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Note

Forbidding a set difference of size 1

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ABSTRACT

How large can a family $\mathcal{A} \subset \mathcal{P}[n]$ be if it does not contain A, B with $|A \setminus B| = 1$? Our aim in this paper is to show that any such family has size at most $\frac{2+o(1)}{n}\binom{n}{\lfloor n/2 \rfloor}$. This is tight up to a multiplicative constant of 2. We also obtain similar results for families $\mathcal{A} \subset \mathcal{P}[n]$ with $|A \setminus B| \neq k$, showing that they satisfy $|\mathcal{A}| \leq \frac{C_k}{n^k}\binom{n}{\lfloor n/2 \rfloor}$, where C_k is a constant depending only on k.

1. Introduction

A family $\mathcal{A} \subset \mathcal{P}[n] = \mathcal{P}(\{1, \ldots, n\})$ is said to be a *Sperner family* or *antichain* if $A \not\subset B$ for all distinct $A, B \in \mathcal{A}$. Sperner's theorem [9], one of the earliest results in extremal combinatorics, states that every Sperner family $\mathcal{A} \subset \mathcal{P}[n]$ satisfies

$$|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}. \tag{1}$$

(We remark that this paper is self-contained; for background on Sperner's theorem and related results see [2].)

Kalai [5] noted that the Sperner condition can be rephrased as follows: $\mathcal{A}$ does not contain two sets A and B such that, in the unique subcube of $\mathcal{P}[n]$ spanned by A and B, A is the bottom point and B is the top point. He asked: what happens if we forbid A and B to be at a different position in this subcube? In particular, he asked how large $\mathcal{A} \subset \mathcal{P}[n]$ can be if we forbid A and B to be at a 'fixed ratio' $p : q$ in this subcube. That is, we forbid A to be $p/(p+q)$ of the way up this subcube and B to be $q/(p+q)$ of the way up this subcube. Equivalently, $q|A \setminus B| \neq p|B \setminus A|$ for all distinct $A, B \in \mathcal{A}$. Note that the Sperner condition corresponds to taking $p = 0$ and $q = 1$. In [8], we gave an asymptotically tight answer for all ratios $p : q$, showing that one cannot improve on the 'obvious' example, namely the $q - p$ middle layers of $\mathcal{P}[n]$.

Theorem 1.1 ([8]). *Let p, q be coprime natural numbers with $q \geq p$. Suppose $\mathcal{A} \subset \mathcal{P}[n]$ does not contain distinct A, B with $q|A \setminus B| = p|B \setminus A|$. Then*

$$|\mathcal{A}| \leq (q - p + o(1))\binom{n}{\lfloor n/2 \rfloor}. \tag{2}$$

Up to the $o(1)$ term, this is best possible. Indeed, the proof of Theorem 1.1 in [8] also gives the exact maximum size of such $\mathcal{A}$ for infinitely many values of n.

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Another natural question considered in [8] asks how large a family $\mathcal{A} \subset \mathcal{P}[n]$ can be if, instead of forbidding a fixed ratio, we forbid a 'fixed distance' in these subcubes. For example, how large can $\mathcal{A} \subset \mathcal{P}[n]$ be if A is not at distance 1 from the bottom of the subcube spanned with B for all $A, B \in \mathcal{A}$? Equivalently, $|A \setminus B| \neq 1$ for all $A, B \in \mathcal{A}$. Here the following family $\mathcal{A}^*$ provides a lower bound: let $\mathcal{A}^*$ consist of all sets A of size $\lfloor n/2 \rfloor$ such that $\sum_{i \in A} i \equiv r \pmod n$ where $r \in \{0, \ldots, n-1\}$ is chosen to maximise $|\mathcal{A}^*|$. Such a choice of r gives $|\mathcal{A}^*| \geq \frac{1}{n}\binom{n}{\lfloor n/2 \rfloor}$. Note that if we had $|A \setminus B| = 1$ for some $A, B \in \mathcal{A}^*$, since $|A| = |B|$, we would also have $|B \setminus A| = 1$ – letting $A \setminus B = \{i\}$ and $B \setminus A = \{j\}$ we then have $i - j \equiv 0 \pmod n$ giving $i = j$, a contradiction.

