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Perarnau, Guillem; Serra, Oriol

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On the tree-depth of random graphs ☆

G. Perarnau *, O. Serra

Departament de Matemàtica Aplicada IV, Universitat Politècnica de Catalunya, Jordi Girona 1-3, 08034 Barcelona, Spain

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ABSTRACT

Tree-depth is a parameter introduced under several names as a measure of sparsity of a graph. We compute asymptotic values of the tree-depth of a random graph on n vertices where each edge appears independently with probability p. For dense graphs, $np \to +\infty$, the tree-depth of a random graph G is *aas* $\text{td}(G) = n - O(\sqrt{n/p})$. Random graphs with $p = c/n$, have *aas*linear tree-depth when $c > 1$, the tree-depth is $\Theta(\log n)$ when $c = 1$ and $\Theta(\log \log n)$ for $c < 1$. The result for $c > 1$ is derived from the computation of tree-width and provides a more direct proof of a conjecture by Gao on the linearity of tree-width recently proved by Lee, Lee and Oum (2012) [15]. We also show that, for $c = 1$, every width parameter is *aas*constant, and that random regular graphs have linear tree-depth.

1. Introduction

An elimination tree of a connected graph G is a rooted tree on the set of vertices such that there are no edges in G between vertices in different branches of the tree. The natural elimination scheme provided by this tree is used in many graph algorithmic problems where two non adjacent subsets of vertices can be managed independently. One good example is the Cholesky decomposition of symmetric matrices (see [24,7,16,22]). Given an elimination tree, a distributed algorithm can be designed which takes care of disjoint subsets of vertices in different parallel processors. Starting from the furthest level from the root, it proceeds by exposing at each step the vertices at a given depth. Then the algorithm runs using the information coming from its children subtrees, which have been computed in previous steps. Observe that the vertices treated in different processors are independent and thus, there is no need to share information among them. Depending on the graph, this distributed algorithm can be more efficient than the sequential one. In fact, its complexity is given by the height of the elimination tree used by the algorithm. This motivates the study of the minimum height of an elimination tree of G. This natural parameter has been introduced under numerous names in the literature: rank function [21], vertex ranking number (or ordered coloring) [6], weak coloring number [13], but its study was systematically undertaken by Nešetřil and Ossona de Mendez under the name of *tree-depth*.

The tree-depth $\text{td}(G)$ of a graph G is a measure introduced by Nešetřil and Ossona de Mendez [19] in the context of bounded expansion classes. The notion of tree-depth is closely connected to tree-width. The tree-width of a graph can be seen as a measure of closeness to a tree, while the tree-depth takes also into account the diameter of the tree (see Section 2 for a precise definition and [20] for an extensive account of the meaning and applications of this parameter).

Bounded expansion classes are defined in terms of shallow minors and its connection to tree-depth is highlighted by the following result. The k-th chromatic number of a graph $\chi_k(G)$ is defined as the minimum number of colors needed to color a graph in such a way that each subgraph H induced by any set of i classes of colors, $i \le k$, satisfy $\text{td}(H) \le i$. When we look

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* Corresponding author. Fax: +34 647642897.
E-mail addresses: guillem.perarnau@ma4.upc.edu (G. Perarnau), oserra@ma4.upc.edu (O. Serra).

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at a bounded number of chromatic classes we see a simple structure. Observe that $\chi_1(G)$ is the ordinary chromatic number and $\chi_2(G)$ is the so-called star chromatic number [1]. The main theorem in this context states that a class of graphs $\mathcal{C}$ has bounded expansion if and only if $\limsup_{G\in\mathcal{C}} \chi_k(G) < +\infty$ for any fixed $k > 0$. This gives another motivation to study the tree-depth.

Randomly generated graphs have been widely used as benchmarks for testing distributed algorithms and therefore it is useful to characterize the elimination tree of such graphs. The main goal of this paper is to give asymptotically tight values for the minimum height of an elimination tree of a random graph.

A *random graph* $G \sim \mathcal{G}(n, p)$ has n vertices and every pair of vertices is chosen independently to be an edge with probability p.

For any graph property $\mathcal{P}$, we say that $\mathcal{P}$ holds *asymptotically almost sure* (*aas*) in $G \sim \mathcal{G}(n, p)$, if

$$\lim_{n\to+\infty} \Pr(G \text{ satisfies } \mathcal{P}) = 1.$$

Throughout the paper, all the results and statements concerning random graphs must be understood in the asymptotically almost sure sense. We will occasionally make use of the $\mathcal{G}(n, m)$ model of random graphs, where a labeled graph with n vertices and m edges is chosen with the uniform distribution. As it is well-known, the two models are closely connected and most of the statements can be transferred from one model to the other one.

Our first result gives the value of tree-depth for dense random graphs.

