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# Elementary doctrines as coalgebras[⋆]

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**Abstract**

Lawvere's hyperdoctrines mark the beginning of applications of category theory to logic. In particular, existential elementary doctrines proved essential to give models of non-classical logics. The clear connection between (typed) logical theories and certain **Pos**-valued functors is exemplified by the embedding of the category of elementary doctrines into that of primary doctrines, which has a right adjoint given by a completion which freely adds quotients for equivalence relations.

We extend that result to show that, in fact, the embedding is 2-functorial and 2-comonadic. As a byproduct we apply the result to produce an algebraic way to extend a first order theory to one which eliminates imaginaries, discuss how it relates to Shelah's original, and show how it works in a wider variety of situations.

*Keywords:* elementary doctrine, 2-comonad, quotient completion, elimination of imaginaries
*2020 MSC:* 03G30, 18C50, 18C20, 03B10, 03B20, 03C45

## 1. Introduction

Lawvere's hyperdoctrines mark the beginning of applications of category theory in logic, and they provide a very clear algebraic tool to work with syntactic theories and their extensions in logic, see [12, 10]. Lawvere's basic intuition in the categorical approach to logic was to rely on the notion of fibration.

As a very basic instance of this, recall that a ***primary doctrine*** is a functor $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ such that (i) the base category $\mathcal{C}$ has finite products, (ii) for every object $A$ of $\mathcal{C}$, the poset $P(A)$ has finite meets and, (iii) for every arrow $f\colon A \to B$, the monotone function $f^* = P(f)\colon P(B) \longrightarrow P(A)$ preserves finite meets. Indeed, these data amount to the same as a faithful fibration with

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products, see [7]. From this point of view, a logical theory becomes a presentation of a primary doctrine. And conversely, a primary doctrine gives rise to a logical theory in the $\top\wedge$-fragment of first order logic, see [20].

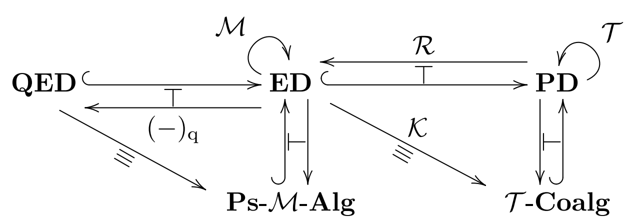
As mathematical structures, primary doctrines come arranged naturally into a 2-category **PD**, which is equivalent to the 2-category of faithful fibrations with fibred products, see [7, 25]. The equivalence restricts between the 2-category of elementary doctrines **ED** which, in turn, correspond to logical theories in the $\top\wedge=$-fragment, and the 2-category of faithful fibrations with equality.

The categorical approach allows the authors of [14] to describe a conservative extension of a logical theory which adds quotient types, in terms of a completion $(-)_{\mathrm{q}}\colon \mathbf{ED} \longrightarrow \mathbf{QED}$, where **QED** is the subcategory of **ED** on elementary doctrines with quotients. That completion is pseudo-monadic, see [26], and in [19] the second author shows that in fact the quotient completion provides a right adjoint $\mathcal{R}\colon \mathbf{PD} \longrightarrow \mathbf{ED}$ to the embedding of elementary doctrines into primary doctrines. Write $\mathcal{T}\colon \mathbf{PD} \longrightarrow \mathbf{PD}$ for the 2-comonad induced by that adjunction.

In the present paper, we extend that analysis to show that the comparison 2-functor

$$\mathbf{ED} \xrightarrow{\ \mathcal{K}\ } \mathcal{T}\text{-}\mathbf{Coalg}$$

is an isomorphism, hence the embedding $\mathbf{ED} \hookrightarrow \mathbf{PD}$ is 2-comonadic. It follows that the associated monad on the 2-category of coalgebras $\mathcal{T}$-**Coalg** coincides with the monad $\mathcal{M}$ on **ED** generated by $(-)_{\mathrm{q}}$, as depicted in the diagram below.




As an application, we consider the elimination of imaginaries in the sense of Poizat [22]. Given a model of a theory, an imaginary element is an equivalence class with respect to a definable equivalence relation in the model. Roughly speaking, a theory eliminates imaginaries if all imaginary elements are definable, see Section 4 for details.

We show that, by viewing first order theories as presentations of certain primary doctrines, the functor $\mathcal{R}$ associates to a theory another theory which eliminates imaginaries. Moreover, if the original theory has equality, then the extension has the same models as the original theory. The categorical standpoint allows us to compare the construction we provide with another construction of a theory eliminating imaginaries given by Shelah in the context of classical model theory [24], and to observe that the one described in the present paper applies to a broader class of theories.

In Section 2 we recall the main definitions and fix notation. We also provide a simple characterization of elementary doctrines which is useful for the proof of the main theorem. Section 3 contains the statement of the main theorem about the comonadicity of $\mathcal{R}$ and its proof. In Section 4 we present an application of the main theorem to the elimination of imaginaries. This is compared to Shelah's $T^{\mathrm{eq}}$ from [24] in Section 5.

## 2. Preliminaries

We recall some notions and constructions from [19] following very closely the notations introduced there: a ***primary doctrine*** is a functor $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ from a category $\mathcal{C}$ with finite products into the category of posets such that, for every object $A$ in the base category $\mathcal{C}$, the fibre $P(A)$ is an inf-semilattice, and for every arrow $f\colon A \to A'$, the reindexing $f^*\colon P(A') \longrightarrow P(A)$ preserves finite meets. This amounts to the same data as a functor $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{ISL}$ like in [19].

When necessary to avoid confusion, we may decorate with an index $A$ the order $\leq$, the meet $\wedge$ and the top element $\top$ of the fibre $P(A)$.

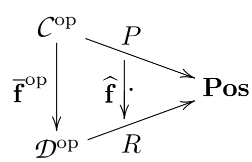
There is a large variety of examples for which we refer the reader to [7, 15, 20]. Throughout the section we shall exemplify definitions considering the following one.

**Example 2.1.** Given an inf-semilattice $\mathsf{H}$ consider the functor $\mathsf{H}^{(-)}\colon \mathcal{S}et^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ that sends a set $A$ to the poset $\mathsf{H}^A$ of functions from $A$ to $\mathsf{H}$ with the pointwise order, and a function $f\colon A \to B$ to $\mathsf{H}^f := - \circ f$. The contravariant functor $\mathsf{H}^{(-)}$ is a primary doctrine.

For $\mathsf{H}$ the two-element boolean algebra $\mathsf{B}$, the doctrine $\mathsf{B}^{(-)}$ is isomorphic to the contravariant powerset functor $\mathscr{P}\colon \mathcal{S}et^{\mathrm{op}} \longrightarrow \mathbf{Pos}$.

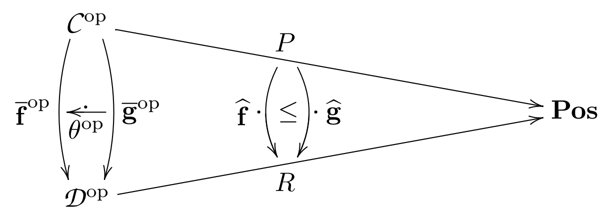
Primary doctrines are the objects of the 2-category $\mathbf{PD}$ where

**the 1-cells** are pairs $\mathbf{f} = (\overline{\mathbf{f}}, \widehat{\mathbf{f}})$ where $\overline{\mathbf{f}}\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a product-preserving functor and $\widehat{\mathbf{f}}\colon P \dot{\to} R \circ (\overline{\mathbf{f}})^{\mathrm{op}}$ is a natural transformation as in the diagram




and, for every object $A$ in the base $\mathcal{C}$, the monotone function $\widehat{\mathbf{f}}_A\colon P(A) \longrightarrow R(\overline{\mathbf{f}}(A))$ preserves finite meets.