In [8], we showed that any such family $\mathcal{A} \subset \mathcal{P}[n]$ satisfies $|\mathcal{A}| \leq \frac{C}{n}2^n = O(\frac{1}{n^{1/2}}\binom{n}{\lfloor n/2 \rfloor})$ for some absolute constant $C > 0$. We conjectured that the family $\mathcal{A}^*$ constructed in the previous paragraph is asymptotically maximal (Conjecture 5 of [8]). In Section 2, we prove that this is true up to a factor of 2.

Theorem 1.2. *Suppose that $\mathcal{A} \subset \mathcal{P}[n]$ is a family of sets with $|A \setminus B| \neq 1$ for all $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leq \frac{2+o(1)}{n}\binom{n}{\lfloor n/2 \rfloor}$.*

One could also ask what happens if we forbid a fixed set difference of size k, instead of 1 (where we think of k as fixed and n as varying). This turns out to be harder. In [8] we noted that the following family $\mathcal{A}_k^* \subset \mathcal{P}[n]$ gives a lower bound of $\frac{1}{n^k}\binom{n}{\lfloor n/2 \rfloor}$: supposing n is prime, let $\mathcal{A}_k^*$ consist of all sets A of size $\lfloor n/2 \rfloor$ which satisfy $\sum_{i \in A} i^d \equiv 0 \pmod n$ for all $1 \leq d \leq k$. In Section 3 we prove that this is also best possible up to a multiplicative constant.

Theorem 1.3. *Let $k \in \mathbb{N}$. Suppose that $\mathcal{A} \subset \mathcal{P}[n]$ with $|A \setminus B| \neq k$ for all $A, B \in \mathcal{P}[n]$. Then $|\mathcal{A}| \leq \frac{C_k}{n^k}\binom{n}{\lfloor n/2 \rfloor}$, where C_k is a constant depending only on k.*

Our notation is standard. We write $[n]$ for $\{1, \ldots, n\}$, and $[a, b]$ for the interval $\{a, \ldots, b\}$. For a set X, we write $\mathcal{P}(X)$ for the power set of X and $X^{(k)}$ for the collection of all k-sets in X. We often suppress integer-part signs.

2. Proof of Theorem 1.2

Our proof of Theorem 1.2 uses Katona's averaging method (see [6]), but modified in a key way. Ideally here, as in the proof of Sperner's theorem or Theorem 1.1, we would find configurations of sets covering $\mathcal{P}[n]$ so that each configuration has at most $C/n^{3/2}$ proportion of its elements in $\mathcal{A}$, for any family $\mathcal{A}$ satisfying $|A \setminus B| \neq 1$ for $A, B \in \mathcal{A}$. Then, provided that these configurations cover $\mathcal{P}[n]$ uniformly, we could count incidences between elements of $\mathcal{A}$ and these configurations to get an upper bound on $|\mathcal{A}|$.

However, we do not see how to find such configurations. So instead our approach is as follows. Rather than insisting that each of the sets in our configuration is well-behaved (in the sense above), we will only require that *most* of them have at most $C/n^{3/2}$ proportion of their elements in $\mathcal{A}$. It turns out that this can be achieved, and that it is good enough for our purposes.

Proof. We will prove the proposition under the assumption that n is even—this can easily be removed. To begin with, remove all elements in $\mathcal{A}$ of size smaller than $n/2 - n^{2/3}$ or larger than $n/2 + n^{2/3}$. By Chernoff's inequality (see Appendix A of [1]), we have removed at most $o\left(\frac{1}{n}\binom{n}{n/2}\right)$ sets. Let $\mathcal{B}$ denote the remaining sets in $\mathcal{A}$. It suffices to show that $|\mathcal{B}| \leq \frac{2+o(1)}{n}\binom{n}{n/2}$.

We write $I = [1, n/2 + n^{2/3}]$ and $J = [n/2 + n^{2/3} + 1, n]$ so that $[n] = I \cup J$. Let us choose a permutation $\sigma \in S_n$ uniformly at random. Given this choice of σ, for all $i \in I, j \in J$ let $C_{i,j} = \{\sigma(1), \ldots \sigma(i)\} \cup \{\sigma(j)\}$. Let $\mathcal{C}_j = \{C_{i,j} : i \in I\}$, and call these sets 'partial chains'. Also let $\mathcal{C} = \bigcup_{j \in J} \mathcal{C}_j$.

