepsilon, \delta, p, i)$ and $\lambda^*(\varepsilon, \delta, p, i)$ are independent of $p \in \mathbb{R}^2$ and of $0 \leqslant i \leqslant n-1$, so that the parameters p and i can be dropped.*

(2) *For any $p \in \mathbb{R}^2$, $0 \leqslant i \leqslant n-1$ and $\delta \in (-\varepsilon, \varepsilon)$ we have that $\lambda = \lambda(\varepsilon, \delta)$ is the unique $\lambda \geqslant 0$ satisfying $p + \delta e_{i+1} - \lambda(e_{i+1} - e_i) \in \partial B_\varepsilon(p)$, and $\lambda^* = \lambda^*(\varepsilon, \delta)$ is the unique $\lambda^* \geqslant 0$ satisfying $p + \delta e_{i+1} + \lambda^*(e_{i+1} - e_i) \in \partial B_\varepsilon(p)$.*

(3) *For any $\delta \in [-\varepsilon, \varepsilon]$ it holds that $\lambda(\varepsilon, -\delta) = \lambda^*(\varepsilon, \delta)$.*

(4) *For any $\delta \in [-\varepsilon, \varepsilon]$ and $c > 0$ it holds that $\lambda(c\varepsilon, \delta) = c\lambda(\varepsilon, \delta/c)$.*

(5) *For any $\delta \in [0, \varepsilon)$ we have $\lambda(\varepsilon, \delta) = \delta + \lambda^*(\varepsilon, \delta)$.*

(6) *For any $\delta \in [0, \varepsilon)$ we have $\lambda(\varepsilon, \delta) > \delta$ and $\lambda(\varepsilon, \delta) \geqslant \lambda^*(\varepsilon, \delta)$, and equality in the latter inequality occurs only if $\delta = 0$.*

Proof. (1) The statement follows from translation invariance of the definition of $\lambda(\varepsilon, \delta, p, i)$ and $\lambda^*(\varepsilon, \delta, p, i)$ and rotational symmetry of $\partial B_r(p)$ around p by $2\pi/n$.

(2) Notice that, from the definition of $\lambda(\varepsilon, \delta, p, i)$, the point $p + \delta e_i + \lambda(e_{i+1} - e_i) = p + (\delta - \lambda)e_i + \lambda e_{i+1}$ is symmetric to the point $p + \delta e_{i+1} - \lambda(e_{i+1} - e_i) = p + \lambda e_i + (\delta - \lambda)e_{i+1}$ with respect to the line through p in the direction $(e_i + e_{i+1})$, as is $B_\varepsilon(p)$.

Similarly, the point $p + \delta e_i - \lambda^*(e_{i+1} - e_i) = p + (\delta + \lambda^*)e_i - \lambda^* e_{i+1}$ from the definition of $\lambda^*(\varepsilon, \delta, p, i)$ is symmetric to $p + \delta e_{i+1} + \lambda^*(e_{i+1} - e_i) = p + (\delta + \lambda^*)e_{i+1} - \lambda^* e_i$ with respect to the line through p in the direction $(e_i + e_{i+1})$, as is $B_\varepsilon(p)$.

(3) If $\delta = \pm\varepsilon$, then this follows from Definition 3.17. Hence assume $|\delta| < \varepsilon$.

Let $\lambda = \lambda(\varepsilon, -\delta) = \lambda(\varepsilon, -\delta, 0, 0)$ and $\lambda^* = \lambda^*(\varepsilon, \delta) = \lambda^*(\varepsilon, \delta, 0, 0)$. Then consider the points $P = -\delta e_0 + \lambda(e_1 - e_0) \in \partial B_\varepsilon(0)$ and $P^* = \delta e_0 - \lambda^*(e_1 - e_0) \in \partial B_\varepsilon(0)$. Notice that the point $-P = \delta e_0 - \lambda(e_1 - e_0)$ also belongs to $\partial B_\varepsilon(0)$, and so

$$\delta e_0 - t(e_1 - e_0) \in \partial B_\varepsilon(0) \tag{3.6}$$

is satisfied with both $t = \lambda$ and $t = \lambda^*$. However, there exists a unique $t > 0$ such that (3.6) is satisfied, so we conclude $\lambda = \lambda^*$.

(4) Let $\lambda = \lambda(c\varepsilon, \delta)$, i.e. $p + \delta e_0 + \lambda(e_1 - e_0) \in \partial B_{c\varepsilon}(p)$. Then $p + \frac{1}{c}\big(\delta e_0 + \lambda(e_1 - e_0)\big) = p + (\delta/c)e_0 + \frac{1}{c}\lambda(e_1 - e_0) \in \partial B_\varepsilon(p)$. Hence $\lambda(\varepsilon, \delta/c) = \frac{1}{c}\lambda = \frac{1}{c}\lambda(c\varepsilon, \delta)$.

(5) For $\delta = 0$ the statement follows from (3), so assume $\delta \in (0, \varepsilon)$. Let $\lambda = \lambda(\varepsilon, \delta, 0, 0)$ and $\lambda^* = \lambda^*(\varepsilon, \delta, 0, 0)$. Notice that as $0 < \delta < \varepsilon$, the closed ball $\bar{B}_\delta(0)$ is a subset of the open ball $B_\varepsilon(0)$, hence $\delta e_1 = 0 + \delta e_0 + \delta(e_1 - e_0) \in B_\varepsilon(0)$. The latter implies $\lambda > \delta$. Also, $\delta e_0 + \lambda(e_1 - e_0) \in \partial B_\varepsilon(0)$ by the definition of λ, and $\delta e_0 + \lambda(e_1 - e_0) = \delta e_1 + (\lambda - \delta)(e_1 - e_0)$ with $\lambda - \delta > 0$, hence, by part (2) of the present lemma, we have $\lambda - \delta = \lambda^*$.

(6) This follows from (5) as $\delta \geqslant 0$ and $\lambda^*(\varepsilon, \delta) \geqslant 0$, and the fact that $\lambda^*(\varepsilon, \delta) = 0$ only if $\delta = \varepsilon$. This value is excluded from the interval $\delta \in [0, \varepsilon)$, hence $\lambda(\varepsilon, \delta) > \delta$ for all $\delta \in [0, \varepsilon)$. □

Now we can prove that the N-fold winding mapping is a 1-co-Lipschitz mapping under the norm $\|\cdot\|_n$.

THEOREM 3.19. *For any even $n \geqslant 4$ and $N \geqslant 1$, the N-fold winding mapping f_n^N, considered as a mapping $(\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$, is 1-co-Lipschitz.*

Proof. This is clear in the trivial case $N = 1$, so we only need to prove the statement for $N \geqslant 2$.

For convenience of notation, we write f for f_n^N. By Corollary 3.16, f is continuous, hence by Lemma 2.3 it is enough to show that for any $p \in \mathbb{R}^2$ there exists $\varepsilon_0 > 0$ such that

$$f(B_\varepsilon(p)) \supseteq B_\varepsilon(f(p)) \tag{3.7}$$

for any $\varepsilon \in (0, \varepsilon_0)$. Here and below all balls are considered with respect to the norm $\|\cdot\|_n$.

Notice that if $p = 0$, this is trivially satisfied for any $\varepsilon_0 > 0$ as $f(B_r(0)) = B_r(0) = B_r(f(0))$ for any $r > 0$.

Assume therefore

$$p \neq 0. \tag{3.8}$$

Let $p = (r_0, \ell_0)$ be given in its length coordinates. This implies $f(p) = (r_0, \ell_1)$, where the angle component ℓ_1 is defined by

$$\ell_1 = N\ell_0 \pmod{\mathscr{L}_n} \in [0, \mathscr{L}_n). \tag{3.9}$$

To prove (3.7), it is enough to find $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any $y = (r, m_1) \in B_\varepsilon(f(p))$, there is an $x \in B_\varepsilon(p)$ such that $f(x) = y$. While ε_0 will be defined later, depending on the values of ℓ_0 and ℓ_1, we explain now how the point x will be defined. Let

$$m_1' = \begin{cases} m_1 + (N-1)\mathscr{L}_n, & \text{if } \ell_0 = 0, \ell_1 = 0 \text{ and } m_1 > \frac{n-1}{n}\mathscr{L}_n, \\ m_1 - \mathscr{L}_n, & \text{if } \ell_0 \neq 0, \ell_1 = 0 \text{ and } m_1 > \frac{n-1}{n}\mathscr{L}_n, \\ m_1, & \text{otherwise,} \end{cases} \tag{3.10}$$

and define

$$m_0 = \frac{1}{N}\big(m_1' + (N\ell_0 - \ell_1)\big) \qquad \text{and} \qquad x = (r, m_0). \tag{3.11}$$

We divide the proof into two cases. In each case, we show that $m_0 \in [0, \mathscr{L}_n)$, i.e. the definition of x in terms of its length coordinates is valid. By (3.11) and Definition 3.12, it is clear that $\|f(x)\|_n = r = \|y\|_n$ and the angle component of $f(x)$ is equal to the angle component of y. Indeed, use (3.10) and (3.9) to deduce $m_1' = m_1 \pmod{\mathscr{L}_n}$ and $N\ell_0 = \ell_1 \pmod{\mathscr{L}_n}$, then use (3.11) to get

$$Nm_0 \pmod{\mathscr{L}_n} = (m_1' + N\ell_0 - \ell_1) \pmod{\mathscr{L}_n} = (m_1 + N\ell_0 - \ell_1) \pmod{\mathscr{L}_n} = m_1.$$

Thus $f(x) = (r, m_1) = y$. Therefore, it is enough to define $\varepsilon_0 > 0$ and to verify that $m_0 \in [0, \mathscr{L}_n)$ and that $x \in B_\varepsilon(p)$ whenever $y \in B_\varepsilon(f(p))$, for all $\varepsilon \in (0, \varepsilon_0)$.

In the proof, we will use Notation 3.4 which introduced $\|\cdot\|_n$-unit vectors e_i and $e_{i,j}$, where $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant N-1$.

Case I. $\ell_0 = 0$.

This means that p belongs to the positive side of the x-axis of $\mathbb{R}^2$, hence $p = r_0 e_0$ and $f(p) = p$, so $\ell_1 = \ell_0 = 0$ by (3.9).

By Corollary 3.11, find $\varepsilon_0 > 0$ such that for any $y \in B_{\varepsilon_0}(f(p))$ the difference in angle between $f(p)$ and y is less than $\mathscr{L}_n/(2n)$. Pick any $\varepsilon \in (0, \varepsilon_0)$ and assume $y = (r, m_1) \in B_\varepsilon(f(p))$. Then $\rho(m_1, \ell_1) = \rho(m_1, 0) < \mathscr{L}_n/(2n)$ means that either $m_1 \in [0, \mathscr{L}_n/(2n))$ or $m_1 \in (\mathscr{L}_n - \mathscr{L}_n/(2n), \mathscr{L}_n)$.

If $m_1 \in [0, \mathscr{L}_n/(2n))$, then by (3.10), $m_1' = m_1$ and hence, by (3.11), $m_0 = m_1/N \in [0, \mathscr{L}_n/n)$ so that $x, p \in \overline{\mathcal{R}}_{0,1}$. We can apply Lemma 3.13 (5) to get $\varepsilon > \|y - p\|_n = \|f(x) - f(p)\|_n = N\|x - p\|_n$, hence $\|x - p\|_n < \varepsilon$, i.e. $x \in B_\varepsilon(p)$.

If $m_1 \in (\mathscr{L}_n - \frac{1}{2n}\mathscr{L}_n, \mathscr{L}_n)$, then by (3.10), $m_1' = m_1 + (N-1)\mathscr{L}_n \in (N\mathscr{L}_n - \frac{1}{2n}\mathscr{L}_n, N\mathscr{L}_n)$ and hence, by (3.11), $m_0 = m_1'/N \in (\mathscr{L}_n - \frac{\mathscr{L}_n}{2Nn}, \mathscr{L}_n)$ so that $x, p \in \overline{\mathcal{R}}_{n-1,N-1}$. We can then again apply Lemma 3.13 (5) to get $\varepsilon > \|y - p\|_n = \|f(x) - f(p)\|_n = N\|x - p\|_n$, hence $\|x - p\|_n < \varepsilon$, i.e. $x \in B_\varepsilon(p)$.

Case II. $\ell_0 \neq 0$.