Theorem 1.1. *Let $G \sim \mathcal{G}(n, p)$ be a random graph with $np \to +\infty$, then G satisfies aas*

$$\mathrm{td}(G) = n - O\left(\sqrt{\frac{n}{p}}\right).$$

Theorem 1.1 implies that random graphs with a superlinear number of edges have a tree structure similar to the one of the complete graph. Actually our proof of Theorem 1.1 provides the same result for tree-width. To our knowledge, the tree-width of a dense random graph has not been studied before.

Nešetřil and Ossona de Mendez showed that a sparse random graph $G(n, c/n)$ belongs *aas* to a bounded expansion class (see [20, Theorem 13.4]). Our main result is the computation of the tree-depth of sparse random graphs.

Theorem 1.2. *Let $G \sim \mathcal{G}(n, p)$ be a random graph with $p = \frac{c}{n}$, with $c > 0$,*

(1) *if $c < 1$, then aas* $\mathrm{td}(G) = \Theta(\log\log n)$
(2) *if $c = 1$, then aas* $\mathrm{td}(G) = \Theta(\log n)$
(3) *if $c > 1$, then aas* $\mathrm{td}(G) = \Theta(n)$.

Henceforth we denote the logarithm in base two by log while we will use ln for the natural logarithm.

The last part of this theorem is closely related to a conjecture of Gao announced in [11] on the linear behavior of tree-width for random graphs with $c = 2$, inspired by the results of Kloks in [14]. This conjecture has been recently proved by Lee et al. [15]. They show that the tree-width is linear for any $c > 1$ as a corollary of their result on the rank-width of random graphs. Here we give a proof of Theorem 1.2.3 which also provides a proof of Gao's conjecture, giving an explicit lower bound on the tree-width. Our proof uses, as the one in [15], the same deep result of Benjamini et al. [2] on the existence of a linear order expander in a sparse random graph for $c > 1$.

The paper is organized as follows. In Section 2, we define the notion of tree-depth and give some useful results concerning this parameter. Section 3 contains the proof of Theorem 1.1, which uses the relation connecting tree-width with balanced partitions. Finally Theorem 1.2 will be proved in Section 4. For $c < 1$ the result follows from the fact that the random graph is a collection of trees and unicyclic graphs of logarithmic order, which gives the upper bound, and that there is one of these components with large diameter with respect to its order, providing the lower bound. For $c = 1$ we show that the giant component in the random graph has just a constant number of additional edges exceeding the order of a tree, which gives the upper bound, and rely on a result of Nachmias and Peres [18] on the concentration of the diameter of the giant component to obtain the lower bound. Finally, as we have already mentioned, for $c > 1$ the result follows readily from the existence of an expander of linear order in a sparse random graph for $c > 1$, a fact proved in Benjamini et al. [2].

2. Tree-depth

Let T be a rooted tree. The *height* of T is defined as the number of vertices of the longest rooted path. The *closure* of T is the graph that has the same set of vertices and an edge between every pair of vertices such that one is an ancestor of the other in the rooted tree.

The tree T is an *elimination tree* of a connected graph G if G is a subgraph of its closure. The *tree-depth* of a connected graph G is defined to be the minimum height of an elimination tree of G. Some examples are shown in Fig. 1.

This definition can be extended to non-connected graphs. Suppose that G has connected components $H_1, \dots, H_s$. Then,

$$\mathrm{td}(G) = \max_{1\le i\le s} \mathrm{td}(H_i). \tag{1}$$

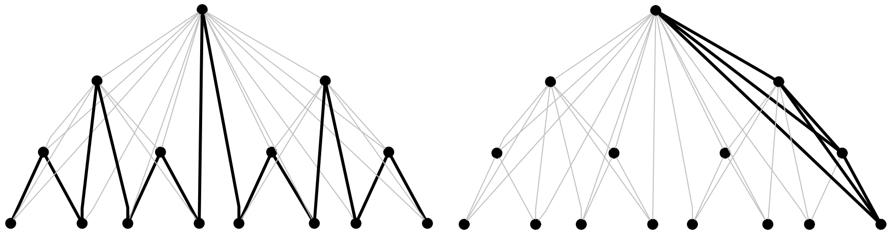

Fig. 1. The path with 15 vertices and the complete graph K_4 have tree-depth 4.

It is clear from the definition that the tree-depth of a graph G with n vertices satisfies $\mathrm{td}(G) \leq n$, and that the equality is satisfied only for the complete graph K_n. Note that the following inequality holds,

$$\mathrm{td}(G \setminus v) \geq \mathrm{td}(G) - 1 \tag{2}$$

for every vertex $v \in V(G)$.

For any tree T, it can be checked by induction that,

$$\mathrm{td}(T) \leq \lfloor \log n \rfloor + 1. \tag{3}$$

As shown by Dvǒrák et al. [8], equality holds for a relatively wide class of trees. In particular, it holds if T is a path P_n with n vertices,

$$\mathrm{td}(P_n) = \lfloor \log n \rfloor + 1. \tag{4}$$

As a simple consequence of the previous equation and the fact that the tree-depth is monotonically increasing under the subgraph ordering, we have that if G contains a path of length t, then $\mathrm{td}(G) \geq \log t$. In particular, if G has diameter d,

$$\mathrm{td}(G) \geq \log d. \tag{5}$$

The following inequalities relate the tree-width and tree-depth of a graph (see e.g. [3])

$$\mathrm{tw}(G) \leq \mathrm{td}(G) \leq \mathrm{tw}(G)(\log n + 1). \tag{6}$$

Note that there are classes of graphs that have bounded tree-width but unbounded tree-depth, for example trees. On the other hand, if a class of graphs has bounded tree-depth, then it also has bounded tree-width.

