Coupled and splitting bedload sediment transport models based on a modified flux-wave approach

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Abstract

Numerical modeling of free-surface flow over a mobile bed with predominantly bedload sediment transport can be done by solving the shallow water and Exner equations using coupled and splitting approaches. The coupled method uses a coupling of the governing equations at the same time step leading to a non-conservative solution. The splitting method solves the Exner and the shallow water equations in a separate manner, and is only capable of modeling weak free-surface and bedload interactions. In the current study, an extended version of a Godunov-type wave propagation algorithm is presented for modeling of morphodynamic systems using both coupled and splitting approaches. In the introduced coupled method the entire morphodynamic system is solved in the form of a conservation law. For the splitting technique, a new wave Riemann decomposition is defined which enables the scheme to be utilized for mild and strong interactions. To consider the bedload sediment discharge within the Exner equation, the Smart and Meyer-Peter & Müller formulae are used. It was found that the coupled solution gives accurate predictions for all investigated flow regimes including propagation over a dry-state using a Courant-Friedrichs-Lewy (CFL) number equal to 0.6. Furthermore, the splitting method was able to model all flow regimes with a lower CFL number of 0.3.

1. Introduction

The interaction between water flow and sediment transport is a contemporary hydraulic engineering problem studied for many problems such as dam-break and multiple fluvial systems. Morphodynamic models should at least be able to accurately evaluate the morphological evolution of the bed as well as the variations of the water surface. These models can be methodologically categorized into coupled and decoupled solutions. For decoupled solutions, only the simplified conservation equations are considered, and it is assumed that the rate of morphological variation is less important than the hydrodynamic changes (Cao et al., 2002). For coupled solutions of the flow field, sediment transport and morphological evolutions are interrelated, and the amount of bed changes is considerable (Cao et al., 2004). Physically coupled models are typically based on the Shallow Water Equations (SWEs) which can be used to compute the hydrodynamics of the flow, and the Exner equation which can be used to evaluate the bedload sediment transport (Castro Díaz et al., 2008; Delis & Papoglou, 2008), and the coupled models are considered as capacity or equilibrium sediment transport approaches. These sets of equations form a nonlinear system of hyperbolic conservation laws that are numerically solved using coupled or splitting methods (Canestrelli et al., 2010; Serrano-Pacheco et al., 2012).

Generally, capacity or equilibrium models such as the Exner equation are considered in the capacity regimes, and mainly evaluated based upon local hydrodynamic conditions (Cao et al., 2011, 2012, 2016). Non-capacity or non-equilibrium models are obtained through mass exchange with the bed, comprise spatial and temporal lag effects between transport capacity and flow conditions (Cao et al., 2016) and can provide more accurate results for strong interaction taking place between the free-surface and bedload sediment flow (Wu et al., 2004, 2018).

Splitting methods solve the SWEs and the Exner equation in an entirely separate manner (Wu et al., 2004). In this approach, the Exner equation is first solved and the resulting bed profile update

is used as a source term for the SWEs. This method is relatively easy to code and allows minimization of the computational costs, since no additional computation is needed for the hydrodynamic phase. However, one major limitation of these methods is that they can only be applied in cases of weak or mild sediment transport and surface wave interactions. This is mainly due to the assumption of a constant total depth which is not sufficient for modeling strong interactions taking place between the free-surface and bed sediment (Wu, 2007).

The coupled approach solves the entire set of governing equations, i.e. the hydrodynamic and morphodynamic equations, simultaneously at each time step, thus, is more stable than the splitting approaches (Benkhaldoun & Seaïd, 2011; Canestrelli et al., 2010; Castro Díaz et al., 2008; Hudson & Sweby, 2005). Moreover, this approach is capable of readily approximating strong interactions (Cordier et al., 2011). However, a main drawback of the coupled method is that a conservative formulation is not available, and therefore, the solution is only developed based upon non-conservative schemes which are notorious for producing incorrect shock wave speeds in some situations (Canestrelli et al., 2010; Hudson & Sweby, 2005; Siviglia et al., 2008)

Various numerical methods have been developed for solving coupled morphodynamic systems, among which the finite-volume method has been widely used (e.g. Castro Díaz et al., 2008; Delis & Papoglou, 2008; Serrano-Pacheco et al., 2012). Over recent decades, different non-conservative formulations of finite-volume Godunov based methods have been extended to the coupled solution of morphodynamic systems. Wu and Wang (2008) implemented a one dimensional (1D) sediment transport model which handles both suspended and bedload sediment discharge within the computations. Fraccarollo et al. (2003) devised a modified version of the Harten-Lax-van Leer (HLL) scheme for intense sediment transport simulations. A more accurate model was introduced by Rossati and Fraccarollo (2006) who used a well-balanced coupled method with a new strategy

for the treatment of non-conservative fluxes. This approach was further developed by Murrillo and García-Navarro (2010) who defined a coupled Jacobian matrix (CJM) method using a triangular mesh for the simulation of bedload sediment transport. Although the proposed model was accurate, it was proven to be computationally expensive for real morphodynamic problems (Juez et al., 2014). To overcome this issue, Juez et al. (2014) introduced a first-order scheme which mainly solves the two dimensional (2D) morphodynamic and hydrodynamic phases in a weakly coupled form. A strategy was developed to consider all required waves involved in the Riemann problem, similar to the coupled solution. From a numerical point of view, Godunov methods provide more reliable results for morphodynamic systems, and are able to precisely capture the shocks and other discontinuities within the solution. The main shortcoming of these solvers is the existence of non-conservative fluxes which cause complexity in eigenvalue calculations (Canestrelli et al., 2010; Hudson & Sweby, 2005; Siviglia et al., 2008).

The wave propagation algorithm has been successfully applied for modeling various fluid flows including gas-dynamic problems (Bale et al., 2002), flooding over complex bathymetry deviations with wet/dry front propagation (George, 2008) and water hammer problems (Mahdizadeh et al., 2018; Mahdizadeh, 2019). The methods used in these studies are generally Godunov-type methods, which in contrast to other Riemann solvers that directly use interface fluxes, re-average the wave arising in the Riemann solutions into neighbouring finite volume computational cells (George, 2008).

To the authors' best knowledge, the wave propagation algorithm has not yet been extended for modeling bedload sediment transport dynamics. Therefore, the main aim of this paper is to devise an extended morphodynamic solver based on a Godunov-type wave propagation algorithm using both coupled and splitting strategies.