Let $\mathcal{R}_{n,N}(p) = (\epsilon, i, j)$, where $\epsilon = 0$ or 1, $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant N-1$. This means $p \in \mathcal{R}_{i,j} \cup \mathbb{R}_+ e_{i,j}$, hence, by Lemma 3.13 (1), we have $f(p) \in \mathcal{U}_s \cup \mathbb{R}_+ e_s$ for $s = (Ni + j) \pmod{n}$, in other words

$$f(p) \in [r_0 e_s, r_0 e_{s+1}). \tag{3.12}$$

Let

$$\varepsilon_1 = \frac{1}{8nN} \min\big\{d_{f(p)}, d_p, r_0\ell_0, r_0(\mathscr{L}_n - \ell_0)\big\}, \tag{3.13}$$

where d_p and $d_{f(p)}$ are defined as in Lemma 3.10 (2) (of course, $p \neq 0$ and $f(p) \neq 0$ by (3.8)).

Let $\varepsilon_2 = \Delta > 0$ be given by (3.2) or (3.3), depending on p, so that the conclusions of Lemma 3.14 are satisfied.

Let now $\varepsilon_3 > 0$ be such that for any $v \in B_{\varepsilon_3}(0)$, the difference in angle between $p + v$ and p is less than $\mathscr{L}_n/(4nN)$, and the difference in angle between $f(p) + v$ and $f(p)$ is less than $\mathscr{L}_n/(4nN)$. Existence of such ε_3 follows from Corollary (3.11).

Finally, let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and $\varepsilon \in (0, \varepsilon_0)$; pick any $y \in B_\varepsilon(f(p))$ with length coordinates (r, m_1). We define $x = (r, m_0)$ according to (3.11).

As we have that $\varepsilon < \varepsilon_1 < d_{f(p)}$, we see, by Lemma 3.10 (2) applied to the pair of points $f(p) = (r_0, \ell_1)$ and $y = (r, m_1)$, that

$$a := \rho(m_1, \ell_1) \leqslant 2\varepsilon/r_0 < \phi_{f(p)} < \mathscr{L}_n/(2n), \tag{3.14}$$

and also

$$f(p)/\|f(p)\|_n = (1, \ell_1) \text{ and } y/\|y\|_n = (1, m_1) \text{ are on the same side of } \partial B_1(0). \tag{3.15}$$

Moreover, Lemma 3.10 (2) implies that

$$\text{if } f(p) \in (r_0 e_s, r_0 e_{s+1}), \text{ then } y \in \mathcal{U}_s \cap [re_s, re_{s+1}] = (re_s, re_{s+1}). \tag{3.16}$$

Our aim now is to show that $m_0 \in [0, \mathscr{L}_n)$. Recall that $\ell_0 \neq 0$.

If $m_1' = m_1$, then by (3.10) either $\ell_1 \neq 0$, or $\ell_1 = 0$ and $m_1 \leqslant \frac{n-1}{n}\mathscr{L}_n$. In the former case, we use $\ell_1 \neq 0$, $a = \rho(\ell_1, m_1) < \phi_{f(p)}$ from (3.14) and Lemma 3.10 (3) to conclude $m_1 \neq 0$ too, thus $|m_1 - \ell_1| = \rho(m_1, \ell_1) = a$ by Definition 3.8. In the latter case, we recall that by (3.14) we have $a = \rho(m_1, \ell_1) = \rho(m_1, 0) = \min\{m_1, \mathscr{L}_n - m_1\} < \mathscr{L}_n/(2n)$. Notice that $0 \leqslant m_1 \leqslant \frac{n-1}{n}\mathscr{L}_n$ implies $\mathscr{L}_n - m_1 \geqslant \frac{\mathscr{L}_n}{n} > \frac{\mathscr{L}_n}{2n}$, so we must have $a = \rho(m_1, \ell_1) = m_1 = |m_1 - \ell_1|$.

We now further use (3.14) and $\varepsilon < \varepsilon_1 < \frac{1}{2}\min\{r_0\ell_0, r_0(\mathscr{L}_n - \ell_0)\}$ to get $a \leqslant 2\varepsilon/r_0 < \min\{\ell_0, \mathscr{L}_n - \ell_0\}$, hence, by (3.11),

$$m_0 = \ell_0 + (m_1 - \ell_1)/N \in (\ell_0 - a, \ell_0 + a) \subseteq (0, \mathscr{L}_n). \tag{3.17}$$

If $m_1' = m_1 - \mathscr{L}_n$, then by (3.10) we have $\ell_1 = 0$ and $m_1 \in (\frac{n-1}{n}\mathscr{L}_n, \mathscr{L}_n)$. This time $a = \rho(m_1, \ell_1) = \rho(m_1, 0) = \min\{m_1, \mathscr{L}_n - m_1\} = \mathscr{L}_n - m_1$, hence, using again that $\varepsilon < \varepsilon_1 < \frac{1}{2}r_0\ell_0$ and (3.14), we get $a \leqslant 2\varepsilon/r_0 < \ell_0$, thus

$$m_0 = \ell_0 + (m_1 - \mathscr{L}_n)/N = \ell_0 - \tfrac{a}{N} \in (0, \mathscr{L}_n). \tag{3.18}$$

We have thus shown in (3.17) and (3.18) that $m_0 \in (0, \mathscr{L}_n)$, thus $x = (r, m_0)$ indeed gives the length coordinates. It is left to show that $x = (r, m_0) \in B_\varepsilon(p)$.

We now use (3.12), (3.15) and (3.16) to conclude that either

$$p \in (r_0 e_{i,j}, r_0 e_{i,j+1}),\ \ f(p) \in (r_0 e_s, r_0 e_{s+1}) \text{ and } y \in (re_s, re_{s+1}), \tag{3.19}$$

or $f(p) = r_0 e_s$, in which case

$$p = r_0 e_{i,j} \ \text{ and } \ y \in [re_s, re_{s+1}], \ \text{ or} \tag{3.20}$$

$$p = r_0 e_{i,j} \ \text{ and } \ y \in [re_{s-1}, re_s]. \tag{3.21}$$

Here in case $s = 0$ we let $e_{-1} = e_{n-1}$.

Denote $\delta = r - r_0$. In case of (3.19), as $|\delta| < \varepsilon < \varepsilon_2$, we have by Lemma 3.14 (1)–(3)

$$\|f(p + \delta e_{i,j})\|_n = \|p + \delta e_{i,j}\|_n = \|p\|_n + \delta = r = \|y\|_n$$

and

$$p + \delta e_{i,j} \in (re_{i,j}, re_{i,j+1}) \text{ and } f(p) + \delta e_s = f(p + \delta e_{i,j}) \in \mathcal{U}_s. \tag{3.22}$$

Together with (3.19), this implies that the points y and $f(p + \delta e_{i,j})$ are on the same side $[re_s, re_{s+1}]$ of $\partial B_r(0)$ whenever $p \in (r_0 e_{i,j}, r_0 e_{i,j+1})$.

Now, in case $p = r_0 e_{i,j}$, i.e. if (3.21) or (3.20) holds, we have $f(p + \delta e_{i,j}) = re_s$, so it is clear that y is on the same side of $\partial B_r(0)$ as its endpoint re_s. Therefore, in any case, y and $f(p + \delta e_{i,j})$ are on the same side of $\partial B_r(0)$.

Recall that by (3.17) and (3.18), and as $N \geqslant 2$, we have $|m_0 - \ell_0| < a$ and $m_0 \neq 0$, whereas by (3.14) and (3.13), we have $a = \rho(m_1, \ell_1) < 2\varepsilon/r_0 \leqslant \ell_0/(4nN) < \mathscr{L}_n/(4nN)$. This means that the difference in angle between points x and p satisfies $\rho(m_0, \ell_0) = |m_0 - \ell_0| < a < \mathscr{L}_n/(4nN)$. As we also have that $|\delta| < \varepsilon < \varepsilon_3$, we get that the difference in angle between points p and $p + \delta e_{i,j}$ is also smaller than $\mathscr{L}_n/(4nN)$. Hence, by Lemma 3.9 (1), the difference in angle between points x and $p + \delta e_{i,j}$ is smaller than $\mathscr{L}_n/(2nN)$.

This, together with the fact that their images, $f(x) = y$ and $f(p + \delta e_{i,j})$, are on the same side of $\partial B_r(0)$, implies, by Lemma 3.13 (3), that the points x and $p + \delta e_{i,j}$ are in the same region $\overline{\mathcal{R}}_{*,*}$. Hence by Lemma 3.13 (5), we have $\|y - f(p + \delta e_{i,j})\|_n = N\|x - (p + \delta e_{i,j})\|_n$.

To finish the proof that $x \in B_\varepsilon(p)$, we now consider three cases given by (3.19), (3.21) and (3.20).

If (3.19) is satisfied, then

$$x, p + \delta e_{e,j} \in (re_{i,j}, re_{i,j+1}) \subset \mathcal{R}_{i,j}, \tag{3.23}$$

so by (3.22) we may apply Lemma 3.13 (4) to $f(p)$ and $q_2 = y$ to get

$$y = f(p) + \delta e_s + t(e_{s+1} - e_s) \tag{3.24}$$

for some $t \in \mathbb{R}$. By (3.23) and Lemma 3.14 (3) we can now apply Lemma 3.13 (4) with $p_1 = p$ and $q_2 = y$ to get

$$x = p + \delta e_{i,j} + \frac{t}{N}(e_{i+1} - e_i) = p + \delta e_i + (\delta \frac{j}{N} + \frac{t}{N})(e_{i+1} - e_i). \tag{3.25}$$

If (3.20) is satisfied, then (3.24) and (3.25) hold, and $t \in [0, \lambda(\varepsilon, \delta))$. If (3.21) is satisfied, then (3.24) would need to be modified, so we will leave this case until later. Meanwhile, consider the case when (3.24) and (3.25) are satisfied with $t \geqslant 0$.

If $t \geqslant 0$, then, as $y \in B_\varepsilon(f(p))$, we have $0 \leqslant t < \lambda(\varepsilon, \delta)$ by Definition 3.17 and Lemma 3.18 (1).

Apply then $\lambda(\varepsilon, \delta) > \delta$ for all $\delta \in (-\varepsilon, \varepsilon)$ (this follows from Lemma 3.18 (6) for $\delta \geqslant 0$ and is trivial for $\delta < 0$) to get

$$-\lambda(\varepsilon, |\delta|) < -|\delta| \leqslant -|\delta|\frac{j}{N} \leqslant \delta\frac{j}{N} + \frac{t}{N} < \lambda(\varepsilon, \delta)\frac{j}{N} + \frac{\lambda(\varepsilon, \delta)}{N} = \lambda(\varepsilon, \delta)\frac{j+1}{N} \leqslant \lambda(\varepsilon, \delta). \tag{3.26}$$

If $\delta\frac{j}{N} + \frac{t}{N} \geqslant 0$, we have $x \in B_\varepsilon(p)$ as a consequence of (3.25) and because the right-hand side inequality of (3.26) implies $\delta\frac{j}{N} + \frac{t}{N} < \lambda(\varepsilon, \delta)$. If $\delta\frac{j}{N} + \frac{t}{N} < 0$, then $\delta < 0$, so the left-hand side inequality of (3.26) and Lemma 3.18 (3) imply $\delta\frac{j}{N} + \frac{t}{N} > -\lambda(\varepsilon, -\delta) = -\lambda^*(\varepsilon, \delta)$, hence $x \in B_\varepsilon(p)$ by (3.25) and the definition of $\lambda^*(\varepsilon, \delta)$ (Definition 3.17 and Lemma 3.18 (1)). This finishes the case (3.20), and (3.19) with $t \geqslant 0$.

If (3.19) is satisfied and $t < 0$, let $t' = -t > 0$. If we also have $\delta \leqslant 0$, then let $\delta' = -\delta$, so from (3.25) and (3.24), it follows $x = p + \delta' e_{i',j'} + \frac{t'}{N}(e_{i'+1} - e_{i'})$ and $y = f(x) = f(p) + \delta' e_{s'} + t'(e_{s'+1} - e_{s'})$, where $i' = (i + n/2) \pmod n$, $j' = (j + n/2) \pmod n$ and $s' = (s + n/2) \pmod n$, so that $e_{i'} = -e_i$, $e_{i'+1} = -e_{i+1}$, $e_{i',j'} = -e_{i,j}$ $e_{s'} = -e_s$ and $e_{s'+1} = -e_{s+1}$. As $y \in B_\varepsilon(f(p))$, we conclude $0 < t' < \lambda(\varepsilon, \delta')$, and (3.24) and (3.25) are satisfied with non-negative t' and δ', instead of t and δ, respectively. Using $0 < t' < \lambda(\varepsilon, \delta')$ and (3.26), we conclude that is satisfied too with t' and δ'. It follows $x \in B_\varepsilon(p)$.