Some of our results use the relation (6) between tree-depth and tree-width. Bounds on the tree-width can be obtained through its connection with balanced separators. Kloks [14] defines a partition (A, S, B) of a graph G with n vertices to be a *balanced k-partition* if $|S| = k + 1$, S is a separator (there are no edges between A and B) and

$$\frac{1}{3}(n-k-1) \leq |A|, \qquad |B| \leq \frac{2}{3}(n-k-1). \tag{7}$$

He states the following result connecting balanced partitions and tree-width.

Lemma 2.1 (*Kloks [14]*). *Let G be a graph with n vertices and $\mathrm{tw}(G) \leq k$, $k \leq n-4$. Then G has a balanced k-partition.*

We will often use the non existence of a balanced partition to provide lower bounds on the tree-width of G.

3. Tree-depth of dense random graphs

This section is devoted to prove Theorem 1.1.

Proof of Theorem 1.1. By definition it is clear that $\mathrm{td}(G) \leq n$, thus we only need to prove the lower bound. It will be derived from the analogous one for tree-width through inequality (6). For this we will show that a random graph $G \sim \mathcal{G}(n, p)$ with $np \to +\infty$, *aas* contains no balanced separator of order less than $n - 3\sqrt{\frac{\ln 3}{2}}\sqrt{\frac{n}{p}}$. The result will follow from Lemma 2.1.

Assume $p = c(n)/n$ and set any function $f(c) > 3\sqrt{\frac{\ln 3}{2c}}$. Suppose that there exists a balanced k-partition (A, S, B) of G with $k \leq (1 - f(c))n$. By the definition of balanced k-partitions, we have $|S| = k + 1$, and $|A|, |B| \geq \frac{f(c)n}{3}$. As $|A| + |B| = n - k \geq f(c)n$, we have

$$|A| \cdot |B| \geq \frac{2f(c)^2}{9}n^2.$$

Let $X_{(A,S,B)}$ denote the event that (A, S, B) is a balanced k-partition of G with $k \leq (1 - f(c))n$. We have

$$\Pr\left(X_{(A,S,B)}\right) = (1-p)^{|A|\,|B|} \leq (1-p)^{\frac{2f(c)^2}{9}n^2}.$$

Let $\mathcal{B}$ denote the set of balanced k-partitions with $k \leq (1-f(c))n$. By using the trivial bound $|\mathcal{B}| \leq 3^n$, the number of labeled partitions of $[n]$ into three sets, we get:

$$\begin{aligned} \Pr\left(\cup_{(A,S,B)\in\mathcal{B}} X_{(A,S,B)}\right) &\leq \sum_{(A,S,B)\in\mathcal{B}} \Pr\left(X_{(A,S,B)}\right) \\ &\leq 3^n (1-p)^{\frac{2f(c)^2}{9}n^2} \\ &\leq \exp\left\{(\ln 3)n - \frac{2f(c)^2}{9}n^2 p\right\}, \end{aligned} \tag{8}$$

where we have used that $1-x \leq e^{-x}$. By using $p = c(n)/n$ and $f(c) > 3\sqrt{\frac{\ln 3}{2c}}$, the expression (8) tends to 0 for $n \to +\infty$.

Since there is no set of size $(1-f(c))n$ separating G, from inequality (6) and Lemma 2.1 we have $\text{td}(G) \geq \text{tw}(G) > (1-f(c))n$. The above inequality is valid for all $f(c) > 3\sqrt{\frac{\ln 3}{2c}}$ and thus, we have that *aas* $\text{td}(G) \geq n - O(\sqrt{n/p})$. □

Observe that the argument in the above proof can be used to deduce that the tree-width of sparse random graphs with $p(n) = c/n$ is linear in n for sufficiently large c and obtain a lower bound for the constant. From (8) we need

$$f(c) > \sqrt{\frac{9\ln 3}{2c}}$$

and since the tree-width is $(1-f(c))n$, we also need $f(c) < 1$. These two conditions imply that for any $c > \frac{9\ln 3}{2} \approx 4.94$, one has that

$$\text{td}(\mathcal{G}(n, c/n)) \geq \text{tw}(\mathcal{G}(n, c/n)) \geq (1-f(c))n = \Omega(n). \tag{9}$$

In the next section we will prove that *aas* $\text{td}(\mathcal{G}(n, c/n))$ is linear in n for any $c > 1$.

4. Tree-depth of sparse random graphs

In this section Theorem 1.2 will be proved. We prove each one of the three cases separately in different subsections.