**the 2-cells** $\theta\colon \mathbf{f} \Rightarrow \mathbf{g}$ are natural transformations $\theta\colon \overline{\mathbf{f}} \xrightarrow{\cdot} \overline{\mathbf{g}}$ such that




so that, for every $A$ in $\mathcal{C}$ and every $\alpha$ in $P(A)$, one has $\widehat{\mathbf{f}}_A(\alpha) \leq R_{\theta_A}(\widehat{\mathbf{g}}_A(\alpha))$.

A primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is ***elementary*** when, for every object $A$ in $\mathcal{C}$, there is an object $\delta^P_A$ in $P(A \times A)$ such that, for every object $X$ in $\mathcal{C}$, the functor

$$\begin{array}{ccc} P(X \times A) & \xrightarrow{Æ^P_{X,A}} & P(X \times A \times A) \\ \alpha & \longmapsto & \langle \mathrm{pr}_1, \mathrm{pr}_2\rangle^*(\alpha) \wedge \langle \mathrm{pr}_2, \mathrm{pr}_3\rangle^*(\delta^P_A) \end{array}$$

is left adjoint to the reindexing $\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2\rangle^*\colon P(X \times A \times A) \longrightarrow P(X \times A)$ along the arrow $\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2\rangle\colon X \times A \to X \times A \times A$.

We shall drop the superscript from $\delta^P_A$ when the doctrine $P$ is clear from the context.

**Example 2.2.** Given an inf-semilattice $\mathsf{H}$, the primary doctrine $\mathsf{H}^{(-)}\colon \mathcal{Set}^{\mathrm{op}} \longrightarrow$ **Pos** is elementary if and only if $\mathsf{H}$ has a least element. Indeed, if $\mathsf{H}$ has a least element $\bot$, the object $\delta_A \in \mathsf{H}^{A \times A}$ can be taken as the function

$$(x, x') \longmapsto \begin{cases} \top, \text{ if } x = x' \\ \bot, \text{ otherwise.} \end{cases}$$

Conversely suppose $\mathsf{H}^{(-)}$ is elementary and consider the set $A := \{0, 1\}$. For every $h \in \mathsf{H}$ let $\alpha_h \in \mathsf{H}^A$ be the function that maps 0 to $\top$ and 1 to $h$. Then

$$\delta_A(0,1) = \top \wedge \delta_A(0,1) = \alpha_h \circ \mathrm{pr}_1(0,1) \wedge \delta_A(0,1) \leq \alpha_h \circ \mathrm{pr}_2(0,1) = h$$

showing that $\delta_A(0,1)$ is a bottom element. In particular, the powerset functor $\mathscr{P}$ is elementary.

The 2-category **ED** is the 2-full subcategory of **PD** on the elementary doctrines where a 1-cell $\mathbf{f}\colon P \to R$ of **PD** is in **ED** if the $b_A$'s commute with the left adjoints, in the sense that there are commutative diagrams

$$\begin{array}{ccccc} P(X \times A) & \xrightarrow[\widehat{\mathbf{f}}_{X \times A}]{} & R(\overline{\mathbf{f}}(X \times A)) & \xrightarrow[R(\langle \overline{\mathbf{f}}\mathrm{pr}_1, \overline{\mathbf{f}}\mathrm{pr}_2\rangle^{-1})]{\sim} & R(\overline{\mathbf{f}}X \times \overline{\mathbf{f}}A) \\ \downarrow{\scriptstyle Æ^P_{X,A}} & & & & \downarrow{\scriptstyle Æ^R_{\overline{\mathbf{f}}X, \overline{\mathbf{f}}A}} \\ P(X \times A \times A) & \xrightarrow[\widehat{\mathbf{f}}_{X \times A \times A}]{} & R(\overline{\mathbf{f}}(X \times A \times A)) & \xrightarrow[R(\langle \overline{\mathbf{f}}\mathrm{pr}_1, \overline{\mathbf{f}}\mathrm{pr}_2, \overline{\mathbf{f}}\mathrm{pr}_3\rangle^{-1})]{\sim} & R(\overline{\mathbf{f}}X \times \overline{\mathbf{f}}A \times \overline{\mathbf{f}}A) \end{array}$$

We shall say that a primary doctrine $P$ is ***first-order*** if each fibre is a Heyting algebra, reindexing preserves the structure and reindexing along a product projection has left and right adjoints satisfying the Beck-Chevalley condition. First-order doctrines are the objects of the category **FOD**, which is the 2-full subcategory of **PD** on those 1-cells $\mathbf{f}$ such that every component of $\widehat{\mathbf{f}}$ is a homomorphism of Heyting algebras commuting with the left and right adjoints. Elementary first-order doctrines are what Pitts in [21] calls first order hyperdoctrines. We shall use the two names interchangeably and denote the category of (first order) hyperdoctrines as **HD**, which is the pullback of **FOD** and **ED** in **PD**.

**Remark 2.3.** In an elementary first-order doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$, for every arrow $f\colon A \to B$ in the base category, the reindexing functor $f^*\colon P(B) \longrightarrow P(A)$ has a left adjoint $\exists^P_f\colon P(A) \longrightarrow P(B)$ which is obtained from those for projections and for parametrised diagonals, see [21, Remark 4.6].

**Example 2.4.** For a given inf-semilattice $\mathsf{H}$, the doctrine $\mathsf{H}^{(-)}$ from Example 2.1 is elementary first-order if and only if $\mathsf{H}$ is complete.

It is well-known, see [7, 25], that the 2-category of primary doctrines is equivalent to the 2-category of faithful fibrations with fibred products, and that the 2-category of elementary doctrines is equivalent to the 2-category of faithful fibrations with equality. Yet we have no reference for the following characterization of elementary doctrines which will be useful for our future purposes.

**Proposition 2.5.** *Let $P\colon \mathcal{C}^{op} \longrightarrow$ **Pos** be a primary doctrine. The following are equivalent:*

(i) *$P$ is elementary.*

(ii) *for each object $C$ in $\mathcal{C}$, there is an object $\mathfrak{d}_C$ in $P(C \times C)$ such that*
  (a) $\mathrm{pr}_1^*(\beta) \wedge \mathfrak{d}_C \leq \mathrm{pr}_2^*(\beta)$ *for every $C$ and every $\beta$ in $P(C)$;*
  (b) $\top_C \leq \langle \mathrm{id}_C, \mathrm{id}_C\rangle^*(\mathfrak{d}_C)$ *for every $C$;*
  (c) $\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*(\mathfrak{d}_C) \wedge \langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^*(\mathfrak{d}_D) \leq \mathfrak{d}_{C\times D}$ *for every $C$ and $D$.*

*Proof.* (i)⇒(ii) is well-known choosing $\delta_C$ for $\mathfrak{d}_C$.
(ii)⇒(i) We want to show that choosing the object $\delta^P_A$ to be $\mathfrak{d}_A$ in $P(A \times A)$ makes the primary doctrine $P$ elementary, in other words we must check that, for every objects $X$ in $\mathcal{C}$, the functor

$$\begin{array}{ccc} P(X \times A) & \longrightarrow & P(X \times A \times A) \\ \alpha & \longmapsto & \langle \mathrm{pr}_1, \mathrm{pr}_2\rangle^*(\alpha) \wedge \langle \mathrm{pr}_2, \mathrm{pr}_3\rangle^*(\mathfrak{d}_A) \end{array}$$

is left adjoint to reindexing along the arrow $\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_2\rangle\colon X \times A \to X \times A \times A$ of $\mathcal{C}$. Note first that, for $\gamma$ in $P(C \times C)$,

$$\langle \mathrm{pr}_1, \mathrm{pr}_1\rangle^*(\gamma) \wedge \mathfrak{d}_C \leq \gamma \tag{1}$$

since, using first (c), then (a), in $P(C \times C \times C \times C)$ one has that

$$\begin{aligned}\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle^*(\gamma) \wedge \langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*(\mathfrak{d}_C) \wedge \langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^*(\mathfrak{d}_C) &\leq \langle \mathrm{pr}_1, \mathrm{pr}_2\rangle^*(\gamma) \wedge \mathfrak{d}_{C\times C}\\ &\leq \langle \mathrm{pr}_3, \mathrm{pr}_4\rangle^*(\gamma).\end{aligned}$$