If $t' = -t > 0$ and $\delta \geqslant 0$, then $y \in B_\varepsilon(f(p))$ implies, using (3.24) and the definition of $\lambda^*(\varepsilon, \delta)$, that $0 < t' < \lambda^*(\varepsilon, \delta)$. The expression (3.25) then becomes

$$x = p + \delta e_i + \left(\delta\frac{j}{N} - \frac{t'}{N}\right)(e_{i+1} - e_i)$$

and instead of (3.26) we have, using again Lemma 3.18 (6),

$$-\lambda^*(\varepsilon,\delta) < -t' \leqslant \frac{-t'}{N} \leqslant \delta\frac{j}{N} - \frac{t'}{N} < \delta\frac{j}{N} \leqslant \delta < \lambda(\varepsilon,\delta), \tag{3.27}$$

which implies $x \in B_\varepsilon(p)$. Here we again use the right-hand side estimate in case $\delta\frac{j}{N} - \frac{t'}{N} \geqslant 0$, and the left-hand side estimate in case $\delta\frac{j}{N} - \frac{t'}{N} < 0$. This finishes (3.19) completely.

Finally, consider the case (3.21). In this case

$$y = f(p) + \delta e_s - t(e_s - e_{s-1}) \text{ with } t \geqslant 0, \tag{3.28}$$

so $y \in B_\varepsilon(f(p))$ implies $0 \leqslant t < \lambda(\varepsilon,\delta)$ by Lemma 3.18 (2). If $j = 0$, i.e. $p = r_0 e_i$, we have $x = p + \delta e_i - \frac{t}{N}(e_i - e_{i-1}) \in [re_{i-1}, re_i]$, otherwise $x = p + \delta e_{i,j} - \frac{t}{N}(e_{i+1} - e_i) \in [re_i, re_{i+1}]$.

In the former case ($p = r_0 e_i$), we have $x \in B_\varepsilon(p)$ by Lemma 3.18 (2) as $0 \leqslant t/N \leqslant t < \lambda(\varepsilon,\delta)$. If $p \neq r_0 e_i$ (i.e. $j \geqslant 1$), then write again expression for x:

$$x = p + \delta e_{i,j} - \frac{t}{N}(e_{i+1} - e_i) = p + \delta e_i + \left(\delta\frac{j}{N} - \frac{t}{N}\right)(e_{i+1} - e_i).$$

We consider the following three possibilities. The first one is when $\delta \geqslant 0$ and $\delta\frac{j}{N} - \frac{t}{N} \geqslant 0$, in this case we use Lemma 3.18 (6) to get $0 \leqslant \delta\frac{j}{N} - \frac{t}{N} \leqslant \delta < \lambda(\varepsilon,\delta)$, hence $x \in B_\varepsilon(p)$. A second possibility is $\delta \geqslant 0$ and $\delta\frac{j}{N} - \frac{t}{N} < 0$, in this case use $t < \lambda(\varepsilon,\delta)$, $j \geqslant 1$ and Lemma 3.18 (5) to get $0 \leqslant \frac{t}{N} - \delta\frac{j}{N} < \frac{\lambda(\varepsilon,\delta)}{N} - \delta\frac{1}{N} = \lambda^*(\varepsilon,\delta)\frac{1}{N} \leqslant \lambda^*(\varepsilon,\delta)$, hence $x \in B_\varepsilon(p)$. The last possibility is $\delta < 0$, in which case let $\delta' = -\delta > 0$, then $x = p - \delta' e_i - \left(\delta'\frac{j}{N} + \frac{t}{N}\right)(e_{i+1} - e_i) = p + \delta' e_{i'} + \left(\delta'\frac{j}{N} + \frac{t}{N}\right)(e_{i'+1} - e_{i'})$, where we set $i' = (i + n/2) \pmod n$. Let also $s' = (s + n/2) \pmod n$, then by (3.28)

$$y = f(p) + \delta e_s - t(e_s - e_{s-1}) = f(p) + \delta' e_{s'} + t(e_{s'} - e_{s'-1}).$$

As $y \in B_\varepsilon(f(p))$, we have $0 \leqslant t \leqslant \lambda^*(\varepsilon,\delta') < \lambda(\varepsilon,\delta')$ by Lemma 3.18 (2) and (6), and $\delta' > 0$. This, together with $0 < \delta' < \lambda(\varepsilon,\delta')$ from Lemma 3.18 (6), implies that

$$0 \leqslant \delta'\frac{j}{N} + \frac{t}{N} < \lambda(\varepsilon,\delta')\frac{j+1}{N} \leqslant \lambda(\varepsilon,\delta'),$$

hence $x = p + \delta' e_{i'} + \left(\delta'\frac{j}{N} + \frac{t}{N}\right)(e_{i'+1} - e_{i'}) \in B_\varepsilon(p)$.

This finishes the proof of (3.21). □

§4. *Polygonal norms with* $4m+2$ *sides.* In this section, we show that there are non-Euclidean norms on the plane for which, as in the Euclidean case, for each $N \geqslant 1$, there exists an N-fold Lipschitz quotient mapping f satisfying $c/L = 1/N$. Indeed, we show that this is satisfied for all polygonal n-norms with $n = 4m+2$; the example of such a mapping for $\|\cdot\|_n$ would be the N-fold winding mapping f_n^N defined in Definition 3.12. This section is devoted to the proof of this result.

We already know, from Therorem 3.19, that f_n^N is an N-fold 1-co-Lipschitz mapping. We also know that any N-fold Lipschitz quotient mapping satisfies $c/L \leqslant 1/N$, see Theorem 2.10. It remains to show that for any $n = 4m+2$, the N-fold winding map $f_n^N : (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$ is an N-Lipschitz mapping. We first prove two additional properties of $\lambda(\varepsilon,\delta)$ and $\lambda^*(\varepsilon,\delta)$ (see Definition 3.17 and Lemma 3.18 (1)) that are satisfied only in case $n = 4m+2$ and show that it is enough to check that f_n^N is N-Lipschitz on $\bigcup_{0\leqslant i\leqslant n-1}(\bigcup_{0\leqslant j\leqslant N-1} \mathcal{R}_{i,j})$, where $\mathcal{R}_{i,j}$ are defined in Notation 3.4.

LEMMA 4.1. *Let $n = 4m + 2$, $m \geqslant 1$ and $\varepsilon > 0$. Then:*
(1) *$\lambda(\varepsilon, \cdot)$ is monotone increasing on $[-\varepsilon, 0]$ and is monotone decreasing on $[0, \varepsilon]$;*
(2) *$N\lambda(\varepsilon, \delta) \leqslant \lambda(N\varepsilon, \delta)$ and $N\lambda^*(\varepsilon, \delta) \leqslant \lambda^*(N\varepsilon, \delta)$ for all $\delta \in [-\varepsilon, \varepsilon]$ and $N \in \mathbb{N}$.*

Proof. (1) By Lemma 3.18 (1), we may use that $\lambda(\varepsilon, \delta) = \lambda(\varepsilon, \delta, 0, 0)$. Letting $\lambda = \lambda(\varepsilon, \delta)$, we can see that $P = \delta e_0 + \lambda(e_1 - e_0) \in \partial B_\varepsilon(0)$ runs vertically upwards along the boundary of the regular n-gon from $-\varepsilon e_0 = \varepsilon e_{2m+1}$ to εe_{m+1}, as $\delta \in [-\varepsilon, 0]$ increases, hence $\lambda(\varepsilon, \cdot)$ increases on $[-\varepsilon, 0]$. Let $d = \|e_1 - e_0\|_n$, then $\lambda(\varepsilon, \cdot)|_{[0,\varepsilon d]}$ is constant and on $[\varepsilon d, \varepsilon]$, we have that $\lambda(\varepsilon, \cdot)$ decreases; hence $\lambda(\varepsilon, \cdot)$ (non-strictly) decreases on $[0, \varepsilon]$.

(2) Use Lemma 3.18 (4) and the first statement of the present lemma to get $\lambda(N\varepsilon, \delta) = N\lambda(\varepsilon, \delta/N) \geqslant N\lambda(\varepsilon, \delta)$. The inequality for λ^* then follows by Lemma 3.18 (3). □

The purpose of the next lemma is to allow us to verify only the pointwise Lipschitzness of a mapping at points of a certain subset of the plane, in order to ensure that the mapping is Lipschitz everywhere. The choice of the subset, $\bigcup_{\substack{0 \leqslant i \leqslant n-1 \\ 0 \leqslant j \leqslant N-1}} \mathcal{R}_{i,j}$, is dictated by the fact that the N- fold winding mapping f_n^N is linear in each of the regions $\mathcal{R}_{i,j}$—we will later show that f_n^N is pointwise N-Lipschitz at each $x \in \mathcal{R}_{i,j}$.

LEMMA 4.2. *Let $L > 0$, $f\colon (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, |||\cdot|||)$ be continuous and pointwise L-Lipschitz at every $x \in \bigcup_{\substack{0 \leqslant i \leqslant n-1 \\ 0 \leqslant j \leqslant N-1}} \mathcal{R}_{i,j}$. Then f is L-Lipschitz on the whole $(\mathbb{R}^2, \|\cdot\|)$.*

Proof. By Lemma 2.2 we know that f is (globally) L-Lipschitz on each of the open regions $\mathcal{R}_{i,j}$. By continuity of f, it is L-Lipschitz on the closure of each region $\mathcal{R}_{i,j}$, see Lemma 2.1. As the closure of $\mathcal{P} = \bigcup_{\substack{0 \leqslant i \leqslant n-1 \\ 0 \leqslant j \leqslant N-1}} \mathcal{R}_{i,j}$ is the whole $\mathbb{R}^2$, by Lemma 2.2, it remains to show that f is pointwise L-Lipschitz at the points x on the boundary of $\mathcal{P}$. Indeed, if $x \in \partial\mathcal{P}$ and is not zero, then for r small enough $B_r(x)$ intersects only $\overline{\mathcal{R}}_{i,j}$ and $\overline{\mathcal{R}}_{i,j+1}$ for some i and j while $x \in \overline{\mathcal{R}}_{i,j} \cap \overline{\mathcal{R}}_{i,j+1}$, hence pointwise L-Lipschitzness of f in each of these regions implies $|||f(x) - f(y)||| \leqslant L\|x - y\|$ for any $y \in B_r(x)$. If $x = 0$, then for any r, we have that $B_r(0)$ intersects all $\overline{\mathcal{R}}_{i,j}$ while $0 \in \partial\mathcal{R}_{i,j}$ for all pairs (i, j). Hence, pointwise L-Lipschitzness of f in each $\mathcal{R}_{i,j}$ implies $|||f(0) - f(y)||| \leqslant L\|y\|$ for any $y \in B_r(0)$. □

THEOREM 4.3. *For $n = 4m + 2$, $m \geqslant 1$ the Lipschitz constant $Lip(f_n^N)$ of the N-fold winding mapping f_n^N under $\|\cdot\|_n$ is equal to N.*

Proof. For simplicity of notation, denote f_n^N by f. It is clear that $\mathrm{Lip}(f) \geqslant N$ from considering points $x = e_0$ and $y = e_{0,1}$: $f(x) = x$ and $f(y) = e_1$, hence $\|f(y) - f(x)\|_n = N\|y - x\|_n$. Therefore, by Lemma 4.2, it is enough to show that f_n^N is pointwise N-Lipschitz at all points $p \in \mathcal{P} := \bigcup_{\substack{0 \leqslant i \leqslant n-1 \\ 0 \leqslant j \leqslant N-1}} \mathcal{R}_{i,j}$.

Let $p \in \mathcal{P}$, say $p \in \mathcal{R}_{i,j}$ and define $s = (Ni + j) \pmod n$. Denote the length coordinates of p as (r_0, ℓ_0). Take $\varepsilon_0 = \Delta > 0$ defined by (3.2) such that the conclusions of Lemma 3.14 hold and let $\varepsilon \in (0, \varepsilon_0)$. Take $x \in B_\varepsilon(p)$, denote the length coordinates of x as (r, ℓ). Let $\delta := r - r_0$, note that $\delta \in (-\varepsilon, \varepsilon)$ and $x \in \mathcal{R}_{i,j}$ hence $x \in (re_{i,j}, re_{i,j+1})$ and $f(x) \in \mathcal{U}_s$. By Lemma 3.13 (1) $f(B_\varepsilon(p)) \subseteq \mathcal{U}_s$. We show that $f(B_\varepsilon(p)) \subseteq B_{N\varepsilon}(p)$.