In the proposed approach, the bedload sediment transport model applies an interaction parameter which is variable in time and space and depends upon the fluid depth. Moreover, to accurately compute the bedload sediment discharge, the solver uses the Smart (1984) formula which has been proven to perform well in reproducing experimental data. In comparison to other accurate and novel coupled morphodynamic solvers, such as the coupled Jacobian matrix (CJM), the proposed method developed based upon the wave propagation formula provides equally accurate results but uses more straightforward and simpler formulations. Additionally, a new wave decomposition method is introduced for the splitting approach which significantly enhances the efficiency of the solver compared to traditional splitting solvers in particular for modeling supercritical and transcritical flow regimes. This new feature allows the solver to calculate an accurate wave speed for mild and strong interactions where the total water depth is not constant.

The proposed method will generalize the shallow water solver introduced by Mahdizadeh et al. (2011, 2012) for bedload sediment transport modeling. To obtain a basic understanding, the problem is considered 1D. The morphodynamic solver defined herein treats any source terms within the flux differencing of the finite-volume neighboring cells and is well-balanced for both splitting and coupled techniques. Moreover, the method can cope with the difficulties stated for upwind solvers developed based on non-conservative fluxes.

The paper is structured as follows: Section 2 provides the governing equations for 1D morphodynamic systems with different sediment transport formulae. This is followed by a brief explanation of the wave propagation algorithm. Section 3 describes the methods including the flux-wave approach for a coupled system. Additionally, the development of a new modified splitting solver and the corresponding wave discretization for sediment transport equations are provided. In Section 4, the proposed method is tested and validated by comparing its results with analytical and

reference solutions. Finally, the paper concludes with a summary of the proposed method and a conclusion of the findings.

2. Governing equations

The 1D SWEs coupled with a bedload sediment transport model can be expressed as follows (Lyn & Altinakar, 2002):

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \tag{1}$$

$$\frac{\partial (hu)}{\partial t} + \frac{\partial (hu^2 + gh^2/2)}{\partial x} = -gh\frac{\partial B}{\partial x} - \frac{\tau_b}{\rho} \tag{2}$$

$$(1-p)\frac{\partial B}{\partial t} + \frac{\partial q_b}{\partial x} = 0 \tag{3}$$

where u is the depth-averaged velocity and h is the water depth over the bed surface elevation (B), p is the porosity of the sediment layer, q_b is the bedload sediment discharge, g is the acceleration due to gravity, x is the longitudinal position, t is time and τ_b is the bed shear stress calculated by:

$$\tau_b = \rho gh S_f \tag{4}$$

where ρ is the water density, and S_f is the bed friction coefficient which can be evaluated by Manning's equation:

$$S_f = \frac{n_m^2 u^2}{h^{4/3}}, \tag{5}$$

where n_m is Manning's roughness coefficient. The bedload sediment transport discharge can be obtained from:

$$q_b = A_g u^{m_g} \tag{6}$$

where m_g is a constant which normally takes a value between $1 \leq m_g \leq 4$, and A_g is the interaction parameter which mainly depends on the sediment properties, is calculated through experimental

data, and can be expressed as $A_g = K.\psi$. Where ψ is obtained using different bedload sediment

formulae as summarized in Table 1, and K can be obtained from:

$$K = \frac{g^{1/2} n_m^3}{(G_s - 1) h^{1/2}}$$ (7)

where $G_s = \rho_s / \rho$ is the specific gravity and ρ_s is the sediment density. The parameter θ_c is the

critical Shields parameter and θ implies the dimensionless bed shear stress defined as:

$$\theta = \frac{n_m^2}{(G_s - 1) d_m h^{1/3}} u^2$$ (8)

where d_m is the median grain size of bed materials.

Table 1. Expression of function ψ based on different bedload sediment transport formulations.

Reference	Formula	
Meyer-Peter & Müller (1948)	$\psi = 8(1 - \theta_c / \theta)^{3/2}$	(9)
Smart (1984)	$\psi = 4 \left(\dfrac{d_{90}}{d_{30}} \right)^{0.2} \left(\dfrac{S_0^{0.6} h^{1/6}}{n_m} \right) (1 - \theta_s / \theta)$	(10)

In Smart's (1984) formula, described in Eq. (10), d_{30} and d_{90} correspond to diameters where 30%

and 90% of the bed material sample is finer by weight. Furthermore, S_0 is the bed slope obtained

as $S_{0x} = -\partial B / \partial x$, and θ_s is Smart's critical Shield parameter which can be obtained from:

$$\theta_s = \theta_c \cos \varphi \left(1 - \frac{\tan \varphi}{\tan \chi} \right)$$ (11)

where χ is the angle of repose for saturated bed materials, φ is the angle of the bed slope, and θ_c is the critical bed shear stress considered as $\theta_c = 0.047$. For cases where the interaction parameter A_g takes a constant value, Eq. (6) is called the Grass (1991) formula. For Grass-type formula, based upon the test cases given in Cordier et al. (2011), a small value of the interaction parameter constant, in the range of $0 \leq A_g < 0.01$, indicates a rather weak interaction and for the values above that mild or strong interactions takes place.

With $\eta = \dfrac{1}{1-p}$, the 1D morphodynamic system presented in Eqs. (1-3) may be written in the form of a conservation law as:

$$U_t + F(U)_x = S(U,x) \tag{12}$$

where

$$U = \begin{bmatrix} h \\ hu \\ B \end{bmatrix} \tag{13}$$

$$F(U) = \begin{bmatrix} hu \\ hu^2 + 1/2gh^2 \\ \eta q_b \end{bmatrix} \tag{14}$$

$$S = \begin{bmatrix} 0 \\ -gh\dfrac{\partial B}{\partial x} - \dfrac{\tau_b}{\rho} \\ 0 \end{bmatrix} \tag{15}$$

In order to evaluate the sediment transport discharge in the current study, the empirical relations in Table 1 are used. The Jacobian matrix of the flux in Eqs. (13-15) can be written as:

$$A(U) = \begin{bmatrix} 0 & 1 & 0 \\ c^2 - u^2 & 2u & 0 \\ \sigma & \delta & 0 \end{bmatrix} \qquad (16)$$

where for the case of the Grass formula, the coefficients in the Jacobian matrix, c, σ and δ are

obtained from $c = \sqrt{gh}$, $\delta = \dfrac{\eta}{h} A_g m_g u^{m_g - 1}$, and $\sigma = -u\delta$. As it can be seen, the Jacobian matrix is

singular and creates difficulties for any numerical solver leading to non-physical results. This

problem can be easily rectified by using the product rule and rewriting the sediment variation term,