By reindexing that inequality along $\langle \mathrm{pr}_1, \mathrm{pr}_1, \mathrm{pr}_1, \mathrm{pr}_2\rangle \colon C \times C \to C \times C \times C \times C$, condition (b) yields (1). For the left adjoint to reindexing along the diagonal $\langle \mathrm{id}_C, \mathrm{id}_C\rangle \colon C \to C \times C$, consider the assignment $\alpha \mapsto \mathrm{pr}_1^*(\alpha) \wedge \mathfrak{d}_C$ where $\mathrm{pr}_1 \colon C \times C \to C$. If $\alpha \leq \langle \mathrm{id}_C, \mathrm{id}_C\rangle^*(\gamma)$, then by (1)

$$\mathrm{pr}_1^*(\alpha) \wedge \mathfrak{d}_c \leq \mathrm{pr}_1^*(\langle \mathrm{id}_C, \mathrm{id}_C\rangle^*(\gamma)) \wedge \mathfrak{d}_C \leq \langle \mathrm{pr}_1, \mathrm{pr}_1\rangle^*(\gamma) \wedge \mathfrak{d}_C \leq \gamma.$$

If $\mathrm{pr}_1^*(\alpha) \wedge \mathfrak{d}_C \leq \gamma$, then (b) yields

$$\alpha \leq \alpha \wedge \langle \mathrm{id}_C, \mathrm{id}_C\rangle^*(\mathfrak{d}_C) = \langle \mathrm{id}_C, \mathrm{id}_C\rangle^*(\mathrm{pr}_1^*(\alpha) \wedge \mathfrak{d}_C) \leq \langle \mathrm{id}_C, \mathrm{id}_C\rangle^*(\gamma).$$

Taking $C$ as $X \times A$, it follows that $\mathfrak{d}_{X\times A} \leq \langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*(\mathfrak{d}_X)$ since by (b)

$$\top \leq \langle \mathrm{id}_X, \mathrm{id}_X\rangle^*(\mathfrak{d}_X) = \langle \mathrm{id}_{X\times A}, \mathrm{id}_{X\times A}\rangle^*(\langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*(\mathfrak{d}_X)).$$

Similarly, $\mathfrak{d}_{X\times A} \leq \langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^*(\mathfrak{d}_A)$. So the left adjoint to reindexing along the diagonal $\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_1, \mathrm{pr}_2\rangle \colon X \times A \to X \times A \times X \times A$ sends an object $\gamma$ in $P(X \times A \times X \times A)$ to $\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle^*(\gamma) \wedge \langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*(\mathfrak{d}_X) \wedge \langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^*(\mathfrak{d}_A)$. The conclusion follows easily. □

**Remark 2.6.** It follows from [14, Remark 2.2] that, once a product

$$A \xleftarrow{\mathrm{pr}_1} A \times A \xrightarrow{\mathrm{pr}_2} A$$

is chosen, there is a unique object object $\delta_A$ in $P(A \times A)$ which satisfies conditions (a)-(c) in Proposition 2.5.

**Corollary 2.7.** *Let $P$ and $R$ be elementary doctrines. A 1-cell $\mathbf{f} \colon P \to R$ of* **PD** *is in* **ED** *if and only if*

$$\widehat{\mathbf{f}}_{A\times A}(\delta_A^P) = \langle \overline{\mathbf{f}}\mathrm{pr}_1, \overline{\mathbf{f}}\mathrm{pr}_2\rangle^*(\delta_{\overline{\mathbf{f}}A}^R)$$

*for every object $A$.*

In [19] the second author proves that the inclusion $\mathbf{ED} \hookrightarrow \mathbf{PD}$, whose functor we shall denote by $\mathcal{L}$ when needed, has a right adjoint; for the sake of completeness, in the following we give a brief summary of that construction. For $P \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ a primary doctrine, let $\mathcal{E}_P$ be the category determined by the following data.

**An object in $\mathcal{E}_P$** is a pair $(A, \rho)$ where $A$ is an object in $\mathcal{C}$ and $\rho$ is in $P(A \times A)$ such that

(a) $\top_A \leq \langle \mathrm{id}_A, \mathrm{id}_A\rangle^*(\rho)$;
(b) $\rho \leq \langle \mathrm{pr}_2, \mathrm{pr}_1\rangle^*(\rho)$;

(c) $\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle^*(\rho) \wedge \langle \mathrm{pr}_2, \mathrm{pr}_3\rangle^*(\rho) \leq \langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*(\rho)$.

**An arrow in $\mathcal{E}_P$** $f\colon (A,\rho) \to (B,\sigma)$ is an arrow $f\colon A \to B$ in $\mathcal{C}$ such that $\rho \leq (f \times f)^* (\sigma)$.

It is customary to refer to conditions (a), (b), and (c) above as reflexivity, symmetry and transitivity, respectively, and to say that the $\rho$-component of an object of $\mathcal{E}_P$ is a ***P*-equivalence relation over** $A$.

**Remark 2.8.** We haste to note that the usual construction of a "category of partial equivalence relations" can be obtained as an appropriate quotient of a category of the form $\mathcal{E}_P$ which forces the equality of arrows in $\mathcal{C}$ to coincide with the equality of the primary doctrine $P$, see [16].

A terminal object in $\mathcal{E}_P$ can be computed as the pair $(T, \top_{T\times T})$ on a(ny) terminal object $T$ in $\mathcal{C}$; a product of $(A,\rho)$ and $(B,\sigma)$ in $\mathcal{E}_P$ can be taken as

$$(A,\rho) \xleftarrow{\mathrm{pr}_1} (A \times B, \rho \boxtimes \sigma) \xrightarrow{\mathrm{pr}_2} (B,\sigma)$$

where $\rho \boxtimes \sigma := \langle \mathrm{pr}_1, \mathrm{pr}_3\rangle^*(\rho) \wedge \langle \mathrm{pr}_2, \mathrm{pr}_4\rangle^*(\sigma)$.

The category $\mathcal{E}_P$ is the base of a primary doctrine $P^{\mathcal{R}}\colon {\mathcal{E}_P}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ determined as follows: the poset $P^{\mathcal{R}}(A,\rho)$ is the sub-poset of $P(A)$ on the ***descent data*** for $\rho$, *i.e.*

$$P^{\mathcal{R}}(A,\rho) = \{\alpha \in P(A) \mid \mathrm{pr}_1^*(\alpha) \wedge \rho \leq \mathrm{pr}_2^*(\alpha)\} \subseteq P(A).$$

It is easy to check that $P^{\mathcal{R}}(A,\rho)$ is a sub-inf-semilattice and that, for $f\colon (A,\rho) \to (B,\sigma)$ in $\mathcal{E}_P$, the function $f^*$ maps $P^{\mathcal{R}}(A,\rho)$ into $P^{\mathcal{R}}(B,\sigma)$. So $P^{\mathcal{R}}\colon {\mathcal{E}_P}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is indeed a primary doctrine, and it is elementary with $\delta^{P^{\mathcal{R}}}_{(A,\rho)} = \rho$ by Proposition 2.5.

