

Tilted Sperner families

Leader, Imre; Long, Eoin

DOI:

[10.1016/j.dam.2012.02.024](https://doi.org/10.1016/j.dam.2012.02.024)

License:

Other (please provide link to licence statement)

Document Version

Publisher's PDF, also known as Version of record

Citation for published version (Harvard):

Leader, I & Long, E 2014, 'Tilted Sperner families', *Discrete Applied Mathematics*, vol. 163, no. 2, pp. 194-198.
<https://doi.org/10.1016/j.dam.2012.02.024>

[Link to publication on Research at Birmingham portal](#)

General rights

Unless a licence is specified above, all rights (including copyright and moral rights) in this document are retained by the authors and/or the copyright holders. The express permission of the copyright holder must be obtained for any use of this material other than for purposes permitted by law.

- Users may freely distribute the URL that is used to identify this publication.
- Users may download and/or print one copy of the publication from the University of Birmingham research portal for the purpose of private study or non-commercial research.
- User may use extracts from the document in line with the concept of 'fair dealing' under the Copyright, Designs and Patents Act 1988 (?)
- Users may not further distribute the material nor use it for the purposes of commercial gain.

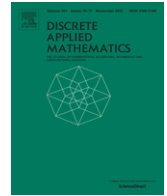
Where a licence is displayed above, please note the terms and conditions of the licence govern your use of this document.

When citing, please reference the published version.

Take down policy

While the University of Birmingham exercises care and attention in making items available there are rare occasions when an item has been uploaded in error or has been deemed to be commercially or otherwise sensitive.

If you believe that this is the case for this document, please contact UBIRA@lists.bham.ac.uk providing details and we will remove access to the work immediately and investigate.



Tilted Sperner families

Imre Leader^{*}, Eoin Long

Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, Cambridge CB3 0WB, United Kingdom

ARTICLE INFO

Article history:

Received 25 January 2011

Accepted 22 February 2012

Available online 7 April 2012

Keywords:

Extremal combinatorics

Sperner families

ABSTRACT

Let \mathcal{A} be a family of subsets of an n -set such that \mathcal{A} does not contain distinct sets A and B with $|A \setminus B| = 2|B \setminus A|$. How large can \mathcal{A} be? Our aim in this note is to determine the maximum size of such an \mathcal{A} . This answers a question of Kalai. We also give some related results and conjectures.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

A set system $\mathcal{A} \subseteq \mathcal{P}[n] = \mathcal{P}(\{1, \dots, n\})$ is said to be an *antichain* or *Sperner family* if $A \not\subseteq B$ for all distinct $A, B \in \mathcal{A}$. Sperner's theorem [5] says that any antichain \mathcal{A} has size at most $\binom{n}{\lfloor n/2 \rfloor}$. (See [2] for general background.)

Kalai [3] noted that the antichain condition may be restated as: \mathcal{A} does not contain A and B such that, in the subcube of the n -cube spanned by A and B , they are the top and bottom points. He asked what happens if we 'tilt' this condition. For example, suppose that we instead forbid A, B such that A is $1/3$ of the way up the subcube spanned by A and B ? Equivalently, \mathcal{A} cannot contain two sets A and B with $|A \setminus B| = 2|B \setminus A|$.

An obvious example of such a system is any level set $[n]^{(i)} = \{A \subset [n] : |A| = i\}$. Thus we may certainly achieve size $\binom{n}{\lfloor n/2 \rfloor}$. The system $[n]^{(\lfloor n/2 \rfloor)}$ is not maximal, as we may for example add to it all sets of size $\lfloor n/4 \rfloor - 1$ —but that is a rather small improvement. Kalai [3] asked if, as for Sperner families, it is still true that our family \mathcal{A} must have size $o(2^n)$.

Our aim in this note is to verify this. We show that the middle layer is asymptotically best, in the sense that the maximum size of such a family is $(1 + o(1)) \binom{n}{\lfloor n/2 \rfloor}$. We also find the exact extremal system, for n even and sufficiently large. We give similar results for any particular 'forbidden ratio' in the subcube spanned.