$h\dfrac{\partial B}{\partial x}$, as shown in Eq. (17):

$$h\frac{\partial B}{\partial x} = \frac{\partial (Bh)}{\partial x} - B\frac{\partial h}{\partial x} \qquad (17)$$

Substituting Eq. (17) in Eqs. (13-15) yields a modified system represented by:

$$U = \begin{bmatrix} h \\ hu \\ B \end{bmatrix} \qquad (18)$$

$$F(U) = \begin{bmatrix} hu \\ hu^2 + 1/2gh^2 + gBh \\ \eta q_b \end{bmatrix} \qquad (19)$$

$$S = \begin{bmatrix} 0 \\ gB\dfrac{\partial h}{\partial x} - \dfrac{\tau_b}{\rho} \\ 0 \end{bmatrix} \qquad (20)$$

and now the associated Jacobian matrix becomes:

$$A(U) = \begin{bmatrix} 0 & 1 & 0 \\ g(B+h) - u^2 & 2u & gh \\ \sigma & \delta & 0 \end{bmatrix} \qquad (21)$$

which gives independent eigenvectors. The values of δ and σ for the Meyer-Peter & Müller and

Smart formulae are given in Table 2. In the case of the Meyer-Peter & Müller and Smart formulae,

the effect of a non-constant interaction parameter A_g, which is now variable in time and space,

should be considered in the eigenvector computations.

Table 2. The derivatives of bedload sediment discharges for the for the Smart (1984) and Meyer-Peter & Müller (1948) bedload sediment transport formulae for the Jacobian matrix of Eq. (21) with respect to the vector of unknowns

	Meyer-Peter & Müller	Smart
σ	$-\dfrac{28\sqrt{g}\,n_m^{\,3}\eta u^3\omega}{(G_s-1)h^{3/2}}$	$-\dfrac{4\left(\dfrac{d_{90}}{d_{30}}\right)^{0.2}\sqrt{g}\,S_0^{\,0.6}\eta u\left(-3d_m\left(G_s-1\right)h^{1/3}\theta_s+10n_m^{\,2}u^2\right)}{3(G_s-1)h^{4/3}}$
δ	$\dfrac{24\sqrt{g}\,n_m^{\,3}u^2\eta\omega}{(G_s-1)h^{3/2}}$	$\dfrac{4\left(\dfrac{d_{90}}{d_{30}}\right)^{0.2}\sqrt{g}\,S_0^{\,0.6}\eta\left(3n_m^{\,2}u^2-d_m\left(G_s-1\right)h^{1/3}\theta_s\right)}{(G_s-1)h^{4/3}}$

ω is a coefficient required for the calculation of the Jacobian matrix and can be obtained as: $\omega=\left(1-\dfrac{d_m\left(G_s-1\right)h^{1/3}\theta_c}{n_m^{\,2}u^2}\right)^{1/2}$

It should be stated that the derivatives in Table 2 contain the necessary mathematical expressions for avoiding miscalculation of eigenvalues and eigenvectors of the Jacobian matrix. The eigenvalues of a given Jacobian matrix can be calculated through the roots of a third-degree polynomial defined by:

$$P(\lambda)=\lambda^3+a_1\lambda^2+a_2\lambda+a_3 \tag{22}$$

where the coefficients a_1, a_2 and a_3 are determined by:

$$a_1=-2u \tag{23}$$
$$a_2=u^2-g(h+B+h\delta) \tag{24}$$
$$a_3=-gh\sigma \tag{25}$$

The roots of this polynomial can be computed as:

$$\lambda_1=2\sqrt{-Q}\,\cos(\tfrac{1}{3}\mu)-\tfrac{1}{3}a_1 \tag{26}$$

$$\lambda_2=2\sqrt{-Q}\,\cos(\tfrac{1}{3}(\mu+2\pi))-\tfrac{1}{3}a_1 \tag{27}$$

$$\lambda_3=2\sqrt{-Q}\,\cos(\tfrac{1}{3}(\mu+4\pi))-\tfrac{1}{3}a_1 \tag{28}$$

where Q, μ and R are obtained from:

$$Q=\frac{1}{9}\left(3a_2-a_1^2\right) \tag{29}$$

$$\mu = \arccos\left(\frac{R}{\sqrt{-Q^3}}\right) \tag{30}$$

$$R = \frac{1}{54}\left(9a_1a_2 - 27a_3 - 2a_1^3\right) \tag{31}$$

It should be noted that the polynomial in Eq. (22) gives the real eigenvalues provided that

$Q^3 + R^2 < 0$. It can be proven that for the sediment discharge formulae provided in Eq. (6), the

roots of the polynomial are always real, and hence, the system represented by Eqs. (18-20) is

strictly hyperbolic (Castro Díaz et al., 2008; Cordier et al., 2011). The corresponding eigenvectors

are calculated as:

$$r_k = \begin{bmatrix} 1 \\ \lambda_k \\ \dfrac{u^2 - g(h+B) + (\lambda_k - 2u)\lambda_k}{gh} \end{bmatrix} \qquad k = 1,2,3 \tag{32}$$

which will be used with the wave propagation algorithm as explained in the next section.

3. The wave propagation algorithm

The 1D Godunov-type wave propagation algorithm can be written as (LeVeque, 1998):

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{A}^+\Delta U_{i-1/2} + \mathcal{A}^-\Delta U_{i+1/2}\right) - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}\right) \tag{33}$$

where U_i^n is the approximation to the cell-average at time $n\Delta t$, Δt is the computational time step,

Δx is the computational space step, and U_i^{n+1} indicates the updated version for the vector of

unknowns. $\tilde{F}_{i+1/2}^n$ and $\tilde{F}_{i-1/2}^n$ are the second-order correction terms needed to achieve a high-

resolution method using total variation diminishing (TVD) limiters defined in the next Section. If