Denote by ℓ_1, ℓ_2, ℓ_3 and ℓ_4 the angle components of length coordinates of points $p + \delta e_i$, $p + \delta e_{i,j}$, $p + \delta e_{i,j+1}$ and $p + \delta e_{i+1}$, respectively. Note that for $\delta \geqslant 0$ we necessarily have $\ell_1 \leqslant \ell_2 \leqslant \ell_3 \leqslant \ell_4$, and for $\delta \leqslant 0$ we have $\ell_4 \leqslant \ell_3 \leqslant \ell_2 \leqslant \ell_1$. Also note that by Lemma 3.14 (1) and (2), all these four points have $\|\cdot\|_n$-norm equal to r and all belong to $\mathcal{R}_{i,j}$, hence all are on the same straight line segment $(re_{i,j}, re_{i,j+1})$ and $x \in (re_{i,j}, re_{i,j+1})$ too.

Case 1: $\delta \geqslant 0$ and $\ell \geqslant \ell_2$.

The assumption $\ell \geqslant \ell_2$ implies that $\|x-(p+\delta e_i)\|_n \geqslant \|x-(p+\delta e_{i,j})\|_n$ as $\ell_2 \geqslant \ell_1$ in this case. As x, $p+\delta e_i$, $p+\delta e_{i,j}$ all have the same norm and belong to $\mathcal{R}_{i,j}$, we use Lemma 3.13 (5) and Lemma 3.14 (3) to get

$$\|f(x)-(f(p)+\delta e_s)\|_n = N\|x-(p+\delta e_{i,j})\|_n \leqslant N\|x-(p+\delta e_i)\|_n < N\lambda(\varepsilon,\delta).$$

The last inequality is satisfied since $x \in B_\varepsilon(p) \cap [re_i, re_{i+1}]$ and $\ell \geqslant \ell_2 \geqslant \ell_1$ implies $x = (p+\delta e_i) + t(e_{i+1}-e_i)$ for some $t \in [0, \lambda(\varepsilon,\delta))$. Using now Lemma 4.1 (2), we conclude $\|f(x)-(f(p)+\delta e_s)\|_n \leqslant \lambda(N\varepsilon,\delta)$, which implies, as $f(x) \in f(p)+\delta e_s + \mathbb{R}_+(e_{s+1}-e_s)$, that $f(x) \in B_{N\varepsilon}(f(p))$.

Case 2: $\delta \geqslant 0$ and $\ell \leqslant \ell_2$.

In this case, we have $\|x-(p+\delta e_{i+1})\|_n \geqslant \|x-(p+\delta e_{i,j+1})\|_n$ as $\ell \leqslant \ell_2 \leqslant \ell_3 \leqslant \ell_4$. Using again Lemma 3.13 (5) and Lemma 3.14 (4), we get

$$\|f(x)-(f(p)+\delta e_{s+1})\|_n = N\|x-(p+\delta e_{i,j+1})\|_n \leqslant N\|x-(p+\delta e_{i+1})\|_n < N\lambda(\varepsilon,\delta).$$

The last inequality is satisfied by Lemma 3.18 (2), since $x \in B_\varepsilon(p) \cap [re_i, re_{i+1}]$ and $\ell \leqslant \ell_2 \leqslant \ell_4$ implies $x = (p+\delta e_{i+1}) - t(e_{i+1}-e_i)$ for some $t \in [0, \lambda(\varepsilon,\delta))$. Using now Lemma 4.1 (2), we conclude $\|f(x)-(f(p)+\delta e_{s+1})\|_n \leqslant \lambda(N\varepsilon,\delta)$, which implies, as $f(x) \in f(p)+\delta e_{s+1} - \mathbb{R}_+(e_{s+1}-e_s)$, that $f(x) \in B_{N\varepsilon}(f(p))$, again by Lemma 3.18 (2).

Case 3: $\delta < 0$ and $\ell \geqslant \ell_3$.

The assumption $\ell \geqslant \ell_3$ implies that $\|x-(p+\delta e_{i+1})\|_n \geqslant \|x-(p+\delta e_{i,j+1})\|_n$ as $\ell_3 \geqslant \ell_4$ in this case. Using Lemma 3.13 (5) and Lemma 3.14 (4), we get

$$\|f(x)-(f(p)+\delta e_{s+1})\|_n = N\|x-(p+\delta e_{i,j+1})\|_n \leqslant N\|x-(p+\delta e_{i+1})\|_n \leqslant N\lambda^*(\varepsilon,\delta).$$

The last inequality is satisfied by Lemma 3.18 (2), since $x \in B_\varepsilon(p) \cap [re_i, re_{i+1}]$ and $\ell \geqslant \ell_3 \geqslant \ell_4$ implies $x = (p+\delta e_{i+1}) + t^*(e_{i+1}-e_i)$ for some $t^* \in [0, \lambda^*(\varepsilon,\delta))$. Using now Lemma 4.1 (2), we conclude $\|f(x)-(f(p)+\delta e_{s+1})\|_n \leqslant \lambda^*(N\varepsilon,\delta)$, which implies, as $f(x) \in f(p)+\delta e_{s+1} + \mathbb{R}_+(e_{s+1}-e_s)$, that $f(x) \in B_{N\varepsilon}(f(p))$.

Case 4: $\delta < 0$ and $\ell \leqslant \ell_3$.

The assumption $\ell \leqslant \ell_3$ implies that $\ell \leqslant \ell_3 \leqslant \ell_2 \leqslant \ell_1$ in this case. Hence $\|x-(p+\delta e_i)\|_n \geqslant \|x-(p+\delta e_{i,j})\|_n$, therefore using again Lemma 3.13 (5) and Lemma 3.14 (3), we get

$$\|f(x)-(f(p)+\delta e_s)\|_n = N\|x-(p+\delta e_{i,j})\|_n \leqslant N\|x-(p+\delta e_i)\|_n \leqslant N\lambda^*(\varepsilon,\delta).$$

The last inequality is satisfied since $x \in B_\varepsilon(p) \cap [re_i, re_{i+1}]$, and $\ell \leqslant \ell_3 \leqslant \ell_2 \leqslant \ell_1$ implies $x = (p+\delta e_i) - t^*(e_{i+1}-e_i)$ for some $t^* \in [0, \lambda^*(\varepsilon,\delta))$. Using now Lemma 4.1 (2), we conclude $\|f(x)-(f(p)+\delta e_s)\|_n \leqslant \lambda^*(N\varepsilon,\delta)$, which implies, as $f(x) \in f(p)+\delta e_s - \mathbb{R}_+(e_{s+1}-e_s)$, that $f(x) \in B_{N\varepsilon}(f(p))$. □

COROLLARY 4.4. *For $n = 4m+2$ and $m \geqslant 1$, the ratio of the co-Lipschitz and Lipschitz constants of the N-fold winding mapping, f_n^N, under $\|\cdot\|_n$, is equal to $1/N$.*

Proof. Follows from Theorem 3.19 and Theorem 4.3. □

COROLLARY 4.5. *Let $n = 4m+2$ and $m \geqslant 1$. If $\|\cdot\|$ is a norm on $\mathbb{R}^2$ whose unit ball is a linear image of a regular polygon with n sides, then there exists a Lipschitz quotient mapping $g\colon (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ with ratio of constants $c/L = 1/N$.*

Proof. Let f_n^N be the N-fold winding mapping for the n-norm $\|\cdot\|_n$. From Remark 3.3, Theorem 4.3 and Theorem 3.19, it follows that the mapping $g\colon (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$, defined as $g = U \circ f_n^N \circ U^{-1}$, is a Lipschitz quotient mapping with ratio of constants equal to $1/N$. Here U is an isomorphism $(\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|)$. $\square$

§5. *Polygonal norms with* $4m$ *sides.* In the previous chapter, we have shown that for any polygonal norm with $4m+2$ sides and for any norm whose unit ball is a linear image of a regular polygon with $4m+2$ sides, there exists an N-fold Lipschitz quotient mapping with ratio of constants equal to $1/N$. Now we are going to show that for all remaining polygonal n-norms on the plane, i.e. whenever n is divisible by 4, every N-fold Lipschitz quotient has ratio of constants strictly less than $1/N$, whenever $N \geqslant 2$. We first show in Theorem 5.12 that for $n = 4m$ and each $N \geqslant 2$ there is no N-fold Lipschitz quotient mapping, under the norm $\|\cdot\|_n$, which achieves the $1/N$ ratio of constants, and then concludes in Theorem 5.13 that the same is true for all polygonal n-norms. We then show in Proposition 5.14 that the ratio of constants c_N/L_N of the N-fold winding map for the n-norm $\|\cdot\|_n$ is bounded above by $\rho_{N,n} = \frac{1}{N+(N-1)\tan^2\frac{\pi}{n}}$. It is our conjecture that for any Lipschitz quotient map $f\colon (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$, the ratio of its co-Lipschitz and Lipschitz constants does not exceed $\rho_{N,n}$, see Conjecture 5.15.

Throughout the section we will be working with the n-norm $\|\cdot\|_n$, and with the Euclidean norm denoted by $|\cdot|$, as usual. Recall Notation 3.4 which we will be using in the next lemma. Also, recall the notion of length coordinates introduced in Definition 3.7.

LEMMA 5.1. *Let $n = 4m$, $m \geqslant 1$ be integers. For $r > 0$ and $0 < a < r\|e_1 - e_0\|_n$, let $P_1 = (r, \mathscr{L}_n - a/r)$ and $P_2 = (r, a/r)$ be two points on the sides $[re_{n-1}, re_0]$ and $[re_0, re_1]$ of $\partial B_r(0)$, respectively, given in their length coordinates. Then $\|P_1 - P_2\|_n = 2a\cos^2(\pi/n)$.*

Proof. Notice that $\|P_i - re_0\|_n = a$ for $i = 1, 2$ by the definition of the angle component of length coordinates. As $[P_1, P_2]$ is a vertical line segment and $\mathcal{D} = [re_m, re_{3m}]$ is a vertical diameter of $\partial B_r(0)$, we get

$$\frac{\|P_1 - P_2\|_n}{\|\mathcal{D}\|_n} = \frac{|P_1 - P_2|}{|\mathcal{D}|},$$

so $\|P_1 - P_2\|_n = |P_1 - P_2|$, as $\|\mathcal{D}\|_n = |\mathcal{D}|$.

On the other hand, applying Remark 3.5 (1) and Lemma 3.6 (1), we get

$$|re_0 - P_i| = \|re_0 - P_i\|_n \frac{|e_0 - e_1|}{\|e_0 - e_1\|_n} = a\frac{2\sin\frac{\pi}{n}}{2\tan\frac{\pi}{n}} = a\cos(\pi/n).$$

Hence $\|P_1 - P_2\|_n = |P_1 - P_2| = 2|re_0 - P_1|\sin(\frac{(n-2)\pi}{2n}) = 2a\cos^2(\pi/n)$. $\square$

Notation 5.2. Recall Notation 3.4. For any $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant N-1$ let $w_{Ni+j} = e_{i,j}$. Denote the angle between $\mathbb{R}_+ w_k$ and $\mathbb{R}_+ w_{k+1}$ as α_k. Of particular importance for us will be α_0. Note that the values of α_k depend on n and N; we will, however, suppress these indices for convenience of notation.

LEMMA 5.3. *Let $n = 4m$, $m \geqslant 1$ and $N \geqslant 2$ be integers. Let further α_0 be the angle $\angle w_0 0 w_1$, as defined in Notation 5.2. Then*

$$\tan\alpha_0 = \frac{\frac{2}{N}\tan\frac{\pi}{n}}{1 + (1 - \frac{2}{N})\tan^2\frac{\pi}{n}} \leqslant \frac{2}{N}\tan\frac{\pi}{n}. \tag{5.1}$$

In particular, if $n = 4$, then $\tan\alpha_0 = \frac{1}{N-1}$.