What happens if, instead of forbidding a particular ratio, we instead forbid an absolute distance from the bottom point? For example, for distance 1 this would correspond to the following: our set system \mathcal{A} must not contain sets A and B with $|A \setminus B| = 1$. How large can \mathcal{A} be?

Here the situation is rather different, as for example one cannot take an entire level. We give a construction that has size about $\frac{1}{n} \binom{n}{\lfloor n/2 \rfloor}$, which is about (a constant fraction of) $1/n^{3/2}$ of the whole cube. But we are not able to show that this is optimal: the best upper bound that we are able to give is $2^n/n$. However, if we strengthen the condition to \mathcal{A} not having A and B with $|A \setminus B| \leq 1$ then we are able to show that the greatest family has size $\frac{1}{n} \binom{n}{\lfloor n/2 \rfloor}$, up to a multiplicative constant.

^{*} Correspondence to: Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, United Kingdom. Fax: +44 1223337920.

E-mail addresses: I.Leader@dpmms.cam.ac.uk (I. Leader), E.P.Long@dpmms.cam.ac.uk (E. Long).

2. Forbidding a fixed ratio

In this section we consider the problem of finding the maximum size of a family \mathcal{A} of subsets of $[n]$ which satisfies $p|A \setminus B| \neq q|B \setminus A|$ for all $A, B \in \mathcal{A}$ where $p : q$ is a fixed ratio. Initially we will focus on the first non-trivial case 1:2 (note that 1:1 is trivial as then the condition just forbids two sets of the same size in \mathcal{A}) and then at the end of the section we extend these results to any given ratio.

As mentioned in the Introduction, for the ratio 1:2 we actually obtain the extremal family when n is even and sufficiently large. This family, which we will denote by \mathcal{B}_0 , is a union of level sets: $\mathcal{B}_0 = \cup_{i \in I} [n]^{(i)}$. Here the set I is defined as follows: $I = \{a_i : i \geq 0\} \cup \{b_i : i \geq 0\}$, where $a_0 = b_0 = \frac{n}{2}$ and a_i and b_i are defined inductively by taking $a_i = \lceil \frac{a_{i-1}}{2} \rceil - 1$ and $b_i = \lfloor \frac{b_{i-1} + n}{2} \rfloor + 1$ for all i . For example, if $n = 2^k$ then $I = \{2^{k-1}\} \cup \{2^i - 1 : 0 \leq i \leq k-1\} \cup \{2^k - 2^i + 1 : 0 \leq i \leq k-1\}$. Noting that for any sets A and B with either (i) $|A| = l$ where $l < \frac{n}{2}$ and $|B| > 2l$ or (ii) $|A| = l$ where $l > \frac{n}{2}$ and $|B| < 2l - n$ we have $|A \setminus B| \neq 2|B \setminus A|$, we see that \mathcal{B}_0 satisfies the required condition. Our main result is the following.

Theorem 1. Suppose \mathcal{A} is a set system on ground set $[n]$ such that $|A \setminus B| \neq 2|B \setminus A|$ for all distinct $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leq (1 + o(1)) \binom{n}{\lfloor n/2 \rfloor}$. Furthermore, if n is even and sufficiently large then $|\mathcal{A}| \leq |\mathcal{B}_0|$, with equality if and only if $\mathcal{A} = \mathcal{B}_0$.

The main step in the proof of Theorem 1 is given by the following lemma. The proof is a Katona-type (see [4]) averaging argument.

Lemma 2. Let \mathcal{A} be a set system on $[n]$ such that $|A \setminus B| \neq 2|B \setminus A|$ for all distinct $A, B \in \mathcal{A}$. Then

$$\sum_{j=l}^{2l} \frac{|\mathcal{A}_j|}{\binom{n}{j}} \leq 1$$

for all $l \leq \frac{n}{3}$ and

$$\sum_{j=2k-n}^k \frac{|\mathcal{A}_j|}{\binom{n}{j}} \leq 1$$

for all $k \geq \frac{2n}{3}$, where $\mathcal{A}_j = \mathcal{A} \cap [n]^{(j)}$.