$\tilde{F}_{i\pm1/2}^n = 0$, then, the first-order Godunov method is obtained. $\mathcal{A}^+\Delta U_{i-1/2}$ and $\mathcal{A}^-\Delta U_{i+1/2}$, represent the

right-going and left-going fluctuations, respectively, and can be obtained by solving the Riemann problems at each cell interface. It should be noted that the terms $\mathcal{A}^\pm \Delta U_{i\pm 1/2}$ are independent symbols not related to the flux Jacobian, $A(U)$, in Eq. (21), and can be defined for the cell interfaces as follows:

$$\mathcal{A}^- \Delta U_{i-1/2} = \sum_{k:s_{i-1/2}<0} \xi_{k,i-1/2} \tag{34}$$

$$\mathcal{A}^+ \Delta U_{i-1/2} = \sum_{k:s_{i-1/2}>0} \xi_{k,i-1/2} \tag{35}$$

where $\xi_{k,i-1/2}$ is called the flux-wave which is computed by multiplying a particular coefficient $\beta_{k,i-1/2}$ into the relevant eigenvector $r_{k,i-1/2}$, propagating with a wave speed $s_{k,i-1/2}$, where for a given eigenvector presented in Eq. (32) is equal to λ_k. Therefore, the flux-wave takes the form of $\xi_{k,i-1/2} = \beta_{k,i-1/2} r_{k,i-/2}$ which is evaluated by the flux-wave approach explained in the subsequent section.

To improve the order of accuracy for the wave propagation algorithm, the second-order correction terms $\tilde{F}^n_{i-1/2}$ can be used. These terms are mainly determined by the flux-waves at the cell interface $x_{i-1/2}$ given by (LeVeque, 2002):

$$\tilde{F}_{i-1/2} = \frac{1}{2} \sum_{k=1}^{M_w} \text{sgn}(s_{k,i-1/2})\left(1 - \frac{\Delta t}{\Delta x}\left|s_{k,i-1/2}\right|\right)\tilde{\xi}_{k,i-1/2} \tag{36}$$

where $\tilde{\xi}_{k,i-1/2}$ is a limited version of the flux-wave, $\xi_{k,i-1/2}$, evaluated as $\tilde{\xi}_{k,i-1/2} = \Phi(\varepsilon)\xi_{k,i-1/2}$ where $\Phi(\varepsilon)$ implies the limiter function and $\varepsilon_{i-1/2} = \xi_{k,I-1/2} / \xi_{k,i-1/2}$. The index I is used to represent the upwind side at the cell interface $x_{i-1/2}$:

$$I = \begin{cases} i-1 & if & s_{k,i-1/2} > 0 \\ i+1 & if & s_{k,i-1/2} < 0 \end{cases} \tag{37}$$

Then $\Phi(\varepsilon)$ is computed using a choice of limiter (LeVeque, 2002) with the monotonized centered

limiter used (LeVeque, 2002; Mahdizadeh, 2010) in the current study:

$$\Phi(\varepsilon) = \max(0, \min((1+\varepsilon)/2, 2, 2\varepsilon)) \tag{38}$$

4. Flux-wave method for the morphodynamic system

4.1. Coupled system

The flux-wave formula for the wave propagation algorithm, originally introduced by Bale et al.

(2002), handles the source terms within the flux-differencing of adjacent finite-volume cells. This

method was later modified by Mahdizadeh et al. (2011, 2012), and here is further extended for the

solution of a 1D coupled morphodynamic system. In comparison to the 1D SWEs, an additional

discretization regarding the sediment flux should be implemented for the flux-wave computations.

The 1D flux-wave formula can be expressed as (Bale et al., 2002; LeVeque, 2002):

$$\boldsymbol{F}(\boldsymbol{U}_i) - \boldsymbol{F}(\boldsymbol{U}_{i-1}) - \boldsymbol{S}_{i-1/2}\Delta x = \sum_{k=1}^{M_w} \boldsymbol{\xi}_{k,i-1/2} \tag{39}$$

where $\boldsymbol{F}(\boldsymbol{U}_i)$ and $\boldsymbol{F}(\boldsymbol{U}_{i-1})$ are the fluxes at the right and left side of the cell interface ($i-1/2$), M_w

denotes the number of waves, which for the morphodynamic system in this work is equal to 3, and

Δx indicates the cell length. To obtain the relevant flux-waves, $\boldsymbol{\xi}_{k,i-1/2}$, the coefficients $\beta_{k,i-1/2}$ for

each cell interface ($i-1/2$) must be calculated. This can be achieved by defining the term $\Delta\phi$

and substituting it for the left-side of Eq. (39) such that:

$$\Delta\varphi = \begin{bmatrix} \Delta\varphi_1 \\ \Delta\varphi_2 \\ \Delta\varphi_3 \end{bmatrix} = \begin{bmatrix} h_i u_i - h_{i-1} u_{i-1} \\ (h_i u_i^2 + 1/2 g h_i^2 + g B_i h_i) - (h_{i-1} u_{i-1}^2 + 1/2 g h_{i-1}^2 + g B_{i-1} h_{i-1}) - g(\frac{B_i + B_{i-1}}{2})(h_i - h_{i-1}) + \frac{\tau_{b(i)} + \tau_{b(i-1)}}{2\rho} \\ \upsilon_i q_{bi} - \upsilon_{i-1} q_{b(i-1)} \end{bmatrix} \quad (40)$$

with these definitions, and substituting the eigenvectors from Eq. (32) into the flux-wave formula, the resulting equation becomes:

$$\begin{bmatrix} 1 & 1 & 1 \\ \tilde{\lambda}_1 & \tilde{\lambda}_2 & \tilde{\lambda}_3 \\ \tilde{s}_1 & \tilde{s}_2 & \tilde{s}_3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \Delta\varphi_1 \\ \Delta\varphi_2 \\ \Delta\varphi_3 \end{bmatrix} \quad (41)$$

where $\tilde{s}_k$ is given by:

$$\tilde{s}_k = \frac{\tilde{u}^2 - g(\tilde{h} + B) + (\tilde{\lambda}_k - 2\tilde{u})\tilde{\lambda}_k}{g\tilde{h}} \quad (42)$$

variables $\tilde{u}$ and $\tilde{h}$ are calculated on the basis of the wave speed formulas as explained in (Mahdizadeh et al., 2011) and the eigenvalues, $\tilde{\lambda}_k$, are evaluated through Eqs. (26-28), where the terms u and h can be substituted by $\tilde{u}$ and $\tilde{h}$, respectively. In the case of a wet-state, $\tilde{u}$ and $\tilde{h}$ can be simply evaluated using the Roe formula (LeVeque, 2002; Roe, 1981):

$$\tilde{u} = \frac{\sqrt{h_{i-1}} u_{i-1} + \sqrt{h_i} u_i}{\sqrt{h_{i-1}} + \sqrt{h_i}} \text{ and } \tilde{h} = \frac{1}{2}(h_{i-1} + h_i). \quad (43)$$

and hence, δ can be re-written as:

$$\delta = \frac{A_g \upsilon(\sqrt{h_{i-1}} + \sqrt{h_i})}{\sqrt{h_{i-1}} u_{i-1} + \sqrt{h_i} u_i} \sum_{k=0}^{m_g - 1} u_i^k u_{i-1}^{m_g - k - 1} \quad (44)$$