4.1. Proof of Theorem 1.2.1

Let $G \sim \mathcal{G}(n, p = c/n)$ with $0 < c < 1$. Our goal is to show that $\text{td}(G) = \Theta(\log\log n)$.

We will first prove the upper bound. A *unicyclic* graph (*unicycle*) is a connected graph that has the same number of vertices and edges, that is, the graph contains exactly one cycle.

Lemma 4.1. *If each connected component of G is either a tree or a unicycle, then $\text{td}(G) \leq \log n_c + 2$, where n_c is the cardinality of the largest connected component of G.*

Proof. As it has been remarked, $\text{td}(G)$ equals the tree-depth of the largest connected component of the graph. After deleting at most one vertex, a connected component becomes a tree. The result follows by using (2) and (3). □

One of the central results of Erdős and Rényi [9] states that, if $0 < c < 1$, then G is composed of trees and unicycles. Moreover the order of the largest component in the random graph is $\Theta(\log n)$ (see e.g. [4, Corollary 5.11, Corollary 5.8]). Using Lemma 4.1 with $n_c = \Theta(\log n)$, we have

$$\text{td}(G) = O(\log\log n).$$

We next show the lower bound. Recall that the diameter of the graph provides a lower bound on the tree-depth of G by inequality (5). Hence our proof for this case will be completed if we show that the random graph G *aas* contains a tree T of order $\Theta(\log n)$ and sufficiently large diameter.

The diameter of $\mathcal{G}(n, p)$ in the subcritical phase has been studied by Łuczak in [17] where the author establishes it when $np \to 0$ and also provides an upper bound when $p = c/n$, $0 < c < 1$. Here we give a simple proof for a lower bound in the latter case, which is not tight but suffices to show the asymptotic behavior of the tree-depth in this range.

Observe that every labeled tree on k vertices has the same probability to appear in G as a connected component, since each tree with fixed order contains the same number of edges. Let H_k be the random variable that indicates the height of a random labeled rooted tree with k vertices. A classical result of Rényi and Szekeres [23] states that H_k satisfies

$$\mathbb{E}(H_k) \sim \sqrt{2\pi k} \quad \text{and} \quad \mathbf{Var}(H_k) \sim \frac{\pi(\pi-3)}{3}k.$$

Notice that the diameter D_k of a random labeled unrooted tree with k vertices satisfies $H_k \leq D_k \leq 2H_k$ and thus, $\mathbb{E}(D_k) = \Theta(\sqrt{k})$. Unfortunately, the variance of D_k is also large and a second moment argument does not provide concentration of

D_k around its expected value. The diameter of an individual tree can be even constant as $k \to +\infty$. However, the number of trees with k vertices in G is large enough to ensure that *aas* there is at least one with sufficiently large diameter.

To count the number of trees of each size it is more convenient to use the random model $\mathcal{G}(n, m(n))$ where a graph is chosen uniformly at random from all the labeled graphs with $m(n)$ edges. For this model, Erdős and Rényi [9] proved that if $m(n) = \Omega(n)$, the random variable X_k counting the number of trees of order k in $\mathcal{G}(n, m(n))$ follows a normal distribution with expected value and variance M_k, where

$$M_k = n\frac{k^{k-2}}{k!}\left(\frac{2m}{n}\right)^{k-1} e^{-\frac{2km}{n}}.$$

Moving back to the random graph model $\mathcal{G}(n, p)$ with $p = c/n$, and noting that the expected number of edges is $\binom{n}{2} p \sim \frac{cn}{2}$, one can translate the former result to an analogous one,

$$M_k = n\frac{k^{k-2}}{k!}c^{k-1}e^{-kc}. \tag{10}$$

We are interested in $X = X_{\log n}$, the number of trees of order $\log n$, for which,

$$M = M_{\log n} = \frac{n^{\log\log n - \alpha}}{c(\log^2 n)(\log n)!}$$

where $\alpha = c - 1 - \log c$.

Observe that $M \to +\infty$ when $n \to +\infty$. By applying Chebyshev's inequality to the variable X, it follows that *aas* the number of tree components of order $\log n$ is $X = (1 - o(1))M \to +\infty \quad (n \to +\infty)$, that is, *aas* there are many trees of order $\log n$.

Denote by $D = D_{\log n}$ and define $\overline{D}$ to be the mean value of the diameters over all the components of order $\log n$. Clearly $\mathbb{E}(\overline{D}) = \mathbb{E}(D) = \Theta(\sqrt{\log n})$. Observe that the random variables measuring the diameter of each component, conditioned to the fact that the component is a tree on $\log n$ vertices, are independent. Thus, the variance of $\overline{D}$ is smaller than the variance of the diameter of each individual tree. Indeed, $\mathbf{Var}(\overline{D}) = o(\log n) = o(\mathbb{E}(X)^2)$. The Chebyshev inequality on $\overline{D}$ directly implies that $\overline{D} = \Theta(\sqrt{\log n})$ *aas*. Hence there is a tree T in G with diameter $d = \Omega(\sqrt{\log n})$.