Proof. Notice that from the right triangle $0w_1W$, where $W = \frac{e_0+e_1}{2}$ is the middle point of the straight line segment $[e_0, e_1]$,

$$\frac{\tan(\frac{\pi}{n} - \alpha_0)}{\tan\frac{\pi}{n}} = \frac{\frac{N}{2} - 1}{\frac{N}{2}} = 1 - \frac{2}{N},$$

hence

$$\tan\alpha_0 = \tan(\tfrac{\pi}{n} - (\tfrac{\pi}{n} - \alpha_0)) = \frac{\tan\frac{\pi}{n} - \tan(\frac{\pi}{n} - \alpha_0)}{1 + \tan\frac{\pi}{n}\tan(\frac{\pi}{n} - \alpha_0)} = \frac{\frac{2}{N}\tan\frac{\pi}{n}}{1 + (1 - \frac{2}{N})\tan^2\frac{\pi}{n}} \leqslant \tfrac{2}{N}\tan\tfrac{\pi}{n}.$$

□

LEMMA 5.4. *Let $n = 4m$, $m \geqslant 1$ and $N \geqslant 2$ be integers and let $r > 0$. Using Notation 5.2, consider the intersection point sw_1 between the line $\mathbb{R}_+w_1$ and the vertical line through re_0. Then*

$$s = r(1 + \tan\tfrac{\pi}{n}\tan\alpha_0), \tag{5.2}$$

$$\|re_0 - sw_1\|_n = r\tan\alpha_0, \tag{5.3}$$

$$\|re_0 - se_1\|_n = r\tan\tfrac{\pi}{n}(2 + \tan\tfrac{\pi}{n}\tan\alpha_0). \tag{5.4}$$

Proof. As the straight line segment $[sw_1, re_0]$ is vertical and n is divisible by 4, we conclude that $\|re_0 - sw_1\|_n = |re_0 - sw_1| = r\tan\alpha_0$, proving (5.3).

Since $w_1 \in [e_0, e_1]$, the angle between $w_1 - e_0$ and the x-axis is equal to $\frac{\pi}{2} - \frac{\pi}{n}$, hence

$$s - r = |se_0 - re_0| = \frac{|re_0 - sw_1|}{\tan(\frac{\pi}{2} - \frac{\pi}{n})} = r\tan\alpha_0\tan\tfrac{\pi}{n},$$

which implies (5.2).

Now let $d := \|re_0 - se_1\|_n$, consider the polygon $\partial B_d(re_0)$, containing se_1, and denote by Q its $\frac{n}{4}$th vertex. Let H be the intersection point between the horizontal line through se_1 and the vertical line through re_0, notice that

$$\angle HE_1E_0 = \tfrac{\pi}{2} - \tfrac{\pi}{n}, \tag{5.5}$$

where $E_i = se_i$, $i = 0, 1$.

Using (5.5), (5.2) and $|se_0 - se_1| = 2s\sin\frac{\pi}{n}$ from Remark 3.5 (1), we get

$$|re_0 - H| = \cos\tfrac{\pi}{n}|se_0 - se_1| = (2\sin\tfrac{\pi}{n}\cos\tfrac{\pi}{n})s, \text{ and} \tag{5.6}$$

$$\begin{aligned}|se_1 - H| &= \tan\tfrac{\pi}{n}|re_0 - H| - |se_0 - re_0| = 2\sin^2\tfrac{\pi}{n}s - (s - r)\\ &= 2r\sin^2\tfrac{\pi}{n}(1 + \tan\alpha_0\tan\tfrac{\pi}{n}) - r\tan\alpha_0\tan\tfrac{\pi}{n}.\end{aligned} \tag{5.7}$$

Finally, as $\angle E_1QH = \frac{\pi}{2} - \frac{\pi}{n}$, we use (5.7) to get

$$|H - Q| = \tan\tfrac{\pi}{n}|se_1 - H| = 2r\sin^2\tfrac{\pi}{n}\tan\tfrac{\pi}{n}(1 + \tan\alpha_0\tan\tfrac{\pi}{n}) - r\tan\alpha_0\tan^2\tfrac{\pi}{n}.$$

We therefore obtain, using (5.6) and (5.2),

$$\begin{aligned}d &= |re_0 - Q| = |re_0 - H| + |H - Q|\\ &= (2\sin\tfrac{\pi}{n}\cos\tfrac{\pi}{n})s + 2r\sin^2\tfrac{\pi}{n}\tan\tfrac{\pi}{n}(1 + \tan\alpha_0\tan\tfrac{\pi}{n}) - r\tan\alpha_0\tan^2\tfrac{\pi}{n}\end{aligned}$$

$$= (2r\sin\tfrac{\pi}{n}\cos\tfrac{\pi}{n})(1+\tan\alpha_0\tan\tfrac{\pi}{n}) + 2r\sin^2\tfrac{\pi}{n}\tan\tfrac{\pi}{n}(1+\tan\alpha_0\tan\tfrac{\pi}{n}) - r\tan\alpha_0\tan^2\tfrac{\pi}{n}$$

$$= 2r\sin\tfrac{\pi}{n}(1+\tan\alpha_0\tan\tfrac{\pi}{n})(\cos\tfrac{\pi}{n}+\sin\tfrac{\pi}{n}\tan\tfrac{\pi}{n}) - r\tan\alpha_0\tan^2\tfrac{\pi}{n}$$

$$= 2r\tan\tfrac{\pi}{n}(1+\tan\alpha_0\tan\tfrac{\pi}{n}) - r\tan\alpha_0\tan^2\tfrac{\pi}{n}$$

$$= 2r\tan\tfrac{\pi}{n} + r\tan\alpha_0\tan^2\tfrac{\pi}{n},$$

which proves (5.4). □

Note that in the following lemma, we assume $m \geqslant 2$ as $\sec(2\pi/n)$ is undefined for $n = 4$.

LEMMA 5.5. *Let $n = 4m$, $m \geqslant 2$ be integers. For $u > 0$ define*

$$a_0 := u\cos(2\pi/n) \ \ \text{and} \ \ a_1 := u\sec(2\pi/n) \tag{5.8}$$

and let $a \in [a_0, a_1]$. Assume further $p \in \mathbb{R}_+e_0$ and $q \in \mathbb{R}_+e_1$. Then

(1) *ae_1 belongs to the $(\frac{n}{4}+1)$th side of $\partial B_{\|ue_0-ae_1\|_n}(ue_0)$;*
(2) *$\|ue_0 - q\|_n \geqslant \|ue_0 - ae_1\|_n$, whenever $\|q\|_n \geqslant a$;*
(3) *$\|p - ae_1\|_n \geqslant \|ue_0 - ae_1\|_n$, whenever $\|p\|_n \geqslant u$;*
(4) *$\|p - q\|_n \geqslant \|ue_0 - ae_1\|_n$, whenever $\|p\|_n \geqslant u$ and $a_1 \geqslant \|q\|_n \geqslant a$.*

Proof. (1) Notice that the constants a_0, a_1 are defined by (5.8) in such a way that $a_1e_1 - ue_0$ is vertical and its angle with $a_0e_1 - ue_0$ is equal to $\frac{2\pi}{n}$. Consider $T \in [ue_0, a_1e_1]$ such that the triangle formed by ue_0, a_0e_1 and T is isosceles, hence the Euclidean distance $|T - ue_0| = |a_0e_1 - ue_0| = u\sin(2\pi/n)$. On the other hand, the orthogonal projection of $a_1e_1 - a_0e_1$ on e_0 is equal to $(a_1 - a_0)\cos(2\pi/n)$, and the line through a_0e_1 and T is at π/n with the horizontal direction, thus

$$|a_0e_1 - T| = (a_1 - a_0)\cos(2\pi/n)/\cos(\pi/n) = u(1-\cos^2(2\pi/n))/\cos(\pi/n)$$
$$= 2u\sin(2\pi/n)\sin(\pi/n) = 2\sin(\pi/n)|ue_0 - T|.$$

Hence it is readily verified by Remark 3.5 (1) that a_0e_1 is the $(\frac{n}{4}+1)$th vertex of $\partial B_{\|ue_0-a_0e_1\|_n}(ue_0)$.

For an arbitrary $a \in [a_0, a_1]$, we now have that the line through ae_1 parallel to $a_0e_1 - T$ intersects $[ue_0, a_1e_1]$ between T and a_1e_1, hence the statement follows.

(2) Notice that if $q = A := ae_1$, then (2) is satisfied trivially. Assume thus $\|q\|_n > a$. Let $d = \|ue_0 - ae_1\|_n$, consider the polygon $\partial B_d(ue_0)$, denote its $(\frac{n}{4})$th vertex by Q and let H denote the intersection point between the horizontal line through A and the vertical line through ue_0.

Since $a_0 \leqslant a \leqslant a_1$, by (1) we know that A belongs to the $(\frac{n}{4}+1)$th side of the polygon $\partial B_d(ue_0)$, therefore $\angle HAQ = \pi/n$. On the other hand, since $q \in \mathbb{R}_+e_1$ and $\|q\|_n \geqslant a$, we have $\angle HAq = \frac{2\pi}{n} > \frac{\pi}{n}$. Thus, $q \notin B_d(ue_0)$, since q belongs to the closed half plane above the line AQ implying (2).

(3) Assume that $p \in \mathbb{R}_+e_0$ is such that $\|p\|_n \geqslant u$ and let $d' = \|p - ae_1\|_n$. Let H' denote the intersection point between the horizontal line through A and the vertical line through p. Also denote by Q' the $(\frac{n}{4})$th vertex of the polygon $\partial B_{d'}(p)$.

Notice that if $0 \leqslant \angle Q'pA \leqslant 2\pi/n$, then A belongs to the $(\frac{n}{4}+1)$th side of the polygon $\partial B_{d'}(p)$. In this case, we have that A, Q, Q' are on the same straight line and

$$\angle pOA = \tfrac{2\pi}{n}\ ;\ \angle AQH = \angle AQ'H' = \tfrac{\pi}{2} - \tfrac{\pi}{n} \ \text{and} \ \angle H'AQ' = \angle HAQ = \tfrac{\pi}{n}. \tag{5.9}$$

It is also clear that if $\angle Q'pA > 2\pi/n$, then A is not on the $(\frac{n}{4}+1)$th side of $\partial B_{d'}(p)$, so instead of the last two equations in (5.9), we have:

$$\angle H'AQ' \geqslant \angle HAQ = \tfrac{\pi}{n}.$$

So Q is below the line AQ'. In both cases we have:

$$d' = \|p - A\|_n = |p - Q'| \geqslant |ue_0 - Q| = \|ue_0 - ae_1\|_n = d,$$

as p and ue_0 belong to the x-axis and Q' is higher than Q, implying (3).

(4) Assume that $q \in \mathbb{R}_+e_1$ is such that $a \leqslant \|q\|_n \leqslant a_1$. Use (3) with $a = \|q\|_n$, to get $\|p - q\|_n \geqslant \|ue_0 - q\|_n$. Joining this with (2) for an arbitrary $a \in [a_0, a_1]$, we get the desired inequality. □

COROLLARY 5.6. *Let $n = 4m$, $m \geqslant 1$ be integers. For any $r > 0$, if $p \in \mathbb{R}_+e_0$ and $q \in \mathbb{R}_+e_1$ are such that $\|p\|_n, \|q\|_n \geqslant r$, then $\|p - q\|_n \geqslant \frac{1}{n} r\mathscr{L}_n$.*

Proof. For $n = 4$, $m = 1$ we know that $\|\cdot\|_4$ is the ℓ_1-norm, so we can easily calculate distances, indeed for $p = (x, 0)$ and $q = (0, y)$, given in Cartesian coordinates, with $x, y \geqslant r$ we have:

$$\|p - q\|_4 = |x| + |y| \geqslant 2r = \frac{1}{4} r\mathscr{L}_4.$$

Assume now that $m \geqslant 2$ and let $p \in \mathbb{R}_+e_0$ and $q \in \mathbb{R}_+e_1$ be as in hypothesis of the corollary. Then $r' := \min\{\|q\|_n, \|p\|_n\} \geqslant r$. As the unit ball $B_1(0)$ is symmetrical with respect to the bisector of $\angle e_1 0 e_0$, we may assume without loss of generality that $r' = \|p\|_n$. Hence we can apply Lemma 5.5 (2), using $a = u = r'$ as in this case $p = ue_0$, $a \in [a_0, a_1]$ trivially and $\|q\|_n \geqslant r' = a$. We get:

$$\|p - q\|_n = \|ue_0 - q\|_n \geqslant \|ue_0 - ae_1\|_n = r'\|e_0 - e_1\|_n = r'\frac{\mathscr{L}_n}{n} \geqslant \frac{1}{n} r\mathscr{L}_n.$$

□

Notation 5.7. In subsequent statements, we will use the following notation. For any $r \geqslant 0$ and $0 \leqslant i \leqslant n-1$ we denote $\mathcal{D}_i = \mathbb{R}_+e_i$ and $\mathcal{D}_i^r = [r, +\infty)e_i$; we also let $\mathbb{D}^r = \bigcup_{i=0}^{n-1} \mathcal{D}_i^r$.

For a point $p \in \mathbb{R}^2$ we will also use notation $\mathrm{dist}_n(p, S)$, where $S = \mathcal{D}_i, \mathcal{D}_i^r$ or $\mathbb{D}^r$ to denote the quantity $\inf\{\|p - x\|_n : x \in S\}$.