For calculating the β_k coefficients, at each time step, the linear system presented in Eq. (41) should be solved, and in this study the Lower-Upper (LU) decomposition method with partial pivoting (Press et al., 1992) is utilized. These coefficients are later used to calculate the flux-waves, $\xi_{k,i-1/2}$

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, eventually required for evaluating the right- and left-going fluctuations, $\mathcal{A}^{\pm}\Delta U_{i\pm1/2}$ as expressed in Eqs. (34-35). Therefore, in summary, the modified coupled numerical solver applied in the current study, is based on a Godunov-type wave propagation algorithm where the fluxes comprising sediment discharge are treated within the flux-wave approach. Additionally, a new choice of source term decomposition is proposed which permits the flux term, $F(U)$, to be written in a well-balanced form.

To ensure the method's stability, the Courant-Friedrichs-Lewy (CFL) condition (Courant et al., 1928), which has relatively similar stability conditions to the SWEs, should be applied. The major difficulty is that for some numerical simulations, the value of $\tilde{\lambda}$ is greater than the characteristic wave speeds created by the SWEs. This problem can be fixed by defining a new choice of CFL number:

$$\text{CFL} = \frac{\max\left(\tilde{\lambda}\right)\Delta t}{\Delta x} \tag{45}$$

where $\tilde{\lambda} = \max\left[\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3\right]^T$. It will be shown in Section 5 that the numerical solver is not sensitive to the choice of the CFL number, and an approximate CFL number close to 1 can be used without impacting the solution's accuracy.

4.2. Splitting system

The 1D decoupled form of the morphodynamic system including the shallow water and the Exner equations can be expressed as:

$$U_t + F(U)_x = S \tag{46}$$
$$B_t + v\left(q_b\right)_x = 0 \tag{47}$$

where

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$$U = \begin{bmatrix} u \\ hu \end{bmatrix} \tag{48}$$

$$F(U) = \begin{bmatrix} hu \\ hu^2 + 1/2gh^2 \end{bmatrix} \tag{49}$$

$$S = \begin{bmatrix} 0 \\ -gh\dfrac{\partial B}{\partial x} - \dfrac{\tau_b}{\rho} \end{bmatrix} \tag{50}$$

As previously mentioned, the splitting solution solves the SWEs and the Exner equation in a separate manner. This means that the bedload sediment transport formula is first solved for a particular time step, and the obtained bed profile is inserted as a fixed source term into the momentum equation for the shallow water flow. The SWEs are then solved using the modified flux-wave approach given in (Mahdizadeh et al. (2011) for both wet and dry states. To evaluate the sediment discharge equation, the following wave decomposition should be done:

$$F(U_i) - F(U_{i-1}) = v\left(q_{b(i)} - q_{b(i-1)}\right) = \xi_{i-1/2} \tag{51}$$

where $q_{b(i)}$ and $q_{b(i-1)}$ are the amount of sediment discharge for cells i and $i-1$ computed using the bedload sediment discharge equations listed in Table 1. In order to incorporate sediment transport into the flux-differencing approach, the depth-average velocity, u, in the bedload sediment transport formula should be substituted by $u = q(z_s - B)^{-1}$, where $z_s = h + B$ and q is the constant water discharge per unit width (steady-state condition). Thus, the bedload-sediment transport equation can be rewritten as:

$$q_b = A_g q^{m_g} (z_s - B)^{-m_g} \tag{52}$$

and the wave speed can be computed as:

$$s_{i-1/2} = \frac{\partial q_b}{\partial B} = A_g m_g q^{m_g} (z_s - B)^{-m_g - 1} \tag{53}$$

By placing a modified form of sediment discharge into Eq. (51), the flux-wave, $\check{\zeta}_{i-1/2}$, can be simply obtained, and the bed variation update for the next time step can be computed as:

$$B_i^{n+1} = B_i^n - \frac{\Delta t}{\Delta x}(\mathcal{A}^+ \Delta U_{i-1/2} + \mathcal{A}^- \Delta U_{i+1/2}) - \frac{\Delta t}{\Delta x}(\tilde{F}_{i+1/2}^n - \tilde{F}_{i-1/2}^n) \tag{54}$$

As discussed in the Introduction Section, any powerful shallow water solver, without any limitation, can be appropriately applied for hydrodynamic calculations. However, as will be shown within the numerical results section, the splitting technique achieved based on Eq. (52) only is valid for weak or rather mild sediment and water interactions, where the total depth of fluid plus sediment, z_s, can be assumed to remain constant. In conventional methods, the wave speed is required for calculating the bedload sediment flux, and is normally obtained by differentiating the sediment discharge, q_b, with respect to B according to Eq. (53). For transcritical and supercritical flows, the interaction between the sediment discharge and the free-surface flow is rather considerable, and the assumption of a constant total depth does not stand. A major difficulty that arises here is that the bedload sediment discharge equations are independent of the bed profile variation, and the derivative $\dfrac{\partial q_b}{\partial B}$ which gives the wave speed, $s_{i-1/2}$, is no longer valid. de Vries (1973) has suggested the following wave speed for each finite volume cell interface for the case of transcritical or supercritical regimes:

$$s_{i-1/2} = \frac{gu\delta}{c^2 - u^2} \tag{55}$$

where c and δ have been defined in Eq (16). Although, the defined formulation has been proposed for supercritical flow conditions, as is shown later, it cannot be accurately applied to near transcritical zones as it leads to non-physical results. To overcome this problem, and to develop a

more generalized splitting solver suitable for use in both transcritical and supercritical flow, a new decomposition of the variable waves, $W_{k,i-1/2}$, and flux-waves, $\xi_{k,i-1/2}$ initially introduced in Bale et al. (2002) for the solution of gas dynamics problems, is utilized:

$$\begin{bmatrix} U_i - U_{i-1} \\ F(U_i) - F(U_{i-1}) \end{bmatrix} = \sum_{i=1}^{2m} \begin{bmatrix} W_{k,i-1/2} \\ \xi_{k,i-1/2} \end{bmatrix} \tag{56}$$