For the same reason the construction extends to a 2-functor[1] $\mathcal{R}\colon \mathbf{PD} \longrightarrow \mathbf{ED}$ since, for a 1-cell $\mathbf{f}\colon P \to R$ of elementary doctrines, each functor

$$P(A\times A) \xrightarrow[\widehat{\mathbf{f}}_{A\times A}]{} R(\overline{\mathbf{f}}(A\times A)) \xrightarrow[R(\langle \overline{\mathbf{f}}\mathrm{pr}_1, \overline{\mathbf{f}}\mathrm{pr}_2\rangle^{-1})]{\sim} R(\overline{\mathbf{f}}A \times \overline{\mathbf{f}}A)$$

turns $P$-equivalence relations into $R$-equivalence relations and preserves descent data; the action of $\mathcal{R}$ on a 1-cell $\mathbf{f} = (\overline{\mathbf{f}}, \widehat{\mathbf{f}})$ is $\mathbf{f}^{\mathcal{R}} = (\overline{\mathbf{f}^{\mathcal{R}}}, \widehat{\mathbf{f}^{\mathcal{R}}})$ where $\overline{\mathbf{f}^{\mathcal{R}}}$ is

$$\begin{array}{ccc}
\mathcal{E}_P & \xrightarrow{\overline{\mathbf{f}^{\mathcal{R}}}} & \mathcal{E}_R \\
(A,\rho) & \longmapsto & (\overline{\mathbf{f}}A, R(\langle \overline{\mathbf{f}}\mathrm{pr}_1, \overline{\mathbf{f}}\mathrm{pr}_2\rangle^{-1})(\widehat{\mathbf{f}}_{A\times A}(\rho))) \\
\downarrow g & \longmapsto & \overline{\mathbf{f}}g \downarrow \\
(A',\rho') & \longmapsto & (\overline{\mathbf{f}}A', R(\langle \overline{\mathbf{f}}\mathrm{pr}_1, \overline{\mathbf{f}}\mathrm{pr}_2\rangle^{-1})(\widehat{\mathbf{f}}_{A'\times A'}(\rho')))
\end{array}$$

[1]The value $\mathcal{R}(P) = P^{\mathcal{R}}$ is denoted as $P_{\mathcal{D}}$ in [19]. In fact, in the following we use the two notations $P^{\mathcal{R}}$ and $\mathcal{R}(P)$ for the action of a 2-functor interchangeably, with the hope to improve readability.

and, for $(A,\rho)$ in $\mathcal{E}_P$, the $(A,\rho)$ component of $\widehat{\mathbf{f}^{\mathcal{R}}}$ is

$$\begin{array}{ccc} P^{\mathcal{R}}(A,\rho) & \xrightarrow{\left(\widehat{\mathbf{f}^{\mathcal{R}}}\right)_{(A,\rho)}} & R^{\mathcal{R}}(\overline{\mathbf{f}^{\mathcal{R}}}(A,\rho)) \\ \alpha & \longmapsto & \widehat{\mathbf{f}}_A(\alpha). \end{array}$$

The action of $\mathcal{R}$ on a 2-cell $\theta\colon \mathbf{f} \Rightarrow \mathbf{g}\colon P \to R$ is simply $\left(\theta^{\mathcal{R}}\right)_{(A,\rho)} = \theta_A$.

**Examples 2.9.** The following examples are from [11, 23].

(a) Consider the positive real line $[0,\infty)$ with the opposite of the natural order, so 0 is the top element and there is no bottom element. Hence the primary doctrine $[0,\infty)^{(-)}$, as in Example 2.1, is not elementary by Example 2.2. The base category $\mathcal{E}_{[0,\infty)^{(-)}}$ of the elementary doctrine

$$\left([0,\infty)^{(-)}\right)^{\mathcal{R}} \colon \mathcal{E}^{\mathrm{op}}_{[0,\infty)^{(-)}} \longrightarrow \mathbf{Pos}$$

is the category of pseudo ultrametric spaces; it consists of pairs $(X,d)$ where $d\colon X \times X \to [0,\infty)$ satisfies all the conditions of an ultrametric space but for the identity of indiscernibles

$$d(x_1,x_2) = 0 \quad \text{if and only if} \quad x_1 = x_2$$

weakened to just $d(x_1,x_1) = 0$.

(b) The closed unit interval $[0,1]$ with the opposite of the natural order is a complete Heyting algebra. Hence the doctrine $[0,1]^{(-)}$ is elementary first-order and the category $\mathcal{E}_{[0,1]^{(-)}}$ is the category of 1-bounded pseudo ultrametric spaces.

As announced previously, there is an adjunction $\mathbf{ED} \underset{\hookrightarrow}{\overset{\mathcal{R}}{\underset{\top}{\leftarrow}}} \mathbf{PD}$. The component of the unit $\mathbf{j}$ on the elementary doctrine $E\colon \mathcal{B}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is $\mathbf{j}_E = (\overline{\mathbf{j}_E}, \widehat{\mathbf{j}_E})$

$$\begin{array}{ccc} \mathcal{B}^{\mathrm{op}} & \xrightarrow{\quad E \quad} & \\ {\scriptstyle (\overline{\mathbf{j}_E})^{\mathrm{op}}}\big\downarrow & \widehat{\mathbf{j}_E}\,\big\downarrow\cdot & \mathbf{Pos} \\ \mathcal{E}_E^{\mathrm{op}} & \xrightarrow{\quad E^{\mathcal{R}} \quad} & \end{array}$$

where $\overline{\mathbf{j}_E}(f\colon A \to A') = f\colon (A,\delta^E_A) \to (A',\delta^E_{A'})$ and $\left(\widehat{\mathbf{j}_E}\right)_A\colon E(A) \to E^{\mathcal{R}}(\overline{\mathbf{j}_E}(A))$ is the identity since $E^{\mathcal{R}}(A,\delta^E_A) = E(A)$. The component of the counit $\mathbf{u}$ on a primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ is $\mathbf{u}_P = (\overline{\mathbf{u}_P}, \widehat{\mathbf{u}_P})$

$$\begin{array}{ccc} {\mathcal{E}_P}^{\mathrm{op}} & \xrightarrow{\quad P^{\mathcal{R}} \quad} & \\ {\scriptstyle (\overline{\mathbf{u}_P})^{\mathrm{op}}}\big\downarrow & \widehat{\mathbf{u}_P}\,\big\downarrow\cdot & \mathbf{Pos} \\ \mathcal{C}^{\mathrm{op}} & \xrightarrow{\quad P \quad} & \end{array}$$

where $\overline{\mathbf{u}_P}\colon \mathcal{E}_P \longrightarrow \mathcal{C}$ is the first projection mapping $f\colon (A,\rho) \to (B,\sigma)$ to $f\colon A \to B$ and $(\widehat{\mathbf{u}_P})_A\colon P^{\mathcal{R}}(A,\rho) \hookrightarrow P(A)$ is the inclusion.

## 3. Elementary doctrines are coalgebras

In this section we prove the main result of the paper: elementary doctrines are coalgebras for the comonad associated to the 2-adjunction in the following proposition.

**Proposition 3.1.** *The adjunction* $\mathbf{ED} \underset{\longleftarrow \atop \mathcal{R}}{\overset{\top}{\hookrightarrow}} \mathbf{PD}$ *is a 2-adjunction.*

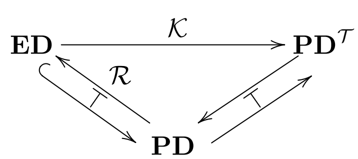
*Proof.* We must show that, for every elementary doctrine $E\colon \mathcal{B}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$ and every primary doctrine $P\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$, the functor

$$\mathbf{ED}(E, P^{\mathcal{R}}) \xrightarrow{\ \mathbf{u}_P \circ -\ } \mathbf{PD}(E, P)$$

is an isomorphism. Since one already knows from [19] that the bijection on the objects sends $\mathbf{f}\colon E \longrightarrow P^{\mathcal{R}}$ to $\mathbf{u}_P \circ \mathbf{f}\colon E \longrightarrow P$, it follows that it is an isomorphism because it is fully faithful. □

Write $\mathcal{T}\colon \mathbf{PD} \longrightarrow \mathbf{PD}$ for the composite $\mathcal{LR}$ so that the 2-comonad induced by the 2-adjunction is $(\mathcal{T}, \mathbf{u}, \mathbf{j}\mathcal{R})$.

**Theorem 3.2.** *The canonical comparison*




*is a 2-isomorphism.*

The comparison 2-functor $\mathcal{K}\colon \mathbf{ED} \longrightarrow \mathbf{PD}^{\mathcal{T}}$ maps an elementary doctrine $E$ to the coalgebra $\mathbf{j}_E = (\overline{\mathbf{j}_E}, \widehat{\mathbf{j}_E})\colon E \longrightarrow \mathcal{T}E$.

The rest of the section is devoted to proving the theorem.