Proof. We only prove the first inequality, as the proof of the second is identical. Pick a random ordering of $[n]$ which we denote by $(a_1, a_2, \dots, a_{\lceil \frac{2n}{3} \rceil}, b_1, \dots, b_{\lfloor \frac{n}{3} \rfloor})$. Given this ordering, let $C_i = \{a_j : j \in [2i]\} \cup \{b_k : k \in [i+1, l]\}$ and let $\mathcal{C} = \{C_i : i \in [0, l]\}$. Consider the random variable $X = |\mathcal{A} \cap \mathcal{C}|$. Since each set $B \in [n]^{(i)}$ is equally likely to be C_{i-l} we have $\mathbb{P}[B \in \mathcal{C}] = \frac{1}{\binom{n}{i}}$. Thus by linearity of expectation we have

$$\mathbb{E}(X) = \sum_{i=l}^{2l} \frac{|\mathcal{A}_i|}{\binom{n}{i}}. \quad (1)$$

On the other hand, given any C_i, C_j with $i < j$ we have $|C_i \setminus C_j| = 2|C_j \setminus C_i|$ and so \mathcal{A} can contain at most one of these sets. This gives $\mathbb{E}(X) \leq 1$. Together with (1) this gives the claimed inequality

$$\sum_{i=l}^{2l} \frac{|\mathcal{A}_i|}{\binom{n}{i}} \leq 1. \quad \square$$

Proof of Theorem 1. We first show $|\mathcal{A}| \leq (1 + o(1)) \binom{n}{\lfloor n/2 \rfloor}$. By standard estimates (see e.g. Appendix A of [1]) we have $|\lfloor n/2 \rfloor^{(\leq \alpha n)} \cup \lfloor n/2 \rfloor^{(\geq (1-\alpha)n)}| = o\left(\binom{n}{\lfloor n/2 \rfloor}\right)$ for any fixed $\alpha \in [0, \frac{1}{2})$, so it suffices to show that $|\bigcup_{i=\frac{3n}{5}}^{\frac{2n}{5}} \mathcal{A}_i| \leq \binom{n}{\frac{n}{2}}$. But this follows immediately from Lemma 2 by taking $l = \lfloor \frac{n}{3} \rfloor$.

We now prove the extremal part of the claim in Theorem 1. We first show that the maximum of $f(x) = \sum_{i=0}^n x_i$ subject to the inequalities

$$\sum_{j=l}^{2l} \frac{x_j}{\binom{n}{j}} \leq 1, \quad l \in \left\{0, 1, \dots, \left\lfloor \frac{n}{3} \right\rfloor\right\} \quad (2)$$

and

$$\sum_{j=2k-n}^k \frac{x_j}{\binom{n}{j}} \leq 1, \quad k \in \left\{\left\lceil \frac{2n}{3} \right\rceil, \dots, n\right\} \quad (3)$$

from Lemma 2 occurs when $x_{n/2} = \binom{n}{\frac{n}{2}}$. Indeed, suppose otherwise. At least one of these inequalities involving $x_{n/2}$ must occur with equality, as otherwise we can increase $x_{n/2}$ slightly, increase the value of $f(x)$ and still satisfy (2) and (3). Pick $j > \frac{n}{2}$ as small as possible such that $x_j > 0$. Let $y_{n/2} = x_{n/2} + \epsilon \binom{n}{n/2}$, $y_j = x_j - \epsilon \binom{n}{j}$ and $y_i = x_i$ for all other i . As $f(y) > f(x)$ one of the (2) or (3) must fail. If ϵ is sufficiently small only the inequalities involving $y_{n/2}$ and not y_j can be violated. Choose $k < n/2$ maximal such that $y_k > 0$ and y_k does not occur in any inequality involving y_j . Note that we must have $j - k \geq \frac{n}{4}$. Decrease y_k by $\epsilon \binom{n}{k}$. Since the only increased variable $y_{n/2}$ always occurs with one of y_j or y_k , it follows that $y = (y_0, \dots, y_n)$ satisfies (2) and (3).