Now, for any choice of $\sigma \in S_n$, at most one of the partial chains of $\mathcal{C}$ can contain an element of $\mathcal{B}$. Indeed, suppose both $C_{i_1,j_1} = C_{i_1} \cup \{\sigma(j_1)\}$ and $C_{i_2,j_2} = C_{i_2} \cup \{\sigma(j_2)\}$ lie in $\mathcal{A}$ for distinct $j_1, j_2 \in J$. Since C_{i_1} and C_{i_2} are elements of a chain, without loss of generality we may assume $C_{i_1} \subset C_{i_2}$. But then $(C_{i_1} \cup \{\sigma(j_1)\}) \setminus (C_{i_2} \cup \{\sigma(j_2)\}) = \{\sigma(j_1)z\}$, which contradicts $|A \setminus B| \neq 1$ for all $A, B \in \mathcal{B}$.

Note that the above bound alone does not guarantee the upper bound on $|\mathcal{A}|$ stated in the theorem, since a fixed partial chain $\mathcal{C}_i$ may contain many elements of $\mathcal{A}$. We now show that this cannot happen too often.

For $i \in I$ and $j \in J$, let $X_{i,j}$ denote the random variable given by

$$X_{i,j} = \begin{cases} 1 & \text{if } C_{i,j} \in \mathcal{B} \text{ and } C_{k,j} \notin \mathcal{B} \quad \text{for } k < i; \\ 0 & \text{otherwise.} \end{cases}$$

From the previous paragraph, we have

$$\sum_{i,j} X_{i,j} \leq 1 \tag{3}$$

where both here and below the sum is taken over all $i \in I$ and $j \in J$. Taking expectations on both sides of (3) this gives

$$\sum_{i,j} \mathbb{E}(X_{i,j}) \leq 1. \tag{4}$$

Rearranging we have

$$\sum_{i,j} \mathbb{E}(X_{i,j}) = \sum_{i,j} \sum_{B \in \mathcal{B}} \mathbb{P}(C_{i,j} = B \text{ and } C_{k,j} \notin \mathcal{B} \text{ for } k < i). \quad (5)$$

We now bound $\mathbb{P}(C_{i,j} = B \text{ and } C_{k,j} \notin \mathcal{B} \text{ for } k < i)$ for sets $B \in \mathcal{B}$. Note that we can only have $C_{i,j} = B$ if $|B| = i + 1$. Furthermore, for such B, since $C_{i,j}$ is equally likely to be any subset of $[n]$ of size $i + 1$, we have $\mathbb{P}(C_{i,j} = B) = 1/\binom{n}{i+1}$. We will show that for all such B

$$\mathbb{P}(C_{i,j} = B \text{ and } C_{k,j} \notin \mathcal{B} \text{ for } k < i) = (1 - o(1))\mathbb{P}(C_{i,j} = B). \quad (6)$$

To see this, note that given any set $D \subset [n]$, there is at most one element $d \in D$ such that $D - d \in \mathcal{B}$. Indeed, $|(D - d') \setminus (D - d)| = 1$ for any distinct choices of $d, d' \in D$. Recalling that $C_{k,j} = C_{i,j} - \{\sigma(k + 1), \dots, \sigma(i)\}$ for all $k < i$ and that $\sigma(k+1)$ is chosen uniformly at random from the $k+1$ elements of $C_{k+1,j} - \{\sigma(j)\}$, we see that for $k+1 \geq n/2 - n^{2/3}$ we have

$$\mathbb{P}(C_{k,j} \notin \mathcal{B} | C_{k+1,j}, \dots, C_{i,j}) \geq \left(1 - \frac{1}{k+1}\right) \geq \left(1 - \frac{1}{n/2 - n^{2/3}}\right). \quad (7)$$

Also, since $\mathcal{B}$ contains no sets of size less than $n/2 - n^{2/3}$, for $k + 1 < n/2 - n^{2/3}$ we have

$$\mathbb{P}(C_{k,j} \notin \mathcal{B} | C_{k+1,j}, \dots, C_{i,j}) = 1. \quad (8)$$

But now by repeatedly applying (7) and (8) we get that for any B of size $i + 1 \in [n/2 - n^{2/3}, n/2 + n^{2/3}]$ we have

$$\begin{aligned} \mathbb{P}(C_{i,j} = B \text{ and } C_{k,j} \notin \mathcal{B} \text{ for } k < i) &\geq \left(1 - \frac{1}{n/2 - n^{2/3}}\right)^{(i-n/2-n^{2/3})} \mathbb{P}(C_{i,j} = B) \\ &\geq \left(1 - \frac{1}{n/2 - n^{2/3}}\right)^{2n^{2/3}} \mathbb{P}(C_{i,j} = B) \\ &= (1 - o(1))\mathbb{P}(C_{i,j} = B). \end{aligned}$$