where $F(U_i)$ and $F(U_{i-1})$ are the fluxes at the right and left side of cell interface, $x_{i-1/2}$, and U_i and U_{i-1} denote the vector of unknowns at the same locations. Satisfying this equation has some useful properties, since its solution comprises both waves and flux differencing, which ultimately leads to a unique wave and extraction of a wave speed independent of the derivative $\dfrac{\partial q_b}{\partial B}$. For the wave propagation algorithm, each wave can be related to the flux-wave through the flowing equation (LeVeque 1998, 2002):

$$\xi_{i-1/2} = s_{i-1/2} W_{i-1/2} \tag{57}$$

where $s_{i-1/2}$ again represents the wave speed. To solve the transport equation, Eq. (56) is rewritten as:

$$\begin{bmatrix} B_i - B_{i-1} \\ q_{bi} - q_{b(i-1)} \end{bmatrix} = \begin{bmatrix} W_{i-1/2} \\ \xi_{i-1/2} \end{bmatrix} \tag{58}$$

where B_i and B_{i-1} are values of sediment bed profiles at cells i-1 and i, respectively, and $q_{b(i)}$ and $q_{b(i-1)}$ are the bedload sediment discharges computed by the equations listed in Table 1. To evaluate waves within each computational finite volume cell interface, only the differences between adjacent bed profile cells are required. The flux-wave is then obtained in a similar manner

to the approach introduced in Section 3.1. Therefore, on the basis of Eq. (57) the wave speed at cell interface $x_{i-1/2}$ can be calculated as:

$$s_{i-1/2} = \xi_{1,i-1/2} / W_{1,i-1/2} \qquad (59)$$

In cases where the differences between the bed profile of adjacent cells tend to zero, i.e. $B_i - B_{i-1} = 0$, it can be assumed that the interaction between the sediment and water surface is rather weak and Eq. (52) can be appropriately used. Additionally, and in particular for the sediment transport equation, the flux-wave approach directly incorporates the sediment fluxes into the flux-wave computation which is independent of the bed differencing. The effect of bedload sediment only emerges in the wave speed evaluation required to determine the flux-wave direction.

5. Numerical results

To validate the suitability of the proposed modified flux-wave method using both modified coupled flux-wave (CFW) and modified splitting flux-wave (SFW) techniques, different test cases were examined. First, the well-balanced property of the proposed method was examined. Second, the method was used to simulate parabolic bedload transportation over a flat bed. Next, dam-break waves over a step type sediment bed were modeled. Then, the movement of a sediment hump with a defined transcritical initial condition was considered. Finally, a dam-failure by overtopping was simulated. The morphodynamic model was solved using an in-house FORTRAN code on an Intel Core (i7-4790) 3.6 GHz processor with 16GB of RAM. It should be stated that for all test-cases reported in this Section, a high-resolution wave propagation algorithm with the monotonized central (MC) limiter was used.

5.1. Quiescent water over smooth and discontinuous bed profile

The shallow water equations solvers with variable bed capabilities should mainly satisfy the well-balanced or C-property to provide a balance between the effects of source terms and flux-gradient for steady-state problems. In order to verify the C-property of the modified flux-wave formula, a still water condition was created over two different bed profiles given by the following equations:

$$B(x,0) = 5\exp(-\frac{2}{5}(x-5)^2) \tag{60}$$

and

$$B(x,0) = \begin{cases} 4 & \text{if} \quad 4 \leq x \leq 8 \\ 0 & \text{Otherwise} \end{cases} \tag{61}$$

where the entire computational domain was set between $0 \leq x \leq 10$. As for the initial conditions of both test cases, the water surface and discharge were defined as $z_s = h + B = 10$ and $hu = 0$.

To examine the capability of the defined flux-wave formula in maintaining still water conditions, the computation was done until time $t = 10$ s. The Euclidean norm calculated between the obtained numerical results using 3000 computational cells, and the quiescent initial condition for both coupled and splitting techniques are listed in Table 3. These results clearly show that still water initial conditions have been preserved during the simulations and confirm that the C-property condition is satisfied.

Table 3. Verification of well-balanced property using the Euclidean norm for both the CFW and SFW approaches.

	$\|e\|_2$	Smooth bed Eq. (60)	Discontinuous bed Eq.(61)
CFW	z_s	5.51366E-016	0
	hu	2.298789E-016	1.23354E-021
SFW	z_s	2.512147E-016	0
	hu	6.594368 E-016	6.501008E-016

5.2. Parabolic bed surface transportation

The purpose of using this test case was to compare the proposed morphodynamic solver with the exact solution proposed by Hudson and Sweby (2005) using the Grass (1981) formula with a

constant A_g value. In this test case, a channel of 1000 m length was considered, and the water depth and discharge were set at constant values of 10 m and 10 m³/s, respectively. The initial condition for the sediment layer profile was defined by:

$$B(x,0) = \begin{cases} \widehat{B}\sin^2\left(\dfrac{\pi(x-300)}{200}\right) & \text{if} \quad 300 \le x \le 500 \\ 0 & \text{Otherwise} \end{cases} \tag{62}$$

The exact solution for the defined test case was achieved by assuming a constant water depth and sediment discharge for the entire computational domain, and calculating x_0 from the following equation (Hudson, 2001):

$$x = x_0 + A_g \upsilon m_g q^{m_g} t \begin{cases} \left(z_s - \sin^2\left(\dfrac{\pi(x_0-300)}{200}\right)\right)^{-(m_g+1)} & \text{if} \quad 300 \le x_0 \le 500 \\ z_s^{-(m_g+1)} & \text{Otherwise} \end{cases} \tag{63}$$

Equation (63) cannot be stated in terms of x_0 and the exact solution of the sediment layer $B(x,t)$ was calculated by substituting x_0 into Eq. (62). This solution is only valid until the time the characteristics of the hyperbolic first cross, which can be computed by equating dx/dx_0 to zero. By setting the values $A_g = 0.001$, $m_g = 3$, $\eta = 0.4$, and $\widehat{B} = 1$, the calculated threshold time was found to be $t = 238079$ s (Hudson, 2001).