By using (1) and (5), the desired lower bound follows:

$$\text{td}(G) \geq \text{td}(T) = \Omega(\log d) = \Omega\left(\log\left(\sqrt{\log n}\right)\right) = \Omega(\log\log n).$$

This completes the proof of this case. □

4.2. Proof of Theorem 1.2.2

Now we look at the critical point where $c = 1$.

It was shown by Erdős and Rényi also in [9] that $p = 1/n$ is the threshold probability for the existence of a polynomial size component, the so called *giant component* of the random graph. For this particular probability, *aas* the largest component has order $O(n^{2/3})$.

We first prove the upper bound. We show that each component is similar to a tree. For convenience we use the definition of a (k, ℓ)-component given in [12]. A (k, ℓ)*-component*, $\ell \geq -1$, is a connected component with k vertices and $k + \ell$ edges. Thus, $(k, -1)$-components are trees and $(k, 0)$-components correspond to unicyclic graphs. A *complex component* is a (k, ℓ)-component with $\ell > 0$.

Proposition 4.2. *Let $G \sim \mathcal{G}(n, p = 1/n)$. Then, for any $\varepsilon > 0$, there exists $\ell_0 = \ell_0(\varepsilon)$ such that the probability that there exists a complex component with $\ell > \ell_0$ is at most ε.*

Proof. Let $Y(k, \ell)$ denote the number of (k, ℓ)-components of $G(n, 1/n)$. The expected value of $Y(k, \ell)$ is

$$\mathbb{E}(Y(k, \ell)) = \binom{n}{k} C(k, \ell)\left(\frac{1}{n}\right)^{k+\ell}\left(1 - \frac{1}{n}\right)^{\binom{k}{2}-(k+\ell)+k(n-k)} \leq \frac{C(k, \ell)}{n^\ell k!}e^{-k(1+o(1))},$$

where $C(k, \ell)$ is the number of connected labeled graphs with k vertices and $k + \ell$ edges and $k/n \to 0$. Bóllobas [4, Corollary 5.21] obtained the following sharp bound for $C(k, \ell)$,

$$C(k, \ell) \leq O\left(\ell^{-\ell/2}k^{k+(3\ell-1)/2}\right).$$

By combining the previous expressions and using Stirling's formula, for $k = O(n^{2/3})$ one gets

$$\mathbb{E}(Y(k, \ell)) \leq \frac{\ell^{-\ell/2}}{k}\left(\frac{k^{3/2}}{n}\right)^{\ell} \leq O\left(\frac{\ell^{-\ell/2}}{n^{2/3}}\right),$$

if $\ell \geq 2$.

Let $Y(\ell) = \sum_k Y(k,\ell)$ denote the total number of connected components with excess ℓ. Since connected components have order at most $O(n^{2/3})$, we have

$$\mathbb{E}(Y(\ell)) = \sum_{k=0}^{O(n^{2/3})} \mathbb{E}(Y(k,\ell)) \le O\left(\ell^{-\ell/2}\right).$$

Let $Y_{\ell_0} = \sum_{\ell \ge \ell_0} Y(\ell)$ denote the total number of complex components with excess at least ℓ_0. We have

$$\mathbb{E}(Y_{\ell_0}) = \sum_{\ell\ge\ell_0} \mathbb{E}(Y(\ell)) \le \sum_{\ell\ge\ell_0} O\left(\ell^{-\ell/2}\right) \le O\left({\ell_0}^{-\frac{\ell_0-1}{2}}\right).$$

By Markov's inequality

$$\Pr(Y_{\ell_0} \ge 1) \le \mathbb{E}(Y_{\ell_0}) < \varepsilon,$$

for some $\ell_0 = \ell_0(\varepsilon)$ large enough.

This completes the proof. □

Proposition 4.2 shows that *aas* there is no complex component with excess larger than, say, $\log\log n$. Thus, every component C has *aas* $k = O(n^{2/3})$ vertices and at most $k + \ell(C)$ edges, where $\ell(C) = O(\log\log n)$. Note that we can delete $\ell(C)$ vertices, turning C into a tree of order $(k - \ell(C)) = O(n^{2/3})$. Let ℓ_m be the maximum excess of edges among all the components. The tree-depth satisfies

$$\mathrm{td}(G) \le \ell_m + O\left(\log\left(n^{2/3}\right)\right) = O(\log\log n) + O(\log n) = O(\log n),$$

giving the upper bound.

The lower bound is obtained by an argument on the diameter of a giant component. Each giant component C is a tree decorated with a constant number $\ell(C)$ of extra edges. Observe that adding an edge to a connected component can at most halve its diameter. Since the expected diameter of a random tree of size k is $\Theta(\sqrt{k})$, it follows from Proposition 4.2 that the expected diameter of a giant component is $\Theta(n^{1/3})$. Here, however, there are too few giant components, and we cannot use the same argument as for the previous case $c < 1$ to conclude that the graph contains a giant component with the expected value of the diameter. The concentration of this random variable follows from a more general statement due to Nachmias and Peres [18] on the diameter of the largest component of a random graph with $p = 1/n$.