**Lemma 3.3.** *The comparison 2-functor* $\mathcal{K}\colon \mathbf{ED} \longrightarrow \mathbf{PD}^{\mathcal{T}}$ *is surjective on objects.*

*Proof.* Consider an object $(P, \mathbf{g})$ in $\mathbf{PD}^{\mathcal{T}}$. We first show that the fact that the coalgebra map $\mathbf{g}\colon P \to P^{\mathcal{T}}$ is a section of the counit $\mathbf{u}_P\colon P^{\mathcal{T}} \longrightarrow P$ suffices to conclude that $P$ is elementary. Since $\mathbf{u}_P\mathbf{g} = \mathrm{id}_P$, for every object $A$ one has $\overline{\mathbf{g}}A = (A, \delta^{P^{\mathcal{R}}}_{\overline{\mathbf{g}}A})$ for some $P$-equivalence relation $\delta^{P^{\mathcal{R}}}_{\overline{\mathbf{g}}A} \in (A \times A)$, and also

$$(\widehat{\mathbf{u}_P})_{\overline{\mathbf{g}}A}\widehat{\mathbf{g}}_A = \mathrm{id}_{PA}. \tag{2}$$

For every $f\colon A \to B$ in $\mathcal{C}$, it is $\overline{\mathbf{u}_P\mathbf{g}}(f) = f$; hence

$$f\colon (A, \delta^{P^{\mathcal{R}}}_{\overline{\mathbf{g}}A}) \longrightarrow (B, \delta^{P^{\mathcal{R}}}_{\overline{\mathbf{g}}B}) \,. \tag{3}$$

Since $(\widehat{\mathbf{u}_P})_{\overline{\mathbf{g}}A}\colon P^{\mathcal{R}}(A, \delta^{P^{\mathcal{R}}}_{\overline{\mathbf{g}}A}) \longrightarrow PA$ is the inclusion, it follows from (2) that

$$P^{\mathcal{R}}(A, \delta^{P^{\mathcal{R}}}_{\overline{\mathbf{g}}A}) = PA \tag{4}$$

and

$$\widehat{\mathbf{g}}_A = \mathrm{id}_{PA}. \tag{5}$$

The identity (4) amounts to say that every object in $PA$ is descent data for $\delta^{P^{\mathcal{R}}}_{\overline{\mathbf{g}}A}$, establishing condition (a) in Proposition 2.5. Condition (b) holds since $\delta^{P^{\mathcal{R}}}_{\overline{\mathbf{g}}A}$ is a $P$-equivalence relation. Condition (c) is a consequence of the fact that $\overline{\mathbf{g}}$ preserves products. Indeed, for $A$ and $B$ in $\mathcal{C}$, the underlying arrow of the iso $\langle \overline{\mathbf{g}}\mathrm{pr}_1, \overline{\mathbf{g}}\mathrm{pr}_2\rangle\colon \overline{\mathbf{g}}(A \times B) \to \overline{\mathbf{g}}A \times \overline{\mathbf{g}}B$ is the identity on $A \times B$ because of (3). Hence the underlying arrow of its inverse is the identity too, that is to say $\delta^{P^{\mathcal{R}}}_{\overline{\mathbf{g}}(A\times B)} \geq \delta^{P^{\mathcal{R}}}_{\overline{\mathbf{g}}A} \boxtimes \delta^{P^{\mathcal{R}}}_{\overline{\mathbf{g}}B}$. Hence $P$ is elementary by Proposition 2.5, and $\delta^P_A = \delta^{P^{\mathcal{R}}}_{\overline{\mathbf{g}}A}$ by Remark 2.6. So $\overline{\mathbf{g}} = \overline{\mathbf{j_P}}$ which, together with (5), yields $\mathbf{g} = \mathbf{j}_P$ as required. □

**Lemma 3.4.** *For elementary doctrines $E, E'$, the functor*

$$\mathcal{K}_{E,E'}\colon \mathbf{ED}(E, E') \longrightarrow \mathbf{PD}^{\mathcal{T}}(\mathbf{j}_E, \mathbf{j}_{E'}).$$

*is an isomorphism.*

*Proof.* The functor $\mathcal{K}_{E,E'}$ is clearly faithful. It is also full since $\mathbf{ED}$ is a 2-full subcategory of $\mathbf{PD}$. For any 1-cell $\mathbf{f}\colon E \to E'$, it is $\mathcal{K}\mathbf{f} = \mathbf{f}$, hence $\mathcal{K}$ is clearly 1-faithful and we are left to show that a homomorphism of coalgebras $\mathbf{f}\colon \mathbf{j}_E \to \mathbf{j}_{E'}$ is also a 1-cell $\mathbf{f}\colon E \to E'$ in $\mathbf{ED}$. By Corollary 2.7, it is enough to show that, for every object $A$,

$$\widehat{\mathbf{f}}_{A\times A}\left(\delta^E_A\right) = \langle \overline{\mathbf{f}}\mathrm{pr}_1, \overline{\mathbf{f}}\mathrm{pr}_2\rangle^* \left(\delta^{E'}_{\overline{\mathbf{f}}A}\right).$$

Since $\mathbf{f}$ is a homomorphism of coalgebras we have

$$\overline{\mathbf{f}^{\mathcal{T}}} \circ \overline{\mathbf{j}_E} = \overline{\mathbf{j}_{E'}} \circ \overline{\mathbf{f}} \qquad \text{and} \qquad \widehat{\mathbf{f}}_A = \left(\widehat{\mathbf{j}_{E'}}\right)_{\overline{\mathbf{f}}A} \circ \widehat{\mathbf{f}}_A = \left(\widehat{\mathbf{f}^{\mathcal{T}}}\right)_{\overline{\mathbf{j}_E}A} \circ \left(\widehat{\mathbf{j}_E}\right)_A = \left(\widehat{\mathbf{f}^{\mathcal{T}}}\right)_{\overline{\mathbf{j}_E}A}.$$

Notice that $\mathbf{f}^{\mathcal{T}}$ is a 1-cell in $\mathbf{ED}$ and that $\delta^{E^{\mathcal{R}}}_{\overline{\mathbf{j}_E}(A)} = \left(\widehat{\mathbf{j}_E}\right)_{A\times A}\left(\delta^E_A\right) = \delta^E_A$, and similarly $\delta^{E'^{\mathcal{R}}}_{\overline{\mathbf{j}_E}(\bar{\mathbf{f}}A)} = \delta^{E'}_{\bar{\mathbf{f}}A}$. Hence

$$\begin{aligned}\widehat{\mathbf{f}}_{A\times A}(\delta^E_A) &= \left(\widehat{\mathbf{f}^{\mathcal{T}}}\right)_{\overline{\mathbf{j}_E}A\times\overline{\mathbf{j}_E}A}\left(\delta^{E^{\mathcal{R}}}_{\overline{\mathbf{j}_E(A)}}\right)\\ &= \langle\overline{\mathbf{f}^{\mathcal{T}}}(\overline{\mathbf{j}_E}\mathrm{pr}_1),\overline{\mathbf{f}^{\mathcal{T}}}(\overline{\mathbf{j}_E}\mathrm{pr}_2)\rangle^*\left(\delta^{E'^{\mathcal{R}}}_{\overline{\mathbf{j}_{E'}}(\bar{\mathbf{f}}A)}\right)\\ &= \langle\overline{\mathbf{f}^{\mathcal{T}}}(\overline{\mathbf{j}_E}\mathrm{pr}_1),\overline{\mathbf{f}^{\mathcal{T}}}(\overline{\mathbf{j}_E}\mathrm{pr}_2)\rangle^*\left(\delta^{E'}_{\bar{\mathbf{f}}A}\right)\\ &= \langle\bar{\mathbf{f}}\mathrm{pr}_1,\bar{\mathbf{f}}\mathrm{pr}_2\rangle^*\left(\delta^{E'}_{\bar{\mathbf{f}}A}\right)\end{aligned}$$

where in the first and the last steps we used the fact that $\overline{\mathbf{j}_E}$ preserves chosen products since $\delta^E_{A\times A} = \delta^E_A \boxtimes \delta^E_A$ by Remark 2.6. □

This concludes the proof of Theorem 3.2.