Figure 1 shows the numerical results obtained with both first-order and high resolution CFW approaches along with the exact solution for the bed profile and velocity. The results were evaluated at times $t = 238079$ s and 5339 s (which correspond to $\widehat{B}=1$ and 5, respectively) with 200 computational cells. As can be seen, a very good agreement is achieved between the CFW

results and the exact solution for both values of $\hat{B}$, in particular at the top of the hump motion where a main discrepancy is observed in the predictions of the numerical methods developed based on non-conservative fluxes (see Castro Díaz et al., 2008). However, the first-order CFW gives rather diffusive results in particular for the longer time ($t = 238079$ s) which verifies that the first-order scheme is not able to accurately predict the bedload sediment movement even for small interactions which occur between the free-surface and bedload sediment as time increases.

The numerical results of the SFW were evaluated with the same computational cells (Fig. 2). It is observed that the numerical solution nearly coincides with the exact solution for both bed and velocity, and similar results to those of the coupled approach were obtained, and again, a rather considerable discrepancy was observed between the first-order and the exact solutions at time $t = 238079$ s. In terms of CPU time, the splitting method took only 16 s to reach the time $t = 238079$ s, whereas for the coupled method, the elapsed CPU time was much larger (68 s). This is mainly attributed to the time required for solving the linear system in Eq. (41) which is implemented at each time step for the coupled technique. Note that for the presented test case no difference was observed between the splitting methods performed based on either Eq. (52) or Eq. (58) (SFW) and both methods converged to the same solution.

5.3. 1D dam-break test case

The capability of the defined numerical solver for modeling dam-break waves was tested using experimental data collected at UCL (Spinewine & Zech, 2007). The experiments were done in a 6 m long channel where a middle gate was used to cause dam-break conditions. The channel bed was covered with uniform sand with $d_{50} = 1.82mm$, density $\rho_s = 2683 kg\,m^{-3}$, porosity $\eta = 0.47$, friction angle of $\varphi = 30$, and a Manning's coefficient of $n_m = 0.0165$ s.m$^{-1/3}$ was considered. The

initial conditions for the dam-break test-case are listed in Table 4. This test-case comprises a left moving rarefaction and right moving shock waves along with a hydraulic jump that appears at the location of the gate when it is removed. In all simulations, the cell space in the *x*-direction was chosen to be $\Delta x = 0.1$ and the Smart (1984) sediment discharge formula was used. The ability to incorporate bed slope variation into the bedload sediment discharge calculations and good agreement with experimental data (Juez et al., 2014; Murillo & García-Navarro, 2010), were the main reasons for choosing the Smart (1984) sediment discharge formula.

Fig. 3 compares the free-surface numerical results of the CFW and SFW approaches with measured experimental data along with the coupled Jacobian method (CJM) provided by Murrillo and García-Navarro (2010) for the given test case. As can be seen, both the coupled (CFW) and splitting schemes (SFW) developed based on the flux-wave method can accurately model the front shock and the left moving rarefaction waves at all times, the simulation results approximately coincide with the experimental results and CJM approach and no obvious discrepancy is observed between the CFW and SFW approaches.

Table 4. Initial condition for the 1D dam-break test case.

Test	h_L	h_R	B_L	B_R
B	0.25	0.1	0.1	0.0

Figure. 4 illustrates the bed elevation variations calculated using the CFW and SFW methods for test case B along with the CJM results. As can be observed, the CFW and SFW approaches produce rather similar results to the CJM method. However, a discrepancy between the SFW results and the experimental measurements is still observed at the place of shock for the bed elevation highlighted in Fig. 5. As can be seen in this figure, this difference is less obvious for the CFW and

CJM methods developed based upon the coupled strategy, in particular at latter times. This can be attributed to the choice of the Exner equation which is considered as a capacity model. In terms of CPU time, the coupled method took 0.0156 s whilst the splitting technique required 0.0127 s to reach time $t = 1.5$ s.

5.4. Sediment profile hump propagation with initial subcritical flow

A third test case, borrowed from Cordier et al. (2011), was used to investigate the ability of the numerical solver in modeling the sediment hump propagation with different values of A_g which results in different sediment and water surface interaction regimes. The initial condition for the bed profile, discharge and depth-average velocity were chosen as:

$$\begin{cases} hu(x,0) = 0.5, & (64) \\ B(x,0) = 0.1 + 0.1exp^{-(x-5)^2} & (65) \\ \dfrac{u^2}{2} + g(h+B) = 6.386, & (66) \end{cases}$$

Fig. 6 shows the initial conditions for the water surface and bed profiles, computed by solving a nonlinear equation for the water depth, h. To produce a rather mild interaction for the initial conditions, first, the bedload sediment discharge formula Eq. (6) with $A_g = 0.005$ was used. The numerical modeling results of both coupled and splitting solutions, using 200 numerical cells are shown in Fig. 7 at time $t = 10$ s. As can be seen, both methods provide approximately identical results in particular for the free-surface and velocity, and the only difference is observed at the shock front location for the bed profile where the coupled method gives a sharper shock front as shown in Fig. 8. The CPU time for the splitting approach was 0.056s, which again is less than that for the coupled method (0.0936s).

Figure 9 depicts the comparison between the first and second-order SFW and CFW schemes using the method presented by Cordier et al. (2011). As can be observed, Cordier's method completely coincides with the first-order wave propagation algorithm for both coupled and splitting solutions. Fig. 10 shows the results for both techniques with a different A_g value set at 0.07. This A_g value produces a rather strong interaction between the bed elevation and free-surface flow, but as shown in Fig. 10b, the new modified splitting technique (SFW) provided in Eq. (58) gives approximately similar results to that of the simultaneous solution. However, there are still some differences between the two approaches, particularly at the start of the sediment bed profile as shown in Fig. 11a. It should be noted that choosing a CFL number greater than 0.3 for the splitting method (SFW) results in much more instability, and finally, causes divergence in the solutions. Therefore, to run all the test cases with a fixed CFL number, this number has been chosen as the reference CFL for the SFW approach. These findings also verify that the defined splitting method is sensitive to the choice of the CFL number when modeling strong interactions or supercritical flow and requires a lower CFL number than the one necessary for propagation over a dry-state (Section 5.5).

Figure 12 shows the comparison among the sediment profile hump propagation simulations by the SFW and CFW methods and the solutions provided by Cordier et al. (2011). As can be seen, the first-order CFW gives very good agreement with the coupled solution presented in Cordier et al. (2011), although being more diffusive than the CFW (second-order scheme). For the splitting solution, as shown in Fig. 11b, the SFW achieves better agreement with the coupled solution than that for Cordier et al. (2011).