Theorem 4.3 (*[18]*). *Let C be the largest component of a random graph in $\mathcal{G}(n, 1/n)$. Then, for any $\varepsilon > 0$, there exists $A = A(\varepsilon)$ such that*

$$\Pr(\mathrm{diam}(C) \notin (A^{-1}n^{1/3}, An^{1/3})) < \varepsilon.$$

From Theorem 4.3 we know that, for instance, $\Pr(\mathrm{diam}(C) < n^{1/3-\varepsilon}) = o(1)$. By using (5), we get that *aas*

$$\mathrm{td}(G) = \Omega(\log n^{1/3-\varepsilon}) = \Omega(\log n).$$

This concludes the proof of the case $c = 1$. □

The result provided by Proposition 4.2 can be also applied to give a good upper bound for the tree-width of $\mathcal{G}(n, 1/n)$,

Proposition 4.4. *Let $G \sim \mathcal{G}(n, p = 1/n)$. Then, for any $\varepsilon > 0$, there exists $\ell_0 = \ell_0(\varepsilon)$ such that*

$$\Pr(\mathrm{tw}(G) > \ell_0) < \varepsilon.$$

Proof. Since a tree has tree-width one and adding an edge to a graph increases its tree-width by at most one unit, a complex component with excess ℓ has tree-width at most $\ell + 1$.

By Proposition 4.2, the probability of having a component with excess larger than ℓ_0 is at most ε. Hence, $\Pr(\mathrm{tw}(G) \ge \ell_0 + 1) = \Pr(Y_{\ell_0} \ge 1) < \varepsilon$. □

Thus, the tree-width of $G \sim \mathcal{G}(n, 1/n)$ is bounded by any function on n that tends to infinity.

Proposition 4.4 and Theorem 1.2.2 show that, for $p = 1/n$, *aas* the second inequality in (6) is asymptotically tight.

We also note that Proposition 4.4 applies to any width parameter $\mathrm{w}(G)$ which can be upper bounded by a function of the tree-width. Examples of such parameters are branch-width, path-width, rank-width and clique-width.

4.3. Proof of Theorem 1.2.3

On the range of $p = c/n$ for any constant $c > 1$, we will prove that the tree-width is linear *aas*, implying the same result for the tree-depth. We recall that (9) already shows that the tree-width of $\mathcal{G}(n, c/n)$ is linear for any $c > 4.94$.

Kloks [14] studied the tree-width of a random graph G with $p = c/n$ and proved that it is linear for any $c > 2.36$. Gao [11] showed that the lower bound can be improved to $c > 2.162$, and conjectured that the threshold for having linear tree-width occurs for some $1 < c < 2$. As a side result of their study on the rank-width of random graphs, Lee et al. [15] settled the conjecture by showing that the tree-width is linear for any $c > 1$.

Here we will give a more direct proof of the linearity of tree-width for random graphs with $c > 1$ which provides an explicit lower bound for the linear constant. Our approach uses the same deep result of Benjamini, Kozma and Wormald, Theorem 4.5, as in [15].

Recall that the (edgewise) *Cheeger constant* of a graph G is defined as

$$\Phi(G) = \min_{\substack{X \subseteq V \\ 0<|X|\le n/2}} \frac{e(X, V \setminus X)}{d(X)} \tag{11}$$

where $e(X, V \setminus X)$ is the number of edges with exactly one end point in X and $d(X) = \sum_{x \in S} d(x)$. For any fixed $\alpha > 0$, G is an *α-edge-expander* if $\Phi(G) > \alpha$.

The recent proof of Benjamini et al. [2] for the value of the mixing time of a random walk on the giant component of a random graph with $p = c/n$, $c > 1$, relies on the existence of an α-edge-expander connected subgraph of linear size in the giant component. Theorem 4.5 is a direct consequence of [2, Theorem 4.2], which ensures the existence *aas* of a certain subgraph $R_N(G)$ of the giant component which is an α-edge-expander for some sufficiently small α (which has some additional properties). The fact that this subgraph has linear order arises in the proof of this theorem (see [2, p. 19]).

Theorem 4.5 ([2]). *Let $G \sim \mathcal{G}(n, p)$ with $p = c/n$, $c > 1$. Aas there exist $\alpha, \delta > 0$ and a subgraph $H \subseteq G$ such that H is an α-edge-expander and $|V(H)| \ge \delta n$.*

Let H be the subgraph obtained in Theorem 4.5. By Lemma 2.1, there exists a balanced partition (A, S, B) in H, where S is a vertex separator with $|S| = \mathrm{tw}(H) + 1$ and $|A| \ge (\delta n - |S|)/3$.