**Remark 3.5.** It is also possible to prove that the comonad $\mathcal{T}$ is KZ, see [9].

## 4. Elimination of imaginaries

Recall from Poizat [22] that a structure $\mathcal{A}$ in a possibly multi-sorted language $\mathsf{L}$ has ***uniform elimination of imaginaries*** if, for every formula $\rho$ with at most the free variables $x$ and $x'$, which we may write $\rho(x,x')$, such that

$$\mathcal{A} \models \text{‘}\rho \text{ is symmetric and transitive’},$$

there is a formula $\phi(x,y)$ such that

$$\begin{gathered}\mathcal{A} \models \text{‘}\phi \text{ is a functional relation’}\\ \mathcal{A} \models \forall x{:}A\,\forall x'{:}A\ [\rho(x,x') \leftrightarrow \exists y{:}B\ (\phi(x,y)\wedge\phi(x',y))]\,.\end{gathered}$$

A theory $T$ in a possibly multi-sorted language $\mathsf{L}$ with equality has ***uniform elimination of imaginaries*** if every model of $T$ has uniform elimination of imaginaries. A result in [6, Theorem 4.4.2] ensures that $T$ has uniform elimination of imaginaries precisely when, for every formula $\sigma(x,y)$ such that

$$T \vdash \text{‘}\sigma \text{ is reflexive, symmetric and transitive’} \tag{6}$$

there is a formula $\phi(x,z)$ such that

$$T \vdash \forall y{:}A\,\exists! z{:}B\,\forall x{:}A\,(\sigma(x,y)\leftrightarrow\phi(x,z))\,. \tag{7}$$

The syntactic characterization gives an interesting extension of the notion of uniform elimination of imaginaries to any theory in intuitionistic first order logic.

**Definition 4.1.** Let $T$ be an intuitionistic first order theory. We say that $T$ has ***uniform elimination of imaginaries*** if, for every formula $\sigma(x, y)$ such that (6) holds, there is a formula $\phi(x, z)$ such that (7) holds.

We can apply the results in the previous section to complete any intuitionistic theory $T$ to one with uniform elimination of imaginaries as follows.

Recall from [15, 20, 21] how theories and doctrines are related. As detailed in [15, Example 2.2], for a given theory $T$, consider the first-order doctrine $\mathsf{P}_T$ ***associated*** to $T$ whose base category consists of contexts and term substitutions, and whose fibre over a context is the Lindenbaum-Tarski algebra of well-formed formulas in that context. As in [4, Section 8.2.1], consider also the category $\mathrm{Mod}(T)$ of models $\mathfrak{M}$ of $T$ and elementary homomorphisms $f: \mathfrak{M} \to \mathfrak{N}$, *i.e.* $f$ is a homomorphism on the underlying algebras such that for each well-formed formula $\alpha$ in $\mathsf{L}$, the map $f$ preserves the interpretation of $\alpha$, *i.e.* for every $\langle m_1, \ldots, m_i \rangle$,

$$\text{if } \langle m_1, \ldots, m_i \rangle \in \alpha^{\mathfrak{M}}, \text{ then } \langle f(m_1), \ldots, f(m_i) \rangle \in \alpha^{\mathfrak{N}}.$$

It is easy to see that the category $\mathrm{Mod}(T)$ is equivalent to the hom-category $\mathbf{FOD}(\mathsf{P}_T, \mathscr{P})$ on the 1-cell into the first-order doctrine $\mathscr{P}: \mathcal{S}\!\mathit{et}^{\mathrm{op}} \longrightarrow \mathbf{Pos}$

$$\mathrm{Mod}(T) \cong \mathbf{FOD}(\mathsf{P}_T, \mathscr{P}).$$

Conversely, for a first-order doctrine $P$, let $\mathrm{Th}_P$ be the internal language of $P$ as in [21]. Similarly, $\mathrm{Mod}(\mathrm{Th}_P) \cong \mathbf{FOD}(P, \mathscr{P})$.

One word of warning: when $T$ has equality, one can consider models where the equality predicate is interpreted as the diagonal. If $\mathrm{Mod}_=(T)$ denotes the full subcategory of $\mathrm{Mod}(T)$ on models where the equality predicate is interpreted as the diagonal, and if $P$ is an elementary first-order doctrine (*i.e.* a first order hyperdoctrine in the sense of [21]), then the isomorphisms above restrict to isomorphisms

$$\mathrm{Mod}_=(T) \cong \mathbf{HD}(\mathsf{P}_T, \mathscr{P}) \qquad\qquad \mathrm{Mod}_=(\mathrm{Th}_P) \cong \mathbf{HD}(P, \mathscr{P}).$$

**Theorem 4.2.** *Given an intuitionistic theory $T$, consider the doctrine $(\mathsf{P}_T)^{\mathcal{R}}$ and write $\overline{T}$ for the theory $\mathrm{Th}_{(\mathsf{P}_T)^{\mathcal{R}}}$ associated to the doctrine $(\mathsf{P}_T)^{\mathcal{R}}$. Then $\overline{T}$ has uniform elimination of imaginaries in the sense of Definition 4.1.*

*Proof.* The sorts of $\overline{T}$ are of the form $(a, \rho)$ where $a{:}A$ and $\rho$ is a formula in the variables $a, a'$ with also $a'{:}A$. So a formula $\sigma(x, x')$, where $x, x'{:}(a, \rho)$, is such that

$$\overline{T} \vdash \text{`}\sigma \text{ is reflexive, symmetric and transitive'}$$

precisely when $[\sigma(x, x')] \in (\mathsf{P}_T)^{\mathcal{R}}((a, \rho) \times (a, \rho))$ is a $(\mathsf{P}_T)^{\mathcal{R}}$-equivalence relation. Hence $\sigma$ is descent data for $\rho$ and we can pick $(a, \sigma)$ as $B$ and $\sigma$ as $\phi$ in (7). So the consequence in (7) becomes

$$\overline{T} \vdash \forall y{:}(a, \rho)\, \exists! z{:}(a, \sigma)\, \forall x{:}(a, \rho)\, (\sigma(x, y) \leftrightarrow \sigma(x, z)) \tag{8}$$

which is clearly provable—note that the two occurrences of $\sigma$ in (8) are in different fibres of $(\mathsf{P}_T)^{\mathcal{R}}$. $\square$

Recall that, for a set $A$ and an equivalence relation $R$ on $A$, the inf-semilattice $\mathscr{P}^{\mathcal{R}}(A,R)$ consists of those subsets $S$ of $A$ which are invariant with respect to $R$, *i.e.*

$$\text{if } a' \; R \; a \in S, \text{ then } a' \in S.$$

The base category $\mathcal{E}_{\mathscr{P}}$ of the doctrine $\mathscr{P}^{\mathcal{R}}$ is the category of equivalence relations and relation-preserving functions. Consider the 1-cell $\mathbf{q} = (\overline{\mathbf{q}}, \widehat{\mathbf{q}})\colon \mathscr{P}^{\mathcal{R}} \longrightarrow \mathscr{P}$ in $\mathbf{HD}$ whose functor $\overline{\mathbf{q}}\colon \mathcal{E}_{\mathscr{P}} \longrightarrow \mathcal{S}et$ takes $f\colon (A,R) \to (A',R')$ to the induced function $f'\colon A/R \to A'/R'$ on the quotient sets, while the monotone function $(\widehat{\mathbf{q}})_{(A,R)}\colon \mathscr{P}^{\mathcal{R}}(A,R) \longrightarrow \mathscr{P}(A/R)$ sends $S \in \mathscr{P}^{\mathcal{R}}(A,R)$ to the set $\left\{[a] \in A/R \;\middle|\; a \in S\right\}$. It is easy to check that $(\widehat{\mathbf{q}})_{(A,R)}$ is an isomorphism.