Figure 13 shows the splitting results obtained using the sediment discharge formula provided in Eq. (52). As can be observed, compared to the coupled solution, this choice of wave speed causes

a significant discrepancy which is mainly due to the fact that the solution cannot be accurately implemented for flow regimes of near supercritical conditions. The CPU time for the current splitting method was 0.0312 s whilst the coupled method took only 0.01 s.

5.5. Sediment hump propagation with initial transcritical flow without shock

This test case was considered to investigate the validity of the proposed method in modeling sediment and surface interactions under transcritical conditions without a shock. In order to create the transcritical regime the following conditions originally defined in Cordier et al. (2011) were used:

$$
\begin{cases}
hu(x,0) = 0.6 & (67) \\
B(x,0) = 0.1 + 0.1exp^{-(x-5)^2} & (68) \\
h(x,0) + B(x,0) = 0.4 & (69)
\end{cases}
$$

The upstream boundary imposed a discharge equal to $hu = 0.6 \, m^2/s$, whilst the downstream depth was fixed at $h = 0.6$ m in the case of supercritical flow; and no boundary condition was required for the supercritical flow. With these conditions, the solver was run with $A_g = 0$ until the steady-state solution was achieved.

The global relative error for the steady-state solution can be defined as :

$$
R = \sqrt{\sum_i \left(\frac{h_i^n - h_i^{n-1}}{h_i^n} \right)} \tag{70}
$$

where h^n and h^{n-1} are the fluid depth at time levels n and $n-1$, respectively. The regime is considered to be steady-state when the value of R reaches approximately zero. Fig. 14 shows the steady-state results used as an initial condition for the morphodynamic solver.

As can be observed, using 200 cells results in the discharge perfectly matching the theoretical

discharge which is equal to $hu = 0.6\,m^2/s$ according to Eq. (70). To compare the models under

a transcritical regime, a small $A_g = 0.0005$ was chosen for both solutions, since the considered

steady-state initial condition dictates Froude numbers greater than 1, and the solution is no longer

affected by the choice of A_g. Fig. 15 shows the results of the flux-wave method computed using

the CFW and SFW techniques at time $t = 15$ s. As can be observed, both solutions provide nearly

identical results which verifies that the SFW method with a new choice of wave speeds, provided

in Eq. (58), can be efficiently applied for the transcritical flow regime which contains no shocks.

The CPU running time for the modified splitting method was 0.327 s including the steady-state

process and the coupled method required 0.63 s of CPU time.

5.6. Dam failure caused by overtopping

This test case was first introduced by Tingsanchali and Chinnarasri (2001) and was used here to

evaluate the wet/dry front modeling capability of the defined SFW and CFW solvers for problems

where morphological changes occur over a dry state. The tests were performed in a channel 35 m

long, 1 m deep, and 1 m wide, a dam with a height of 0.8 m and crest width of 0.3 m was

considered. The downstream slope of the dam was set at $1V:3H$ whilst the downstream slope was

varied but initially fixed at $1V:2.5H$. It should be stated that the downstream face of the dam was

also covered with a material type called Sand I with Manning's coefficient equal to $n_m = 0.018$,

$d_{30} = 0.52\,mm$, the mean grain size $d_m = 1.13\,mm$, and $d_{90} = 3.8\,mm$ corresponding to the conditions

of test *C*-2 in the original paper (Tingsanchali & Chinnarasri, 2001). As a boundary condition, an

inflow discharge equal to 1.23 L/s was imposed onto the left boundary and an extrapolation

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boundary condition was utilized for the left boundary. Both results of both solvers were calculated with a cell length of 0.01 m and CFL = 0.5.

Figure 16 shows the numerical results for the bed profile along with the overtopping discharge and water surface level at the crest obtained using the SFW and CFW solvers with choices of both the Meyer–Peter and Müller (1948) and Smart (1984) formulae at time t = 30 and 60 s, respectively. These results have also been compared with the non-capacity turbid model provided in Wu (2008). As can be seen, both methods with the choice of the Smart (1984) sediment discharge formula can accurately follow the experimental data in particular at time t = 60 s, rather same results were obtained from the two solvers. This indicates that both approaches can precisely model a rapidly varying flow over a dry and erodible bed. For the Meyer-Peter and Müller (1948) bedload discharge formula, some discrepancy is seen, in particular at the dam crest where the overtopping initiated. This may be due to fact that in the Meyer-Peter and Müller (1948) formula the bed slope is not incorporated into the sediment discharge computations. In Fig. 16c the relevant water reservoir level is depicted which demonstrates a good agreement between the model outputs and measured data in particular for the Smart (1984) formula. These results are also in good agreement with the Turbid flow model. Fig. 16d exhibits the over-topping discharge computed by the CFW and SFW approaches along with the measured data after 120 s just upstream of the breach. It is evident, the prediction of discharge at the location of the dam crest using even the Smart (1984) formula is far from the experimental measurements for both CFW and SFW approaches. This inaccurate computation of the bedload sediment discharge can be revised using a non-capacity model. In contrast, the Turbid model developed based upon a non-capacity formulation accurately follows experimental measurements.

6. Conclusions

In the current paper, a new numerical scheme based on a modified flux-wave method is presented for solving 1D bedload morphodynamic systems. The proposed approach includes both coupled and splitting solutions developed using a wave propagation algorithm. The modified coupled solution (CFW) solves a fully coupled system in a well-balanced form using a less sophisticated approach compared to other novel coupled morphodynamic solvers, whilst preserving computational accuracy. For the modified splitting approach (SFW), a new decomposition of wave speed was defined and shown to significantly improve the effectiveness of the method for modeling strong interaction regimes based on an unsteady formulation. The accurate performance of the numerical solver was demonstrated by comparison with exact solutions and alternative simulations for different test cases taken from the literature. First the parabolic bed surface transportation over a flatbed was simulated using both the CFW and SFW schemes and very good agreements were obtained. For the dam break test case, the CFW and SFW approaches using the Smart (1984) bedload sediment discharge formula provided very close results with the observed measurements for both wet and dry states. The defined CFW and SFW approaches were then used to simulate mild and strong free-surface and bedload interactions, and both models were found to produce nearly identical results. Finally, the defined solvers were tested for modeling dam failure caused by overtopping with choices of Smart (1984) and Meyer-Peter and Müller (1948) sediment discharge formule, and again, good agreement was achieved with experimental data in particular for the Smart (1984) formula.

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