Now combining (6) with (4) and (5) we obtain

$$\begin{aligned} 1 &\geq \sum_{i,j} \mathbb{E}(X_{i,j}) \\ &= \sum_{i,j} \sum_{B \in \mathcal{B}} \mathbb{P}(C_{i,j} = B \text{ and } C_{k,j} \notin \mathcal{B} \text{ for } k < i) \\ &= \sum_{i,j} \sum_{B \in \mathcal{B}^{(i+1)}} (1 - o(1))\mathbb{P}(C_{i,j} = B) \\ &= (1 - o(1)) \sum_{i,j} \frac{|\mathcal{B}^{(i+1)}|}{\binom{n}{i+1}} \\ &= (1 - o(1))|J| \sum_{i} \frac{|\mathcal{B}^{(i+1)}|}{\binom{n}{i+1}}. \end{aligned}$$

Since $|J| = n/2 - n^{2/3}$, this shows that

$$\frac{2 + o(1)}{n} \geq \sum_{i} \frac{|\mathcal{B}^{(i+1)}|}{\binom{n}{i+1}}$$

giving $|\mathcal{B}| \leq \frac{2+o(1)}{n} \binom{n}{n/2}$, as required. $\square$

3. Proof of Theorem 1.3

The proof of Theorem 1.3 will use the following result of Frankl and Füredi [4].

Theorem 3.1 (*Frankl–Füredi*)**.** *Let $r, k \in \mathbb{N}$ with $0 \leq k < r$. Suppose that $\mathcal{A} \subset [n]^{(r)}$ with $|A \cap B| \neq k$ for all $A, B \in \mathcal{A}$. Then $|\mathcal{A}| \leq d_r n^{\max(k, r-k-1)}$ where d_r is a constant depending only on r.*

We will also make use of the Erdős–Ko–Rado theorem [3].

Theorem 3.2 (*Erdős–Ko–Rado*)**.** *Suppose that $k \in \mathbb{N}$ and that $2k \leq n$. Then any family $\mathcal{A} \subset [n]^{(k)}$ with $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$ satisfies $|\mathcal{A}| \leq \binom{n-1}{k-1}$.*

We are now ready for the proof of the main result. Given a set $U \subset [n]$ and a permutation $\sigma \in S_n$, below we write $\sigma(U) = \{\sigma(u) : u \in U\}$.

Proof of Theorem 1.3. We will assume for convenience that n is a multiple of $3k$—this assumption can easily be removed. To begin, remove all elements in $\mathcal{A}$ of size smaller than $n/2 - n^{2/3}$ or larger than $n/2 + n^{2/3}$. By Chernoff's inequality (see Appendix A of [1]), we have removed at most $o\left(\frac{1}{n^k}\binom{n}{n/2}\right)$ sets. Let $\mathcal{B}$ denote the remaining sets in $\mathcal{A}$. For each $l \in [0, k-1]$, let

$$\mathcal{B}_l = \{B \in \mathcal{B} : |B| \equiv l \,(\mathrm{mod}\ k)\}.$$

To prove the theorem it suffices to prove that for all $l \in [0, k-1]$ we have $|\mathcal{B}_l| \le \frac{c'}{n^k}\binom{n}{n/2}$, where $c' = c'(k) > 0$. We will show this when $l = 0$ as the other cases are similar.

Let $I = [1, n/3]$ and $J = [n/3 + 1, n]$ so that $[n] = I \cup J$. Let us choose a permutation $\sigma \in S_n$ uniformly at random. Given this choice of σ, for all $i \in [n/3k]$ and $S \in J^{(n/3)}$ let

$$C_{i,S} = \sigma(\{1, \dots, ik\}) \cup \sigma(S).$$

Let $\mathcal{C}_S = \{C_{i,S} : i \in [n/3k]\}$ and call these sets 'partial chains'. We write

$$\mathcal{D} = \left\{S \in \binom{J}{n/3} : \mathcal{C}_S \cap \mathcal{B}_0 \neq \emptyset\right\} \subset \binom{J}{n/3}.$$