LEMMA 5.8. *Let $n = 4m$, $m \geqslant 1$, $0 \leqslant i \leqslant n-1$ be integers. If $P \in \mathbb{R}^2$ is such that $\|P\|_n \sin(2\pi/n) \geqslant 2\mathrm{dist}_n(P, \mathcal{D}_i)$, then the point $Q = \|P\|_n e_i$ satisfies $Q \in \mathcal{D}_i$, $\|Q\|_n = \|P\|_n$ and $\|P - Q\|_n \leqslant \sec^2(\pi/n)\mathrm{dist}_n(P, \mathcal{D}_i)$.*

Proof. As the norm is invariant with respect to the rotation by $2\pi/n$, we may assume without loss of generality that $i = 0$. Denote $r = \|P\|_n$. Let $Q_1 = re_1$ and $Q = re_0$. The statement is straightforward if $m = 1$, i.e. $\|\cdot\|_n$ is the ℓ_1-norm.

Assume therefore $m \geqslant 2$. As $d := \mathrm{dist}_n(Q_1, \mathcal{D}_0)$ is equal to the Euclidean distance from Q_1 to $\mathcal{D}_0$, which, by Remark 3.5 (1), is equal to $2r\sin(\pi/n)\cos(\pi/n) = \|P\|_n \sin(2\pi/n) \geqslant 2\mathrm{dist}_n(P, \mathcal{D}_0)$, we conclude that either $P \in [re_0, re_1]$ or $P \in [re_{n-1}, re_0]$. Let $H \in \mathcal{D}_0$ be the point such that PH is vertical, then $|P - H| = \|P - H\|_n \leqslant \mathrm{dist}_n(P, \mathcal{D}_0)$, thus $|P - Q| \leqslant \sec(\pi/n)\mathrm{dist}_n(P, \mathcal{D}_0)$. Then by Lemma 3.6 (1) $\|P - Q\|_n = |P - Q|\sec(\pi/n) \leqslant \sec^2(\pi/n)\mathrm{dist}_n(P, \mathcal{D}_0)$. □

The next proposition shows that if an N-fold Lipschitz quotient mapping has the ratio of its co-Lipschitz and Lipschitz constants with respect to $\|\cdot\|_n$ equal to $1/N$, then it must map corners re_i of its ball of radius r centred at the origin close to the rays $\mathbb{R}_+e_j$, for all r big enough. Eventually we will use it to show that any such Lipschitz quotient mapping is 'close' to the N-fold winding map whose ratio of constants is strictly less than $1/N$, leading to contradiction. We will assume in the next proposition that the Lipschitz quotient map keeps the orientation, see Remark 2.7, in the sense that its decomposition $f = P \circ h$ consists of a monic polynomial P and an orientation preserving homeomorphism h.

PROPOSITION 5.9. *Let $n = 4m$ for some $m \geqslant 1$, and let $f\colon (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$ be an L-Lipschitz and c-co-Lipschitz N-fold mapping which satisfies $f(0) = 0$. If $c/L = \frac{1}{N}$, then there exist positive constants κ (defined by (5.13)) and R such that for all $\rho > R$ and $p \in \mathbb{D}^\rho$, it holds $\mathrm{dist}_n(f(p), \mathbb{D}^{\rho'}) \leqslant \kappa$, where $\rho' = c(\rho - M)$ and $M = \max\{\|z\|_n : f(z) = 0\}$.*

Proof. Consider the mapping $g(x) := f(x)/L$; note that g is a 1-Lipschitz, $1/N$-co-Lipschitz mapping that maps zero to zero. It is then enough to find R_1 and κ_1 which work for g and $c = 1/N$, and to let $R = R_1$, $\kappa = L\kappa_1$. Indeed, this is because $g(z) = 0$ if and only if $f(z) = 0$, so that $M = \max\{\|z\|_n : f(z) = 0\} = \max\{\|z\|_n : g(z) = 0\}$, and $\|f(p) - Lp'\|_n = L\|g(p) - p'\|_n$ for $p \in \mathbb{D}^\rho$ and $p' \in \mathbb{D}^{\frac{1}{N}(\rho - M)}$.

Let R_1 be given by Theorem 2.8 applied to g. Let $\rho > R_1$ and $p \in \mathbb{D}^\rho$; set $\rho' = \frac{1}{N}(\rho - M)$. As $p = re_i$ is a vertex of $\partial B_r(0)$ for $r = \|p\|_n \geqslant \rho$, we may perform a rotation of an integer multiple of $2\pi/n$ without affecting the Lipschitz and co-Lipschitz constants of g, and assume without loss of generality that $p = re_0$.

Set

$$\kappa_1 = \frac{2nM}{\sin(2\pi/n)}$$

and let $a := \mathrm{dist}_n(g(re_0), \mathbb{D}^{\rho'})$. If $a = 0$, there is nothing to prove as $\kappa_1 \geqslant 0$. Assume $a > 0$, i.e. $g(re_0) \notin \mathbb{D}^{\rho'}$; we first show that

$$a < r\|e_0 - e_1\|_n. \tag{5.10}$$

By Theorem 2.8 (1), we know that $r' := \|g(re_0)\|_n \geqslant \frac{1}{N}(r - M) \geqslant \rho'$, therefore, since $g(re_0)$ is not in $\mathbb{D}^{\rho'}$, we have that $g(re_0)$ lies between two of the lines $\mathcal{D}_k$, say $g(re_0)$ lies in one of the regions $\mathcal{U}_s$, $0 \leqslant s \leqslant n-1$. Hence, $g(re_0) \in (r'e_s, r'e_{s+1})$, so that

$$0 < a \leqslant \|g(re_0) - r'e_s\|_n < \|r'e_{s+1} - r'e_s\|_n = r'\|e_1 - e_0\|_n.$$

Since g is a 1-Lipschitz mapping and $g(0) = 0$, we have $r' = \|g(re_0)\|_n \leqslant \|re_0\|_n = r$, therefore (5.10) follows.

Let, as in Lemma 5.1, P_1 and P_2 be on the sides $[re_{n-1}, re_0]$ and $[re_0, re_1]$ of $\partial B_r(0)$, respectively, such that $\|P_i - re_0\|_n = a$. Let $\gamma : [0, \rho\mathscr{L}_n] \to \partial B_\rho(0)$ be the 1-Lipschitz parametrization of the boundary of the polygon $B_\rho(0)$ with starting point at P_1 so that $\gamma(0) = P_1$, $\gamma(a) = re_0$, $\gamma(2a) = P_2$ and $\mathrm{Ind}_0\,\gamma = 1$. Let $q_i = g(P_i)$; as $\|P_i\|_n = r \geqslant \rho$, it follows, by Theorem 2.8 (1), that $\|q_1\|_n, \|q_2\|_n \geqslant \rho' = \frac{1}{N}(\rho - M)$, and, by 1-Lipschitzness of g and Lemma 5.1 that

$$\|q_1 - q_2\|_n \leqslant \|P_1 - P_2\|_n = 2a\cos^2(\pi/n). \tag{5.11}$$

Let $\mathcal{U} = \overline{\mathcal{U}_s} \setminus B_{\rho'}(0)$. Since g and γ are 1-Lipschitz, for any $t \in [0, 2a]$

$$\|g(\gamma(t)) - g(re_0)\|_n = \|g(\gamma(t)) - g(\gamma(a))\|_n \leqslant |t - a| \leqslant a = \mathrm{dist}_n(g(re_0), \mathbb{D}^{\rho'}),$$

so as $g(re_0) \in \mathcal{U}$ and $\|g(\gamma(t))\|_n \geqslant \rho'$ for all $t \in [0, \rho\mathcal{L}_n]$, we conclude that $(g \circ \gamma)([0, 2a]) \subseteq \mathcal{U}$. Moreover, since the region $\mathcal{U}$ is convex, it follows $[q_1, q_2] = [(g \circ \gamma)(0), (g \circ \gamma)(2a)] \subseteq \mathcal{U}$. Let $\phi : [0, 2a] \to [q_1, q_2]$ be a linear parametrisation, and define

$$\Gamma(t) = \begin{cases} \phi(t), & \text{if } 0 \leqslant t \leqslant 2a; \\ (g \circ \gamma)(t), & \text{if } 2a \leqslant t \leqslant \rho\mathcal{L}_n. \end{cases}$$

We thus have that $\mathrm{Ind}_0 \Gamma = \mathrm{Ind}_0(g \circ \gamma) = N$ and $\|\Gamma(t)\|_n \geqslant \rho'$ for all t. Hence from Lemma 2.9, we infer that

$$\|q_1 - q_2\|_n + \mathrm{length}_n((g \circ \gamma)|_{[2a, \rho\mathcal{L}_n]}) \geqslant N(\rho'\mathcal{L}_n) = N\frac{1}{N}(\rho - M)\mathcal{L}_n = (\rho - M)\mathcal{L}_n.$$

Therefore, using again that g and γ are 1-Lipschitz, and (5.11), we get

$$\rho\mathcal{L}_n - 2a \geqslant \mathrm{length}_n(g \circ \gamma|_{[2a, \mathcal{L}_n\rho]}) \geqslant \mathcal{L}_n(\rho - M) - 2a\cos^2(\pi/n).$$

Hence we conclude that $0 < a \leqslant \frac{\mathcal{L}_n M}{2\sin^2(\pi/n)} = \frac{2nM}{\sin(2\pi/n)}$, where the last equality follows from Lemma 3.6 (1), therefore

$$\mathrm{dist}_n(g(p), \mathbb{D}^{\frac{1}{N}(\rho - M)}) = a \leqslant \frac{2nM}{\sin(2\pi/n)} = \kappa_1. \tag{5.12}$$

To finish the proof of this proposition is now enough to define the constant

$$\kappa := L\frac{2nM}{\sin(2\pi/n)}. \tag{5.13}$$

$\square$

LEMMA 5.10. *Let $n = 4m$ for some $m \geqslant 1$, $N \geqslant 2$ and let $f : (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$ be an N-Lipschitz and 1-co-Lipschitz N-fold mapping which satisfies $f(0) = 0$. Then for any $T > 0$ and $\varepsilon > 0$ there exist positive constants $R_0, \kappa_1 > T$ such that for each $\rho \geqslant R_0$, if*

$$\mathrm{dist}_n(f(\rho w_0), \mathcal{D}_0^{\rho - M}) < \kappa_1, \tag{5.14}$$

then

$$\mathrm{dist}_n(f(\rho w_k), \mathcal{D}_{k \pmod n}^{\rho - M})) < (1 + \varepsilon)\kappa_1 \tag{5.15}$$

for all $1 \leqslant k \leqslant nN - 1$, where w_k are defined in Notation 5.2.

Proof. Let $M = \max\{\|z\|_n : f(z) = 0\}$ and let R be the maximum of values given by Theorem 2.8 and Proposition 5.9 for f. Define

$$\kappa_1 = \max\{T, NM\mathcal{L}_n/\varepsilon\}, \tag{5.16}$$

and

$$R_0 = 1 + \max\left\{R, T, \frac{\kappa_1}{\sin(\pi/(2n))} + M\right\}, \tag{5.17}$$

let $\rho \geqslant R_0$, and assume, as in the hypothesis of the present lemma, that (5.14) is satisfied. Consider $\gamma : [0, \rho\mathcal{L}_n] \to \partial B_\rho(0)$ to be a 1-Lipschitz parametrisation of $\partial B_\rho(0)$ with $\gamma(0) = \gamma(\rho\mathcal{L}_n) = \rho e_0 = \rho w_0$.