**Proposition 4.3.** *For every first-order doctrine $P$, the functor*

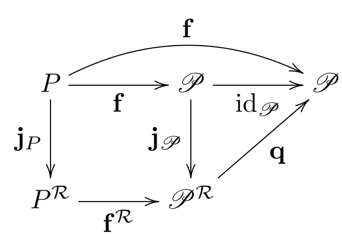
$$\begin{array}{ccccc} \mathbf{FOD}(P,\mathscr{P}) & \xrightarrow{\mathcal{R}_{P,\mathscr{P}}} & \mathbf{HD}(P^{\mathcal{R}},\mathscr{P}^{\mathcal{R}}) & \xrightarrow{\mathbf{q}\circ -} & \mathbf{HD}(P^{\mathcal{R}},\mathscr{P}) \\ \wr\Vert & & & & \Vert\wr \\ \mathrm{Mod}(\mathrm{Th}_P) & & \longrightarrow & & \mathrm{Mod}_{=}(\mathrm{Th}_{P^{\mathcal{R}}}) \end{array} \tag{9}$$

*applies the category of models of $P$ into the category of models of $P^{\mathcal{R}}$. Moreover, if $P$ is elementary, then the functor precomposition with $\mathbf{j}_P\colon P \to P^{\mathcal{R}}$*

$$\begin{array}{ccc} \mathbf{HD}(P^{\mathcal{R}},\mathscr{P}) & \xrightarrow{-\circ\mathbf{j}_P} & \mathbf{HD}(P,\mathscr{P}) \\ \wr\Vert & & \Vert\wr \\ \mathrm{Mod}_{=}(\mathrm{Th}_{P^{\mathcal{R}}}) & \longrightarrow & \mathrm{Mod}_{=}(\mathrm{Th}_P) \end{array} \tag{10}$$

*is an equivalence of categories.*

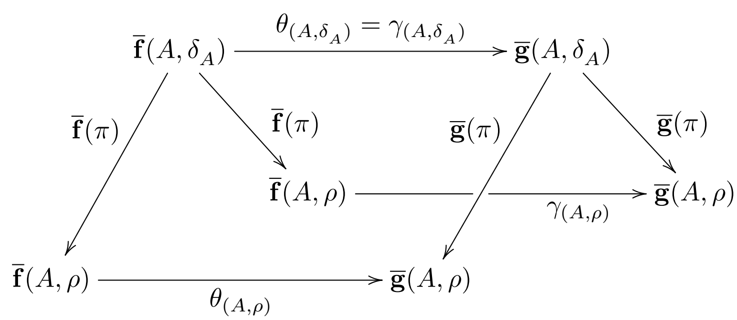
*Proof.* Let $P$ be a hyperdoctrine. Since every 1-cell $\mathbf{f}\colon P \to \mathscr{P}$ in $\mathbf{ED}$ factors as



the composition

$$\mathbf{HD}(P,\mathscr{P}) \hookrightarrow \mathbf{FOD}(P,\mathscr{P}) \xrightarrow{(\mathbf{q}\circ -)\mathcal{R}_{P,\mathscr{P}}} \mathbf{HD}(P^{\mathcal{R}},\mathscr{P}) \xrightarrow{-\circ\mathbf{j}_P} \mathbf{HD}(P,\mathscr{P})$$

is naturally isomorphic to the identity. This shows that $-\circ\mathbf{j}_P$ is full and essentially surjective. It remains to prove its faithfulness. Let $\mathbf{f},\mathbf{g}\colon P^{\mathcal{R}} \to \mathscr{P}$ be in $\mathbf{HD}$, consider two parallel 2-cells $\theta,\gamma\colon \mathbf{f} \Rightarrow \mathbf{g}$ and suppose $\theta\circ\mathbf{j}_P = \gamma\circ\mathbf{j}_P$. So the following diagram of sets and functions commutes




where clearly $\pi = \mathrm{id}_A\colon (A,\delta_A) \to (A,\rho)$ is such that $\top_{(A,\rho)} \leq \exists_\pi^{P^{\mathcal{R}}}(\top_{(A,\delta_A)})$ where $\exists_\pi^{P^{\mathcal{R}}}$ is the left adjoint to $\pi^* = P^{\mathcal{R}}(\pi)$ as in Remark 2.3. Since $\mathbf{f}$ is in $\mathbf{HD}$, the functor $\widehat{\mathbf{f}}_{(A,\rho)}$ preserves the top and left adjoints. Hence

$$\overline{\mathbf{f}}(A,\rho) = \top_{\overline{\mathbf{f}}(A,\rho)} \subseteq \exists^{\mathscr{P}}_{\overline{\mathbf{f}}(\pi)}(\top_{\overline{\mathbf{f}}(A,\delta_A)}) = \overline{\mathbf{f}}(\pi)\big[\overline{\mathbf{f}}(A,\delta_A)\big]$$

which proves that $\overline{\mathbf{f}}(\pi)$ is surjective. Therefore $\theta_{(A,\rho)} = \gamma_{(A,\rho)}$. □

Reading Proposition 4.3 for a doctrine of the form $\mathsf{P}_T$ where $T$ is a multi-sorted theory (not necessarily with equality), one sees that the functor (9) ensures that every model of the original theory $T$ can be turned functorially into a model of the theory $\overline{T} = \mathrm{Th}_{(\mathsf{P}_T)^{\mathcal{R}}}$, which uniformly eliminates imaginaries by Theorem 4.2. Moreover, when $T$ has equality, the functor in (10) ensures that every model of $T$ can be expanded to a model of $\overline{T}$, and every model of $\overline{T}$ is completely determined by its reduct to $T$.

## 5. Comparison with Shelah's $T^{\mathrm{eq}}$

The construction of $\overline{T}$ in Section 4 is a radically different, simpler characterisation of Shelah's $(-)^{\mathrm{eq}}$ in [24] than the one given in [5]. To see this, let $T$ be an intuitionistic theory in a possibly multi-sorted language $\mathsf{L}$. In [5] it is proved that, if $T$ is classical (*i.e.* $T \vdash \alpha \vee \neg\alpha$ for all well-formed formulas $\alpha$ in $\mathsf{L}$), has equality and

$$\begin{aligned} &T \vdash \exists x{:}A\ x = x && \text{for every sort } A \text{ in } \mathsf{L} \\ &T \vdash \exists x{:}A_0\, \exists y{:}A_0\ x \neq y && \text{for some sort } A_0 \text{ in } \mathsf{L} \end{aligned} \tag{11}$$

then $T^{\mathrm{eq}}$ coincides with the theory associated to the pretopos completion of the syntactic category of the theory $T$ as in [18]. Recall from [18, Section 8.2] that the syntactic category $\mathbf{R}_T$ of $T$ consists of

**objects:** pairs $\langle \vec{x}, \phi\rangle$ where $\vec{x}$ is a context in $\mathsf{L}$ and $\phi$ is a well-formed formula in context $\vec{x}$;

**arrows:** an arrow $[\theta]\colon \langle \vec{x}, \phi \rangle \twoheadrightarrow \langle \vec{y}, \psi \rangle$ is an equivalence class of formulas in a context $\vec{x'}, \vec{y'}$ with appropriate distinct variables, such that $\theta$ is a functional relation.

It is denoted as $\mathbb{T}$ in [5, Section 4] and computed as $\mathcal{EF}_{(\mathsf{P}_T)_c}$ in [13, Section 3].

From Theorem 4.2, we can obtain a similar result for general intuitionistic first order theories. First we need a strengthening of [1, Lemma 2.2(i)] when an ex/reg completion produces a pretopos. In order to state the result, recall that an object $B$ in a regular category is ***well-supported*** if the unique arrow $B \longrightarrow\!\!\!\triangleright 1$ is regular epic.

**Proposition 5.1.** *Let $\mathcal{A}$ be a coherent category, that is, a regular category with pullback-stable unions of subobjects. Suppose that*

(i) *for every object $A$ in $\mathcal{A}$, there is a mono $m\colon A \rightarrowtail B$ into a well-supported object $B$;*

(ii) *there is a decidable object $D$ in $\mathcal{A}$ such that the complement of its diagonal is well-supported.*

*Then the ex/reg completion of $\mathcal{A}$*

$$\mathcal{A} \xrightarrow{\Gamma_{ex/reg}} \mathcal{A}_{ex/reg}$$

*is also the pretopos completion of $\mathcal{A}$ as a coherent category.*

*Proof.* Is in the Appendix. □

**Remark 5.2.** The two conditions in the hypotheses of Proposition 5.1 easily compare with those in (11): condition 5.1(i) is just a categorical reformulation of the first in (11) read in the syntactic category. More interestingly, condition 5.1(ii) has a global requirement about the object $D$ that cannot be spotted in the classical case.