We claim that $f(y) > f(x)$. Indeed, we must have either $|j - \frac{n}{2}| \geq \frac{n}{8}$ or $|k - \frac{n}{2}| \geq \frac{n}{8}$. Without loss of generality assume that $|k - \frac{n}{2}| \geq \frac{n}{8}$. Then since $\binom{n}{n/2} > \binom{n}{(n/2)+1} + \binom{n}{3n/8}$ for sufficiently large n we have

$$f(y) = f(x) + \epsilon \binom{n}{n/2} - \epsilon \binom{n}{j} - \epsilon \binom{n}{k} > f(x) + \epsilon \binom{n}{n/2} - \epsilon \binom{n}{(n/2)+1} - \epsilon \binom{n}{3n/8} > f(x).$$

Therefore we must have $x_{n/2} = \binom{n}{n/2}$, as claimed.

Now, by the inequalities (2) and (3) we have $x_j = 0$ for all $\frac{n}{4} \leq j \leq \frac{3n}{4}$ with $j \neq \frac{n}{2}$. From here it is easy to see by a weight transfer argument that $f(x)$ has a unique maximum when $x_i = \binom{n}{i}$ for $i \in I$ and $x_i = 0$ otherwise. For a set system \mathcal{A} these values of $x_i = |\mathcal{A}_i|$ can only be achieved if $\mathcal{A} = \mathcal{B}_0$, as claimed. \square

We remark that the statement of Theorem 1 does not hold for all even n , as can be seen for example by taking $n = 4$ and $\mathcal{A} = \mathcal{P}[n] \setminus \{n\}^{(2)}$.

We now extend Theorem 1 from the ratio 1 : 2 to any given ratio $p : q$. Let $p : q$ be in its lowest terms and $p < q$. If $A \in [n]^{(i+a)}$ and $B \in [n]^{(i)}$ satisfy $p|A \setminus B| = q|B \setminus A|$ then we have $p(a+b) = q(b)$ where $b = |B \setminus A|$. But then $pa = (q-p)b$ and since p and q are coprime we must have that $(q-p)|a$. Therefore any family $\mathcal{A} = \bigcup_{i \in I} [n]^{(i)}$, where I is an interval of length $q-p$, satisfies $p|A \setminus B| \neq q|B \setminus A|$ for all $A, B \in \mathcal{A}$. Taking $\lfloor \frac{n}{2} \rfloor \in I$ gives $|\mathcal{A}| = (q-p + o(1)) \binom{n}{\lfloor n/2 \rfloor}$. Our next result shows that this is asymptotically best possible.

Theorem 3. Let $p, q \in \mathbb{N}$ be coprime with $p < q$. Let \mathcal{A} be a set system on ground set $[n]$ such that $p|A \setminus B| \neq q|B \setminus A|$ for all distinct $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leq (q-p + o(1)) \binom{n}{\lfloor n/2 \rfloor}$.

The following lemma performs an analogous role to that of Lemma 2 in the proof of Theorem 1.

Lemma 4. Let \mathcal{A} be a set system on $[n]$ such that $p|A \setminus B| \neq q|B \setminus A|$ for all distinct $A, B \in \mathcal{A}$. Then

$$\sum_{j \in J_k} \frac{|\mathcal{A}_j|}{\binom{n}{j}} \leq 1$$

where $J_k = \{l : \lceil \frac{pn}{p+q} \rceil \leq l \leq \lfloor \frac{qn}{p+q} \rfloor, l \equiv k \pmod{(q-p)}\}$ for $0 \leq k \leq q-p-1$.

Proof. We only sketch the proof, as it is very similar to the proof of Lemma 2. For convenience we assume $n = (p+q)m$ (this assumption is easily removed). Fix $k \in [0, q-p-1]$ and let $k' \equiv k - pm \pmod{(q-p)}$ where $k' \in [0, q-p-1]$. Pick a random ordering of $[n]$ which we denote by $(a_1, a_2, \dots, a_{qm}, b_1, \dots, b_{pm})$. Given this ordering let $C_i = \{a_j : j \in [qi + k']\} \cup \{b_j : j \in [pi + 1, pm]\}$ and let $\mathcal{C} = \{C_i : i \in [0, m-1]\}$. (Here if $k' = 0$ we additionally adjoin C_m to \mathcal{C} .) By choice of k' , we have $|C_i| \in J_k$ for all $i \in [0, m-1]$.