By Theorem 2.8 (2), $\mathrm{Ind}_0(f \circ \gamma) = N$, hence there exists a continuous parametrisation $\theta : [0, \rho\mathcal{L}_n] \to \mathbb{R}$ of the Euclidean argument of $(f \circ \gamma)([0, \rho\mathcal{L}_n])$ which satisfies

$$\theta(0) = \arg(f \circ \gamma)(0);$$
$$\theta(t) \pmod{2\pi} = \arg(f \circ \gamma)(t) \text{ for all } t \in [0, \rho\mathscr{L}_n];$$
$$\theta(\rho\mathscr{L}_n) = \theta(0) + 2\pi N.$$

As the co-Lipschitz constant of f is equal to 1, we have $\|(f \circ \gamma)(0)\|_n = \|f(\rho w_0)\|_n \geqslant \rho - M$ by Theorem 2.8 (1), thus $\|f(\rho w_0)\|_n > \frac{\kappa_1}{\sin(\pi/(2n))}$ by (5.17), and so, by (5.14), we get that $\theta(0) \in (-\pi/(2n), \pi/(2n))$. For each $1 \leqslant k \leqslant Nn - 1$, let us define the values

$$\begin{aligned} t_k &= \sup\left\{t \in [0, \rho\mathscr{L}_n] : \theta(t) = k\frac{2\pi}{n}\right\}, \text{ and} \\ s_k &= \frac{k}{N}\frac{\rho\mathscr{L}_n}{n}. \end{aligned} \tag{5.18}$$

Notice that $0 < t_1 < \cdots < t_{nN-1} < \rho\mathscr{L}_n, \theta(t_k) = k\frac{2\pi}{n}$ and $\theta(t) > k\frac{2\pi}{n}$ for any $t \in (t_k, \rho\mathscr{L}_n]$. This, in particular, implies (5.19) below, while the definition of s_k implies (5.20), for all $1 \leqslant k \leqslant nN - 1$:

$$(f \circ \gamma)(t_k) \in \mathcal{D}^{\rho-M}_{k \pmod n}, \tag{5.19}$$

$$\gamma(s_k) = \rho w_k. \tag{5.20}$$

We also conclude, for each $1 \leqslant k \leqslant nN - 2$,

$$N(t_{k+1} - t_k) \geqslant \operatorname{length}_n((f \circ \gamma)[t_k, t_{k+1}]) \geqslant \frac{(\rho - M)\mathscr{L}_n}{n}. \tag{5.21}$$

Indeed, the left-hand side inequality follows from the fact that $(f \circ \gamma)$ is N-Lipschitz, and to prove the right-hand side inequality, we use (5.19) and Theorem 2.8 (1) to get $(f \circ \gamma)(t_i) \in \mathcal{D}^{\rho-M}_{i \pmod n}$, $i = k, k+1$, and then apply Corollary 5.6. Using the fact that $\theta(0) \in (-\pi/(2n), \pi/(2n))$ and

$$\operatorname{dist}_n((f \circ \gamma)(0), \mathcal{D}^{\rho-M}_0) = \operatorname{dist}_n(f(\rho w_0), \mathcal{D}^{\rho-M}_0) < \kappa_1,$$

we similarly conclude that

$$\begin{aligned} Nt_1 &\geqslant \operatorname{length}_n((f \circ \gamma)[0, t_1]) > \frac{(\rho - M)\mathscr{L}_n}{n} - \kappa_1, \text{ and} \\ N(\rho\mathscr{L}_n - t_{nN-1}) &\geqslant \operatorname{length}_n\big((f \circ \gamma)[t_{nN-1}, \rho\mathscr{L}_n]\big) > \frac{(\rho - M)\mathscr{L}_n}{n} - \kappa_1. \end{aligned} \tag{5.22}$$

Let $1 \leqslant j \leqslant nN - 1$. Summing up the inequalities (5.21) over $1 \leqslant k \leqslant j - 1$ and over $j \leqslant k \leqslant nN - 2$ with, respectively, first or second inequality from (5.22), we get

$$Nt_j \geqslant \operatorname{length}_n\big(f \circ \gamma[0, t_j]\big) > j\frac{(\rho - M)\mathscr{L}_n}{n} - \kappa_1, \text{ and}$$

$$N(\rho\mathscr{L}_n - t_j) \geqslant \operatorname{length}_n\big(f \circ \gamma[t_j, \rho\mathscr{L}_n]\big) > (nN - j)\frac{(\rho - M)\mathscr{L}_n}{n} - \kappa_1.$$

Therefore, using $N \geqslant 2$ and $\kappa_1 \geqslant M\mathscr{L}_n$ from (5.16), we get, using the definition of s_j from (5.18),

$$t_j > \frac{j}{N}\frac{(\rho - M)\mathscr{L}_n}{n} - \frac{\kappa_1}{N} = s_j - \left(\frac{jM\mathscr{L}_n}{nN} + \frac{\kappa_1}{N}\right) \geqslant s_j - \left(M\mathscr{L}_n + \frac{\kappa_1}{N}\right)$$

$$
\begin{aligned}
t_j < \rho\mathscr{L}_n - (n - \frac{j}{N})\frac{(\rho - M)\mathscr{L}_n}{n} + \frac{\kappa_1}{N} &= M\mathscr{L}_n + s_j - \frac{(jM)\mathscr{L}_n}{nN} + \frac{\kappa_1}{N} \\
&\leqslant s_j + M\mathscr{L}_n + \frac{\kappa_1}{N}.
\end{aligned}
$$

We thus conclude that

$$
\left|t_j - s_j\right| < M\mathscr{L}_n + \frac{\kappa_1}{N}. \tag{5.23}
$$

Hence, using (5.19) and (5.20), and, additionally, that $(f \circ \gamma)$ is N-Lipschitz, we conclude that for $1 \leqslant k \leqslant nN - 1$,

$$
\operatorname{dist}_n(f(\rho w_k), \mathcal{D}^{\rho-M}_{k \pmod n}) \leqslant \|(f\circ\gamma)(s_k) - (f\circ\gamma)(t_k)\|_n \leqslant N|s_k - t_k| < \kappa_1 + NM\mathscr{L}_n,
$$

and we get (5.15) using (5.16). □

Remark 5.11. One can see that if, instead introducing κ_1 in (5.16), we simply assume $\operatorname{dist}_n(f(\rho w_0), \mathcal{D}^{\rho-M}_0) < T$ in (5.14), we would get, instead of (5.15), that

$$
\operatorname{dist}_n(f(\rho w_0), \mathcal{D}^{\rho-M}_0) < T + NM\mathscr{L}_n.
$$

With the last two results in hand, we are now able to show that the ratio of constants of any N-fold Lipschitz quotient mapping under a $4m$-norm is strictly less than $1/N$.

THEOREM 5.12. *Let $n = 4m$ for some $m \geqslant 1$, and $N \geqslant 2$. If $f : (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$ is an L-Lipschitz and c-co-Lipschitz N-fold mapping, then $c/L < 1/N$.*

Proof. By Theorem 2.10, we know that $c/L \leqslant 1/N$. Hence if the conclusion of the present theorem is not satisfied, then $c/L = 1/N$. Without loss of generality, we may assume further that $f(0) = 0$, $c = 1$ and $L = N$ (replace $f(z)$ by $(f(z) - f(0))/c$), thus we may apply Proposition 5.9 and Lemma 5.10.

By Theorem 2.8 and Proposition 5.9 there exists $R' > 0$ and a constant κ such that whenever $r \geqslant R'$ and $\|p\|_n = r$, we have $\|f(p)\|_n \geqslant c(r - M) = r - M$ and, moreover, there exists $j = j(r) \in \{0, 1, \ldots, n-1\}$ such that

$$
\operatorname{dist}_n(f(re_0), \mathcal{D}^{r-M}_{j(r) \pmod n}) < \kappa. \tag{5.24}
$$

Let $R = R' + 3n\kappa/\mathscr{L}_n + M$. Since we may perform a rotation of the image by $-j(R)\frac{2\pi}{n}$ radians without changing the Lipschitz and co-Lipschitz constants of f, we can assume without loss of generality that $j(R) = 0$, i.e. $\operatorname{dist}_n(f(Re_0), \mathcal{D}^{R-M}_0) < \kappa$. As the length of a side of $\partial B_{r-M}(0)$ is $(r - M)\mathscr{L}_n/n > 2\kappa$ for $r \geqslant R$ and $f(re_0) : [R, \infty) \to \mathbb{R}^2$ is continuous, we use (5.24) to conclude by Corollary 5.6 that $j(r) = j(R) = 0$ for all $r \geqslant R$, i.e.

$$
\operatorname{dist}_n(f(re_0), \mathcal{D}^{r-M}_0) < \kappa \quad \text{for all} \quad r \geqslant R. \tag{5.25}
$$

By Lemma 5.10 this implies that there exist $R_0 > R$ and $\kappa_1 > \kappa$ such that for any $r > R_0$ and any $1 \leqslant k \leqslant nN - 1$ we have $\operatorname{dist}_n(f(rw_k), \mathcal{D}^{r-M}_k) < 2\kappa_1$; for the definition of κ_1 see (5.16), where we set $T = \kappa$ and $\varepsilon = 1$.

Now we set the new constants

$$
\begin{aligned}
\kappa_2 &= 2\kappa_1 \sec^2(\pi/n); \\
\delta &= 1 + \max\left\{\frac{2n}{\mathscr{L}_n}\kappa_2 \sec^2(\pi/n), M + \frac{2}{\tan(\pi/n)}\kappa_2\right\};
\end{aligned} \tag{5.26}
$$

$$R^* = \max\left\{\frac{3\kappa_2}{\sin(2\pi/n)} + 2M, \frac{4\delta}{\tan^2(\pi/n)\tan\alpha_0}, \frac{M+\delta\cos(2\pi/n)}{1-\cos(2\pi/n)}\right\},$$

and pick $r > \max\{R, R_0, R^*\}$.

Consider first the case $n = 4$. In this case the norm $\|\cdot\|_n$ coincides with the ℓ_1-norm. Consider the pair of points re_0 and sw_1, as in Lemma 5.4. Let $f(re_0) = (x_1, y_1)$ and $f(sw_1) = (x_2, y_2)$ be given in Cartesian coordinates. Then $x_1 \geqslant r - M - \kappa \geqslant r - M - \kappa_1$ and $|y_1| \leqslant \kappa_1$, hence $|x_1| - |y_1| \geqslant r - M - 2\kappa_1$; $|x_2| \leqslant 2\kappa_1$ and $y_2 \geqslant s - M - 2\kappa_1$, hence $|y_2| - |x_2| \geqslant s - M - 4\kappa_1$. Thus, we can use (5.26), Lemma 5.4 (5.3) and Lemma 5.3 to get

$$\begin{aligned}\|f(sw_1) - f(re_0)\|_n &= |x_1 - x_2| + |y_1 - y_2| \geqslant (r+s) - 2(M+3\kappa_1)\\ &= r\frac{2N-1}{N-1} - 2(M+3\kappa_1) > r\frac{N}{N-1} = Nr\tan\alpha_0 = N\|re_0 - sw_1\|,\end{aligned}$$

a contradiction since f is N-Lipschitz. Here we used that by (5.2) of Lemma 5.4$s = r(1 + \tan\frac{\pi}{n}\tan\alpha_0) = rN/(N-1)$ as $\tan\frac{\pi}{n} = \tan\frac{\pi}{4} = 1$ and, by Lemma 5.3, $\tan\alpha_0 = 1/(N-1)$, followed by $r > R^* \geqslant 6\kappa_1 + 2M$.

Let now $n > 4$. The remaining proof is organised as follows. We first check that at least one of the two points $P_i = f(rw_i)$, $i = 0, 1$ belongs to $\overline{B}_{r+\delta}(0)$. We then consider two cases, $P_i \in \overline{B}_{r+\delta}(0)$ for $i = 0$ or $i = 1$, and get a contradiction in each of the cases. This completes the proof of this theorem.

Assume first $\|P_i\|_n > r + \delta$ for both $i = 0, 1$. As $\mathrm{dist}_n(P_i, \mathcal{D}_i) \leqslant 2\kappa_1$, by (5.26) we have $2\mathrm{dist}_n(P_i, \mathcal{D}_i) \leqslant \|P_i\|_n \sin(2\pi/n)$, so we may apply Lemma 5.8 to get that $W_i = \|P_i\|_n e_i \in \mathcal{D}_i$ are such that $\|W_i\|_n = \|P_i\|_n > r + \delta$ and $\|P_i - W_i\|_n \leqslant 2\kappa_1 \sec^2(\pi/n) = \kappa_2$, $i = 0, 1$. By Corollary 5.6 we have $\|W_0 - W_1\|_n \geqslant \frac{1}{n}(r+\delta)\mathscr{L}_n$, hence also using (5.26) and $\mathscr{L}_n \leqslant 2n$ (easily seen from Lemma 3.6), we get

$$\begin{aligned} r\frac{\mathscr{L}_n}{n} &= Nr\frac{\mathscr{L}_n}{nN} = Nr\|w_0 - w_1\|_n \geqslant \|f(rw_0) - f(rw_1)\|_n = \|P_0 - P_1\|_n\\ &\geqslant \|W_0 - W_1\|_n - 2\kappa_2 \geqslant (r+\delta)\frac{\mathscr{L}_n}{n} - 2\kappa_2 > r\frac{\mathscr{L}_n}{n},\end{aligned}$$

a contradiction.

Case 1. $m \geqslant 2$ and $P_0 = f(re_0) \in \overline{B}_{r+\delta}(0)$.