Since H is connected we have $\sum_{x \in A} d(x) \ge |A|$ and, since S is a separator, $e(A, V \setminus A) = e(A, S)$. Hence,

$$|A| \le \frac{e(A, S)}{\alpha},$$

implying that

$$e(A, S) \ge \alpha|A| \ge \alpha \frac{\delta n - |S|}{3} = \beta n,$$

where $\beta = \alpha(\delta - \gamma)/3$ and $\gamma = |S|/n$.

Since $e(A, S) \le d(S)$, it suffices to show that any set S of vertices with $d(S) \ge \beta n$, must have linear order. For this we show that there is $\gamma_0 > 0$, which depends only on α, δ and c, such that the probability that a set S with $|S| \le \gamma_0 n$ satisfies $d(S) \ge \beta n$ tends to zero when $n \to +\infty$. We use a union bound on the number of sets S of size γn, with $\gamma < \alpha\delta/(3c + \alpha)$, and the fact that $d(S)$ is a binomial random variable $\mathrm{Bin}(\gamma n^2, c/n)$. We have,

$$\begin{aligned}
\Pr(\exists S : |S| = \gamma n,\ d(S) \ge \beta n) &\le \binom{n}{\gamma n} \sum_{e=\beta n}^{\gamma n^2} \binom{\gamma n^2}{e} p^e (1-p)^{\gamma n^2 - e} \\
(\text{since } \beta n > c\gamma n = \mathbb{E}(d(S)), \text{ if } \gamma < \alpha\delta/(3c+\alpha)) &\le \binom{n}{\gamma n} \gamma n^2 \binom{\gamma n^2}{\beta n} p^{\beta n} (1-p)^{\gamma n^2 - \beta n} \\
&\le \binom{n}{\gamma n} \gamma n^2 \binom{\gamma n^2}{\beta n} p^{\beta n} \\
\left(\text{since } \binom{x}{y} \le \left(\frac{xe}{y}\right)^y\right) &\le \gamma n^2 \left(\left(\frac{e}{\gamma}\right)^{\gamma} \left(\frac{\gamma e c}{\beta}\right)^{\beta}\right)^n .
\end{aligned}$$

Since $\beta = \alpha(\delta - \gamma)/3$, the expression $\left(\frac{e}{\gamma}\right)^{\gamma} \left(\frac{\gamma e c}{\beta}\right)^{\beta}$ tends to 0 when $\gamma \to 0$. Thus, there exists $0 < \gamma_0 < \alpha\delta/(3c + \alpha)$ such that $\left(\frac{e}{\gamma_0}\right)^{\gamma_0} \left(\frac{\gamma_0 e c}{\beta}\right)^{\beta} < 1$. It follows that

$$\Pr(\exists S : |S| \le \gamma_0 n,\ d(S) \ge \beta n) \to 0 (n \to +\infty).$$

Therefore the set S has size at least $\gamma_0 n$ and $\mathrm{tw}(H) \ge \gamma_0 n$. Since the tree-width is monotone with respect to the subgraph relation and $H \subseteq G$, we know that $\mathrm{tw}(H) \le \mathrm{tw}(G)$, and $\mathrm{tw}(G) = \Omega(n)$, concluding the proof of Theorem 1.2. □

Observe that it is not true in general that every set S with $f(n)$ incident edges has size $\Theta(f(n))$. For instance, the maximum degree of $\mathcal{G}(n, c/n)$ $(c > 1)$ is not constant (see e.g. [4]).

Moreover, it can be checked that the above argument gives

$$\mathrm{tw}(G) \ge \frac{(\alpha\delta)^2}{9e^3c^2} n,$$

thus providing an explicit lower bound for $\mathrm{tw}(G)$. In [15] the authors provide the lower bound $\frac{\alpha\delta}{M^2}n$, where M is constant but it is not made explicit.

We finish the paper by showing, with analogous arguments, that the tree-width and tree-depth of random regular graphs is also linear. We consider the configuration model $\mathcal{G}(n, d)$ as a model of random regular graphs (see e.g. [4]).

Proposition 4.6. *There is a constant d_0 such that, for every $d \geq d_0$, aas random d-regular graphs have linear tree-width and linear tree-depth.*

Proof. The Cheeger constant of d-regular graphs can be bounded in terms of the second largest eigenvalue $\lambda_2(G)$ of the adjacency matrix (see e.g. [5, Lemma 2.1]),

$$\Phi(G) \geq \frac{d - \lambda_2(G)}{2d}.$$

Friedman et al. [10] proved that this eigenvalue in d-regular random graphs is *aas* $O(\sqrt{d})$. Therefore,

$$\Phi(G) \geq \frac{d - O(\sqrt{d})}{2d} = \alpha(d) > 0$$

and $G \sim \mathcal{G}(n, d)$ is $\alpha(d)$-edge-expander. Since G is d-regular, any set of vertices X of size at most $n/2$ has a large set of neighbors, i.e. the graph is not only an edge-expander but a vertex-expander. If $N(X) = \{v \in V \setminus X : \exists u \in X,\ u \sim v\}$, then

$$|N(X)| \geq e(X, V \setminus X)/d \geq \Phi(G)|X| \geq \alpha(d)|X|$$

and the graph is an $\alpha(d)$-vertex-expander.