We claim that for any choice of $\sigma \in S_n$, we have

$$|\mathcal{D}| \le \frac{d_{2k}(12k^2)^k}{n^k}\binom{|J|}{n/3}, \tag{9}$$

where d_{2k} is as in Theorem 3.1. Indeed otherwise, by averaging, there exists $T \in J^{(n/3-2k)}$ for which the family

$$\mathcal{D}_T = \left\{U \in (J \setminus T)^{(2k)} : U \cup T \in \mathcal{D}\right\} \subset (J \setminus T)^{(2k)}$$

satisfies $|\mathcal{D}_T| > \frac{d_{2k}(12k^2)^k}{n^k}\binom{|J \setminus T|}{2k}$. This gives that

$$|\mathcal{D}_T| > \frac{d_{2k}(12k^2)^k}{n^k}\binom{|J \setminus T|}{2k} \ge \frac{d_{2k}(12k^2)^k}{n^k}\frac{|J \setminus T|^{2k}}{(2k)^{2k}} = \frac{d_{2k}|J \setminus T|^{2k}}{(n/3)^k} \ge d_{2k}|J \setminus T|^k,$$

since $|J \setminus T| = n/3 + 2k \ge n/3$. However, applying Theorem 3.1 to $\mathcal{D}_T$ with $r = 2k$ we find $U, U' \in \mathcal{D}_T$ with $|U \cap U'| = k$. This then gives $C_{i,U\cup T}, C_{i',U'\cup T} \in \mathcal{B}_0$ for some $i, i' \in [n]$. Without loss of generality, we have $i \le i'$. But then, as $\sigma(\{1, \dots, ik\}) \subset \sigma(\{1, \dots, i'k\})$, we have

$$|C_{i,U\cup T} \setminus C_{i',U'\cup T}| = |\sigma(U) \setminus \sigma(U')| = |U \setminus U'| = |U| - |U \cap U'| = 2k - k = k.$$

However $|A \setminus B| \neq k$ for all $A, B \in \mathcal{B}_0$. This contradiction shows that (9) must hold.

Now the bound (9) shows that for any choice of $\sigma \in S_n$, at most c_k/n^k proportion of the sets $\mathcal{C}_S$ can contain elements of $\mathcal{B}_0$. Note however that any of these partial chains may still contain many elements from $\mathcal{B}_0$. As in the proof of Theorem 1.2, we now show that this cannot happen too often.

For $i \in [n/3k]$ and $S \in J^{(n/3)}$, let $X_{i,S}$ denote the random variable given by

$$X_{i,S} = \begin{cases} 1 & \text{if } C_{i,S} \in \mathcal{B}_0 \text{ and } C_{i',S} \notin \mathcal{B}_0 \text{ for all } i' < i; \\ 0 & \text{otherwise.} \end{cases}$$

From the previous paragraph, we have

$$\sum_{i,S} X_{i,S} \le \frac{d_{2k}(12k^2)^k}{n^k}\binom{|J|}{n/3} \tag{10}$$

where both here and below the sum is taken over all $i \in [n/3k]$ and $S \in J^{(n/3)}$. Taking expectations on both sides of (3) this gives

$$\sum_{i,S} \mathbb{E}(X_{i,S}) \le \frac{d_{2k}(12k^2)^k}{n^k}\binom{|J|}{n/3}. \tag{11}$$

Rearranging we have

$$\sum_{i,S} \mathbb{E}(X_{i,S}) = \sum_{i,S}\sum_{B \in \mathcal{B}_0} \mathbb{P}(C_{i,S} = B \text{ and } C_{i',S} \notin \mathcal{B}_0 \text{ for } i' < i). \tag{12}$$

We now bound $\mathbb{P}(C_{i,S} = B \text{ and } C_{i',S} \notin \mathcal{B}_0 \text{ for } i' < i)$ for sets $B \in \mathcal{B}_0$. Note that we can only have $C_{i,S} = B$ if $|B| = ik + n/3$. Furthermore, for such B, since $C_{i,S}$ is equally likely to be any subset of $[n]$ of size $ik + n/3$, we have $\mathbb{P}(C_{i,S} = B) = 1/\binom{n}{ik+n/3}$. We will prove that for all such B

$$\mathbb{P}(C_{i,S} = B \text{ and } C_{i',S} \notin \mathcal{B}_0 \text{ for } i' < i) = (1 - o(1))\mathbb{P}(C_{i,S} = B). \tag{13}$$