As a consequence of Proposition 5.1, we know that the theory $T^{\mathrm{eq}}$ for $T$ a theory in classical first order logic is the theory associated to the subobject doctrine of $(\mathbf{R}_T)_{\mathrm{ex/reg}}$. In turn the ex/reg completion has a neat algebraic description in terms of doctrines as $\mathcal{C}_{\mathrm{ex/reg}} = \mathcal{EF}_{(\mathrm{Sub}_{\mathcal{C}})^{\mathcal{R}}}$, see [16, Example 3.2].

In conclusion, the characterisation of $T^{\mathrm{eq}}$ in [5] is the theory of the subobject doctrine of the category

$$\mathcal{EF}_{\left(\mathrm{Sub}_{\left(\mathcal{EF}_{(\mathsf{P}_T)_c}\right)}\right)^{\mathcal{R}}}.$$

But there is an equivalence between $(\mathsf{P}_T)^{\mathcal{R}}$ and a subobject doctrine if and only if $\mathsf{P}_T$ validates a rule of choice, see [17, Proposition 4.11].

**Acknowledgements**

We would like to thank John Power for enlightening discussions in the early stages of the research, and we wish to express our gratitude to the late Erik Palmgren for directing us to the connection between the pretopos completion and the elimination of imaginaries. We would also like to express our gratitude to the referee who provided extremely useful comments which helped to improve the original draft of the paper.

## A. Appendix

The appendix is devoted to the proof of Proposition 5.1 which relies on a technical result about coherent categories.

**Proposition A.1.** *Let $\mathcal{A}$ be a coherent exact category. Suppose that*

(i) *for every object $A$ in $\mathcal{A}$, there is a mono $m: A \rightarrowtail B$ into a well-supported object $B$;*

(ii) *there is a decidable object $D$ in $\mathcal{A}$ such that the complement of its diagonal is well-supported.*

*Then $\mathcal{A}$ is a pretopos.*

We defer the proof of the technical result to after the proof of Proposition 5.1.

**Proof of Proposition 5.1**. It is well-known that the ex/reg completion of a coherent category is coherent and the embedding $\Gamma_{\mathrm{ex/reg}}: \mathcal{A} \to \mathcal{A}_{\mathrm{ex/reg}}$ is a coherent functor, see [8, Corollary 3.3.10]. We are left to show that, when $\mathcal{A}$ satisfies conditions 5.1(i)-(ii), the exact category $\mathcal{A}_{\mathrm{ex/reg}}$ is in fact a pretopos, as the required universal property follows from the fact that $\Gamma_{\mathrm{ex/reg}}$ is universal among regular functors into exact categories. To this aim, it is enough to show that the ex/reg completion preserves conditions 5.1(i) and (ii) and apply Proposition A.1.

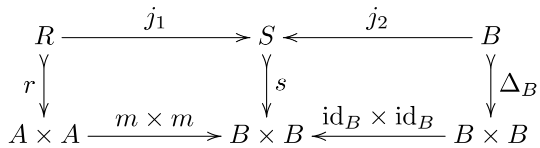
Recall from [3, Section 2.3] that objects of $\mathcal{A}_{\mathrm{ex/reg}}$ are equivalence relations $R \rightarrowtail A \times A$ in $\mathcal{A}$ and that an arrow from $r: R \rightarrowtail A \times A$ to $s: S \rightarrowtail B \times B$ is a relation $f: F \rightarrowtail A \times B$ such that

$$f \cdot r = f = s \cdot f, \quad r \leq f^{\mathrm{o}} \cdot f \quad \text{and} \quad f \cdot f^{\mathrm{o}} \leq s,$$

where $f^{\mathrm{o}}$ is the converse of the relation $f$ and $f \cdot g$ denotes relational composition. The embedding $\Gamma_{\mathrm{ex/reg}}: \mathcal{A} \to \mathcal{A}_{\mathrm{ex/reg}}$ maps an object $A$ in $\mathcal{A}$ to the identity relation $\Delta_A := \langle \mathrm{id}_A, \mathrm{id}_A \rangle$ and an arrow $f: A \to B$ to its graph $\langle \mathrm{id}_A, f \rangle: A \rightarrowtail A \times B$. We know it preserves the regular structure; in particular, it takes a well-supported object to a well-supported object.

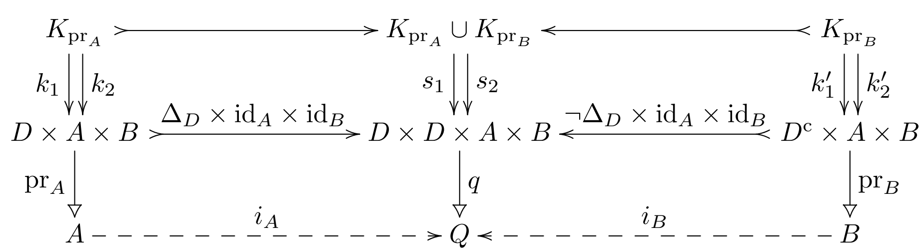
*Ad* (i). Let $r: R \rightarrowtail A \times A$ be an arbitrary object in $\mathcal{A}_{\mathrm{ex/reg}}$. By hypothesis, there is a mono $m: A \rightarrowtail B$ into a well-supported object $B$ in $\mathcal{A}$. In $\mathcal{A}$ consider the join $s: S \rightarrowtail B \times B$ of the subobjects $(m \times m)r: R \rightarrowtail B \times B$ and

$\Delta_A\colon B \rightarrowtail B \times B$ which is clearly an equivalence relation, say $j_1\colon R \rightarrowtail S$ and $j_2\colon B \rightarrowtail S$ are the two inclusions into the join. Moreover the diagram




produces a mono from $r$ into $s$ and a factor from $\Delta_B$ into $s$ of the terminal arrow which ensures that $s$ is well-supported in $\mathcal{A}_{\mathrm{ex/reg}}$.
*Ad* (ii). Immediate since $\Gamma_{\mathrm{ex/reg}}\colon \mathcal{A} \to \mathcal{A}_{\mathrm{ex/reg}}$ preserves finite unions and finite intersections of subobjects. □

**Proof of Proposition A.1**. To see that $\mathcal{A}$ is extensive, consider first that, in a coherent category, any disjoint sum that happens to exist is universal in the sense of [2, Definition 2.10] because a disjoint sum is also a disjoint union of subobjects, and these are stable under pullback. So, by [2, Proposition 2.14] it is enough to prove that $\mathcal{A}$ has disjoint sums. And, by hypothesis (i) it suffices to do so just for well-supported objects. So let $A$ and $B$ be well-supported objects. Let $D$ be a decidable object such that, in the complement $\neg\Delta_D\colon D^{\mathrm{c}} \rightarrowtail D \times D$ of the diagonal $\Delta_D\colon D \rightarrowtail D \times D$, the object $D^{\mathrm{c}}$ is well-supported. In the diagram




where the side columns are kernel pairs of regular epis because $D$, $D^{\mathrm{c}}$, $A$ and $B$ are well-supported. The join is taken in the poset of subobjects of $(D \times D \times A \times B)^2$; it is an equivalence relation as the the (partial) equivalence relations $K_{\mathrm{pr}_A} \rightarrowtail (D \times D \times A \times B)^2$ and $K_{\mathrm{pr}_B} \rightarrowtail (D \times D \times A \times B)^2$ are disjoint. So, taking the coequalizer of $K_{\mathrm{pr}_A} \cup K_{\mathrm{pr}_B} \rightrightarrows D \times D \times A \times B$, all columns are exact; now one easily checks that the bottom row is a disjoint sum. □

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