Again for any C_i and C_j with $i < j$ we have $q|C_i \setminus C_j| = p|C_j \setminus C_i|$, which implies that \mathcal{A} contains at most one element of \mathcal{C} . Using this the rest of the proof is as in Lemma 2. \square

The proof of Theorem 3 is now identical to the proof of Theorem 1 taking Lemma 4 in place of Lemma 2.

For simplicity we have given in Lemma 4 only the inequalities that we needed in order to prove Theorem 3. Further inequalities involving smaller level sets analogous to those in Lemma 2 can also be obtained in a similar fashion. While we have not done so here, we note that it is possible to use these inequalities to again find an exact extremal family for any given ratio $p : q$ as in Theorem 1, provided $q-p$ and n have the opposite parity and n is sufficiently large.

3. Forbidding a fixed distance

In this final section we consider how large a family \mathcal{A} can be if for all $A, B \in \mathcal{A}$ we do not allow A to have a constant distance from the bottom of the subcube formed with B . For 'distance exactly 1' this would mean that we exclude $|A \setminus B| = 1$ for $A, B \in \mathcal{A}$. Here the following family \mathcal{A}^* provides a lower bound: let \mathcal{A}^* consist of all sets A of size $\lfloor n/2 \rfloor$ such that $\sum_{i \in A} i \equiv r \pmod{n}$, where $r \in \{0, \dots, n-1\}$ is chosen to maximise $|\mathcal{A}^*|$. Such a choice of r gives $|\mathcal{A}^*| \geq \frac{1}{n} \binom{n}{\lfloor n/2 \rfloor}$. Note that if we had $|A \setminus B| = 1$ for some $A, B \in \mathcal{A}^*$ then, since $|A| = |B|$, we would also have $|B \setminus A| = 1$. Letting $A \setminus B = \{i\}$ and $B \setminus A = \{j\}$ we then have $i - j \equiv 0 \pmod{n}$, giving $i = j$, a contradiction.

We suspect that this bound is best.

Conjecture 5. Let $\mathcal{A} \subset \mathcal{P}[n]$ be a family which satisfies $|A \setminus B| \neq 1$ for all $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leq (1 + o(1)) \frac{1}{n} \binom{n}{\lfloor n/2 \rfloor}$.

The following gives an upper bound that is a factor $n^{1/2}$ larger than this.

Theorem 6. Let $\mathcal{A} \subset \mathcal{P}[n]$ be a family such that $|A \setminus B| \neq 1$ for all $A, B \in \mathcal{A}$. Then there exists a constant C independent of n such that $|\mathcal{A}| \leq \frac{C}{n} 2^n$.

Proof. An easy estimate gives that the number of subsets of \mathcal{A} in $[n]^{(\leq n/3)} \cup [n]^{(\geq 2n/3)}$ is at most $4 \binom{n}{n/3} = o(\frac{2^n}{n})$. Therefore it suffices to show that $|\mathcal{A}_i| \leq \frac{C}{n} \binom{n}{i}$ for all $i \in [\frac{n}{3}, \frac{2n}{3}]$.

To see this, note that since $|A \setminus A'| \neq 1$ for all $A, A' \in \mathcal{A}$, each $B \in [n]^{(i+1)}$ contains at most one $A \in \mathcal{A}_i$. Double counting, we have

$$\begin{aligned} \frac{n}{3} |\mathcal{A}_i| &\leq (n-i) |\mathcal{A}_i| = |\{(A, B) : A \in \mathcal{A}_i, B \in [n]^{(i+1)}, A \subset B\}| \\ &\leq \binom{n}{i+1} \leq 3 \binom{n}{i} \end{aligned}$$

as required. \square

Our final result gives an upper bound on the size of a family \mathcal{A} in which we forbid ‘distance at most 1’ instead of ‘distance exactly 1’, i.e. where we have $|A \setminus B| > 1$ for all $A, B \in \mathcal{A}$. Again, the family \mathcal{A}^* constructed above gives a lower bound for this problem. In general, if we forbid ‘distance at most k ’ then it is easily seen that the following family \mathcal{A}_k^* gives a lower bound of $\frac{1}{n^k} \binom{n}{\lfloor n/2 \rfloor}$: supposing n is prime, let \mathcal{A}_k^* consist of all sets A of $\lfloor n/2 \rfloor$ which satisfy $\sum_{i \in A} i^d \equiv 0 \pmod{n}$ for all $1 \leq d \leq k$.