By Lemma 2.1, we know that there is a balanced partition (A, S, B) of G, where S is a vertex separator of cardinality $\mathrm{tw}(G) + 1$ and $|A| \geq (n - \mathrm{tw}(G) - 1)/3$. Since $N(A) \subseteq S$, we have

$$\mathrm{tw}(G) = |S| - 1 \geq \alpha(d)|A| - 1 \geq \alpha(d)\frac{n - \mathrm{tw}(G) - 1}{3} - 1.$$

Hence,

$$\mathrm{tw}(G) \geq \frac{\alpha(d)(n-1) - 3}{\alpha(d) + 3} = \Omega(n). \quad \square$$

References

[1] M.O. Albertson, G.G. Chappell, H.A. Kierstead, A. Kündgen, R. Ramamurthi, Coloring with no 2-colored P_4's, Electron. J. Combin. 11 (2004) 13. Research Paper 26 (electronic).
[2] I. Benjamini, G. Kozma, N. Wormald, The mixing time of the giant component of a random graph, 2006.
[3] H.L. Bodlaender, J.R. Gilbert, H. Hafsteinsson, T. Kloks, Approximating treewidth, pathwidth, frontsize, and shortest elimination tree, J. Algorithms 18 (1995) 238–255.
[4] B. Bollobás, Random Graphs, Academic Press Inc., Harcourt Brace Jovanovich Publishers, London, 1985.
[5] F.R.K. Chung, Spectral Graph Theory, in: CBMS Regional Conference Series in Mathematics, vol. 92, American Mathematical Society, 1997.
[6] J.S. Deogun, T. Kloks, D. Kratsch, H. Müller, On vertex ranking for permutation and other graphs, in: STACS 94, Caen, 1994, in: Lecture Notes in Comput. Sci., vol. 775, Springer, Berlin, 1994, pp. 747–758.
[7] I.S. Duff, A.M. Erisman, J.K. Reid, Direct Methods for Sparse Matrices, Oxford University Press, Inc., New York, NY, USA, 1986.
[8] Z. Dvořák, A.C. Giannopoulou, D.M. Thilikos, Forbidden graphs for tree-depth, Eur. J. Comb. 33 (2012) 969–979.
[9] P. Erdős, A. Rényi, On the evolution of random graphs, Magyar Tud. Akad. mat. Kutató Int. Közl. 5 (1960) 17–61.
[10] J. Friedman, J. Kahn, E. Szemerédi, On the second eigenvalue of random regular graphs, in: STOC'89: Proceedings of the Twenty-First Annual ACM Symposium on Theory of Computing, ACM, New York, NY, USA, 1989, pp. 587–598.
[11] Y. Gao, On the threshold of having a linear treewidth in random graphs, in: Computing and Combinatorics, in: Lecture Notes in Comput. Sci., vol. 4112, Springer, Berlin, 2006, pp. 226–234.
[12] S. Janson, T. Łuczak, A. Rucinski, Random Graphs, in: Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.
[13] H.A. Kierstead, D. Yang, Orderings on graphs and game coloring number, Order 20 (2003) 255–264. 2004.
[14] T. Kloks, Treewidth: Computations and Approximations, in: Lecture Notes in Computer Science, vol. 842, Springer-Verlag, Berlin, 1994.
[15] C. Lee, J. Lee, S.i. Oum, Rank-width of random graphs, J. Graph Theory 70 (2012) 339–347.
[16] J.W.H. Liu, The role of elimination trees in sparse factorization, SIAM J. Matrix Anal. Appl. 11 (1990) 134–172.
[17] T. Łuczak, Random trees and random graph, in: Proceedings of the Eighth International Conference Random Structures and Algorithms (Poznan, 1997), 1998, pp. 485–500.
[18] A. Nachmias, Y. Peres, Critical random graphs: diameter and mixing time, Ann. Probab. 36 (2008) 1267–1286.
[19] J. Nešetřil, P. Ossona de Mendez, Tree-depth, subgraph coloring and homomorphism bounds, European J. Combin. 27 (2006) 1022–1041.
[20] J. Nešetřil, P. Ossona de Mendez, Sparsity (Graphs, Structures, and Algorithms), first ed., in: Springer Series: Algorithms and Combinatorics, Springer, 2012.
[21] J. Nešetřil, S. Shelah, On the order of countable graphs, European J. Combin. 24 (2003) 649–663.
[22] A. Pothen, S. Toledo, Elimination structures in scientific computing, in: Handbook on Data Structures and Applications, Chapman and Hall, 2004 (Chapter 59).
[23] A. Rényi, G. Szekeres, On the height of trees, J. Aust. Math. Soc. 7 (1967) 497–507.
[24] R. Schreiber, A new implementation of sparse Gaussian elimination, ACM Trans. Math. Software 8 (1982) 256–276.