To see this, note that given any set $D \subset [n]$ and two sets $E_1, E_2 \in D^{(k)}$ for which $D \backslash E_1, D \backslash E_2 \in \mathcal{B}_0$, we must have $E_1 \cap E_2 \neq 0$—otherwise $|(D\backslash E_1)\backslash(D\backslash E_2)| = k$. Therefore, for $|D| \geq 2k$, by Theorem 3.2, there are at most $\binom{|D|-1}{k-1} = \frac{k}{|D|}\binom{|D|}{k}$ choices of $E \in D^{(k)}$ with $D \backslash E \in \mathcal{B}_0$. Recalling that $C_{i',S} = C_{i,S} - \{\sigma(i'k+1), \ldots, \sigma(ik)\}$ for all $i' < i$ and that $\{\sigma(i'k+1), \ldots, \sigma((i'+1)k)\}$ is chosen uniformly at random among all k-sets in $\{\sigma(1), \ldots, \sigma((i'+1)k)\}$, we see that for $(i'+1)k + n/3 \geq (n/2 - n^{2/3})$ we have

$$\mathbb{P}(C_{i',S} \notin \mathcal{B}_0 | C_{i'+1,S}, \ldots, C_{i,S}) \geq \left(1 - \frac{k}{(i'+1)k}\right) \geq \left(1 - \frac{k}{n/6 - n^{2/3}}\right). \tag{14}$$

Also, since $\mathcal{B}_0$ contains no sets of size less than $n/2 - n^{2/3}$, for $(i'+1)k + n/3 < (n/2 - n^{2/3})$ we have

$$\mathbb{P}(C_{i',S} \notin \mathcal{B}_0 | C_{i'+1,S}, \ldots, C_{i,S}) = 1. \tag{15}$$

But now by repeatedly applying (14) and (15), we get that for any B of size $ik + n/3 \in [n/2 - n^{2/3}, n/2 + n^{2/3}]$ we have

$$\begin{aligned}
\mathbb{P}(C_{i,S} = B \text{ and } C_{i',S} \notin \mathcal{B}_0 \text{ for } i' < i) &\geq \left(1 - \frac{k}{n/6 - n^{2/3}}\right)^{2n^{2/3}/k} \mathbb{P}(C_{i,S} = B) \\
&\geq \left(1 - \frac{k}{n/6 - n^{2/3}}\right)^{2n^{2/3}/k} \mathbb{P}(C_{i,S} = B) \\
&= (1 - o(1))\mathbb{P}(C_{i,S} = B).
\end{aligned}$$

Now combining (13) with (11) and (12) we obtain

$$\begin{aligned}
\frac{d_{2k}(12k^2)^k}{n^k}\binom{|J|}{n/3} &\geq \sum_{i,S} \mathbb{E}(X_{i,S}) \\
&= \sum_{i,S}\sum_{B \in \mathcal{B}_0} \mathbb{P}(C_{i,S} = B \text{ and } C_{i',S} \notin \mathcal{B}_0 \text{ for } i' < i) \\
&= \sum_{i,S}\sum_{B \in \mathcal{B}_0^{(ik+n/3)}} (1 - o(1))\mathbb{P}(C_{i,S} = B) \\
&= (1 - o(1)) \sum_{i,S} \frac{|\mathcal{B}_0^{(ik+n/3)}|}{\binom{n}{ik+n/3}} \\
&= (1 - o(1)) \binom{|J|}{n/3} \sum_{j \in [n]} \frac{|\mathcal{B}_0^{(j)}|}{\binom{n}{j}}.
\end{aligned}$$

But this shows that

$$\frac{d_{2k}(12k^2)^k}{n^k} \geq \sum_{j \in [n]} \frac{|\mathcal{B}_0^{(j)}|}{\binom{n}{j}}$$

giving $|\mathcal{B}_0| \leq \frac{d_{2k}(12k^2)^k}{n^k}\binom{n}{n/2}$, as required. □

4. Concluding remarks

It would be very interesting to determine the true answer in Theorem 1.2, i.e., to remove the factor of 2. This is related to the well-known problem of finding the maximum size of a set system in which no two members are at Hamming distance 2, where there is also a 'gap' of a multiplicative constant 2. Indeed, our proof of Theorem 1.2 can be modified to show that the answers to these two questions are asymptotically equal. See Katona [7] for background on this problem.

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