In this case $\mathrm{dist}_n(f(re_0), \mathcal{D}_0^{r-M}) \leqslant \kappa < \kappa_1$ and $r + \delta \geqslant \|f(re_0)\|_n \geqslant r - M \geqslant r - \delta$, as $\delta \geqslant M$. As $2\kappa_1 \leqslant (r - M)\sin(2\pi/n)$ from (5.26), we may apply Lemma 5.8 to get that the point $Q = \|f(re_0)\|_n e_0 \in \mathcal{D}_0$ is such that $\|Q\|_n = \|f(re_0)\|_n \in (r-\delta, r+\delta)$, hence $\|Q - re_0\|_n < \delta$, and $\|f(re_0) - Q\|_n \leqslant \kappa_1 \sec^2(\pi/n)$. This implies, using (5.26)

$$\|f(re_0) - re_0\|_n \leqslant \delta + \kappa_1 \sec^2(\pi/n) \leqslant 2\delta. \tag{5.27}$$

Now (going back to the domain of f), let sw_1 be the intersection point between $\mathbb{R}_+ w_1$ and the vertical line through re_0, as in Lemma 5.4. As $s > r > R_0$, we get from the choice of R_0 that Lemma 5.10 holds for f and $\varepsilon = 1$, so that $\mathrm{dist}_n(f(sw_1), \mathcal{D}_1^{s-M}) \leqslant 2\kappa_1$, and from Theorem 2.8 (1), $s' = \|f(sw_1)\|_n \geqslant s - M$. Since $s - M \geqslant r - M$, we get from (5.26) that $(s - M)\sin(2\pi/n) \geqslant 4\kappa_1$, so we may apply Lemma 5.8 to get

$$\|f(sw_1) - s'e_1\|_n \leqslant 2\kappa_1 \sec^2(\pi/n). \tag{5.28}$$

Apply (5.27) and (5.28), following by $2\kappa_1 \sec^2(\pi/n) \leqslant \delta$ from (5.26), to get

$$\begin{aligned}\|f(re_0) - f(sw_1)\|_n &\geqslant \|s'e_1 - re_0\|_n - \|re_0 - f(re_0)\|_n - \|s'e_1 - f(sw_1)\|_n \qquad (5.29)\\ &\geqslant \|s'e_1 - re_0\|_n - 2\delta - 2\kappa_1\sec^2(\pi/n) \geqslant \|s'e_1 - re_0\|_n - 3\delta.\end{aligned}$$

We now plan to use Lemma 5.5 (2) for $u = r$, $a = s - M$ and $q = s'e_1$. Let us first verify that its conditions are satisfied. We have already mentioned above that $s' \geqslant s - M$, so it remains to check that

$$\cos(2\pi/n) \leqslant \frac{s-M}{r} \leqslant \sec(2\pi/n). \tag{5.30}$$

Let us now recall the results of Lemma 5.4. By (5.2), $s/r = 1 + \tan\frac{\pi}{n}\tan\alpha_0$. As $M/r \leqslant \tan\frac{\pi}{n}\tan\alpha_0$ by the choice of R^*, we only need to verify that $\frac{s-M}{r} \leqslant \sec(2\pi/n)$. As $N \geqslant 2$, we get from Lemma 5.3 that $0 < \tan\alpha_0 \leqslant \tan\frac{\pi}{n}$, hence

$$\sec\tfrac{2\pi}{n} \geqslant \frac{1}{\cos^2\frac{\pi}{n}} = 1 + \tan^2\tfrac{\pi}{n} \geqslant 1 + \tan\alpha_0\tan\tfrac{\pi}{n} = \frac{s}{r}.$$

This allows us to use Lemma 5.5 (2) to conclude that $\|re_0 - s'e_1\|_n \geqslant \|re_0 - (s-M)e_1\|_n$. This last inequality, used together with (5.29), gives us:

$$\begin{aligned}\|f(re_0) - f(sw_1)\|_n &\geqslant \|re_0 - (s-M)e_1\|_n - 3\delta \geqslant \|re_0 - se_1\|_n - M - 3\delta\\ &\geqslant \|re_0 - se_1\|_n - 4\delta.\end{aligned}$$

We now apply (5.4) of Lemma 5.4 to the above inequality, following by the estimate for R^* from (5.26), then Lemma 5.3 and (5.3) of Lemma 5.4, and conclude that

$$\begin{aligned}\|f(re_0) - f(sw_1)\|_n &\geqslant r\tan\tfrac{\pi}{n}(2 + \tan\tfrac{\pi}{n}\tan\alpha_0) - 4\delta > 2r\tan\tfrac{\pi}{n}\\ &\geqslant Nr\tan\alpha_0 = N\|re_0 - sw_1\|_n,\end{aligned}$$

a contradiction as f is N-Lipschitz.

Case 2. $m \geqslant 2$, $P_0 = f(re_0) \notin \overline{B}_{r+\delta}(0)$ and $P_1 = f(rw_1) \in \overline{B}_{r+\delta}(0)$.

First we note that if we let $u = r + \delta$, $a = r - M$ and a_0, a_1 be defined as in (5.8), then

$$u, a, a_0, a_1 \text{ satisfy the assumptions of Lemma 5.5.} \tag{5.31}$$

Indeed, $a \leqslant r \leqslant u \leqslant a_1$ is trivial and $a_0 \leqslant a$ because $r \geqslant R^*$, see (5.26). By Lemma 5.5 (1), we conclude that the point ae_1 is on the $(\frac{n}{4}+1)$th side of the polygon $B_d(ue_0)$, where $d = \|ue_0 - ae_1\|_n$.

Consider the vertical line through ue_0 and let Q denote the $(\frac{n}{4})$th vertex of $\partial B_d(ue_0)$ that belongs to this vertical line. Let further H be the intersection between the horizontal line through ae_1 and the segment $[ue_0, Q]$. Finally, let $V = ve_0$ be the intersection between the x-axis and the vertical line through ae_1.

Using that $v = a\cos\frac{2\pi}{n}$, we conclude

$$\begin{aligned}|Q - H| &= (u - v)\tan\tfrac{\pi}{n} = \big(u - a\cos\tfrac{2\pi}{n}\big)\tan\tfrac{\pi}{n},\\ |H - ue_0| &= |ae_1 - ve_0| = a\sin\tfrac{2\pi}{n} = a\tan\tfrac{\pi}{n}(\cos\tfrac{2\pi}{n} + 1),\end{aligned}$$

so that

$$d = |Q - H| + |H - ue_0| = (u + a)\tan\frac{\pi}{n}.$$

Hence, as $u + a = 2r + \delta - M$, using Lemma 3.6 (1) and (5.26), we conclude

$$\begin{aligned}\|(r+\delta)e_0 - (r-M)e_1\|_n &= \|ue_0 - ae_1\|_n = d \qquad (5.32)\\ &= (2r + \delta - M)\tan\tfrac{\pi}{n} = r\|e_0 - e_1\|_n + (\delta - M)\tan\tfrac{\pi}{n} \geqslant r\|e_0 - e_1\|_n + 2\kappa_2.\end{aligned}$$

Now, from (5.25) we know that $f(re_0)$ is at most κ-far from the ray $\mathcal{D}_0$ and, in this case, we have $\|f(re_0)\|_n > r+\delta$, so applying again Lemma 5.8, as $(r+\delta)\sin(2\pi/n) \geqslant 2\kappa$, we get that $Q_0 = \|f(re_0)\|_n e_0$ satisfies

$$\|f(re_0) - Q_0\|_n \leqslant \kappa \sec^2(\pi/n) < \kappa_1 \sec^2(\pi/n) < \kappa_2. \tag{5.33}$$

We also use Lemma 5.10 to get $\operatorname{dist}_n(f(rw_1), \mathcal{D}_1) \leqslant 2\kappa_1$, hence as $r - M \leqslant \|f(rw_1)\|_n \leqslant r+\delta$ and $(r-M)\sin(2\pi/n) \geqslant 4\kappa_1$, we use Lemma 5.8 to get that for $Q_1 = \|f(rw_1)\|_n e_1$ we have

$$\|f(rw_1) - Q_1\|_n \leqslant 2\kappa_1 \sec^2(\pi/n) = \kappa_2. \tag{5.34}$$

Recall that in (5.31) we verified the general conditions of Lemma 5.5 for $u = r+\delta$ and $a = r - M$. Let now $p = Q_0$ and $q = Q_1$. It is readily seen that $\|p\|_n = \|f(re_0)\|_n \geqslant u$ and $a_1 \geqslant u \geqslant \|q\|_n = \|f(rw_1)\|_n \geqslant a$, so apply Lemma 5.5 (4) to get

$$\|Q_0 - Q_1\|_n \geqslant \|(r+\delta)e_0 - (r-M)e_1\|_n,$$

hence, using in addition (5.32),

$$\|Q_0 - Q_1\|_n \geqslant r\|e_0 - e_1\|_n + 2\kappa_2.$$

Combining the above inequality with (5.33) and (5.34), we conclude

$$\|f(re_0) - f(rw_1)\|_n > \|Q_0 - Q_1\|_n - 2\kappa_2 \geqslant r\|e_0 - e_1\|_n = N\|re_0 - rw_1\|_n.$$

This is not possible since f is N-Lipschitz. □

We derive now the more general result.

THEOREM 5.13. *Let $n = 4m$ for some $m \geqslant 1$ and $N \geqslant 2$, and let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ whose unit ball is a linear image of a regular n-gon (for example, any polygonal n-norm). Then any N-fold Lipschitz quotient mapping $f\colon (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ has ratio of constants c/L strictly less than $1/N$.*

In particular, this includes the cases of the ℓ_1 and ℓ_∞ norms.

Proof. This follows from Theorem 5.12 and Remark 3.3. □

We have shown then that for every norm whose unit ball is a linear image of a regular polygon with $4m$ sides, every N-fold Lipschitz quotient mapping with $N \geqslant 2$ satisfies $c/L < 1/N$. A natural question is what is the upper bound for this ratio.

PROPOSITION 5.14. *If $n = 4m$ for some $m \geqslant 1$, then the Lipschitz and co-Lipschitz constants, L_N and c_N, of the N-fold winding mapping $f_n^N : (\mathbb{R}^2, \|\cdot\|_n) \to (\mathbb{R}^2, \|\cdot\|_n)$ satisfy $L_N \geqslant N + (N-1)\tan^2(\frac{\pi}{n})$ and $c_N = 1$. Hence,*

$$\frac{c_N}{L_N} \leqslant \frac{1}{N + (N-1)\tan^2(\frac{\pi}{n})}$$

and

$$N \leqslant \frac{L_N}{c_N}\cos^2\frac{\pi}{n} + \sin^2\frac{\pi}{n}.$$

Proof. First notice that $c_N = 1$ by Theorem 3.19. To prove that $L_N \geqslant N + (N-1)\tan^2(\frac{\pi}{n})$, consider the pair of points re_0 and sw_1, as in Lemma 5.4. By Lemma 5.4, we have that $\|re_0 -$

$we_1\|_n = r\tan\alpha_0$, where α_0 is defined in Notation 5.2. As $f_n^N(re_0) = re_0$ and $f_n^N(sw_1) = se_1$, we apply again Lemma 5.4 to get

$$\|f_n^N(re_0) - f_n^N(sw_1)\|_n = \|re_0 - se_1\|_n = r\tan\tfrac{\pi}{n}(2+\tan\tfrac{\pi}{n}\tan\alpha_0).$$

Hence, using (5.1) from Lemma 5.3

$$\begin{aligned}\mathrm{Lip}(f_n^N) &\geqslant \frac{\tan\frac{\pi}{n}(2+\tan\frac{\pi}{n}\tan\alpha_0)}{\tan\alpha_0} = 2\frac{\tan\frac{\pi}{n}}{\tan\alpha_0} + \tan^2\tfrac{\pi}{n} = N(1+(1-\tfrac{2}{N})\tan^2\tfrac{\pi}{n}) + \tan^2\tfrac{\pi}{n}\\ &= N + (N-1)\tan^2\tfrac{\pi}{n}.\end{aligned}$$

The upper estimate for N is then obtained by a simple rearrangement. $\square$

The following conjecture is a generalisation of Theorem 5.13 and Proposition 5.14:

CONJECTURE 5.15. Let $n = 4m$ for $m \geqslant 1$ defines a polygonal n-norm $\|\cdot\|$ on $\mathbb{R}^2$. If $f\colon (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ is an L-Lipschitz and c-co-Lipschitz N-fold mapping, then

$$N \leqslant \frac{L}{c}\cos^2\frac{\pi}{n} + \sin^2\frac{\pi}{n}.$$

The equality is achieved for an appropriately rotated mapping $f = f_n^N$, where f_n^N is the N-fold winding mapping defined by Definition 3.12.

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