Our last result provides an upper bound which matches this up to a multiplicative constant. The proof is again a Katona-type argument. Here the condition $|A \setminus B| > k$ rather than $|A \setminus B| \neq k$ seems to be crucial.

Theorem 7. Let $k \in \mathbb{N}$. Suppose \mathcal{A} is a set system on $[n]$ such that $|A \setminus B| > k$ for all distinct $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leq \frac{(2^k - o(1))}{n^k} \binom{n}{\lfloor n/2 \rfloor}$.

Proof. Consider the family $\partial^{(k)} \mathcal{A}$, the k -shadow of \mathcal{A} , where

$$\partial^{(k)} \mathcal{A} = \{B \in \mathcal{P}[n] : B = A \setminus C \text{ for some } A \in \mathcal{A} \text{ and } C \subset A \text{ with } |C| = k\}.$$

Since \mathcal{A} does not contain A, B with $|A \setminus B| \leq k$, every element of $\partial^{(k)} \mathcal{A}$ is contained in at most one element of \mathcal{A} . Therefore we have

$$|\partial^{(k)} \mathcal{A}| = \sum_{i=0}^n (i)_k |\mathcal{A}_i| \quad (4)$$

where $(i)_k = i(i-1)\cdots(i-k+1)$. Now, since \mathcal{A} does not contain A, B with $|A \setminus B| \leq k$, it follows that $\partial^{(k)} \mathcal{A}$ is an antichain, and so by Sperner’s theorem we have

$$|\partial^{(k)} \mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}. \quad (5)$$

Finally, an estimate of the sum of binomial coefficients (Appendix A of [1]) gives

$$\sum_{i=0}^{\frac{n}{2} - n^{2/3}} |\mathcal{A}_i| \leq \sum_{i=0}^{\frac{n}{2} - n^{2/3}} \binom{n}{i} \leq e^{-n^{1/3}} 2^n. \quad (6)$$

Combining (4)–(6) we obtain

$$\begin{aligned} \binom{n}{\lfloor n/2 \rfloor} &\geq \sum_{i=0}^{\frac{n}{2} - n^{2/3}} (i)_k |\mathcal{A}_i| + \sum_{i=\frac{n}{2} - n^{2/3}}^n (i)_k |\mathcal{A}_i| \\ &\geq \sum_{i=0}^{\frac{n}{2} - n^{2/3}} \left(\frac{n}{2} - n^{2/3}\right)_k |\mathcal{A}_i| - \left(\frac{n}{2} - n^{2/3}\right)_k e^{-n^{1/3}} 2^n + \sum_{i=\frac{n}{2} - n^{2/3}}^n \left(\frac{n}{2} - n^{2/3}\right)_k |\mathcal{A}_i| \\ &= \left(\frac{n}{2} - o(n)\right)^k |\mathcal{A}| - o\left(\binom{n}{\lfloor n/2 \rfloor}\right) \end{aligned}$$

which gives the desired result. \square

Taking $k = 1$ in [Theorem 7](#) we obtain an upper bound which differs by a factor of 2 from the lower bound given by the family \mathcal{A}^* . It would be interesting to close this gap.

Acknowledgment

Research of the second author is supported by a Benefactor Scholarship from St. John's College, Cambridge.

References

- [1] N. Alon, J. Spencer, The Probabilistic Method, third ed., Wiley, 2008.
- [2] B. Bollobás, Combinatorics: Set Systems, Hypergraphs, Families of Vectors and Combinatorial Probability, first ed., Cambridge University Press, 1986.
- [3] G. Kalai, Personal communication, 2010.
- [4] G.O.H. Katona, Two applications of Sperner type theorems, Period. Math. Hungar. 3 (1973) 19–26.
- [5] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1928) 544–548.